CONTINUA WITH σ–FINITE LINEAR MEASURE *)

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In this note we investigate σ–rectifiable continua: those metrizable continua which have σ–finite linear Hausdorff measure with respect to some compatible metric. Let us recall that Eilenberg and Harrold [EH] have given several characterizations of rectifiable continua: those continua which have finite linear Hausdorff measure with respect to some compatible metric. One of their characterizations is that a metrizable continuum X has rectifiable if and only if every subcontinuum of X contains uncountably many local separating points of X. An analysis of this condition in the σ–finite case naturally leads to a transfinite recursion. Via this recursion, we obtain a similar necessary condition and conjecture that this condition is also sufficient. The results given here were first presented at the 1986 Symposium on Topology and its applications in Prague and then at the Oberwolfach conference on measure theory in March 1990.

First, let us assume that X is a continuum with metric ρ and X has σ–finite linear measure with respect to this metric. Thus, there are subsets $X_1, X_2, X_3, \ldots$ of X such that for each n, $X_n \subset X_{n+1}$ and $\mathscr{H}^1(X_n, \rho) < \infty$ and $\bigcup X_n = X$. In what follows, $\mathscr{H}^1(X_n, \rho)$ denotes the linear Hausdorff measure on X with respect to the metric ρ. If it is understood what metric is being used, ρ will be suppressed.

LEMMA 1. *If X is σ-rectifiable, then, for each* $x_0 \in X$ *and for Lebesgue measure almost all* t,
$\{x \mid \rho(x, x_0) = t\}$ *is countable.*

PROOF. Fix x_0. Now, for each n we have the fundamental formula

$$\mathscr{H}^1(X_n) \geq \int_0^{+\infty} \mathscr{H}^0(\{x \in X_n \mid \rho(x_0, x) = t\})\, d\lambda(t).$$

This formula is explicitly used by Eilenberg [E]. Thus, for each

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n, for λ-a.e. t $\{x \in X_n | \rho(x_0,x) = t \}$ is finite, since $\mathscr{H}^0$ is simply counting measure. The lemma follows from this.

COROLLARY 2. *If a continuum X is σ-rectifiable, then X is a rational curve: X has a base such that the boundary of each set in this base is countable.*

Several examples will be given to show the converse does not hold. Next, we obtain some more topological properties of σ-rectifiable curves.

Recall that a point x of a continuum M is a local separating point of M means there is some open neighborhood U of x for which there is a separation of U, $U \setminus \{x\} = U_1 \cup U_2$ and both open sets U_1 and U_2 meet the component of U containing x [W2, p. 61].

LEMMA 3. *If X is σ-rectifiable, then, for each* x_0, *for* λ-a.e. t with $0 < t < \max\{\rho(x_0,x) : x \in X\}$, *there is a local separating point* x *of* X *such that* $\rho(x_0,x) = t$.

PROOF. This follows from Lemma 1 and Whyburn's theorem that every countable set which separates X contains a local separating point of X[W2, p.62].

LEMMA 4. *If X is σ-rectifiable, then* X *is Suslinian: every collection of pairwise disjoint nondegenerate subcontinua of X is countable.*

PROOF. If X were not Suslinian, then there would be uncountably many pairwise disjoint nondegenerate subcontinua of X. But, every nondegenerate continuum has positive $\mathscr{H}^1$ measure. This would contradict the basic fact that for no σ-finite measure does there exist an uncountable collection of pairwise disjoint sets with positive measure.

EXAMPLE 1. Let T be Sierpinski's triangular curve [K, p. 276]. Then T is a regular curve(has a base of open sets each with finite boundary) and is Suslinian. However, T has only countably many local separating points. Thus, there is no metric under which T has σ-finite linear Hausdorff measure.

EXAMPLE 2. The sin 1/x continuum has σ-finite $\mathscr{H}^1$-measure as a subset of R^2 under the usual Euclidean metric. Thus, a σ-rectifiable continuum need not be locally connected. This contrasts with the fact that rectifiable continua must be locally connected.

DEFINITION. A continuum M has property P means every subcontinuum K of M which contains only countably many local separating points of M is nowhere dense in M.

From Lemmas 3 and 4, we have

THEOREM 5. *If the continuum X is σ-rectifiable, then every nondegenerate subcontinuum of X has property P.*

EXAMPLE 3. The converse of theorem 5 does not hold. Consider the planar continuum $X = C \times [0,1] \cup [0,1] \times \{0\}$ where C is the Cantor set. Every nondegenerate subcontinuum of X has property P, but X is not even Suslinian much less σ–rectifiable.

We recall some of Whyburn's results for a general metrizable continuum X. For each point x of X, there is a maximal subcontinuum, $K(x) = K(x,X)$, of X containing x which also contains only countably many local separating points of X [W1, Theorem 2.1]. Let $\mathscr{D}$ be the decomposition of X into the collection of its maximal subcontinua C such that C contains only countably many local separating points of X. Whyburn showed that the decomposition $\mathscr{D}$ is upper semi–continuous [W1, Theorem 4.2] and that the decomposition space $X/\mathscr{D}$ with the quotient topology is a hereditarily locally connected continuum [W1, Theorem 5.5]. Indeed, Whyburn showed that $X/\mathscr{D}$ is a regular curve and every subcontinuum of $X/\mathscr{D}$ contains uncountably many local separating points of $X/\mathscr{D}$ [W1 , Theorem 6.2]. Thus, $X/\mathscr{D}$ is a continuum of finite degree and is, therefore, rectifiable [EH].

From Lemmas 3 and 4, we have:

LEMMA 6. *If X is σ-rectifiable, then there are only countably many nondegenerate elements of $\mathscr{D}$ and each element of $\mathscr{D}$ is nowhere dense in* X.

DEFINITION. The <u>transfinite local separating point decomposition sequence</u> $\{\mathscr{D}_\alpha\}_{\alpha<\omega_1}$ of a continuum X is defined as follows. Set $\mathscr{D}_0 = \mathscr{D}$, the upper semi–continuous decomposition of X into the collection of its maximal subcontinua C such that C contains only countably many local separating points of X. Suppose α is an ordinal and for each $\gamma < \alpha$, a decomposition $\mathscr{D}_\gamma$ of X into continua has been given. If α is a limit ordinal, then let $K \in \mathscr{D}_\alpha$ if and only if $K = \bigcap_{\gamma<\alpha} K_\gamma$, where each $K_\gamma \in \mathscr{D}_\gamma$. If $\alpha = \beta+1$, then let $K \in \mathscr{D}_\alpha$ if and only if there is some $H \in \mathscr{D}_\beta$ such that K is a maximal subcontinuum of H which contains only countably many local separating points of H. For each point x of X and ordinal γ, let $K_\gamma(x) = K_\gamma(x,X)$ be the element of $\mathscr{D}_\gamma$ containing x. Since $K_\gamma(x)$ is a transfinite decreasing sequence of closed subsets of X, there is a smallest countable ordinal $\alpha = \sigma(x)$ such that $K_\alpha(x) = K_{\alpha+1}(x)$. The ordinal $\sigma(x)$ will be called the <u>local separating point index</u> of x. The order of X, $o(X) = \sup\{\sigma(x) \mid x \in X\}$.

In order to prove the next theorem, we first reformulate a result of Whyburn's as the next lemma.

LEMMA 7. *Let* $C \subset Y \subset X$ *be continua. If* C *contains only countably many local separating points of* Y, *then* C *contains only countably many local separating points of* X.

An argument for this lemma may be gathered from the proof of (5.2) of [W1, p. 446].

COROLLARY 8. *Let* Y *be a nondegenerate subcontinuum of the continuum* X. *Then for each* $y \in Y$ *and for each ordinal* α, $K_\alpha(y,Y) \subset K_\alpha(y,X)$.

PROOF. Let $C = K_0(y,Y)$ be the maximal subcontinuum of Y containing y and only countably many local separating points of Y. By lemma 7, C contains only countably many local separating points of X. Thus, $K_0(y,Y) \subset K_0(y,X)$. The proof may be completed by transfinite recursion.

THEOREM 9. *Let* X *be a continuum. Every nondegenerate subcontinuum of* X *has property* P *if and only if for each* x, $K_{\sigma(x)} = \{x\}$.

PROOF. Assume every subcontinuum of X has property P. If $K_{\sigma(x)}(x)$ were not the singleton x, then since $K_{\sigma(x)}(x)$ has property P, the maximal subcontinuum Q of $K_{\sigma(x)}(x)$ containing x and only countably many non-separating points of $K_{\sigma(x)}(x)$ is nowhere dense in $K_{\sigma(x)}(x)$. But, $Q = K_{\sigma(x)+1} \neq K_{\sigma(x)}(x)$. Thus, $K_{\sigma(x)} = \{x\}$.

Now, assume for each x, $K_{\sigma(x)} = \{x\}$. Let Y be a nondegenerate subcontinuum of X. If Y does not have property P, then there is some subcontinuum K of Y with nonempty interior, U, relative to Y and which contains only countably many local separating points of Y and let $y \in Y$. If $y \in U$, then for each α, $K_\alpha(y,Y)$ includes the closure of U with respect to Y. But, from corollary 8, we have the contradiction: $K_{\gamma(y)}(y,Y) = \{y\}$.

QUESTION 2. Is it true that $o(X) < \omega_1$? What if X has property P?

Next, we show the answer to this last question is yes if X is σ—rectifiable.

THEOREM 10. *If X has a metric ρ under which the $\mathscr{H}^1$ measure of X is σ-finite, then, for each* x, $K_{\sigma(x)} = \{x\}$. *Moreover, there is a countable ordinal $\alpha < \omega_1$ such that for each* x, $\sigma(x) \leq \alpha$, *i. e., there is a countable ordinal $\alpha < \omega_1$ such that $\mathscr{D}_\alpha$ is the decomposition of X into singletons.*

PROOF. The first part of the theorem follows from theorems 5 and 7. In order to show the indices are uniformly bounded below ω_1, write $X = \bigcup_{n=1}^{\infty} E_n$, where each E_n is a G_δ set with positive finite $\mathscr{H}^1$ measure. Since X is Suslinian, for each γ, there are only countably many elements of D_γ which are not singletons. By way of contradiction, let us assume each $\mathscr{D}_\alpha$ is not trivial. It follows that there would be some positive integer n and some $\epsilon > 0$ such that for each $\alpha < \omega_1$, $\mathscr{D}_{\alpha,n} = \{K \in \mathscr{D}_\alpha \mid \mathscr{H}^1(E_n \cap K) \geq \epsilon\} \neq \phi$. Since the sets in $\mathscr{D}_{\alpha,n}$ are disjoint, there are not more than $\mathscr{H}^1(E_n)/\epsilon$ members of the collection $\mathscr{D}_{\alpha,n}$. For each $\alpha < \omega_1$, consider the closed set $M_\alpha = \bigcup \mathscr{D}_{\alpha,n}$. If $\alpha < \beta$, then $M_\alpha \supset M_\beta$. Let $x \in \bigcap M_\alpha$. Then for each $\alpha < \omega_1$, $K_\alpha(x)$ would be nondegenerate. Q.E.D.

REMARK. By rather involved constructions one can show that for each $\alpha < \omega_1$ there is a σ—rectifiable X such that the order of X is exactly α.

CONJECTURE. A metrizable continuum X is σ—rectifiable if and only if X is Suslinian and there exists some $\alpha < \omega_1$ such that $\mathscr{D}_\alpha$ is the decomposition of X into singletons.

The next example shows that the connected union of two rectifiable continua need not be rectifiable.

EXAMPLE 4. There is a continuum X and two subcontinua X_1 and X_2 such that each X_i is of finite degree(and therefore possesses finite $\mathscr{H}^1$ measure in some metric ρ_i) and yet $X = X_1 \cup X_2$ is not rectifiable. The continuum described by Kuratowski [K, p. 268] is such an example. (The bottom unit interval contains no local separating points.)

In contrast to this last example, we will show that the connected union of two σ-rectifiable continua is a σ-rectifiable continuum. In order to show this, we first require a more detailed analysis of Hausdorff's theorem concerning the extension of metrics.

DEFINITION. Let d and r be metrics on a set E. Then r is said to be <u>compact Lipschitz</u> with respect to d on E provided that for each compact subset K of E there is a constant c_K such that if x and y are in K, then $r(x,y) \leq c_K d(x,y)$.

THEOREM 11. *Let d be a bounded metric on a space X and let A be a closed subset of X with a bounded compatible metric r defined on A. There is an extension ρ of r to a metric on X which is compact Lipschitz on $X \setminus A$.*

PROOF. We will follow the construction given by Arens [A] of a metric ρ on X which extends r. First, let $\mathcal{G} = \{B(x,d(x,A)/4 \mid x \in X \setminus A\}$. Let $\mathcal{R} = \{U_\lambda : \lambda \in L\}$ be a locally finite refinement of $\mathcal{G}$. Set $f_\lambda(x) = d(x, X \setminus U_\lambda)$ and $s = \Sigma_\lambda f_\lambda$. Thus, s is a continuous map of $X \setminus A$ into the positive real numbers. Set $g_\lambda = f_\lambda / s$. Of course, the family $\{g_\lambda : \lambda \in L\}$ forms a partition of unity subordinated to the cover $\mathcal{R}$. For each λ, choose x_λ such that $U_\lambda \subset B(x_\lambda, d(x_\lambda, X \setminus A)/4$ and choose $a_\lambda \in A$ such that $d(x_\lambda, a_\lambda) < (5/4)d(x_\lambda, A)$.

Fix a point $a_0 \in A$ and consider φ, the usual isometric imbedding of A with metric r into $C_b(A)$, the Banach space of bounded continuous functions on A, given by $\varphi(a)(x) = r(x,a) - r(x,a_0)$. For each a, $\|\varphi(a)\| = r(a,a_0) \leq |A|_r$, the r-diameter of A.

Consider the extension $\overline{\varphi}: X \to C_b(A)$ defined by $\overline{\varphi}(x) = \Sigma_\lambda g_\lambda(x)\varphi(a_\lambda)$, for $x \notin A$. Arens shows $\overline{\varphi}$ is a continuous extension of φ. Moreover, consider the Banach space $L = C_b(A) \times R \times C_b(X)$ with the sup norm: $\|(h,t,k)\| = \max(\|h\|, |t|, \|k\|)$. For each x, let $d_x(y) = d(x,y)$. Define $F : X \to L$ by $F(x) = (\overline{\varphi}(x), d(x,A), d(x,A)d_x)$. Clearly, $F|_A$ is an isometric imbedding of A with metric r into L. Arens shows that F is a homeomorphism and thus, setting $\rho(x,y) = \|F(x) - F(y)\|$, we have an metric extension of r to X compatible with the metric d. Let K be a compact set lying in $X \setminus A$.

Claim. There is a number w_K such that if x and y are in K, then

(*) $$\|\overline{\varphi}(x) - \overline{\varphi}(y)\| \leq w_K d(x,y).$$

Proof. Let $g_1 = g_{\lambda_1},\ldots,g_{\lambda_n} = g_n$ be the elements of the partition of unity which are nonzero at some point of K. Thus, $|\overline{\varphi}(x) - \overline{\varphi}(y)| = |\Sigma(g_i(x) - g_i(y))\overline{\varphi}(a_{\lambda_i})| \leq \|\overline{\varphi}(a_{\lambda_i})\|\Sigma|(g_i(x) - g_i(y))|$.

But, $\|\overline{\varphi}(a_{\lambda_i})\| \leq |A|_r$ and for each j,

$$g_j(x) - g_j(y) = [s(y)(d(x,X\backslash U_{\lambda_j}) - d(y,X\backslash U_{\lambda_j})) + (s(y) - s(x))d(y,X\backslash U_{\lambda_j})]/s(x)s(y).$$

Let $B_K = \min_{z\in K} s(z) > 0$ and $H_K = \max_{z\in K} s(z)$. Then

$$|g_j(x) - g_j(y)| \leq 1/B_K^2[H_K d(x,y) + |X|_d|s(x) - s(y)|]$$

and

$$|s(y) - s(x)| \leq \Sigma_{i=1}^n |d(x,X\backslash U_{\lambda_j}) - d(y,X\backslash U_{\lambda_j})| \leq nd(x,y).$$

Inequality (*) follows from these inequalities.

Note

$$|d(x,A) - d(y,A)| \leq d(x,y)$$

and for each z,

$$|d(x,A)d(x,z) - d(y,A)d(y,z)| \leq 2|X|_d d(x,y).$$

That ρ is d-Lipschitz on K now follows easily from these inequalities and the claim.

THEOREM 12. *Let X be a compact metric space. If X may be expressed as* $\cup_{i=1}^n M_i$ *where each* M_i *is closed and has a metric* r_i *under which* M_i *has σ-finite* $\mathscr{H}^1$ *measure, then X possesses a metric under which it has σ-finite* $\mathscr{H}^1$ *measure. In particular, if X is the union of finitely many continua each with finite degree, then X is σ-rectifiable.*

PROOF. Suppose $X = M_1 \cup M_2$ and let d be a metric on X. First, extend r_1 to a metric ρ_1 on X. Use theorem 8 to extend the metric r_2 on M_2 to a metric ρ_2 on X which is compact Lipschitz with respect to ρ_1 on $X\backslash M_2$. Express $X\backslash M_2 = \cup K_j$, where each K_j is compact. Since each set K_j has σ-finite $\mathscr{H}^1$ measure with respect to the metric r_1 and ρ_2 is r_1–Lipschitz on K_j, each set K_j has σ–finite $\mathscr{H}^1$–measure with respect to the metric ρ_2. Thus, X has σ–finite $\mathscr{H}^1$–measure with respect to the measure ρ_2. The theorem follows by induction.

REMARK. Obviously, this theorem can be placed in a more general context and holds for general Hausdorff measures.

QUESTION 3. Suppose $X = \bigcup_{i=1}^{\infty} M_i$ where each M_i is closed and has a metric r_i under which M_i has σ–finite $\mathscr{H}^1$ measure. Is it true that X possesses a metric under which it has σ–finite $\mathscr{H}^1$ measure?

QUESTION 4. Supppose the continuum X possesses a metric under which it has σ–finite $\mathscr{H}^1$ measure. Is it true that $X = \bigcup_{i=1}^{\infty} M_i$, where each M_i is closed and has finite $\mathscr{H}^1$ measure.

Finally, we review the fact that a rectifiable curve X is a Peano continuum, by showing there is a map of [0,1] onto X with a bound on its arc length. We do this since there is a minor point missing from the argument given in [EH].

We need the the following lemmas concerning dendrites. Recall that a dendrite is a locally connected continuum D which contains no simple closed curve. A dendrite is finite means it has only finitely many end points or non–cut points.

LEMMA 13. *Let D be a finite dendrite with $\mathscr{H}^1(D) < \infty$. If each of x and y is a point of D, then there is a continuous map f of [0,1] onto D such that the arc length, $L_0^1 f \leq 2\mathscr{H}^1(X)$, $f(0) = x$ and $f(1) = y$.*

This lemma may be proven by induction on the number of end points.

THEOREM 14. *Let X be a nongenerate metrizable continuum with metric ρ. Then $0 < \mathscr{H}^1(X,\rho) < \infty$ if and only if there is a continuous map f of the unit interval onto X which has finite arc length.*

PROOF. First, assume X is a rectifiable curve. Let f be a continuous rectifiable map of [0,1] onto X with arc length L. We recall a standard argument that $\mathscr{H}^1(X,\rho) \leq L$. Let g parametrize X by arc length. In other words, let h be the continuous strictly increasing map of $[0,1]$ onto $[0,L]$ given by $h(t) = L_0^t(f)$, the length of f on $[0,t]$. Then $g = f \circ h^{-1}$ and g is a nonexpansive map of $[0,1]$ onto X: for $s,t \in [0,L]$, $\rho(g(s),g(t)) \leq \rho(f(h^{-1}(s),f(h^{-1})) \leq L_{h^{-1}(s)}^{h^{-1}(t)} f = |g(s) - g(t)|$. By the standard inequality concerning the Hausdorff measure of image sets under Lipschitz maps, we have $\mathscr{H}^1(X,\rho) = \mathscr{H}^1(g([0,L]) \leq \mathscr{H}^1 f[0,L] = L$.

Now, let us assume $\mathscr{H}^1(X,\rho) < \infty$. We recall the argument [EH] that the continuum X has finite degree. Fix x_0. We have the fundamental formula

$$\infty > \mathscr{H}^1(X) \geq \int_0^{+\infty} \mathscr{H}^0(\{x \in X_n | \ \rho(x_0,x) = t\}) \, d\lambda(t).$$

Thus, for Lebesgue measure almost all t, the set of points at distance t from x_0 is finite. This implies the continuum X has finite degree. In particular, X is a regular curve in the sense of Menger and Urysohn. This implies that X is locally connected [Ku, p. 283]. Therefore, there is a sequence $\{D_n\}_{n=1}^{\infty}$ of dendrites such that for each n, (1) $D_n \subset D_{n+1}$, (2) $\bigcup_{n=1}^{\infty} D_n$ is dense in X and (3) if C is a component of $D_{n+1} \setminus D_n$, then C meets D_n in exactly one point x(c) and the dendrite $C \cup \{x(c)\}$ has diameter $< 2^{-n}$. Let f_1 be a continuous map of $[0,1]$ onto D_1 such that $L_0^1 \, f_1 \leq 2\mathscr{H}^1(D_1)$. Assume that a continuous map f_k of $[0,1]$ onto D_k has been given with $L_0^1 \, f_k \leq 2\mathscr{H}^1(D_k)$. Let $x_1, \quad ,x_m$ list the points x of D_k, for which there is some component, C, of $D_{k+1} \setminus D_k$, for which $x = x(C)$. For each i, let $K_i = \bigcup\{C : x(C) = x_i\} \cup \{x_i\}$ and let t_i be a number in $[0,1]$ such that $f_k(t_i) = x_i$. Without loss of generality, we can assume $0 \leq t_1 <\ldots< t_m \leq 1$. Choose $0 \leq s_1 \leq t_1 \leq u_1 < s_2 < t_2 < u_2 <\ldots< s_m < t_m \leq u_m$ such that for each i, the diameter of $f_k([s_i, u_i]) < 2^{-(k+1)}$. According to lemma 11, we may define f_{k+1} by letting f_{k+1} be a continuous map of $[s_i,u_i]$ onto K_i such that $f_{k+1}(s_i) = f_{k+1}(u_i) = x_i$, $L_{s_i}^{u_i} \, f_{k+1} \leq 2\mathscr{H}^1(K_i)$ and letting f_{k+1} agree with f_k otherwise

The sequence of continuous maps so defined has the property that for each k, f_k maps [0,1] onto D_k, $L_0^1 f_k \leq 2\mathscr{H}^1(D_k)$ and $\sup_t |f_k(t) - f_{k+1}(t)| < 2^{-k}$. Let f be the uniform limit of the sequence of functions f_k. Then $L_0^1 f \leq 2\mathscr{H}^1(X)$ and it also follows that f maps [0,1] onto X. Q.E.D.

This argument clarifies some points in [EH]. This argument is essentially given in [Wa] for other purposes. It also appears in [Fa, p. 53] and [F–M–P].

During the preparation of this manuscript, D. Fremlin sent me a manuscript which overlaps somewhat with the results presented here [F].

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