RIGOROUS MULTIFRACTAL ANALYSIS

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INTRODUCTION

The circumstance of having a fractal \( K \), together with a probability measure \( \rho \) on the fractal, allows us to think about a "multi-fractal", where, for example, we make use of that measure to specify a dimension,

\[
d_p(x) = \lim_{\epsilon \to 0} \frac{\log \rho(B(x,\epsilon))}{\log \epsilon},
\]

where \( \rho(B(x,\epsilon)) \) denotes the \( \rho \)-measure (one can imagine the measure as specifying a shade of gray between black and white) of a ball of radius \( \epsilon \) centered at \( x \), and \( d_p(x) \) is called the pointwise dimension of \( K \) relative to \( x \). Now define the "sub-fractal",

\[
K_\alpha = \{ x \in K : d_p(x) = \alpha \},
\]

which is the set of points \( x \) from \( K \) relative to which the pointwise dimension value \( \alpha \) is taken. Less restrictive possibilities than eq. (1) also exist, allowing for possibly greater generality in construction of \( K_\alpha \) for some cases. In general, the collection of all these "sub-fractal" \( K_\alpha \)'s, for \( \alpha \geq 0 \), may be thought of as a "multi-fractal." This notion first appeared in Ref. (1) for the context of modeling fluid turbulence. The remarkable theoretical scenario that \( f(\alpha) \), where

\[
f(\alpha) = \dim K_\alpha,
\]

is a smooth function of \( \alpha \), despite the decidedly non-smooth properties of the \( K \) and \( K_\alpha \), was laid out in Ref. 2. Moreover, \( f(\alpha) \) was argued to have a variety of special properties: (1) it is everywhere concave downwards; (2) its peak value is \( \dim K \); (3) \( f(\alpha) \) intersects the \( \alpha \)-axis with infinite slope, at positive and finite values; and (4) the line \( f(\alpha) = \alpha \) is tangent to \( f(\alpha) \) where \( f \) and \( \alpha \) are equal, and this value is the information dimension of \( K \) (or the dimension of the measure \( \rho \)). The \( f(\alpha) \) formalism has been used with success to model data in several contexts; \( f(\alpha) \) curves with one or more of the basic expected properties violated have been found; and the scheme has received widespread application as a means of organizing fractal data.

We have the first rigorous proofs of all the results described above for generalized Cantor sets (Moran fractals\(^3\), with the product measure \( \rho \) defined below.) And, we have proofs of when the limit of eq. (1) exists, with answers to some open, hitherto unanalyzed issues. In particular, is the collection of all \( K_\alpha \) equal to \( K \)? In other words, does the multi-fractal procedure get back the whole fractal? It assuredly does not; however, in the sense of \( \rho \)-measure, it does,

\[
\rho(\bigcup K_\alpha) = \rho(K).
\]

Finally, we have an example where the "fractal" \( K \) is the unit interval, but the collection of the \( K_\alpha \) is a Baire first category set (i.e. is topologically meager).
POSSIBLE MULTIFRACTAL GENERALIZATION

The way the $f(\alpha)$ curve is constructed for Moran fractals in $\mathbb{R}^n$ can be stated a little more precisely than has been possible in the early literature. For a set $(t_1, \ldots, t_n)$ of contracting similarity ratios, let $K$ be a Moran fractal constructed with these ratios from seed set $J$. (In the middle-thirds Cantor set, $n = 2$, $t_1 = t_2 = 1/3$ and $J = \left[0, 1\right]$.) It is important to note for the standard middle-thirds prototype not only are the ratios fixed but the similarity maps implementing the construction are fixed. The latter need not be so for the general Moran case.

Now, fix a probability vector $(p_1, \ldots, p_n)$ and let $\rho$ be the probability measure naturally defined on $K$ via redistribution. In other words, $\rho(J_i) = p_i$, where $J_i$, $i=1$ to $n$, are the sets obtained from $J$ by similarities with contraction ratios $t_i$.

The $J_i$ comprise the first generation of the construction of $K$. The sets obtained in successive generations of the construction are assigned probabilities which are products of the $p_i$'s in the natural way (product measure). The starting point for the $f(\alpha)$ construction is the auxiliary measure $\mu_q$, $q \in \mathbb{R}$, which is an infinite product measure but based on $(p_1 q^\beta(q), \ldots, p_n q^\beta(q))$, where

\[
\begin{align*}
\alpha & = 1.1 \quad 1.15 \quad 1.2 \quad 1.25 \quad 1.3 \quad 1.35 \quad 1.4 \quad 1.45 \\
\gamma & = 0.0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad 1.0 \quad 1.2 \quad 1.4
\end{align*}
\]

\textbf{FIGURE 1.} Generalized multifractal curves for a Moran construction.
\[
\sum_{i=1}^{n} p_i q_i = 1. \tag{5}
\]

A slight generalization of this is usually referred to as the partition function owing to parallels to statistical mechanics. The formula for \( f(\alpha) \) depends on \( \rho \), on the fractal set, and on the function \( \beta(q) \) uniquely solving eq. (5). We have proved the existence of a multi-fractal construction based on the generalized quantity \( \hat{\beta}(q,w) \) specified by the normalization of a slightly different probability vector
\[
\sum_{i=1}^{n} w_i p_i q_i \hat{\beta}(q,w), \quad w_1 > 0, \ldots, w_n > 0, \tag{6}
\]

and where \( w \) denotes the n-tuple \((w_1,...,w_n)\). Note that \( w = 1 \) is the usual case: i.e. \( \hat{\beta}(q,1) = \beta(q) \). The general properties of the \( \tilde{f}(\alpha, w) \) curve that results are no longer those laid out above for the \( f(\alpha) \) curve. We have several results about the generalized scheme. One of these is that the \( \tilde{f}(\alpha, w) \) curve is stationary under variation of \( w \) at \( w_1=...=w_n=1 \). The \( f(\alpha) \) curve is probably an absolute maximum, a conjecture confirmed by initial numerical studies.

In Fig. 1, we show a numerical study giving results of varying the \( w \). The measure and fractal parameter values chosen were \( n = 4, t_1=...=t_4 = 1/3, p_1 = 0.29, p_2 = p_3 = 0.25, p_4 = 0.21 \). The weights are, for the curve marked:

(1) \( w_1=...=w_4 = 1 \); (2) \( w_1 = w_2 = 1, w_3 = w_4 = 0.01 \); (3) \( w_1 = w_2 = 1, w_3 = w_4 = 0 \); and \( w_1 = w_2 = 0, w_3 = w_4 = 1 \). The last two, extreme cases are "forbidden" by the theorems; and case (3) (resp. (4)) is a horizontal translate of the \( f(\alpha) \) curve for a middle-thirds Cantor set having \( p_1 + p_2 \) (resp., \( p_3 + p_4 \)) normalized to one. Note in particular that only the first of the two permissible cases has given a curve concave downwards everywhere. Studies of the generalized multifractal theory are in progress. For example, we don't know yet whether the analogue of eq. (4) holds for a nontrivial weight system; and connections to statistical mechanics have to be investigated.

REFERENCES


