ANSWERS FOR MIDTERM 2

**Problem 1:** Differentiate both sides of the equation with respect to \( x \):

\[
\frac{d}{dx}(y^3 + yx^2 + x^2 - 3y^2) = \frac{d}{dx}(0) = 0
\]

Evaluate \( \frac{d}{dx}(y^3 + yx^2 + x^2 - 3y^2) \), by applying the product rule and the chain rule as appropriate:

\[
\frac{d}{dx}(y^3 + yx^2 + x^2 - 3y^2) = 3y^2 \frac{dy}{dx} + y \cdot 2x + x^2 \frac{dy}{dx} + 2x - 6y \frac{dy}{dx}
\]

\[
= (3y^2 + x^2 - 6y) \frac{dy}{dx} + 2xy + 2x
\]

Now plug in the given values of \( x \) and \( y \), namely \( x = 1 \) and \( y = 1 \), set the above expression equal to zero (because \( \frac{d}{dx}(0) \) is zero), and solve:

\[
(3(1)^2 + 1^2 - 6(1)) \frac{dy}{dx} + 2(1)(1) + 2(1) = 0
\]

\[
-2 \frac{dy}{dx} + 4 = 0
\]

\[
-2 \frac{dy}{dx} = -4
\]

\[
\frac{dy}{dx} = 2
\]

**Problem 2:** The profit is revenue minus cost,

\[
P(x) = R(x) - C(x) = [20x - 0.5x^2] - [10x + 7]
\]

\[
P(x) = 10x - 0.5x^2 - 7
\]

We want to know where the derivative of \( P(x) \) is zero or does not exist:

\[
P'(x) = 10 - x = 0
\]

\[
x = 10
\]

Either the first- or the second-derivative test will show that this is a local maximum. Because it is the only critical point, it must in fact be a global maximum.
**Problem 3:** This is a basic law:

\[ e^a e^b = e^{a+b} \]

**Problem 4:** Writing \( P \) for profit, we have profit = revenue minus cost,

\[ P = R - C = (34x - 0.5x^2) - (8x + 15) = -0.5x^2 + 26x - 15 \]

We are asked to find \( \frac{dP}{dt} \) when \( x = 10 \) and \( \frac{dx}{dt} = 7 \). Differentiate the above expression for \( P \), using the chain rule as appropriate:

\[
\frac{dP}{dt} = \frac{d}{dt}(-0.5x^2 + 26x - 15)
\]

\[ = -x \frac{dx}{dt} + 26 \frac{dx}{dt} - 0 \]

\[ = (26 - x) \frac{dx}{dt} \]

Plugging in \( x = 10 \) and \( \frac{dx}{dt} = 7 \), we get

\[
\frac{dP}{dt} = (26 - 10)7 = 16 \cdot 7 = 112
\]

**Problem 5:** Consider \( 5x^2e^{3x} \) as the product of \( 5x^2 \) and \( e^{3x} \) and use the product rule:

\[
\frac{d}{dx}(5x^2e^{3x}) = 5x^2 \frac{d}{dx}e^{3x} + e^{3x} \frac{d}{dx}(5x^2)
\]

\[ = 5x^2(3e^{3x}) + e^{3x}(10x) \]

\[ = 5xe^{3x}(3x^2 + 2) \]

(Note that we used the chain rule to compute \( \frac{d}{dx}(e^{3x}) = 3e^{3x} \).)
Problem 6:

\[
\frac{x^2}{(x + 2)(x - 3)} = \frac{x^2}{x^2 - x - 6} = \frac{\frac{1}{x^2} \cdot x^2}{\frac{1}{x^2} \cdot x^2 - \frac{1}{x^2} \cdot x - \frac{1}{x^2} \cdot 6} = \frac{1}{1 - \frac{1}{x} - \frac{6}{x^2}}
\]

So

\[
\lim_{x \to \infty} \frac{x^2}{(x + 2)(x - 3)} = \frac{1}{1 - 0 - 0} = 1
\]

Similarly

\[
\lim_{x \to -\infty} \frac{x^2}{(x + 2)(x - 3)} = 1
\]

Problem 7: Because the limit of the function as \(x \to \infty\) and as \(x \to -\infty\) is 1, the graph has a horizontal asymptote at \(y = 1\).

Problem 8: Vertical asymptotes of a rational function occur at \(x\) values that make the denominator equal zero, after the function has been simplified to cancel all common factors from numerator and denominator. The function we’re considering, \(\frac{x^2}{(x + 2)(x - 3)}\), does not have any common factors between numerator and denominator, so there are vertical asymptotes wherever \((x + 2)(x - 3) = 0\); that is, at \(x = -2\) and \(x = 3\).

Problem 9: The vertical asymptotes were computed in Problem 8 as \(x = -2\) and \(x = 3\), so we need the limit as \(x\) approaches \(-2\) from the left and the right, and the limit as \(x\) approaches \(3\) from the left and the right.
For $x$ slightly to the left of $-2$, $x + 2$ and $x - 3$ are both negative, and $x^2$ is always positive (or zero), so \( \frac{x^2}{(x+2)(x-3)} \) is positive. Thus:

\[
\lim_{x \to -2^-} \frac{x^2}{(x+2)(x-3)} = +\infty
\]

For $x$ slightly to the right of $-2$, $x + 2$ is positive but $x - 3$ is still negative, so \( \frac{x^2}{(x+2)(x-3)} \) is negative and

\[
\lim_{x \to -2^+} \frac{x^2}{(x+2)(x-3)} = -\infty
\]

For $x$ slightly to the left of $3$, $x + 2$ is positive but $x - 3$ is still negative, so \( \frac{x^2}{(x+2)(x-3)} \) is negative and

\[
\lim_{x \to 3^-} \frac{x^2}{(x+2)(x-3)} = -\infty
\]

For $x$ slightly to the right of $3$, $x + 2$ and $x - 3$ are both positive, so \( \frac{x^2}{(x+2)(x-3)} \) is positive and

\[
\lim_{x \to 3^+} \frac{x^2}{(x+2)(x-3)} = +\infty
\]

**Problem 10:** Use the quotient rule:

\[
\frac{d}{dx} \left( \frac{x^2}{x^2 - x - 6} \right) = \frac{(x^2 - x - 6) \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(x^2 - x - 6)}{(x^2 - x - 6)^2}
\]

\[
= \frac{(x^2 - x - 6)(2x) - x^2(2 - 1)}{(x^2 - x - 6)^2}
\]

\[
= \frac{-x^2 - 12x}{(x^2 - x - 6)^2}
\]

\[
= \frac{-x(x + 12)}{(x^2 - x - 6)^2}
\]

**Problem 11:** The critical points are where the first derivative either equals zero or does not exist. The first derivative was computed in Problem 10 as

\[
\frac{-x(x + 12)}{(x^2 - x - 6)^2}
\]
This will equal zero when the numerator equals zero:

\[-x(x + 12) = 0\]

\[x = 0 \text{ or } x = -12\]

The other possible critical points are where the first derivative does not exist because the denominator is zero:

\[(x^2 - x - 6)^2 = 0\]

\[x^2 - x - 6 = 0\]

\[(x + 2)(x - 3) = 0\]

\[x = -2 \text{ or } x = 3\]

However these last two points are not critical points by the definition in the book, because the original function is not defined there. Thus the answer is \(x = 0\) and \(x = -12\).

Note however that you will have to take the values \(x = -2\) and \(x = 3\) into account when dividing up the real line into intervals on which the function is increasing or decreasing.

**Problem 12:** The critical points, and points at which the original function is not defined, are \(x = -12\), \(x = -2\), \(x = 0\), and \(x = 3\). So we need to find the sign of the first derivative,

\[\frac{-x(x + 12)}{(x^2 - x - 6)^2}\]

on the intervals \((-\infty, -12)\), \((-12, -2)\), \((-2, 0)\), \((0, 3)\), and \((3, \infty)\).

The denominator \((x^2 - x - 6)^2\) is always positive (or zero), so the sign of the first derivative is determined by the signs of \(-x\) and \(x + 12\). If these two are both positive or both negative, then the first derivative is positive and the function is increasing; if one is positive and the other negative, then the first derivative is negative and the function is decreasing. Summarizing,

\[
\begin{array}{c|c|c|c|c}
(-\infty, -12) & (-12, -2) & (-2, 0) & (0, 3) & (3, \infty) \\
\text{decreasing} & \text{increasing} & \text{increasing} & \text{decreasing} & \text{decreasing}
\end{array}
\]

That is, \(f(x)\) is increasing on \((-12, -2) \cup (-2, 0)\) and decreasing on \((-\infty, -12) \cup (0, 3) \cup (3, \infty)\).
Problem 13:

\[ f(-12) = \frac{(-12)^2}{(-12 + 2)(-12 - 3)} = \frac{144}{(-10)(-15)} = \frac{144}{150} = 0.96 \]

The purpose of including problem 13 was to find out that the value of \( f \) at the critical point (local minimum) \( x = -12 \) was 0.96, slightly less than the horizontal asymptote at \( y = 1 \).

Problem 14: Graph appears on final page. Note the following:

- There’s a horizontal asymptote in both directions at \( y = 1 \), as computed in Problem 7.
- There are vertical asymptotes at \( x = -2 \) and \( x = 3 \), as in Problem 8, and the direction in which the curve blows up at each vertical asymptote is as computed in Problem 9.
- The intervals on which the function is increasing/decreasing are as computed in Problem 12, although you can’t really see from this picture that the function decreases to a local minimum at \( x = -12 \) and then begins to increase again.
```math
In[1]:= Plot[x^2/((x + 2)(x - 3)), {x, -15, 10}]
```

```
```

```
In[1]:= Plot[x^2/((x + 2)(x - 3)), {x, -15, 10}]
```

```
```