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SCALING HAUSDORFF MEASURES

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In this note, we investigate those Hausdorff measures which obey a simple scaling law. Consider a continuous increasing function θ defined on $\mathbb{R}^+$ with $\theta(0)=0$ and let $\mathcal{H}^\theta$ be the corresponding Hausdorff measure. We say that $\mathcal{H}^\theta$ obeys an order α scaling law provided whenever $K \subset \mathbb{R}^m$ and $c>0$, then

$$\mathcal{H}^\theta(cK)=c^\alpha \mathcal{H}^\theta(K); \tag{1}$$

or, equivalently, if T is a similarity map of $\mathbb{R}^m$ with similarity ratio c:

$$\mathcal{H}^\theta(TK)=c^\alpha \mathcal{H}^\theta(K).$$

Clearly, it would be interesting to characterize the functions θ for which $\mathcal{H}^\theta$ obeys a scaling law. We verify that if θ is of the form $\theta(t)=t^\alpha L(t)$, where L is *slowly varying* in the sense of Karamata [5], then (1) holds. Within a particular class of functions θ, we shall prove the converse. Specifically, we prove the following theorem.

THEOREM. *Let θ be a continuous increasing map of $\mathbb{R}^+$ into $\mathbb{R}^+$ with $\theta(0)=0$ such that θ is strictly concave down on a right neighborhood of* 0: *there is some $d>0$ such that if $0 \leqslant x<y<d$ and $0<t<1$,*

$$\theta(tx+(1-t)y)>t\theta(x)+(1-t)\theta(y).$$

Let $0 \leqslant \alpha \leqslant 1$. The following three statements are equivalent:

(i) *there is a slowly varying function L such that $\theta(t)=t^\alpha L(t)$*;

(ii) *if $c>0$, then $\lim_{t\to 0}\theta(ct)/\theta(t)=c^\alpha$*;

and

(iii) *if $K \subset \mathbb{R}^m$ and $c>0$, then $\mathcal{H}^\theta(cK)=c^\alpha \mathcal{H}^\theta(K)$.*

Note that since $\alpha \leqslant 1$, the only important case in (iii) is $m=1$. A complete general characterization even for $\mathbb{R}^1$ remains to be carried out. Since we are interested in the behavior of θ at 0, we take L to be "*slowly varying*" to mean that $L(t)>0$ *for* $t>0$, *L is continuous on the positive reals and for each* $\lambda>0$,

$$\lim_{x\to 0} L(\lambda x)/L(x)=1.$$

Let us note some general facts.

THEOREM 1. *Suppose* (1) *holds. Then, for all* $c>0$,

$$\varliminf_{t\to 0}\frac{\theta(ct)}{\theta(t)} \leqslant c^\alpha. \tag{2}$$

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Proof. Let E be a set with $0<\mathscr{H}^{\theta}(E)<+\infty$ [4, p. 122]. Suppose there is some $c>0$ such that

$$\varliminf_{t\to 0}\frac{\theta(ct)}{\theta(t)}\geqslant Ac^{\alpha},$$

with $A>B>1$. Then for $t<\varepsilon_0$, we have $\theta(ct)>Bc^{\alpha}\theta(t)$. If $\varepsilon<\varepsilon_0$, then $\mathscr{G}$ is an ε-cover of E, if, and only if, $c\mathscr{G}$ is a $c\varepsilon$-cover of cE. Therefore,

$$\begin{aligned}\mathscr{H}^{\theta}_{c\varepsilon}(cE)&=\inf_{\mathscr{G}}\left\{\sum_{\mathscr{G}}\theta(c|G|)\text{: }\mathscr{G}\text{ is an }\varepsilon\text{-cover of }E\right\}\\&\geqslant Bc^{\alpha}\inf_{\mathscr{G}}\left\{\sum_{\mathscr{G}}\theta(|G|)\text{: }\mathscr{G}\text{ is an }\varepsilon\text{-cover of }E\right\}\\&\geqslant Bc^{\alpha}\mathscr{H}^{\theta}_{\varepsilon}(E).\end{aligned}$$

Thus, $c^{\alpha}\mathscr{H}^{\theta}(E)=\mathscr{H}^{\theta}(cE)\geqslant Bc^{\alpha}\mathscr{H}^{\theta}(E)$. Or, $1\geqslant B$, a contradiction.

If the inequality in (2) is an equality, then much more is true.

LEMMA 2. *Suppose that for all* $c>0$

$$\varliminf_{t\to 0}\frac{\theta(ct)}{\theta(t)}=c^{\alpha}. \tag{3}$$

Then, for all $c>0$,

$$\lim_{t\to 0}\frac{\theta(ct)}{\theta(t)}=c^{\alpha}. \tag{4}$$

Proof. Set $t'=ct$. Thus,

$$\varliminf_{t'\to 0}\frac{\theta(t')}{\theta(t'/c)}=c^{\alpha}.$$

Or,

$$\varlimsup_{t'\to 0}\frac{\theta((1/c)t')}{\theta(t')}=(1/c)^{\alpha}.$$

Therefore, for $c>0$,

$$\varlimsup_{t\to 0}\frac{\theta(ct)}{\theta(t)}=c^{\alpha},$$

and the lemma follows.

But the functions which satisfy (4) are simply characterized.

THEOREM 3. *Express* θ *as* $\theta(t)=t^{\alpha}L(t)$. *The following two statements are equivalent:*

(1) $L(t)$ *is slowly varying: for all* $c>0$,

$$\lim_{t\to 0}L(ct)/L(t)=1; \tag{5}$$

(2) *for all* $c>0$,

$$\lim_{t\to 0}\theta(ct)/\theta(t)=c^{\alpha}. \tag{6}$$

Next, we note that the slowly varying perturbations of t^α do obey the scaling law (1).

THEOREM 4. *Suppose $L(t)$ is slowly varying and $\theta(t)=t^\alpha L(t)$. If $K\subset\mathbb{R}^m$ and $c>0$, then*

$$\mathcal{H}^\theta(cK)=c^\alpha\mathcal{H}^\theta(K). \tag{1}$$

Proof. According to Theorem 3, for ε sufficiently small and $t<\varepsilon$, we have

$$\theta(ct)<c^\alpha\theta(t)+M(\varepsilon)\theta(t),$$

where $M(\varepsilon)\to 0$ as $\varepsilon\to 0$. If $\mathcal{G}$ be an (ε/c)-cover of K, then $c\mathcal{G}$ is an ε-cover of cK and

$$\sum_{G\in\mathcal{G}}\theta(c\operatorname{diam}G)\leqslant c^\alpha\sum_{G\in\mathcal{G}}\theta(\operatorname{diam}G)+M(\varepsilon)\sum_{G\in\mathcal{G}}\theta(\operatorname{diam}G).$$

Thus,

$$\mathcal{H}^\theta_{\varepsilon/c}(cK)\leqslant c^\alpha\mathcal{H}^\theta_\varepsilon(K)+M(\varepsilon)\mathcal{H}^\theta_\varepsilon(K). \tag{7}$$

Letting ε go to zero in (7), we have

$$\mathcal{H}^\theta(cK)\leqslant c^\alpha\mathcal{H}^\theta(K).$$

Equation (1) follows from this last inequality.

The main goal of the remainder of this note is to prove a partial converse to Theorem 4.

THEOREM 5. *Let θ be a continuous increasing map of $\mathbb{R}^+$ into $\mathbb{R}^+$ with $\theta(0)=0$ such that θ is strictly concave down on a right neighborhood of 0: there is some $d>0$ such that if $0\leqslant x<y<d$ and $0<t<1$,*

$$\theta(tx+(1-t)y)>t\theta(x)+(1-t)\theta(y).$$

Let $0\leqslant\alpha\leqslant 1$. The following four statements are equivalent.

(i) *There is a slowly varying function L such that*

$$\theta(t)=t^\alpha L(t).$$

(ii) *If $c>0$, then*

$$\lim_{t\to 0}\theta(ct)/\theta(t)=c^\alpha.$$

(iii) *If $K\subset\mathbb{R}^m$ and $c>0$, then*

$$\mathcal{H}^\theta(cK)=c^\alpha\mathcal{H}^\theta(K).$$

(iv) *If $K\subset\mathbb{R}^1$ and $c>0$, then*

$$\mathcal{H}^\theta(cK)=c^\alpha\mathcal{H}^\theta(K).$$

Proof. Theorem 3 demonstrates the equivalence of statement (i) to statement (ii) and Theorem 4 shows statement (ii) implies statement (iii).

Assume statement (iv) holds. Note that the strict concavity of θ on the interval $(0,d]$ implies $\theta(t)/t$ is decreasing on $(0,d]$. If $\lim_{t\to 0}\theta(t)/t$ is finite, then statement two holds with $\alpha=1$ and we have linear scaling: $c\mathscr{H}^\theta(E)=\mathscr{H}^\theta(cE)$. Therefore, we assume from this point on that

$$\lim_{t\to 0}\theta(t)/t=+\infty.$$

In other words, we are assuming θ corresponds to a smaller generalized dimension than linear Hausdorff measure [4, p. 78]. At this point we interrupt the proof of Theorem 5 to verify the following properties of θ. These properties are needed in order to construct a special Cantor set K such that $0<\mathscr{H}^\theta(K)<\infty$. Our construction harks back to a construction given in Hausdorff's original paper [3] and several related constructions by Rogers [4], Dvoretzsky [1] and Fekete. However, our constructed set requires some additional scaling properties.

LEMMA 6. *Let $\{z_n\}_{n=0}^\infty$ be a decreasing sequence converging to 0. Let $\{\varepsilon_n\}_{n=0}^\infty$ decrease to 0 with $\sum_{n=0}^\infty \varepsilon_n<\frac{1}{2}$. Then there is a subsequence $\{x_n\}_{n=0}^\infty$ of $\{z_p\}_{p=0}^\infty$ and a sequence of positive integers $\{m_n\}_{n=1}^\infty$ such that*

$$\theta(x_n)/\theta(x_{n+1})\geqslant 4, \qquad n=0,1,2,\ldots, \tag{8}$$

$$[x_n/x_{n+1}]\geqslant 4, \qquad n=0,1,2,\ldots, \tag{9}$$

$$1<2m_{n+1}\leqslant[x_n/x_{n+1}], \qquad n=0,1,2,\ldots, \tag{10}$$

$$1-\varepsilon_n<m_{n+1}\theta(x_{n+1})/\theta(x_n)\leqslant 1. \tag{11}$$

Proof. Set $x_0=z_0$. Since $\theta(x_0)/\theta(z)\to+\infty$ as $z\to 0$, $[x_0/z]\to+\infty$ as $z\to 0$, and $(\theta(z)/z)/(\theta(x_0)/x_0)\to+\infty$ as $z\to 0$, there is some d_0 such that if $0<z<d_0$, then

$$\theta(x_0)/\theta(z)\geqslant 4,$$

$$\theta(z)/\theta(x_0)<\varepsilon_0/2=\min(\varepsilon_0/2, 1-\varepsilon_0/2)<\tfrac{1}{2},$$

$$[x_0/z]\geqslant 4,$$

and

$$(\theta(z)/z)/(\theta(x_0)/x_0)>3.$$

Set $x_1=z_{n_1}$, where $z_{n_1}<d_0$. Now, we define m_1. Since,

$$\theta(x_1)/\theta(x_0)<\varepsilon_0/2<\tfrac{1}{2}<1-\varepsilon_0$$

and

$$(\theta(x_1)/\theta(x_0))(x_0/x_1)=(\theta(x_1)/\theta(x_0))[x_0/x_1]+(\theta(x_1)/\theta(x_0))(x_0/x_1-[x_0/x_1]),$$

we have

$$(\theta(x_1)/\theta(x_0))[x_0/x_1]>(\theta(x_1)/\theta(x_0))(x_0/x_1)-\theta(x_1)/\theta(x_0)>2.$$

Let m_1 be the integer such that

$$m_1\theta(x_1)/\theta(x_0)\leqslant 1<(m_1+1)\theta(x_1)/\theta(x_0).$$

Certainly, the inequalities of (10) and the second inequality of (11) hold.

Finally, if $m_1\theta(x_1)/\theta(x_0)$ were not to exceed $1-\varepsilon_0$, then

$$\varepsilon_0 \leq 1 - m_1\theta(x_1)/\theta(x_0) < \theta(x_1)/\theta(x_0) < \varepsilon_0/2.$$

This contradiction affirms that the first inequality of (11) holds.

The process can clearly be continued.

Construction. We turn now to the construction of a special Cantor set K based on the sequences $\{m_n\}$ and $\{x_n\}$ of Lemma 6. Let

$$\Omega = \prod_{i=1}^{\infty} \{1, \ldots, m_i\}$$

and

$$\Omega^* = \bigcup_{p=0}^{\infty} \prod_{i=1}^{p} \{1, \ldots, m_i\}.$$

Let

$$\{J_\sigma : \sigma \in \Omega^*\}$$

be the Cantor scheme defined by setting

$$J_\varnothing = [0, x_0],$$

and the recursion: If $J_\sigma = [a(\sigma), b(\sigma)]$ has been defined with $|\sigma| = n$ and $|J_\sigma| = x_n$, then let $\{J_{\sigma * i}\}_{i=1}^{m_{n+1}}$ be subintervals of J_σ, each of length x_{n+1}, such that $J_\sigma \setminus \bigcup J_{\sigma * i}$ consists of $m_{n+1}-1$ pairwise disjoint intervals, each of length

$$g_{n+1} = (x_n - m_{n+1}x_{n+1})/(m_{n+1}-1), \qquad n = 0, 1, \ldots. \tag{12}$$

It follows from condition (10) that $g_n > x_n > g_{n+1}$, $n = 1, 2, 3, \ldots$. Assume these intervals are labelled so that $J_{\sigma * i}$ is to the left of $J_{\sigma * j}$, if $1 \leq i < j \leq m_{n+1}$. Let

$$K = \bigcap_n \left[\bigcup_{\substack{\sigma \in \Omega^* \\ |\sigma| = n}} J_\sigma \right]$$

and let μ be the probability measure defined on K by setting, for each σ with $|\sigma| \geq 1$,

$$\mu(K \cap J_\sigma) = \prod_{i=1}^{|\sigma|} (1/m_i).$$

Set $P_0 = \theta(x_0)$ and for $n \geq 1$,

$$P_n = \left(\prod_{i=1}^{n} m_i \right) \theta(x_n).$$

Note $P_{n+1}/P_n = m_{n+1}\theta(x_{n+1})/\theta(x_n)$. By (10), $P_{n+1} \leq P_n$. Let

$$a_n = 1 - P_{n+1}/P_n = 1 - m_{n+1}\theta(x_{n+1})/\theta(x_n).$$

Then $\sum_{n=1}^{\infty} |a_n| < \infty$. Thus, $S = \sum \log(1-a_n)$ exists and

$$\lim_{n\to\infty} P_n = \lim_{n\to\infty} \prod_{i=1}^{n} (1-a_n) = B = e^S.$$

Thus,

$$0 < \lim_{n\to\infty} \left(\prod_{i=1}^{n} m_i \right) \theta(x_n) = B < \infty.$$

Claim 1. $\mathscr{H}^\theta(K) \leqslant B$. This can be seen from the estimates obtained from the covers of K by $\{J_\sigma : |\sigma| = n\}$

$$\mathscr{H}^\theta(K) \leqslant \lim_{n\to\infty} \left(\prod_{i=1}^{n} m_i \right) \theta(x_n) = B.$$

Claim 2. For all $\sigma, \tau \in \Omega^*$, if $a(\sigma) < b(\tau)$, then

$$\theta(b(\tau) - a(\sigma)) \geqslant B\mu([a(\sigma), b(\tau)]). \tag{13}$$

We will prove this claim by induction. But, first let us check it for some special cases. For example, if $\tau = \sigma$, then

$$\theta(b(\tau) - a(\sigma)) = \theta(x_{|\sigma|}).$$

Since $\prod_{i=1}^{n} m_i \theta(x_n) = P_n \geqslant B$, we have

$$\theta(x_n) = \theta(b(\sigma) - a(\sigma)) \geqslant B\mu([a(\sigma), b(\sigma)]).$$

This is inequality (13) for this case. Another special case arises when one of σ and τ is an immediate successor of the other. Without loss of generality, assume $|\sigma| = n$ and $\tau = \sigma * i$. Then

$$b(\tau) - a(\sigma) = ix_{n+1} + (i-1)g_{n+1}.$$

We have

$$b(\tau) - a(\sigma) = \lambda x_n + (1-\lambda)x_{n+1},$$

where

$$\lambda = (i-1)(g_{n+1} + x_{n+1})/(x_n - x_{n+1}). \tag{14}$$

Thus,

$$\begin{aligned}
\theta(b(\tau) - a(\sigma)) &> \lambda\theta(x_n) + (1-\lambda)\theta(x_{n+1}) \\
&= \lambda\theta(x_n) + [(1-\lambda)/m_{n+1}]m_{n+1}\theta(x_{n+1}) \\
&= \left(1\Big/\prod_{j=1}^{n} m_j\right)\left[\lambda \prod_{j=1}^{n} m_j\theta(x_n) + (1-\lambda)/m_{n+1} \prod_{j=1}^{n+1} m_j\theta(x_{n+1})\right] \\
&\geqslant \left(1\Big/\prod_{j=1}^{n} m_j\right) B(\lambda + (1-\lambda)/m_{n+1}) \\
&= \left(1\Big/\prod_{j=1}^{n+1} m_j\right) B(m_{n+1}\lambda + (1-\lambda)).
\end{aligned}$$

But,

$$m_{n+1}\lambda + (1-\lambda) = \lambda(m_{n+1} - 1) + 1. \tag{15}$$

Since

$$x_n - x_{n+1} = (m_{n+1} - 1)(x_{n+1} + g_{n+1}), \tag{16}$$

we have, from (14), (15) and (16)

$$m_{n+1}\lambda+(1-\lambda)=i.$$

Therefore,

$$\theta(b(\tau)-a(\sigma))\geqslant Bi\Big/\prod_{j=1}^{n+1}m_j=B\mu([a(\sigma),b(\tau)]),$$

which again is inequality (13).

Now, let us continue with the induction argument. Suppose that for all κ, λ with $|\kappa|,|\lambda|\leqslant n$ and $a(\kappa)<b(\lambda)$:

$$\theta(b(\lambda)-a(\kappa))\geqslant B\mu([a(\kappa),b(\lambda)])$$

and $|\sigma|,|\tau|\leqslant n+1$. We will prove inequality (13) for σ and τ. Note that we can and do assume that $|\sigma|,|\tau|\geqslant n$, since, if one of them has length less than n, it can be extended by 1's to a sequence of length n without changing the endpoint coded by the sequence.

Case 1. $|\sigma|=n$, $|\tau|=n+1$. We can assume that $J_{\tau|n}$ lies to the right of J_σ and $i=\tau(n+1)<m_{n+1}$. Let $\hat{\tau}$ be the sequence of length n such that $J_{\hat{\tau}}$ is the interval on level n immediately to the left of $J_{\tau|n}$. There is a gap between $J_{\hat{\tau}}$ and $J_{\tau|n}$ of size at least g_n.

Thus,

$$\begin{aligned} b(\tau)-a(\sigma)&=\lambda w+(1-\lambda)(b(\tau|n)-a(\sigma))\\ &\geqslant b(\hat{\tau})-a(\sigma)+g_n+ix_{n+1}+(i-1)g_{n+1}, \end{aligned}\tag{17}$$

where $w=b(\hat{\tau})-a(\sigma)$ and $W=[a(\sigma),b(\hat{\tau})]$.

By the concavity of θ,

$$\begin{aligned} \theta(b(\tau)-a(\sigma))&\geqslant\lambda\theta(w)+(1-\lambda)\theta(b(\tau|n)-a(\sigma))\\ &\geqslant\lambda B\mu(W)+(1-\lambda)B\mu([a(\sigma),b(\tau|n)])\\ &\geqslant B(\lambda\mu(W)+(1-\lambda)\mu([a(\sigma),b(\tau|n)])). \end{aligned}$$

In order to prove inequality (13), it suffices to show

$$\lambda\mu(W)+(1-\lambda)\mu([a(\sigma),b(\tau|n)])\geqslant\mu(W)+i\left(\prod_{j=1}^{n+1}1/m_j\right).\tag{18}$$

Or, simplifying,

$$\begin{aligned} (1-\lambda)\left[\prod_{j=1}^{n}(1/m_j)\right]&=(1-\lambda)[\mu([a(\sigma),b(\tau|n)])-\mu([a(\sigma),b(\hat{\tau})])]\\ &\geqslant\prod_{j=1}^{n+1}(1/m_j). \end{aligned}$$

In other words, it suffices to show

$$(1-\lambda)\geqslant i/m_{n+1}.\tag{19}$$

Or,

$$(m_{n+1}-i)/m_{n+1}\geqslant\lambda.$$

Solving (17) for λ and estimating, we have

$$\lambda=(b(\tau|n)-b(\tau))/(b(\tau|n)-b(\hat{\tau}))\leq(m_{n+1}-i)(x_{n+1}+g_{n+1})/(x_n+g_n).$$

Thus, for inequality (13) or, equivalently, inequality (19) to hold, it suffices that

$$x_n+g_n\geq m_{n+1}(x_{n+1}+g_{n+1})=x_n+g_{n+1}.$$

But, again, according to (10), $g_n>g_{n+1}$ and inequality (13) holds.

Case 2. $|\sigma|=n+1$, $|\tau|=n+1$. Assume $J_{\sigma|n}$ lies to the left of $J_{\tau|n}$.

If $a(\sigma)=a(\sigma|n)$ or $b(\tau)=b(\tau|n)$, then (13) follows from the preceding cases. So, assume $1<\sigma(n+1)$ and $\tau(n+1)<m_{n+1}$. First, suppose $\sigma(n+1)>\tau(n+1)$. Let σ' be the element of Ω^* of length n such that $J(\sigma')$ is the first interval on level n to the right of $J(\sigma|n)$ and let τ' be the sequence of length n such that $\tau'|n=\tau|n$ and $\tau'(n+1)=\tau(n+1)+m_{n+1}-\sigma(n+1)$. Then

$$\mu([a(\sigma),b(\tau)])=\mu([a(\sigma'),b(\tau')])$$

and

$$\theta(b(\tau)-a(\sigma))\geq\theta(b(\tau')-a(\sigma')).$$

By the preceding case,

$$\begin{aligned}\theta(b(\tau)-a(\sigma))&\geq B\mu([a(\sigma'),b(\tau')])\\&=B\mu([a(\sigma),b(\tau)]).\end{aligned}\tag{13}$$

Finally, suppose $\sigma(n+1)\leq\tau(n+1)$. As before, identify in order the intervals on level $n+1$ inside $J_{\sigma|n}$ beginning with J_σ with the intervals on level $n+1$ inside $J_{\tau|n}$ beginning with the one immediately to the right of J_τ. The intervals in $J_{\tau|n}$ to the right of J_τ will be all matched and some intervals in J_σ will remain. However, our new interval has the same μ measure and the length of the interval has remained unchanged. Moreover, the end points of the new interval are of the form considered in the first case. This completes the argument for Claim 2.

Now, we return to the proof of Theorem 5. Fix $c>0$ and let the sequence $\{z_n\}$ decrease to zero with

$$\lim_{n\to\infty}\frac{\theta(cz_n)}{\theta(z_n)}=\varliminf_{n\to\infty}\frac{\theta(ct)}{\theta(t)}.$$

Consider the set K just constructed. We have

$$\mathcal{H}^\theta(K)=B.$$

According to the assumed scaling property:

$$\mathcal{H}^\theta(cK)=c^\alpha B.$$

Using the scaled covers $\{cJ_\sigma: |\sigma|=n\}$, we have

$$c^\alpha B\leq\varliminf_{n\to\infty}\theta(cx_n)\prod_{i=1}^n m_i.$$

So,

$$c^{\alpha} \leqslant \varliminf_{n\to\infty} \frac{\theta(cx_n)}{\theta(x_n)}.$$

Thus, for each $c>0$,

$$c^{\alpha} \leqslant \varliminf_{n\to\infty} \frac{\theta(cx_n)}{\theta(x_n)} = \varliminf_{t\to 0} \frac{\theta(ct)}{\theta(t)}.$$

According to Lemma 2, this implies statement (ii). The proof of Theorem 5 is completed.

Problem. It remains to characterize the higher dimensional functions such that the corresponding Hausdorff measure scales. Perhaps one could begin by considering those functions θ such that for some positive integer m $\theta(t)/t^{m-1}$ or $\theta(t)^{1/m}$ is concave down.

Problem. There are continuum many non-Hausdorff measures μ in $\mathbb{R}^m$ which obey a scaling law: $\mu(cK)=c^{\alpha}\mu(K)$. For example, let $\mu(E)=0$ if E is meager, and let $\mu(E)=\infty$, otherwise. For each x, with $0\leqslant x\leqslant 1$, the measure $x\mu+(1-x)\mathcal{H}^{\alpha}$ scales correctly and is not a Hausdorff measure. Another group of measures which obey a scaling law are the packing measures [6]. However, although it seems true, we have not been able to prove that $\mathcal{P}^{\alpha}$ or more generally, $\mathcal{P}^{\theta}$ in $\mathbb{R}^m$ is not a Hausdorff measure. Haase [2] has some partial results for general complete separable metric spaces provided one can change the metric to topologically equivalent metrics. This much is true: $\mathcal{P}^{\alpha}\neq\mathcal{H}^{\theta}$, provided there is a number γ such that if $\dim_H(E)<\gamma$, then $\mathcal{H}^{\theta}(E)=0$, and if $\dim_H(E)>\gamma$, then $\mathcal{H}^{\theta}(E)=\infty$.

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