SETS GENERATED BY RECTANGLES

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For any family \( F \) of sets, let \( \mathcal{B}(F) \) denote the smallest \( \sigma \)-algebra containing \( F \). Throughout this paper \( X \) denotes a set and \( \mathcal{P} \) the family of sets of the form \( A \times B \), for \( A \subseteq X \) and \( B \subseteq X \). It is of interest to find conditions under which the following holds:

(1) Each subset of \( X \times X \) is a member of \( \mathcal{B}(\mathcal{P}) \)

The interesting case is when

\[ \omega_1 < \text{Card } X \leq \omega \],

since results for other cases are known.

It is shown in Theorem 9 that (1) is equivalent to

There is a countable ordinal \( \alpha \) such that

(2) each subset of \( X \times X \) can be generated

from \( \mathcal{P} \) is \( \alpha \) Baire process steps.

It is also shown that the two-dimensional statements (1) and (2) are equivalent to the one-dimensional statement

There is a countable ordinal \( \alpha \) such that

(3) \( \text{Card } H = \text{Card } X \), there is a countable

family \( G \) such that each member of \( H \)

can be generated from \( G \) in \( \alpha \) steps.

It is shown in Theorem 5 that the continuum hypothesis (CH) is equivalent to certain statements about rectangles of the form (1) and (2) with \( \alpha = 2 \).

Rao [7, 8] and Kunen [2] have shown that

**THEOREM 1.** If \( \text{Card } X \leq \omega_1 \), (the first uncountable cardinal) then (1) is true and if \( \text{Card } X > \omega \) then (1) is false.

The question of whether (1) is true (without the requirement \( \text{Card } X \leq \omega_1 \)) was raised by Johnson [1] and earlier by Erdös, Ulam, and others (see [8], p. 197). The arguments in Kunen's thesis actually showed that if \( \text{Card } X \leq \omega_1 \), then

(4) Each subset of \( X \times X \) can be generated

from \( \mathcal{P} \) in 2 steps (i.e., each subset is a

member of \( \mathcal{P}^2 \). See definitions in § 2.)

In Theorem 5 we generalize Theorem 1 and Kunen's result (4),
and give a new characterization of CH by showing it to be equivalent to certain statements about rectangles of the form (1) and (4).

If CH is assumed the \( \alpha \) appearing in statements (2) and (3) above is 2 (see Theorem 10). This raises the intriguing (but unanswered) question of whether \( \alpha \) must always be 2 if (1) holds and CH is false.

It would also be interesting to know whether statements (1), (2), and (3) are equivalent to statement (5) below. Clearly (3) implies (5).

If \( H \) is a family of subsets of \( X \) with

\[
(5) \quad \text{Card } H = \text{Card } X, \text{ then there is a countable family } G \text{ for which } H \subseteq \mathcal{P}(G). 
\]

The equivalence of (1) and (2) means for example, (assuming CH), that there is a countable family \( G \) from which all real Borel sets (or analytic sets, or projective sets) can be generated in two steps (i.e., Borel sets \( \subseteq G_\omega \)). This is remarkable in view of the well known result [4, 8] that if \( G \) is a countable basis for the real topology, then the Borel sets cannot be generated from \( G \) in less than \( \omega_1 \) steps.

As a generalization of this well known result we show in Theorem 12 that any countable family \( G \) which is closed to complementation and which generates the Borel sets (i.e., Borel sets \( \subseteq \mathcal{P}(G) \)) must have order \( \omega_1 \). That is

\[ \mathcal{P}(G) \not\subseteq G_\alpha \]

for any countable ordinal \( \alpha \). Thus, even though \( G \) might generate the Borel sets in \( \alpha \) steps (or 2 steps if CH is assumed), the process, nevertheless, continues to produce new members of \( \mathcal{P}(G) \) until we reach \( G_\omega \).

We would like to point out in conjunction with our characterization of CH that Kunen [2] has proved that if Martin's Axiom A holds (see [6]) and Card \( X \leq c \) then (4) holds. He also proved that if \( \omega_1 < \text{Card } X \leq c \) then (1) is independent of ZFC (Zermo-Frankel Axioms + the Axiom of Choice) together with the negation of CH.

2. Notation and definitions. If \( G \) is any family of sets, let \( \mathcal{G}_\alpha \) be the family \( G \), and for each ordinal \( \alpha, \alpha > 0 \), let \( \mathcal{G}_\alpha \) be the family of all countable unions (intersections) of sets in \( \bigcup_{\gamma < \alpha} \mathcal{G}_\gamma \), if \( \alpha \) is odd (even). Limit ordinals will be considered even. (Compare Kuratowski [3].) Thus we have

\[ G_0 = G, \ G_1 = G_\omega, \ G_2 = G_{\omega_1}, \ G_3 = G_{\omega_2}, \ldots, G_\alpha, \ldots \]

Also \( G_\alpha \subseteq G_{\alpha+1} \) for each ordinal \( \alpha \) and \( G_{\omega_1} = G_{\omega_1+1} \), where \( \omega_1 \) is the first uncountable ordinal. If \( \alpha > 0 \), then the family \( G_\alpha \) is closed under countable unions (intersections) if \( \alpha \) is odd (even).
We define the order of $G$ to be the first ordinal $\alpha$, $\alpha > 0$, such that $G_{\alpha+1} = G_\alpha$.

For each $A \subseteq X$ (or $A \subseteq X \times X$), let $A'$ be the complement of $A$ with respect to $X$ (or $X \times X$), and for each family $G$ of subsets of $X$ (or $X \times X$) let $\mathcal{G}(G)$ be the family of complements of $G$. Note that if $\mathcal{G}(G) \subseteq G$, or even if $\mathcal{G}(G) \subseteq G_\omega$, then the family $G_\omega$ is the family $\mathcal{B}(G)$, the $\sigma$-algebra generated by $G$. Thus, since

$$(A \times B)' = A \times B' \cup A' \times X \in \mathcal{B},$$

it follows that $\mathcal{B}_\omega = \mathcal{B}(\mathcal{B})$.

If $G$ is a family of subsets of $X$, let $VG = \{A \times B: A \subseteq X, B \in G\}$, and let $HG = \{A \times B: A \in G, B \subseteq X\}$.

If $Z \subseteq X \times X$ and $x \in X$, let $Z_x$ denote the vertical section of $Z$ at $x$, $Z_x = \{y: (x, y) \in Z\}$.

3. Results. The following lemma is easily proved by transfinite induction.

**Lemma 2.** If $1 \leq \alpha < \omega_1$ and $A \in G_\alpha$, then there is a set $B$ in $G_1$ such that $A \subseteq B$.

**Theorem 3.** If $G$ is a countable family of subsets of $X$, $Z \subseteq X \times X$, and $0 < \alpha < \omega_1$, then $Z \in (VG)_\alpha$ if and only if $Z_x \in G_\alpha$ for each $x \in$ domain $Z$.

**Proof.** By considering the natural projections of the sets involved on the second coordinate axis, it is easily seen that

if $Z \in (VG)_\alpha$, then $Z_x \in G_\alpha$ for each $x \in$ domain $Z$.

Now suppose that $Z_x \in G_\alpha$, for each $x \in$ domain $Z$, and let $G = \{\theta_1, \theta_2, \theta_3, \ldots\}$. We complete the proof by transfinite induction on $\alpha$.

**Case 1.** $\alpha = 1$.

For each $n$, let $A_n = \{x \in$ domain $Z: \theta_n \subseteq Z\}$, and let $Z_n = A_n \times \theta_n$. Then $Z_n \in VG$, for each $n$, and

$$Z = \bigcup_{n=1}^{\infty} Z_n \in (VG)_1.$$

Now suppose $1 < \alpha < \omega_1$, and that the theorem holds for every $\gamma$, $0 < \gamma < \alpha$.

**Case 2.** $\alpha$ is even.
Let \( \{ \gamma_n \}_{n=1}^\infty \) be a sequence of odd ordinals less than \( \alpha \) such that each odd ordinal less than \( \alpha \) appears infinitely often in \( \{ \gamma_n \}_{n=1}^\infty \). For each \( x \in \text{domain } Z \), let
\[
D_1(x), \ D_2(x), \ D_3(x), \ldots
\]
be a sequence such that \( D_i(x) \in G_{\gamma_i} \) for each \( i \), and
\[
Z_x = \bigcap_{i=1}^\infty D_i(x).
\]
This can be done in view of Lemma 2. For each \( i \), let
\[
Z^i = \bigcup_{x \in \text{domain } Z} [x] \times D_i(x).
\]
First note that \( Z = \bigcap_{i=1}^\infty Z^i \). Also each nonempty section \( (Z^i)_x \) of \( Z^i \) is equal to \( D_i(x) \in G_{\gamma_i} \). Hence, by the induction hypothesis, \( Z^i \in (VG)_{\gamma_i} \), for each \( i \), and therefore
\[
Z = \bigcap_{i=1}^\infty Z^i \in (VG)_{\alpha}.
\]
by the definition of the family \( (VG)_a \).

**Case 3.** \( \alpha \) is odd and greater than 1.

For each \( x \in \text{domain } Z \), let \( \{ D_i(x) \}_{i=1}^\infty \) be a sequence of members of \( G_{\gamma_i} \) for which \( Z_x = \bigcup_{i=1}^\infty D_i(x) \), and let \( Z^i = \bigcup_{x \in \text{domain } Z} [x] \times D_i(x) \), for each \( i \).

Again it follows that \( Z^i \in G_{\gamma_i} \), for each \( i \), and
\[
Z = \bigcup_{i=1}^\infty Z^i \in (VG)_{\alpha}.
\]

**Corollary 4.** If \( Z \subseteq X \times X \) is the graph of a function then \( Z \in B \), \( \subseteq B \left( B \right) \).

**Proof.** Let \( G \) be a countable basis for the real topology and note that, for each \( x \in X \), \( Z_x \) is a singleton and hence \( Z \in G \). Thus by Theorem 3, \( Z \in (VG)_{\alpha} \subseteq B \left( B \right) \). Also see [7].

**Theorem 5.** Let \( X \) be the real numbers and let \( G \) be a countable base for the usual topology on \( X \). The following three statements are equivalent:

1. CH holds
2. if \( Z \subseteq X \times X \), then \( Z = A \cap B \), where \( A \in (VG)_{\alpha} \) and \( B \in (HG)_{\alpha} \), and
3. if \( E \subseteq X \times X \), then \( E = C \cup D \), where \( C \in B \left( VG \right) \) and \( D \in B \left( HG \right) \).


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Proof. First, assume CH and suppose \( Z \subseteq X \times X \). As is well known [7], the complement of \( Z \) is the union of two sets \( H \) and \( K \) such that each vertical section of \( H \) is countable and each horizontal section of \( K \) is countable.

Let \( A \) be the complement of \( H \) and let \( B \) be the complement of \( K \). Then each vertical section of \( A \) is a \( G_\delta \) set and by Theorem 3, \( A \in (VG)_\mathbb{R} \). Similarly, \( B \in (HG)_\mathbb{R} \). Of course, \( Z = A \cap B \).

Since \( A \in (VG)_\mathbb{R} \subseteq \mathcal{R} \) and \( B \in (HG)_\mathbb{R} \subseteq \mathcal{R} \), and \( \mathcal{R} \) is closed under finite intersections, \( Z \in \mathcal{R} \). Thus, if CH holds, then the order of \( \mathcal{R} \) is \( \leq 2 \). Since the graph of the identity function, \( f(x) = x \), is not in \( \mathcal{R} \), it follows that the order of \( \mathcal{R} \) is 2.

Now, suppose statement 2 holds and \( E \subseteq X \times X \). Then, the complement of \( E \) can be expressed as the intersection of sets \( A \) and \( B \) with \( A \in (VG)_\mathbb{R} \) and \( B \in (HG)_\mathbb{R} \). It follows that \( A \in (VG)_\mathbb{R} \subseteq \mathcal{B}(VG) \) and \( B \in (HG)_\mathbb{R} \subseteq \mathcal{B}(HG) \). Thus, \( E \) is the union of two sets \( C \) and \( D \), where \( C \in \mathcal{B}(VG) \) and \( D \in \mathcal{B}(HG) \).

Finally, assume statement 3 holds. Let \( T \) be a totally imperfect subset of \( X \) of cardinality \( c \). The existence of such a set can be proven without assuming CH [3, p. 514]. Let \( E = T \times T \) and let \( E = C \cup D \), with \( C \in \mathcal{B}(VG) \) and \( D \in \mathcal{B}(HG) \). Then each vertical section of \( C \) is a subset of \( T \) which is a Borel set. Since an uncountable Borel set contains a perfect set and \( T \) contains no perfect set, we have that each vertical section of \( C \) is countable. Similarly, each horizontal section of \( D \) is countable. But, as is well known [10] this implies CH.

This completes the proof of Theorem 5.

The following two lemmas are well known.

**Lemma 6.** If \( F \) is a family of sets, \( \alpha \) is a countable ordinal, and \( A \in F_\alpha \), then there is a countable subfamily \( J \) of \( F \) for which \( A \in J_\alpha \).

**Lemma 7.** If \( F \) is a family of sets, \( \mathcal{B}(F) \subseteq F \), and \( A \in \mathcal{B}(F) \) then there is a countable subfamily \( J \) of \( F \) and a countable ordinal \( \alpha \) for which \( A \in J_\alpha \).

**Theorem 8.** (a) The following two statements are equivalent: (i) For each subset \( Z \) of \( X \times X \) there is a countable ordinal \( \alpha \) such that \( Z \in \mathcal{R}_\alpha \). (ii) If \( H \) is a family of subsets of \( X \) and \( \text{Card} \, H = \text{Card} \, X \), then there is a countable family \( G \) of subsets of \( X \) and a countable ordinal \( \alpha \) for which \( H \subseteq G_\alpha \).

(b) If \( \alpha \) is a countable ordinal, the following two statements are equivalent: (i) Each subset of \( X \times X \) is a member of \( \mathcal{R}_\alpha \).
(ii) If $H$ is a family of subsets of $X$ and $\text{Card } H = \text{Card } X$ then there is a countable family $G$ of subsets of $X$ for which $H \subseteq G_s$.

Proof. The proof of part (b) is similar to that of part (a) which is given below.

First suppose (i) holds, and suppose that $H$ satisfies the hypotheses of (ii). Define the subset $Z = X \times X$ by letting each member of $H$ be a vertical section of $Z$. More precisely, let $f$ be a 1-1 function from $X$ to $H$ and let

$$Z = \bigcup_{x \in X} [x] \times f(x).$$

By (i) there is a countable ordinal $\alpha$ such that $Z \in R_\alpha$ and hence by Lemma 6, there is a countable subfamily $J$ of $R$ for which $Z \in J_\alpha$. Let

$$G = \{B: A \times B \in J\},$$

note that $Z \in (VG)_\alpha$ and use Theorem 3 to conclude that $H \subseteq G_\alpha$.

Now suppose (ii) holds, and that $Z \subseteq X \times X$. Let $H$ be the family of vertical sections of $Z$, and use (ii) to secure a countable family $G$ and a countable ordinal $\alpha$ for which $H \subseteq G_\alpha$. Thus $Z \in G_\alpha$ for each $x \in \text{domain } Z$ and by Theorem 3

$$Z \in (VG)_\alpha \subseteq R_\alpha.$$

THEOREM 9. The following four statements are equivalent:

(i) Each subset of $X \times X$ is a member of $B(R)$.

(ii) If $H$ is a family of subsets of $X$ and $\text{Card } H = \text{Card } X$ then there is a countable family $G$ and a countable ordinal $\alpha$ for which $H \subseteq G_\alpha$.

(iii) There is a countable ordinal $\alpha$ such that, for each family $H$ of subsets of $X$ with $\text{Card } H = \text{Card } X$, there is a countable family $G$ for which $H \subseteq G_\alpha$.

(iv) There is a countable ordinal $\alpha \geq 2$ such that each subset of $X \times X$ is a member of $R_\alpha$.

Proof. Statements (i) and (ii) are equivalent by Lemma 7 and Theorem 8a. Clearly (iii) implies (ii) and (iv) implies (i). Also by Theorem 8b it follows that (iii) implies (iv). $\alpha$ cannot be equal to 1 in (iv) because by (i) the identity function $f(x) = x$ is not in $R_\alpha$.

We complete the proof by showing that (ii) implies (iii). Since this result is immediate if $X$ is countable we will assume that $\text{Card } X \geq \omega_1$.

Suppose that (ii) holds and that (iii) does not. Then for each $\alpha < \omega_1$, there is a family $H(\alpha)$ of subsets of $X$ for which $\text{Card } H(\alpha) =$
Card $X$ and

\[(1)\]

for each countable $G$, $H(\alpha) \subseteq G_\alpha$.

Let $H' = \bigcup_{\alpha \in \omega_1} H(\alpha)$. Thus Card $H' = \text{Card} X$ and hence by (ii) there is a countable family $G'$ and a countable ordinal $\alpha'$ for which $H' \subseteq G'_{\alpha'}$. But then $H(\alpha') \subseteq H' \subseteq G'_{\alpha'}$, in contradiction of (1).

Therefore (ii) implies (iii).

In part (ii) above the family $G$ can be chosen so that $G_{\omega_1}$ is closed to complementation (i.e., is a $\sigma$-algebra).

In view of condition (ii) of Theorem 9, it is interesting to note that R. Mansfield has shown that if $G$ is a countable family of Lebesgue measurable sets, then $B(G)$ does not contain all analytic sets [5].

As was mentioned in the introduction it would be interesting to know whether the formula "$H \subseteq G_\beta$" in Theorem 9 could be replaced by $H \subseteq \mathcal{B}(G)$. We do not know the answer to this question.

**Theorem 10.** If CH holds, Card $X = c$, $H$ is a family of subsets of $X$, and Card $H = c$, then there is a countable family $G$ for which $H \subseteq G_\beta$.

**Proof.** By Theorem 5 each subset $Z$ of $X \times X$ is a member of $\mathcal{B}$, the desired conclusion now follows from Theorem 8b.

4. Generating Borel sets. Let $R$ be the set of reals, and let $H$ be the family of all Borel subsets of $R$. This family has cardinality $c$. Suppose $G$ is a countable family of subsets of $R$ such that $H \subseteq G_\beta$, and $G_\beta$ is closed to complementation. The next two theorems show that, even if the family $G$ generates all the Borel sets at an early stage, the order of $G$ is $\omega_1$. This is a generalization of the well known result [4, 9] that if $G$ is a countable basis for the real topology then $G$ has order $\omega_1$. Our proof which is a usual "diagonal" type argument, parallels somewhat Lebesgue's proof of that result [3, p. 368].

Let $G = \{V_1, V_2, V_3, \ldots\}$, let $N$ be the set of irrational numbers between 0 and 1 and let $K$ be the family $\{\theta_1, \theta_2, \theta_3, \ldots\}$ of all intersections of the members of $G$ with $N$,

\[\theta_i = V_i \cap N .\]

It will be shown that the order of $K$ is $\omega_1$. It then follows that the order of $G$ is $\omega_1$.

For each $z \in N$, let $(x_0, x_1, x_2, \ldots)$ be the sequence of integers appearing in the continued fraction expansion of $z$. This defines a
reversible transformation from \( N \) onto the set of all sequences of positive integers. Let

\[
\begin{align*}
  x^1 &= (x_1, x_2, x_3, \ldots) \quad \text{(odd indices)} \\
  x^2 &= (x_2, x_3, x_4, \ldots) \\
  x^3 &= (x_3, x_4, x_5, \ldots) \\
  &\vdots \\
  x^n &= (x_{n-1}, x_n, x_{n+1}, \ldots)
\end{align*}
\]

This defines a homeomorphism between \( N \) and \( N^\omega \) (see Kuratowski [3], p. 369). Also note that if \( f \) is a continuous function from \( N \) into \( N \), then the functions \( f_n \) from \( N \) into the space of positive integers are continuous, where

\[
f(z) = (f_1(z), f_2(z), f_3(z), \ldots), \quad \text{or} \quad (f_n(z) = f(z)_n).
\]

Recall that \( K = \{\theta_1, \theta_2, \theta_3, \ldots\} \). The family \( K \), which appears in Theorem 11 is defined in § 2.

**Theorem 11.** For each countable ordinal \( \alpha, \alpha > 0 \), there is a function \( U_\alpha \) from \( N \) onto \( K_\alpha \) such that if \( f \) is a continuous function from \( N \) into \( N \), then the set

\[
A_f = \{z: z \in U_\alpha(f(z))\}
\]

is in \( \mathcal{G}(K) \).

**Proof.** Let \( U_\alpha(z) = \bigcup_{i=0}^{\infty} \theta_{\alpha(i)} \), for each \( z \in N \). Clearly \( U_\alpha \) maps \( N \) onto \( K_\alpha \).

Let \( f \) be a continuous function from \( N \) onto \( N \). We have

\[
A_f = \{z: z \in U_\alpha(f(z))\}
= \{z: z \in \bigcup_{i=1}^{\infty} \theta_{f_n(z)}\}
= \bigcup_{i=1}^{\infty} \{z: z \in \theta_{f_n(z)}\}.
\]

For each \( n \),

\[
\{z: z \in \theta_{f_n(z)}\} = \bigcup_{i=n}^{\infty} \{J_{\alpha(i)} \cap \theta_i\}
\]

where \( J_{\alpha(i)} = \{z: f_n(z) = i\} \). Since each \( f_n \) is continuous it follows that each \( J_{\alpha(i)} \) is open and therefore the set \( A_f \) belongs to \( G_{\alpha(i)} \).

Suppose \( 1 < \alpha < \omega \), and suppose that the function \( U_\gamma \) has been defined for each ordinal \( \gamma \) with \( 1 \leq \gamma < \alpha \). (Induction hypothesis.)

If \( \alpha \) is odd, let
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\[ U_\alpha(z) = \bigcup_{\kappa=1}^\infty U_{\alpha\cdot}(z^\kappa), \text{ for } z \in \mathbb{N}. \]

Clearly \( U_\alpha \) maps \( \mathbb{N} \) onto \( K_\alpha \).

If \( \alpha \) is even, let \( \{\gamma_n\}_{n=1}^\infty \) be a sequence of odd ordinals less than \( \alpha \) such that each odd ordinal less than \( \alpha \) appears infinitely often in \( \{\gamma_n\}_{n=1}^\infty \) and let

\[ U_\alpha(z) = \bigcap_{n=1}^\infty U_{\gamma_n}(z^\gamma), \text{ for } z \in \mathbb{N}. \]

If \( A \in K_\alpha \) (\( \alpha \) is still even), then

\[ A = \bigcap_{n=1}^\infty D_n, \]

where \( D_n \in K_{\gamma_n} \), for each \( n \). For each \( n \), let \( y_n \) be a point of \( \mathbb{N} \) such that

\[ D_n = U_{\gamma_n}(y_n). \]

And let \( z \) be the point mapped by the transformation described by (*) to the point \( (y_n, y_n, y_n, \ldots) \) of \( \mathbb{N}^\omega \). Thus

\[ U_\alpha(z) = A \]

and \( U_\alpha \) maps \( \mathbb{N} \) onto \( K_\alpha \).

This completes the definition of the functions \( U_\alpha \). Now let \( f \) be a continuous function from \( \mathbb{N} \) into \( \mathbb{N} \). It will be shown that if \( \alpha \) is even the set

\[ A_f = \{z: z \in U_\alpha(f(z))\} \]

is in \( G_\omega \). The argument for the case \( \alpha \) is odd is similar.

We have

\[ A_f = \{z: z \in \bigcap_{n=1}^\infty U_{\gamma_n}((f(z))^\gamma)\} \]

\[ = \bigcap_{n=1}^\infty \{z: z \in U_{\gamma_n}((f(z))^\gamma)\}. \]

But, for each \( n \), the function \( z \to (f(z))^\gamma \), being the composition of two continuous functions, is a continuous function from \( \mathbb{N} \) to \( \mathbb{N} \).

Thus by the induction hypothesis, the sets \( \{z: z \in U_{\gamma_n}((f(z))^\gamma)\} \) are in the family \( G_{\omega_1} \). Therefore \( A_f \in G_\omega \).

**Theorem 12.** If \( G \) is a countable family of subsets of real numbers with \( \mathcal{G}(G) \subseteq G \), and each Borel set is a member of \( \mathcal{B}(G) \) then \( G \) has order \( \omega_1 \).
Proof. Let \( \alpha \) be any countable ordinal, and let

\[ I_\alpha = \{ z : z \in U_\alpha(z) \}. \]

Suppose \( I_\alpha \in K_\alpha \), and let \( U_\alpha(z) = I_\alpha \). If \( z \in I_\alpha \) then \( z \in U_\alpha(z) \). But this contradicts the definition of \( I_\alpha \). If \( z \notin I_\alpha \), then \( z \notin U_\alpha(z) = I_\alpha \), \( z \notin I_\alpha \). This contradiction shows that \( I_\alpha \notin K_\alpha \).

Since \( \mathcal{G}(G) = G_{\omega_1} \) (because \( \mathcal{G}(G) \subseteq G \)), and \( I_\alpha = \{ z : z \in U_\alpha(z) \} \in G_{\omega_1} \), by Theorem 11, it follows that \( I_\alpha \in G_{\omega_1} - G_\omega \). Thus \( G_\alpha \neq G_{\omega_1} \), and hence \( G \) has order \( \omega_1 \) [3, p. 371].

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