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INFINITE ITERATED FUNCTION SYSTEMS: THEORY AND APPLICATIONS

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ABSTRACT. In this paper we exposit the theory of infinite iterated function systems consisting of conformal maps. We indicate some of the new phenomena which appear in these systems in contrast to the known behaviour of the limit sets of finite iterated function systems consisting of similarity maps. We also indicate the methods by which the Hausdorff, packing and Minkowski or box counting dimension and measures can be studied. We give several examples involving sets of continued fractions.

Over the past 25 years, the fundamental properties of the limit set generated by a finite iterated function system consisting of similarity maps have been determined. Certainly, Mandelbrot's books and papers has stimulated and directed tremendous interest in this subject. Perhaps Hutchinson's 1981 paper, [Hu], may be considered a starting point for the formal development of the theory although there are several other earlier papers including the seminal 1946 paper of P.A.P. Moran [Mo]. The purpose of this paper is to exposit a generalization of these finite systems, conformal iterated function systems. The two main theoretical aspects of this generalization are that the system may consist of infinitely many maps and that the maps need not be similarity maps but only conformal. We will indicate both the variety of new phenomena that arise in this setting and the main techniques used to analyze various natural geometric measures and dimensions defined on the limit sets of such systems. Finally, we will also give several examples, including various sets of complex and real standard continued fractions which can be readily dealt with within this framework. The examination and theoretical development of limit sets associated with infinite systems is a fairly recent development . Finite conformal systems have been studied in several contexts by several authors including Patterson [Pat], Sullivan [Su1,Su2], Bedford [Be] and Urbański and his collaborators where conformal measure is used as a tool to study the geometry of various objects arising in dynamics(see the references in [MU]). Infinite systems of similarity maps, φ_i, with reduction ratios r_i, were studied by Mauldin

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and Williams in [MW]. We showed in this 1986 paper that there is a self similar measure, m, satisfying $m = \sum_{i=1}^{\infty} r_i^\alpha m \circ \varphi_i^{-1}$, provided $\sum_{i=1}^{\infty} r_i^\alpha = 1$. This was proven even in the random case. Infinite systems of similarity maps were mentioned in several contexts at this conference and are the subject of several recent papers, Riedi [Ri], Riedi and Mandelbrot [RiM], M. Moran [M], Staiger [St], Staiger-Fernau [St-F] and the book of Fernau [F]. The deeper properties of infinite systems of conformal maps are developed in the paper of Mauldin and Urbański [MU]. Most of the results and problems raised here arise from this paper.

First, let me review some basic features concerning the iteration of a finite set of similarity maps and the limit set of such a system, a self-similar fractal.

1. Self-similar fractals

Let X be a compact regular subset of $R^d : \emptyset \neq X = cl(\mathrm{Int}(X))$. Let $S = \{\varphi_i\}_{i \in I}$ be a finite set of contracting similarity maps of R^d. For each i, let r_i be the contraction ratio of φ_i. By a beautiful application of Banach's contraction mapping theorem, Hutchinson [Hu] showed that there is a unique compact set $J \subset R^d$ such that

$$J = \bigcup_{i \in I} \varphi_i(J). \tag{1.1}$$

The set J is the self-similar set or limit set generated by the system S. Of course, J has a single element if the cardinality of I is 1. To avoid trivialities we shall always assume $\mathrm{card}(I) > 1$.

There is one simple function from which the basic geometric measure theories properties of the limit set can be determined provided the system satisfies the open set condition. Let

$$\psi(t) = \sum_{i=1}^{n} r_i^t, \text{ for } t \geq 0. \tag{1.2}$$

Then ψ is a strictly decreasing continuous map, $\psi(0) = card(I)$ and $\psi(t) \to 0$, as $t \to \infty$. The similarity dimension of J is defined to be α where

$$\psi(\alpha) = \sum_{i=1}^{n} r_i^\alpha = 1, \tag{1.3}$$

and it is easily seen that

$$0 \leq H^\alpha(J) \leq \mathrm{diam}(X)^\alpha. \tag{1.4}$$

By another application of Banach's contraction mapping theorem, one can show that there is a unique associated natural invariant probability measure m such that m is supported on $J : m(J) = 1$ and such that m satisfies:

$$m = \sum_{i=1}^{n} r_i^\alpha m \circ \varphi_i^{-1}, \tag{1.5}$$

or equivalently, for $f \in C(J)$,

$$\int_J f dm = \int_J \sum_{i=1}^{n} r_i^\alpha f \circ \varphi_i dm. \tag{1.6}$$

The probability measure m is the self-similar probability measure induced by the system S. In general, the similarity dimension and the measure m may not be related in any reasonable manner to the Hausdorff dimension and measure in the dimension of J. This is because there may be some severe overlaps or redundancies in the system S. However, there is a simple natural condition to impose on S from which detailed analysis of the limit set and measure may be derived. This is the **open set condition (OSC)**: there is a nonempty bounded open set U such that for each $i \in I$,

$$\varphi_i(U) \subset U \text{ and } \varphi_i(U) \cap \varphi_j(U) = \emptyset, \text{ if } i \neq j. \tag{1.7}$$

If we let $X = cl(U)$, then X is a nonempty compact regular subset of R^d, for each i, $\varphi_i(\text{Int(X)}) \subset \text{I}nt(X)$ and $\varphi_i(\text{I}nt(X)) \cap \varphi_j(\text{I}nt(X)) = \emptyset$, if $i \neq j$. For this reason, we could take our original open set U to be the interior of X.

Let me collect the main facts concerning these systems in the following theorem.

Theorem 1.1. *Let $S = \{\varphi_i\}_{i \in I}$ be a finite system of similarity maps of R^d satisfying the open set condition. Then*

$$\dim_H(J) = \dim_P(J) = \underline{\dim}_B(J) = \overline{\dim}_B(J) = h. \tag{1.8}$$

Moreover, there is some $C > 0$ such that for all $x \in J$ and $0 < r < diam(J)$,

$$C^{-1} < \frac{m(B(x,r))}{r^\alpha} < C, \tag{1.9}$$

and there are constants $c_1, c_2 > 0$ such that

$$c_1 H^\alpha \llcorner J = m = c_2 \Pi^\alpha \llcorner J. \tag{1.10}$$

Thus, for a finite *i.f.s.* consisting of similarity maps and satisfying the open set condition, the various notions of dimension all agree and the self-similar probability measure determined by the system is up to a multiplicative constant the same as the Hausdorff or packing measure restricted to the limit set.

The self-similar probability measure m also has several other important properties, worked out in the papers of Bandt and Graf [BG] and Schief [Sch],

$$m(\varphi_i(X) \cap \varphi_j(X)) = 0, \text{ if } i, j \in I, i \neq j \tag{1..11}$$

and for Borel subsets, A, of X

$$m(\varphi_i(A)) = r_i^\alpha m(A). \tag{1.12}$$

2. Measures and dimensions

Let me recall the main geometric measure theoretic properties I am discussing here.

Let $t \geq 0$. The t-dimensional outer Hausdorff measure of A is given by

$$H^t(A) = \lim_{\epsilon \to 0} H^t_\epsilon(A)$$

where

$$H^t_\epsilon(A) = \inf\{\sum_i (\text{diam } A_i)^t)\},$$

and the infinum is taken over all covers $\{A_i : i \geq 1\}$ whose diameters are $\leq \epsilon$.
The t-dimensional prepacking premeasure $\Pi^{*t}(A)$ is given by:

$$\Pi^{*t}(A) = \lim_{\epsilon \to 0} \Pi^{*t}_\epsilon(A),$$

where

$$\Pi^{*t}_\epsilon(A) = \sup\{\sum_i (2r_i)^t\}$$

and the supremum is taken over all ϵ-packings of A, i.e. families $\{B(x_i, r_i)\}_{i=1}^\infty$ of pairwise disjoint open balls centered at points x_i of A with radii $r_i \leq \epsilon$.
The t-dimensional outer packing measure is

$$\Pi^t(A) := \inf_{\cup A_i = A} \left\{ \sum_i \Pi^{*t}(A_i) \right\},$$

We have the fundamental inequality:

$$H^t(A) \leq \Pi^t(A).$$

The Hausdorff dimension of A is defined by:

$$\dim_H(A) = HD(A) := \inf\{t : H_t(A) = 0\} = \sup\{t : H_t(A) = \infty\},$$

and the packing dimension of A is given by

$$\dim_P(A) = PD(A) := \inf\{t : \Pi_t(A) = 0\} = \sup\{t : \Pi_t(A) = \infty\}.$$

Remark. Let me mention here something about the awkwardness of the definition of packing measure and dimension. In a very real sense there is no way to simplify the two stages, prepacking and then packing, involved in the definition. I mean this in the following sense. Although it is possible to give a one stage defintion of packing measure, there will always be some higher order quantification involved. In fact, although Hausdorff dimension regarded as a function on the space of compact subsets of X is Borel measurable, Mattila and I, in [MaM], have shown that the packing dimension is not Borel measurable. The packing dimension function is measurable with respect to the σ-algebra generated by the analytic sets.

We shall also mention the upper and lower box counting or Minkowski dimensions:

$$\underline{\dim}_B(A) = \underline{\mathrm{BD}}(A) = \liminf_{\epsilon \to 0} \frac{\log N(A,\varepsilon)}{-\log \varepsilon}$$

$$\overline{\dim}_B(A) = \overline{\mathrm{BD}}(A) = \limsup_{\epsilon \to 0} \frac{\log N(A,\varepsilon)}{-\log \varepsilon}$$

where $N(A,\epsilon)$ is the minimal # of balls with radius $\leq \epsilon$ needed to cover A.

The following two inequalities concerning these dimensions are the only inequalities which hold in general:

$$\dim_H(A) \leq \dim_P(A) \leq \overline{\dim}_B(A)$$

$$\dim_H(A) \leq \underline{\dim}_B(A)$$

My terminology follows fairly closely that given by Falconer [Fa1,Fa2] and Mattila [Mat]. However, I use Π in connection with packing measure rather than P because P is used in this paper to denote the pressure function.

Remark. S.J. Taylor [Tay] has made a tentative working definition of a fractal. A set J is a fractal(in the sense of Taylor) provided $\dim_H(J) = \dim_P(J)$. There is really no exact definition of a fractal, but for technical purposes, this is a reasonable one. However, we shall see that the limit sets discussed here are not always fractals in this sense. Indeed, one of the main points of our theory is that there is a class of these systems, the regular systems, for which the limit set possesses a natural invariant measure, the conformal measure. For these systems, this measure has stronger stability properties than either the Hausdorff or packing measure.

3. Conformal iterated function systems: the setting

Here is the generalization of the iteration of finitely many similarity maps. Let $X \subset \mathbb{R}^d$ be nonempty, compact and regular: $X = \mathrm{cl}(U)$, where $U = \mathrm{I}nt(X)$. By a conformal iterated function system ($c.i.f.s.$) with seed set X, we mean a family of maps, $S = \{\varphi_i\}_{i \in I}$, where I is a countable set satisfying the following 6 properties:

(1) for each $i \in I$ (the index set I may be infinite), φ_i is an injective map of X into X,

(2) the system S is uniformly contractive on X :

$$\exists s < 1 \ \ |\varphi_i(x) - \varphi_j(y)| \leq s|x-y|,$$

where $|\cdot|$ is the distance function.

Before listing the other properties, let us fix some notation and make some initial comments concerning the limit set of this system.

Notation. For each finite word $\tau = (\tau_1, \dots, \tau_k) \in I^* = \cup_{n=0}^{\infty} I^n$, let

$$\varphi_\tau = \varphi_{\tau_1} \circ \varphi_{\tau_2} \circ \cdots \circ \varphi_{\tau_k}.$$

We denote the length, k, of τ by $|\tau|$. Note that if $\omega \in I^\infty$, then

$$\operatorname{diam}(\varphi_{\omega|n}(X)) \leq s^n(\operatorname{diam}(X)). \tag{3.1}$$

Let π be the coding map from the symbol or coding space, $\Omega = I^\infty$, into X such that

$$\{\pi(\omega)\} = \cap_{n=1}^{\infty} \varphi_{\omega|n}(X). \tag{3.2}$$

Then π is a continuous map from the coding space onto $J := \pi(I^\infty)$. We define the limit set of the system S to be this set J. The set J is "self reflexive" in that it satisfies:

$$J = \bigcup_{\omega \in I^\infty} \bigcap_{n=1}^{\infty} \varphi_{\omega|n}(X) = \bigcup_{i \in I} \varphi_i(J). \tag{3.3}$$

Remarks. First, unlike the iteration of finitely many contraction maps, the set J need not be compact and there are several possible sets W satisfying $W = \cup_{i \in I} \varphi_i(W)$. In fáct, there may be no compact set W satisfying this invariance property. However, it is natural to take for J the image of the coding space, since this definition agrees with the finite case and the set J is the largest set satisfying (3.3). Second, if the alphabet I is infinite, the set J could be very complicated in the descriptive set theoretic sense. Indeed, the middle expression for J in (3.3) shows that J is the kernel of a Suslin scheme, or equivalently, J is the continuous image of $I^{\mathbb{N}}$, where we give I the discrete topology. This is the very definition of an analytic set. So, without some additional assumptions J could be very complicated.

Question 3.1. *Assuming only conditions (1) and (2), is it possible to obtain an analytic non-Borel set as a limit set?*

We assume, in addition, that S is a **conformal system**:

(3) The **open set condition (OSC)** is satisfied: if $U = \mathrm{Int}(X)$, then

$$\varphi_i(U) \subset U \text{ and } \varphi_i(U) \cap \varphi_j(U) = \emptyset,\ i, j \in I,\ i \neq j,$$

(4) There is a connected open set V in $\mathbb{R}^d$ with $V \supset X$ such that each φ_i extends to a $C^{1+\epsilon}$ diffeomorphism on V and is conformal on V : $\varphi_i'(x)$ is a similarity map for each $x \in V$ and $i \in I$.

Condition (4) automatically holds if the maps φ_i are similarities. However, at present we need this condition in the general conformal case. Next, we need some geometric regularity of the seed set X.

(5) Cone condition: $\exists \alpha, l > 0$ such that for every $x \in \partial X$, there is an open cone with vertex x, direction vector u_x, central angle α and altitude l such that

$$Con(x, u_x, \alpha, l) \subset \text{Int}(X).$$

Let us remark the geometric condition (5) can be relaxed as indicated in [MU] to the neighborhood boundedness property given in [GMW].

Under these 5 assumptions, the limit set J cannot be too complicated at least in the descriptive set theoretic sense.

Theorem 3.1. *For each n, the family $\{\varphi_\omega(X) : \omega \in I^n\}$ is pointwise finite in the sense that for each x, there are only finitely many ω of length n such that $x \in \varphi_\omega(X)$. Moreover, the union and intersection can be exchanged in (3.3). Thus,*

$$J = \bigcap_{n=1} \bigcup_{\omega \in I^n} \varphi_\omega(X), \tag{3.4}$$

and the limit set J is always an $F_{\sigma\delta}$ set.

In [MU], we give an example of a system S such that J is not a G_δ set.

Our final assumption is the :

(6) **Bounded Distortion Property (BDP)**: There is a $K \geq 1$ such that

$$|\varphi'_\tau(y)| \leq K|\varphi'_\tau(x)|, \tag{3.5}$$

for $\tau \in I^*$ and $x, y \in V$, where $|\varphi'_\tau(y)|$ means the norm of the linear transformation $\varphi'_t(y)$.

The bounded distortion property is a strong condition in that not only must (3.5) hold for x and y in X, but in some open set V including X. Of course, if the initial family consists of similarities, then there is no distortion and one can take $K = 1$. Also, in the general case, it is not sufficient to simply have a constant K which works for all words of length one. Rather, a constant which works for all finite words is required. Several sufficient conditions are given in [MU] in order that (3.5) be satisfied including some involving only the initial family of maps. For example,

Theorem 3.2. *Suppose there are constants $L \geq 1$ and $\alpha > 0$ such that*

$$||\varphi'_i(x)| - |\varphi'_i(y)|| \leq L\|(\varphi'_i)^{-1}\|^{-1}|y - x|,$$

where the norm $\|\cdot\|$ is the uniform norm of $|\varphi'_i(x)^{-1}|$ taken over X. Then the bounded distortion property holds.

Again, notice that a subsystem of a *c.i.f.s.* is a *c.i.f.s.* Next is a simple initial result concerning the dimension of the limit set.

Theorem 3.3. $\dim_P(J) = \overline{\dim}_B(J) = \overline{\dim}_B(\text{cl}(J))$.

Proof. It suffices to show $\dim_P(J) \geq \overline{\dim}_B(J)$. Fix $t < \overline{\dim}_B(J)$ and let $\{Y_n\}$ be a countable cover of J. Since I^∞ is a complete metric space, there is some q and some $\omega \in I^*$ such that $\pi^{-1}(\overline{Y_q}) \supset \{\omega\} \times I^\infty$. Thus, $\varphi_\omega(J) \subset \pi(\{\omega\} \times I^\infty) \subset \overline{Y_q}$. Since $\Pi^{*t}(J) = \infty$ and φ_ω is bi-Lipschitz, $P^{*t}(Y_q) = \infty$. Thus, $\sum_n \Pi^{*t}(Y_n) = \infty$. Consequently, $\Pi^t(J) = \infty$. *Q.E.D.*

Note. Unlike finite systems, it can happen that $\dim_P(J) > \underline{\dim}_B(J) > \dim_H(J)$.So, J need not be a fractal in the sense of Taylor. Several examples of this are given in [MU].

Before continuing with the development of this theory, let me illustrate this theory with an example, a system generating a set of complex continued fractions which includes the usual standard continued fraction development of the irrational numbers in the unit interval.

4. Example: complex and standard continued fractions

Let $X = \overline{B}(1/2, 1/2)$ be the closed ball with center $1/2$ and radius $1/2$ in $\mathbb{R}^2$, the complex plane. Let $I = \{m + in : (m, n) \in \mathbb{N} \times \mathbb{Z}\}$, I is the set of lattice points with positive first coordinate. For each $b \in I$, let φ_b be the conformal map given by:

$$\varphi_b(z) = \frac{1}{b+z}$$

Let $V = B(1/2, 3/4)$. For each b, $\varphi_b : V \to V$ and the φ_b's map the disk X onto a collection of nonoverlapping subdisks. There are several figures indicating how this system behaves in [GM] and [MU]. The bounded distortion property can be verified in several different ways, for example, by using the Koebe distortion theorem itself. Or, as is shown in [MU], one can apply the theory of continued fractions to represent φ_τ, from which one can show that we can take $K = 4$. Thus, z belongs to the limit set J if and only if

$$z = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cfrac{1}{b_3 + \cdots}}}$$

where each $b_j \in I$. Now, actually the system $\{\varphi_b\}_{b \in I}$ is not a conformal system. It satisfies all six conditions except condition (1). The system is not uniformly contractive because $\varphi_1'(0) = 1$. However, if we consider the new system $\{\varphi_{b_1} \circ \varphi_{b_2}\}_{(b_1, b_2) \in I \times I}$, then this new adjusted system satisfies all six conditions and has the same limit set. This sort of adjustment is a common occurrence in infinite systems.

Let J be the limit set, J is a set of complex standard continued fractions, and let $h = \dim_H(J)$. Gardner and Mauldin showed $1 < h < 2$ [GM]. We

will indicate how this follows from the theory presented here. In fact, we will indicate why $0 = H^h(J)$ and $0 < \Pi^h(J) < \infty$.
Since a subsystem of a *c.i.f.s.* is again a *c.i.f.s.*, we can apply the theory to various sets of standard continued fractions. We will give two examples later, the set of all continued fractions having only even integers in their expansion and the set of all continued fractions having only powers of 2 in their expansion. These two systems have very different measure theoretic properties.

5. Major features of infinite conformal iterated function systems

(1) The limit set J need not be compact.
(2) The system may be regular: J supports a unique "t-conformal" probability measure.
(3) There is a natural pressure function P associated to the system. The system is regular if and only if there is some(and therefore unique) t such that $P(t) = 0$.
(4) For any system, $\dim_H(J) = \inf\{t : P(t) < 0\}$. If the system is regular, then $0 \leq H^t(J) < \infty$ and $0 < \Pi^t(J) \leq \infty$.
(5) If the system consists of finitely many maps, then the system is regular and $m = c_1 H^t \llcorner J = c_2 \Pi^t \llcorner J$.
(6) There is a natural "asymptotic boundary" associated with the system. For regular systems, the exact inequalities in (4) depend on the "pointwise scaling" behaviour on the t-conformal measure mon this boundary.
(7) For regular systems, there is a natural ergodic measure $m^* \sim m$. The Radon-Nikodym dm^*/dm is the unique(up to scalar multiples) fixed point of a Frobenius-Perron operator and is the unique normalized solution of an associated functional equation.
(8) There are some natural approximating forms for estimating the dimension of the limit set J.

6. The pressure function

In the case of a finite *i.f.s.* consisting of similarity maps, the dimension of the limit set can be determined by an analysis of the auxiliary function $\psi(t)$ given in (1.2). We simply determine the value α such that $\psi(\alpha) = 1$. The situation is somewhat more delicate in a *c.i.f.s.*, since the system is infinite and we do not have similarity maps. However, there are some related auxiliary functions which play a corresponding role for a *c.i.f.s.* These functions yield a natural topological pressure function. Instead of deriving this pressure function from the dynamical viewpoint, let us show how it and the other functions naturally arise geometrically. The natural idea is to estimate the H^t measure of J by using the covers of J consisting of the sets on level n generated by the system S. So, consider

$$S_n(t) = \sum_{|\omega|=n} \operatorname{diam}(\varphi_\omega(X))^t.$$

By our assumptions, we find there is a constant $D \geq 1$ such that

$$D^{-1}\|\varphi_\omega'\| \leq \operatorname{diam}(\varphi_\omega(X)) \leq D\|\varphi_\omega'\|.$$

Thus,

$$D^{-t}\sum_{|\omega|=n}\|\varphi_\omega'\|^t \leq S_n(t) \leq D^t\sum_{|\omega|=n}\|\varphi_\omega'\|^t. \tag{6.1}$$

Let us define a sequence of auxiliary functions:

$$\psi_n(t) = \sum_{|\omega|=n}\|\varphi_\omega'\|^t.$$

Note that in case the maps φ_i are similarities with contraction ratio r_i, this sequence has a simple multiplicative structure:

$$\psi_n(t) = (\psi_1(t))^n = (\sum_i r_i^t)^n.$$

In case the maps are not similarities or the system is infinite, the situation is a little more complicated. Notice that the system is infinite if and only if $\psi_1(0) = \infty$.
Let $\theta = \theta_S = \inf\{t : \psi_1(t) < \infty\} \geq 0$ and let $F(S)$ be the set of finiteness of ψ_1. So, $F(S)$ is either $[\theta,\infty)$ or (θ,∞). Let us gather some basic properties of these functions.

Theorem 6.1.
(i) Each function $\psi_n(t)$ is nonincreasing.
(ii) $\psi_n(d) = \sum_i\|\varphi_i'\|^d \leq K^d$.
(iii) ψ_n is strictly decreasing on $[\theta,\infty)$, continuous and log convex on $F(S)$.
(iv) $K^{-t}\psi_n(t)\psi_k(t) \leq \psi_{n+k}(t) \leq \psi_n(t)\psi_k(t)$

Notes. Again, $\psi_1(\theta)$ may be finite or infinite. This dichotomy plays a central role in determining whether J is "dimensionless" in the sense of Hausdorff or not. Also, it can happen that $\theta = d$.

Unlike the case of similarities, the functions ψ_n are not multiplicative, part (iv) shows there is a submultiplicative structure present. Classical analysis of sequences with this property, naturally leads to an examination of the **topological pressure function**:

$$P(t) := \lim_{n\to\infty}\frac{1}{n}log\psi_n(t) = \lim_{n\to\infty}\frac{1}{n}log\sum_{|\tau|=n}\|\varphi_\tau'\|^t. \tag{6.2}$$

Of course, $P(t) = \inf\{log\psi_n(t)/n : n \in \mathbb{N}\}$.

Theorem 6.2.

(i) $P(t)$ is nonincreasing on $[0,\infty)$, strictly decreasing on $[\theta,\infty)$, convex and continuous on $F(S)$.

(ii) $P(d) \le 0$ and $P(0)=\infty$ if and only if I is infinite.

To see that $P(d) \le 0$, recall $U = Int(X)$. By the bounded distortion property

$$L^d(U) \ge \sum_{|\omega|=n} L^d(\varphi_\omega(U)) = \sum_{|\omega|=n} \int_U |\varphi_\omega'(x)| dL^d(x) \ge K^{-D} \sum_{|\omega|=n} \|\varphi_\omega'\| L^d(U).$$

Thus, for all n,

$$\sum_{|\omega|=n} \|\varphi_\omega'\|^d = \psi_n(d) \le K^d. \tag{6.3}$$

From this we get $P(d) \le 0$ and also $\lim_i \mathrm{diam}(\varphi_i(X)) = 0$.
If $t_1 < t_2$, then $\psi_n(t_2) = \sum_{|\omega|=n} \|\varphi_\omega'\|^{t_1} \|\varphi_\omega'\|^{t_2-t_1} \le s^{(t_2-t_1)n}\psi_n(t_1)$. So, if $\psi_1(t_1) < \infty$, then

$$P(t_2) \le (t_2-t_1)\log s + P(t_1). \tag{6.4}$$

Thus, P is strictly decreasing on $[\theta,\infty)$. Also, it follows from the submultiplicative property that

$$-t \log K + \log \psi_1(t) \le P(t) \le \log \psi_1(t). \tag{6.5}$$

Set

$$h = \inf\{t : P(t) < 0\}. \tag{6.6}$$

Then $0 \le h \le d$.

Theorem 6.3. $\dim_H(J) \le h$.

Proof. Suppose $h < t$. For some n, $\psi_n(t) < \exp(nP(t)/2) = c < 1$. Then by the submultiplicative property, for every k, $\psi_{kn}(t) \le c^k$. By (6.1), $H^t(J) = 0$.

It is not necessarily so that $P(h) = 0$. However, $h = \dim_H(J)$ and to show this leads to the idea of a conformal measure.

7. Regular and irregular systems and conformal measures

Definition 7.1. *The system S is* **regular** *means there is some t such that $P(t)=0$. (For example, if I is finite, then $0 < P(0) = \log$ card$(I) < \infty$ and S is regular.) Otherwise, the system S is* **irregular**.

We note that if $\psi_1(\theta) < 1$, then the system is irregular.

Definition 7.2. *A measure m is t-conformal for the system S means*

(7.1) $m(J) = 1$,
(7.2) $m(\varphi_i(A)) = \int_A |\varphi_i'|^t dm$, for all Borel sets A,
(7.3) $m(\varphi_i(X) \cap \varphi_j(X)) = 0$, if $i \ne j$.

Remarks. One can think of a t-conformal measure as the measure which obeys the fundamental theorem of calculus in dimension t. Also, it follows that one can replace i in (7.2) by any $\omega \in I^*$ and in (7.3) i and j can be replaced by any two words $\omega, \tau \in I^*$ such that τ and ω are incompatible.

Remark. If the system consists of finitely many similarity maps, then m is the self-similar measure induced by the system. For an infinite system $\{\varphi_i\}_{i\in I}$, consisting of similarity maps with reduction ratios r_i, it was shown in [MW] that there is as self-similar measure m satisfying $m = \sum_i r_i^t m \circ \varphi_i^{-1}$ provided $\psi_1(t) = 1$. This was even proven in the random setting there. This m is the conformal measure in this case. Of course, if ψ_1 is never equal to 1, there is no conformal measure.

The general idea is to use a Frobenius-Perron operator as follows. Define the positive operator $L = L_t : C(X) \to C(X)$ by

$$L(f)(x) = \sum_i |\varphi_i'(x)|^t f(\varphi_i(x)). \tag{7.4}$$

If the probability measure m is t-conformal, then m is a fixed point of L^*. One can use the Schauder fixed point theorem to show that some probability measure, $\hat{m}$ is fixed by L^*. The more difficult part is to then show this "semi-conformal" measure, $\hat{m}$, is actually conformal.

Theorem 7.1. *If $P(t) = 0$, then there is exactly one probability measure m which is fixed by the dual operator L^*. The measure m is t-***conformal**.

Notice that if m is t-conformal and $U = \mathrm{Int}(X)$, then

$$m(U) \geq \sum_{|\omega|=n} m(\varphi_\omega(U)) = \sum_{|\omega|=n} \int_U |\varphi_\omega'(x)|^t dm(x) \geq K^{-t} \sum_{|\omega|=n} \|\varphi_\omega'\|^t m(U).$$

On the other hand, $m(X) = 1$ can be expressed as

$$\sum_{|\omega|=n} m(\varphi_\omega(U)) = \sum_{|\omega|=n} \int_U |\varphi_\omega'(x)|^t dm(x) \leq K^{-t} \sum_{|\omega|=n} \|\varphi_\omega'\|^t m(X).$$

Thus, for all n,

$$1 \leq \sum_{|\omega|=n} \|\varphi_\omega'\| = \psi_n(t) \leq K^t. \tag{7.5}$$

Conversely, (7.5) implies $P(t) = 0$.

Application 7.1. Inequality (7.5) may be used to estimate the Hausdorff dimension of J, the set of complex continued fractions from Section 4. In [MU], this inequality is used to show $1.2267 < dim_H(J) < 1.89$

The final conclusion here is that the pressure having a zero at t and the existence of a t-conformal measure are equivalent.

Theorem 7.2. There is a t-conformal measure $\iff P(t) = 0 \iff S$ is regular.

In particular, if I is finite, then $P(0) = \log(\mathrm{card}(I)) > 0$ and the pressure function has a zero. In fact, finite conformal systems behave almost exactly as finite systems of similarities:

Theorem 7.3. *If I is finite, then there is a unique number t such that $P(t) = 0$. Indeed,*

$$\dim_H(J) = \dim_P(J) = \underline{\dim}_B(J) = \overline{\dim}_B J = t. \tag{7.6}$$

Moreover, there is some $C > 0$ such that for all $x \in J$ and $0 < r < diam(J)$,

$$C^{-1} < \frac{m(B(x,r))}{r^t} < C, \tag{7.7}$$

and there are constants $c_1, c_2 > 0$ such that

$$c_1 H^t \llcorner J = m = c_2 \Pi^t \llcorner J. \tag{7.8}$$

The inequalities (7.7) were proven by Bedford [Be] and by different means in [MU]. It follows from (7.7) that the measures in (7.8) are equivalent. Also, $H^t \llcorner J$ and $\Pi^t \llcorner J$ are fixed points of the operator L and (7.8) follows from the uniqueness of m.

The existence of a conformal measure for finite systems allows us to characterize the Hausdorff dimension of J.

Theorem 7.4. *For any c.i.f.s. S, $h_S = \dim_H(J) = \inf\{t \geq 0 : P(t) < 0\} = \sup\{\dim_H(J_F) : F$ is a finite subset of $I\}$. If $P(t) = 0$, then $t = \dim_H(J)$.*

Proof. We have seen in Theorem 6.3 $\dim_H(J) \leq h_S$. For each $F \in Fin(I) = \{F : F$ is a finite subset of $I\}$, consider the finite system $\{\varphi_i\}_{i\in F}$, its limit set J_F and pressure function P_F. Then $\dim_H(J) \geq \dim_H(J_F) = h_F$ and $P_F(h_F) = 0$. The net of functions $\{P_F(t) : F \in Fin\}$ increases up to $P(t)$. Let $\eta = \lim_F h_F$. Then for each F, $P_F(\eta) \leq 0$. Thus, $P(\eta) \leq 0$. Therefore, $\eta = h$ and $\dim_H(J) = h$. *Q.E.D.*

There is a simple test for determining when the Hausdorff dimension of a regular limit set is less than the dimension of the ambient Euclidean space. In fact, there is one case in which the conformal measure is easy to determine. In the next theorem, λ is Lebesgue measure and $X_1 = \cup_{i\in I}\varphi_i(X)$.

Theorem 7.5. *If S is a regular c.i.f.s. and $\lambda(Int(X) \setminus X_1) > 0$, then $h = dim_H(J) < d$. Conversely, if $\lambda(X \setminus X_1) = 0$, then S is regular, $\lambda(J) = \lambda(X)$, and $\lambda/\lambda(X)$ is the conformal measure.*

Application 7.2. For J, the set of complex continued fractions, $dim_H(J) < 2$.

Also, in [MU], we find a sufficient condition for the dimension of the conformal measure to be h.

Theorem 7.6. *If the system has finite entropy, equivalently, if* $\sum_{i\in I} -h\log(\|\varphi_i'\|)\|\varphi_i'\|^h < \infty$ *, then* $\dim_H(m) = h$.

Question 7.1. *If a t-conformal measure m exists, is it true that* $\dim_H(m) = t$*?*

8. The natural ergodic measure for regular systems, functional equations

For regular systems, we have not yet indicated why the t-conformal measure is unique. The proof of this is based upon a unique equivalent ergodic measure. There are at least two methods for deriving this measure. First, consider the probability measure, μ, defined on the coding space according to the condition:

$$\mu([\omega]) = \int_X |\varphi_\omega'|^t dm. \tag{8.1}$$

Since, for each n, $1 = \int 1dm = \int L^n(1)dm = \sum_{|\omega|=n} \int_X |\varphi_\omega'|^t dm$, μ does extend to a probability measure on I^∞. Thus, μ is the natural coding measure on the coding space. This measure is fairly easy to analyze since there is no overlap in the coding space — there are no serious geometric considerations. One can now obtain a measure μ^* equivalent to μ, invariant and ergodic with respect to the shift σ on I^∞ as follows. Let LIM be a Banach limit on I^∞ and define μ^* as follows,

$$\mu^*([\omega]) = \text{LIM}_{n\to\infty}\mu(\sigma^{-n}[\omega]). \tag{8.2}$$

It is shown in [MU] that this defines the required measure.

In fact,

Theorem 8.1. *Suppose $P(t) = 0$. There is a unique ergodic measure μ^* on I^∞ which is invariant under the shift σ and which is equivalent to μ. Moreover, $K^{-t} \le d\mu^*/d\mu \le K^t$.*

This measure may be used to show that if $P(t) = 0$, then there is only one probability measure m such that $L_t(m) = m$; there is only one semiconformal measure. I note that in the proof of uniqueness of the fixed point of L_t, the open set condition was not used.

The measure μ^* gives rise to an "ergodic" measure on J equivalent to m in the following sense.
Let $G = \{x \in J : \exists$ a unique point $\omega \in I^\infty$ such that $\pi(\omega) = x\}$. By the conformality of m, $m(G) = 1$. Define the transformation T on G by $T(x) = \varphi_{\omega_1}^{-1}(x) = \pi(\sigma(\omega))$. Then T is a 1-1 transformation of almost all of J onto almost all of J.

Theorem 8.2 *Suppose $P(t) = 0$. Then $\mu \circ \pi^{-1} = m$. Also, if $m^* = \mu^* \circ \pi^{-1}$, then m^* is the unique invariant measure, ergodic with respect to T and with $m^* \sim m$.*

We can also obtain the measure m^* directly without using Banach limits.

Theorem 8.3. *For m-a.e. x, $\lim_{n\to\infty} L^n(1)(x) = g(x)$ exists and $g = dm^*/dm$. In particular,*

$$g(x) = L(g)(x) = \sum_i |\varphi_i'(x)|^h g(\varphi_i(x)), \tag{8.3}$$

for m-a.e. x. Thus, g is a fixed point of the operator L, where we consider L extended to the space of bounded measurable functions. Indeed, g is unique up to scalar multiples.

In many cases, but not always, the operator L is almost periodic and the function g is continuous. In fact, in many cases, g is defined on a larger domain and g is real analytic.

Also, (8.3) may be regarded as a functional equation. For some systems, this equation may be simplified and analyzed. For example, in the case of standard continued fractions: $\varphi_n(x) = 1/(n+x)$, the conformal measure is Lebesgue measure λ, every point has a unique code and m^* is Gauss' measure:

$$m^*(E) = \frac{1}{\log 2}\int_E \frac{1}{1+x} d\lambda, \tag{8.4}$$

and T is the standard geometric representation of the shift. Let me indicate how we can find g. We have $L(g) = g$ and we know from Theorem 7.5, $h = 1$ and the conformal measure is Lebesgue measure on $X = [0,1]$. Also, the domain of g can be extended to $(0, \infty)$. Thus,

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{(x+n)^2} g(\frac{1}{x+n}) = \frac{1}{(x+1)^2} g(\frac{1}{x+1}) + g(x+1), \tag{8.6}$$

for $x > 0$. Thus, g is a solution of the functional equation;

$$g(x) - g(x+1) = \frac{1}{(x+1)^2} g(\frac{1}{x+1}). \tag{8.8}$$

The function $g = \frac{1}{c+x}$, is a continuous solution of (8.8). Since we know that the solution is unique up to a scalar multiple, we get $c = 1/log2$.

Let me give one more example. Consider the set, J, of all standard continued fractions which have only even integers in their expansion. This is the limit set of the system $\{\varphi_i(x) = 1/1+2i\}_{i=1}^{\infty}$. This system is regular, $\psi_1(1/2) = \infty$ and $\psi_1(t) < \infty$ for $1/2 < t$. Let $h = \dim_H(J)$. So, $1/2 < h < 1$ and g is the solution of the functional equation.

$$g(x) - g(x+2) = \frac{1}{(x+2)^{2h}} g(\frac{1}{x+2}). \tag{8.9}$$

What we know is that there is only one value of h such that the functional equation (8.9) has a nontrivial continuous solution. This value of h is the dimension of J. The continuous solution at this value is real analytic. But, we have no idea how to find h.

9. Strong open set condition, asymptotic boundary and scaling behaviour

For finite systems, the conformal measure is, up to a constant, the Hausdorff or packing measure. For infinite systems, there is a bifurcation [MU]. The Hausdorff measure may become zero and the packing measure infinite. However, there are some basic relationship which remain and some techniques for determining how large or small the goemetric measures are.

Theorem 9.1. *If m is t-conformal, then $H^t \ll m$ and dH^t/dm is uniformly bounded. In particular, $H^t(J) < \infty$.*

Theorem 9.2. *If m is a t-conformal measure and either I is finite or $J \cap int(X) \neq \emptyset$, then $m << \Pi^t$. Moreover, $dm/d\Pi^t$ is uniformly bounded away from infinity. In particular, $0 < \Pi^t(J)$.*

We have not been able to prove this theorem without assuming the strong open set condition and we do not know whether the open set condition implies the strong open set condition. This naturally leads to a series of questions.

Questions 9.1. *If the system satisfies the open set condition, then does it satisfy the strong open set condition, i.e., can we choose X such that $J \cap int(X) \neq \emptyset$? In particular, if the system is finite, does the OSC imply the SOSC? Also, if both the Hausdorff measure and packing measure are positive and finite, then does OSC imply SOSC? Perhaps, if there is a t-conformal measure, then the OSC implies the SOSC?*

It turns out that whether the Hausdorff measure is positive or the packing measure is finite depends on the scaling behaviour of the conformal measure m. What I have in mind here are analogues of some theorems concerning the pointwise scaling behaviour of a measure on a compact set and its implications for the Hausdorff and packing measure. The following theorems (with perhaps better inequalities) can be found in the books of Falconer and Mattila.

Theorem 9.3. *Let ν be a measure on a compact $K \subset \mathbb{R}^d$ with $\nu(U) > 0$, for $U \neq \emptyset$ and open relative to K.*
(i) If for all $x \in A \subset K$:

$$\limsup_{r\to 0} \frac{\nu(B(x,r))}{r^t} \geq C \text{ , then } \qquad H^t(A) \leq C^{-1}2^{-t}\nu(A).$$

(ii) If for all $x \in A \subset K$:

$$\limsup_{r\to 0} \frac{\nu(B(x,r))}{r^t} \leq C \text{ , then } \qquad H^t(A) \geq C^{-1}2^{-2t}\nu(A).$$

Similarly,
(iii) If for all $x \in A \subset K$:

$$\liminf_{r\to 0} \frac{\nu(B(x,r))}{r^t} \leq C \text{ , then } \qquad \Pi^t(A) \geq C^{-1}2^{-t}\nu(A).$$

and
(iv) If for all $x \in A \subset K$:

$$\liminf_{r\to 0} \frac{\nu(B(x,r))}{r^t} \geq C \text{ , then } \qquad \Pi^t(A) \leq C^{-1}2^{-t}\nu(A).$$

We want to apply results like these to the conformal measure m to obtain estimates on the Hausdorff and packing measure of J. However, if the system is infinite, J is not compact, and it turns out that we must prove some theorems similar to those just stated but not at a fixed point x of J but rather as we let x approach or actually belong to the "asymptotic boundary" of J.

Definition 9.1. *The asymptotic boundary $X(\infty)$ of J is defined as the set*

$$\{x : \text{every neighborhood of } x \text{ meets infinitely many level one sets, } \varphi_i(X)\}.$$

Of course, $X(\infty) = \emptyset$ if and only if the system is finite. In the case of standard continued fractions the only point of $X(\infty)$ is 0. The following theorems are proven in [MU].
Here are two theorems concerning the scaling behaviour of m as it relates to Hausdorff measure.

Theorem 9.4. *If m is a t-conformal measure and $\exists z_j \in X(\infty)$ and positive numbers r_j such that*

$$\limsup_{j\to\infty} \frac{m(B(z_j, r_j))}{r_j^t} = \infty,$$

then $H^t(J) = 0$.

Application 9.1. *For complex continued fractions:* $H_t(J) = 0$

Theorem 9.5. *Let m be a t-conformal measure. Suppose $\exists L > 0, \gamma \geq 1$ such that for $i \in I$ and $r \geq \gamma diam(\varphi_i(X))$, $\exists y \in \varphi_i(V)$ s.t. $m(B(y,r)) \leq Lr^t$. Then $H^t(J) > 0$.*

Next, we have two theorems corresponding theorems concerning packing measure.

Theorem 9.6. *Let m be t-conformal. Suppose $\exists z_j \in J$ and $r_j > 0$ s. t. $B(z_j, r_j) \subset X$ and*

$$\liminf_{j\to\infty} \frac{m(B(z_j,r_j))}{r_j^t} = 0,$$

then $\Pi^t(J) = \infty$.

Theorem 9.7. *Let m be t-conformal. Suppose $\exists L > 0, \gamma \geq 1$ such that if $i \in I$ and $1 \geq r \geq \gamma diam(\varphi_i(X))$, $\exists y \in \varphi_i(V)$ s.t. $m(B(y,r)) \geq Lr^t$. Then $\Pi^t(J) < \infty$.*

Application 9.2. *For complex continued fractions*

$$0 < \Pi_t(J) < \infty.$$

10. Absolutely regular, hereditary regular and irregular systems

Definition 10.1. *A cofinite subsystem of a c.i.f.s. $\{\varphi_i\}_{i\in I}$ is a system $\{\varphi_i\}_{i\in I\setminus F}$, where F is a finite subset of I. A system is hereditarily regular means every cofinite subsystem is regular. A system is absolutely regular means every subsystem is regular.*

There is a very simple means of determining when a system is hereditarily regular.

Theorem 10.1. *An infinite system S is hereditarily regular if and only if $P(\theta) = \infty \Longleftrightarrow \psi(\theta) = \infty$. If S is hereditarily regular, then $h > \theta$. Moreover. if $\{\varphi_i\}_{i\in I}$ is hereditarily regular, $i_0 \in I$, $I_0 = I\setminus\{i_0\}$ and J_0 is the limit set generated by $\{\varphi_i\}_{i\in I_0}$, then $\dim_H(J_0) < \dim_H(J)$.*

Application 10.1. *For complex continued fractions, $\theta = 1$ and $\psi(\theta) = \infty$. Therefore, $\dim_H(J) > 1$.*

Example 10.2. *Consider the system $\{\varphi_i(x) = 1/(x+2^i)\}_{i\in\mathbb{N}}$. The limit set, J, consists of all continued fractions which have only powers of 2 as partial denominators. It is easy to check that this system is absolutely regular, $\theta = 0$.*

Using scaling behaviour of Section 9, Urbański and I show that $0 < H^h(J) < \infty$ *and* $\Pi^h(J) = \infty$.

Example 10.3. *Consider the system* $\{\varphi_i(x) = 1/(x+2i)\}_{i \in \mathbb{N}}$. *The limit set, J, consists of all continued fractions which have even integers as partial denominators. It is easy to check that this system is hereditarily regular. Again, using the scaling theorems of Section 9, Urbański and I show that* $H^h(J) = 0$ *and* $0 < \Pi^h(J) < \infty$.

Theorem 10.2. *If a system S is irregular, then either measure* $H^g(J)$ *or* $\Pi^g(J)$ *is either 0 or infinite for every gauge function g of the form* $t^h L(t)$, *where* $L(t)$ *is slowly varying.*

Theorem 10.2 is proven by showing that if such a measure were positive and finite then because of the slowly varying property, the operator L_h would have a fixed point and thus a conformal measure would exist. We do not know whether an irregular limit set must be totally dimensionless in the sense of Hausdorff.

Question 10.1. *Let J be the limit set generated by an irregular c.i.f.s. Is it true that for every gauge function g either measure* $H^g(J)$ *or* $\Pi^g(J)$ *is either 0 or infinite?*

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