R. J. GARDNER AND R. DANIEL MAULDIN*

BIJECTIONS OF $\mathbb{R}^n$ ONTO ITSELF

ABSTRACT. We characterize affine and continuous maps within the class of bijections of $\mathbb{R}^n$ onto itself by the preservation of various geometric or topological figures. A characterization of similarity maps of Hilbert space is given.

1. INTRODUCTION

Several years ago, one of us (R. D. M.) rediscovered the following curious theorem: a bijection of $\mathbb{R}^n$ ($n \geqslant 2$, here and throughout) onto itself which maps each circle onto a circle must be an affine map. The result is one of many which characterizes maps by the preservation of particular classes of sets. Indeed, the subject is a classical one. We shall not attempt to disentangle its history here; however, the theorem stated above can be traced to work of Möbius and Darboux (see [9]), in which the fundamental theorem of projective geometry plays a central role.

The fundamental theorem states that a bijection of $\mathbb{R}^n$ onto itself which takes each three collinear points onto three collinear points is affine, and so is itself a result of the type we are considering. We use it to give a very short proof of the theorem concerning circle-preserving maps. The proof suggests the study of bijections of $\mathbb{R}^n$ which leave invariant geometric or topological figures, or which change shapes in some specified way, and we present new results of this sort.

To put these in perspective, here is a sample from the literature. Beckman and Quarles [4] show that maps taking pairs of points separated by unit distance onto congruent pairs must be rigid motions. There are various generalizations of this; see, for example, [12] and [14]. Other papers, such as [13] and [17], characterize affine maps as those which map convex sets onto convex sets. Finally, we mention that certain results in this area are, surprisingly, of interest to physicists working in relativity theory. The 'fundamental theorem of chronogeometry' of A. D. Alexandrov (see [2]) says that bijections of $\mathbb{R}^n$ ($n \geqslant 3$) onto itself which map light-cones onto light-cones are affine. Copious generalizations and related results may be found, both in the Russian literature ([3], [11], [16] and many others) and in Western journals ([5], [6], [15], [18], etc.) the two seemingly almost unaware of each other.

Our short proof mentioned above actually shows that bijections which

* Supported in part by a grant from the National Science Foundation.

Geometriae Dedicata 26 (1988), 323–332.

map each circle *into* a circle are affine. This stronger version is probably not new, but the relaxation of hypothesis raises interesting questions which have largely been ignored. (The only exception we know is [7], where it is shown that Alexandrov's theorem also holds with 'onto' replaced by 'into'.) For example, we show here that there are no bijections of $\mathbb{R}^n$ onto itself which map every quadrilateral into a circle, but there are bijections of $\mathbb{R}^2$ onto itself taking each circle into a quadrilateral. In the latter half of this paper, we also prove and apply some theorems which characterize continuous bijections and indeed similarity maps of Hilbert space.

We thank W. F. Pfeffer for useful discussions on these questions.

2. Results

THEOREM 1. *Let $\mathscr{C}$ be a class of sets in $\mathbb{R}^n$ such that every three non-collinear points are contained in a member of $\mathscr{C}$. Let $\mathscr{D}$ be a class of sets in $\mathbb{R}^n$ such that no member of $\mathscr{D}$ contains three collinear points. If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a bijection such that for each $C \in \mathscr{C}$, there is some $D \in \mathscr{D}$ with $f(C) \subset D$, then f is affine.*

Proof. Let a, b and c be three collinear points and suppose the points $f^{-1}(a)$, $f^{-1}(b)$ and $f^{-1}(c)$ are not collinear. Then there is some $C \in \mathscr{C}$ containing them and a $D \in \mathscr{D}$ such that $f(C) \subset D$. But then D contains a, b and c; a contradiction. By the fundamental theorem of projective geometry, f^{-1} is affine, and therefore f is also.

COROLLARY 2. *If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a bijection mapping every circle into a circle, then f is affine [and therefore, of course, a similarity].*

By a similarity we mean an element of the group generated by isometries and dilations. In [8] (see also [1]) it is shown that one-to-one maps from a region in $\mathbb{R}^2$ into $\mathbb{R}^2$ which sends circles onto circles are restrictions of Möbius transformations. This gives Corollary 2 for $n = 2$ with 'into' replaced by 'onto'. Maps defined on spheres in $\mathbb{R}^n$ sending circles into circles are considered in [10]. Although Corollary 2 is not explicitly stated in [10], it may be deduced from the results given there. The argument given here is shorter.

Many other corollaries to Theorem 1 may be formulated. We shall list only two. Note that by circles, triangles, etc., we refer to their boundaries only.

COROLLARY 3. *If $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a bijection mapping each circle into the boundary of a strictly convex body, then f is affine.*

COROLLARY 4. *There are no bijections* $f: \mathbb{R}^n \to \mathbb{R}^n$ *mapping triangles into circles, or each square into the boundary of some strictly convex body.*

As we mentioned in the Introduction, for $n \geqslant 3$, each bijection of $\mathbb{R}^n$ preserving light-cones (cones given by $\Sigma_{i=1}^{n-1} (x_i - a_i)^2 = (x_n - a_n)^2$, for some $(a_1, \ldots, a_n)$) is affine. For $n = 2$, this is not true. The map $f(x,y) = ((x + y)^3 - (x - y), (x + y)^3 + (x - y))$ preserves light-cones in $\mathbb{R}^2$ (see [15, Example 2.5]).

EXAMPLE 5. Let $\mathscr{E}$ be the class of squares in $\mathbb{R}^2$ whose sides are at 45° to the coordinates axes. Then there is a non-affine bijection of $\mathbb{R}^2$ which maps each $E \in \mathscr{E}$ onto some $E' \in \mathscr{E}$. (It is easy to see that the non-affine map f does this; note that f is continuous, however.)

Theorem 1 and its corollaries say nothing about bijections taking circles onto or into triangles, squares, etc., since the latter cannot serve for class $\mathscr{D}$. Theorem 6 and Example 7 address this question.

THEOREM 6. *Let* $f: \mathbb{R}^n \to \mathbb{R}^n$ *be a bijection which maps each circle onto the boundary of some convex body (of any dimension). Then* f *is affine.*

Proof. As in Theorem 1, we assume there are three collinear points, a, b and c, such that $f^{-1}(a), f^{-1}(b)$ and $f^{-1}(c)$ are not collinear. If C is a circle containing the latter points, then $f(C)$ is the boundary of some convex body, and each of a, b and c lie on a line segment in $f(C)$. Let l be the line through a, b and c and let $d \in l \setminus f(C)$.

Suppose that $f^{-1}(d)$ does not belong to the line through $f^{-1}(a)$ and $f^{-1}(b)$. Then there is a circle D through $f^{-1}(a), f^{-1}(b)$ and $f^{-1}(d)$. Thus, $f(D)$ is the boundary of some convex body containing a and b. But, $C \cap D$ contains only two points, whereas $f(C) \cap f(D)$ contains the line segment $[a, b]$, which is impossible.

Therefore, $f^{-1}(d)$ belongs to the line through $f^{-1}(a)$ and $f^{-1}(b)$. However, exactly the same argument shows that $f^{-1}(d)$ belongs to the line through $f^{-1}(a)$ and $f^{-1}(c)$. Since these two lines intersect only at $f^{-1}(a)$, we have a contradiction, which allows us to deduce that f is affine as in Theorem 1.

EXAMPLE 7. For $n \geqslant 2$, there is a bijection $f: \mathbb{R}^n \to \mathbb{R}^n$ which maps each circle into the union of some two line segments.

Proof. Let l be any fixed line segment in $\mathbb{R}^n$ and f a bijection from $\mathbb{R}^n$ onto $\mathbb{R}^n$ such that $f(l) = \mathbb{R}^n \setminus l$. Note that such a bijection cannot be continuous. If C is a circle that does not meet l, then $f(C) \subset l$. Otherwise, C meets l in either one or two points whose images under f are in $\mathbb{R}^n \setminus l$. Let m be a line segment containing these points. Then $f(C) \subset m \cup l$.

EXAMPLE 8. For $n \geqslant 2$, there is a bijection $f: \mathbb{R}^n \to \mathbb{R}^n$ which maps each circle into a quadrilateral.

Proof. The argument given for Example 7 actually shows that for each circle C, the image of C is a subset of a line segment together with two points not on this segment. Of course any such set in $\mathbb{R}^n$ with $n \geqslant 2$ is a subset of a quadrilateral.

Note that in Example 7 two line segments cannot be replaced by one, since circles can intersect each other in two points.

Example 7 raises the question of whether a bijection of $\mathbb{R}^n$ can 'straighten out' circles, for example, by mapping each circle onto a finite union of line segments. We cannot answer this (see, however, Corollary 12), but prove instead the following weaker result using the continuum hypothesis, CH.

EXAMPLE 9 (CH). For $n \geqslant 3$, there is a bijection $f: \mathbb{R}^n \to \mathbb{R}^n$, which maps each circle onto a curve consisting of countably many line segments.

Proof. Well order all circles in $\mathbb{R}^n$ as $\{C_\alpha: \alpha < c = \omega_1\}$ and all points in $\mathbb{R}^n$ as $\{x_\alpha: \alpha < \omega_1\}$.

We define a bijection f inductively. Let f be any bijection mapping C_1 onto a polygon S_1 which contains the point x_1.

Suppose $\beta < \omega_1$, and we have extended f so that it bijectively maps each C_α, $\alpha < \beta$ onto a curve S_α consisting of countably many line segments.

Let α_β be the least ordinal such that $x_{\alpha_\beta} \notin \bigcup_{\alpha<\beta} S_\alpha$. Also, let $X = \bigcup_{\alpha<\beta} (C_\beta \cap C_\alpha)$ and $Y = f(X)$. To complete the induction we construct a curve S_β, consisting of countably many line segments, containing the point $x_{\alpha\beta}$ and such that $\bigcup_{\alpha<\beta} (S_\beta \cap S_\alpha) = Y$.

Order the countable set $Y \cup \{x_{\alpha_\beta}\}$ as $\{y_n: n < \omega\}$. We are finished if we can find inductively a polygonal arc P_n connecting y_n and y_{n+1} such that

$$P_n \cap [(\bigcup_{\alpha<\beta} S_\alpha) \cup (\bigcup_{m<n} P_m)] = \{y_n, y_{n+1}\}.$$

The set $(\bigcup_{\alpha<\beta} S_\alpha) \cup (\bigcup_{m<n} P_m)$ is contained in a set E which is a countable union of lines. We can choose a hyperplane H containing y_n and y_{n+1}, but not containing any line in E, except possibly the line l joining y_n and y_{n+1}. Now, $(E \cap H) \setminus l$ is a countable set, so we can choose in H lines l_n and l_{n+1}, different from l, and containing y_n and y_{n+1} respectively, which do not meet this countable set. Let $l_n \cap l_{n+1} = \{a\}$. Then the polygonal arc P_n can be taken to be the one consisting of the two line segments $[y_n, a]$ and $[a, y_{n+1}]$.

Our conclusion in several theorems above was that the bijection f was affine. We now seek characterizations of continuous bijections. There are of course plenty of non-affine bijections taking each circle onto a simple closed curve; for example, $f(z) = |z|z$ (z complex) in $\mathbb{R}^2$. This leads naturally to the next theorem.

THEOREM 10. *Let f be a bijection of $\mathbb{R}^2$ onto itself which maps each circle onto a simple closed curve. Then f is continuous.*

Proof. We first prove that if C is a circle, then either $f(\text{int}(C)) = \text{int}(f(C))$ or $f(\text{int}(C)) = \text{ext}(f(C))$. For, if this is not true, there are points x, y in $\text{int}(C)$ with $f(x) \in \text{int}(f(C))$ and $f(y) \in \text{ext}(f(C))$. Then there is a circle C' through x and y with $C' \subset \text{int}(C)$. But $f(C')$ is a simple closed curve and $f(C') \cap f(C) \neq \emptyset$, which contradicts bijectivity.

Suppose that C_1 is a circle with center x_1 and radius r, such that $f(\text{int}(C_1)) = \text{ext}(f(C_1))$. For each $n \geqslant 1$, let C_n be the circle with center x_1 and radius nr. Notice that for each $n \geqslant 1$, $f(C_{n+1}) \subset \text{int}(f(C_n))$. Let $z \in \bigcap_n \text{int}(f(C_n))$. Obviously, z, is not in the range of f, which is a contradiction. Therefore, for each circle C, $f(\text{int}(C)) = \text{int}(f(C))$.

Let (x_n) be a sequence of points with $x_n \to x$, such that $f(x_n) \not\to f(x)$. There is a circle C with center x such that $x_n \in \text{int}(C)$ for each n. Consequently, $f(x_n) \in \text{int}(f(C))$ for each n, so there is a subsequence of $(f(x_n))$ which converges. Without loss of generality, we can assume there is a point $y \neq x$ such that $f(x_n) \to f(y)$. Let D be a circle separating y from x. Clearly, it is impossible that $f(\text{int}(D)) = \text{int}(f(D))$.

We now apply Theorem 10 to obtain a negative result on 'straightening circles' (Corollary 12).

THEOREM 11. *Let f be a bijection of $\mathbb{R}^2$ onto itself such that the image of each circle is a piecewise smooth simple closed curve. If C_0 and C_1 are disjoint circles except for a common point of tangency x_0, then there are two rays l_0 and l_1 emanating from $f(x_0)$ which form common tangent rays to $f(C_0)$ and $f(C_1)$.*

Proof. By Theorem 10, f is continuous. Without loss of generality, we can assume $x_0 = f(x_0)$ and f is orientation-preserving. We give the argument for C_0 and C_1 being tangent externally.

Let C be the circle with center on the line, L, passing through the centers of C_0 and C_1 and such that C passes through the points y_0 of $C_0 \cap L$ and y_1 of $C_1 \cap L$ with $y_0 \neq x_0, y_1 \neq x_0$. Let z be one of the points of intersection of C with the common tangent line to C_0 and C_1 at x_0. Suppose y_0, z, y_1 lie in that order clockwise on C. For each point y on the circular arc $y_0 z y_1$ let C_y be the circle with diameter $x_0 y$. For each y, let C_y^+ and C_y^- be the two semicircular arcs of C_y which lie on the right and left of $[x_0, y]$, respectively, looking out from x_0.

For each y, since $f(C_y)$ is piecewise smooth, there are two line segments l_y^-, l_y^+ containing x_0 as an endpoint, such that l_y^- is the tangent line to $f(C_y^-)$ at x_0 and l_y^+ is the tangent line to $f(C_y^+)$ at x_0. For each y, let $g^-(y)$, $g^+(y)$ be the points of C such that l_y^-, l_y^+ lie on the lines through x_0 and $g^-(y)$, $g^+(y)$ respectively.

If y is between y_0 and y_1 on $y_0 z y_1$, the arcs C_y^-, $C_{y_0}^+$, $C_{y_1}^-$ emanate in that order clockwise from x_0. Thus, $g^-(y) \leqslant g^+(y_0) \leqslant g^-(y_1)$ in the clockwise order on C. It is clear by continuity that as y converges to y_1, $g^-(y)$ converges to $g^-(y_1)$. Thus, $g^+(y_0) = g^-(y_1)$. Similarly, $g^-(y_0) = g^+(y_1)$.

COROLLARY 12. *A bijection $f: \mathbb{R}^2 \to \mathbb{R}^2$ cannot map each circle onto a polygon.*

Proof. Let the circles C_0, C_1 be as in Theorem 11. The polygons $f(C_0)$, $f(C_1)$ have l_0 as a common tangent ray, and so each meet l_0 in a line segment. Thus $f(C_0) \cap f(C_1)$ is infinite; a contradiction.

We do not know if Corollary 12 remains true in higher dimensions. However, the next example shows that Theorem 10 is false for $n \geqslant 3$.

EXAMPLE 13. A bijection of $\mathbb{R}^3$ onto itself which maps each circle continuously onto a simple closed curve, but which is not continuous.

First, let us fix a function $g: \mathbb{R}^2 \to \mathbb{R}$ such that

(i) g is continuous everywhere except at $(0, 0)$.

(ii) $\lim_{(x,y)\to(0,0):(x,y)\in D} g(x, y) = 0$ for all curves D which are either ellipses or straight lines.

To see that such a function exists, let $E = \{(x, y): 0 < x, 0 < y < x^3\}$ and let g be any function such that

(a) $g(x, y) = 0$ for $(x, y) \notin E$;

(b) $g(x, y) = 1$ for $0 < x$ and $y = x^3/2$,

(c) g is continuous on E.

Now (i) is obvious, so we need only check (ii). This is clear if D is a straight line, and also if D is an ellipse, unless D is contained in $\{y: y \geqslant 0\}$ and is tangent to the x-axis at $(0, 0)$. In the latter case D has the equation

$$y = b[1 - \sqrt{1 - x^2/a^2}]$$

near $(0, 0)$. Now,

$$\lim_{x\to 0} \frac{x^3}{b[1 - \sqrt{1 - x^2/a^2}]} = \lim_{x\to 0} \frac{3x^2\sqrt{1 - x^2/a^2}}{bx/a^2} = 0$$

by l'Hôpital's rule, so for small x, D lies outside E, and this proves (ii).

To get the example, define $f:\mathbb{R}^3 \to \mathbb{R}^3$ by $f(x,y,z) = (x,y,z+g(x,y))$. Clearly f is a bijection which is continuous everywhere except on the z-axis.

We claim that f is continuous on every circle C. This is clear unless C meets the z-axis. If C contains $(0,0,z_0)$, then

$$\lim_{\substack{(x,y,z)\to(0,0,z_0)\\(x,y,z)\in C}} f(x,y,z) = (0,0,z_0 + \lim_{\substack{(x,y)\to(0,0)\\(x,y)\in D}} g(x,y)) = (0,0,z_0),$$

where D is the projection of C onto the xy-plane and where we have used property (ii) of g.

In view of Example 13, we state a theorem which characterizes continuous bijections in higher dimensions.

THEOREM 14. *Let f: $\mathbb{R}^n \to \mathbb{R}^n$ be a bijection which maps each simple closed curve onto a compact set. Then f is continuous.*

Proof. Let (x_n) be a convergent sequence of points, $x_n \to x$ say. Then we can find a simple arc I with one endpoint at x such that $x_n \in I$ for each n. Let (S_n) be a sequence of simple closed curves, each containing a subarc of I with one endpoint at x, such that $\bigcap_n S_n = \{x\}$.

For each n, $K_n = f(S_n)$ is a compact set, with $f(x) \in K_n$ and $f(x_m) \in K_n$ for sufficiently large m. Suppose that $f(x_n) \not\to f(x)$. Then there is a subsequence (x_{n_m}) of (x_n) such that $f(x_{n_m}) \to y \neq f(x)$ as $m \to \infty$. Since K_n is compact, $y \in K_n$ for each n. Therefore $f^{-1}(y) \in S_n$ for each n, and $f^{-1}(y) \neq x$, which contradicts $\bigcap_n S_n = \{x\}$.

COROLLARY 15. *Bijections f: $\mathbb{R}^n \to \mathbb{R}^n$ which map simple closed curves onto simple closed curves, or arcs onto arcs, are continuous.*

Finally, we briefly examine the situation in Hilbert spaces. In the finite dimensional spaces a bijection taking circles to ellipses is affine and continuous by Corollary 3.

EXAMPLE 16. Let H be an infinite dimensional Hilbert space. There is a (linear) bijection, f, of H such that the image of each ellipse is an ellipse and yet f is not continuous.

Let H be an infinite dimensional Hilbert space and let f be a linear bijection of H which is not continuous. (For such an f one could take a maximal linearly independent subset, M, of H and let f be the bijection induced by some permutation π of M such that π fails to be continuous at some point of M.) If E is an ellipse, then E lies in a two-dimensional subspace of H. But, f is affine on each finite dimensional subspace of H. So, $f(E)$ is an ellipse.

However, if f preserves circles, the situation is different. Our proof is based on a result of Carathéodory [8].

LEMMA 17. *Let f be an injection of $\mathbb{R}^2$ into $\mathbb{R}^3$ such that f maps circles to circles. Either $f(\mathbb{R}^2)$ is a plane and f is a similarity map or $f(\mathbb{R}^2)$ is a punctured sphere and f is an element of the inversive group.*

THEOREM 18. *Let f be a bijection of a Hilbert space H such that the image of a circle is a circle. Then f is a similarity map.*

Proof. By composing f with a translation, we can assume $f(0) = 0$. We will first show f maps planes to planes. Suppose a and b are linearly independent. Let U be the circle determined by 0, a, and b. Then $f(U)$ lies in the two-dimensional span of $f(a)$ and $f(b)$. Let us assume there is a point $c \in \operatorname{sp}\{a, b\}$ such that $f(c) \notin \operatorname{sp}\{f(a), f(b)\}$ and c is not on the line determined by a and b. Let V be the circle passing through a, b and c. Then $f(V)$ lies in the span of $f(a)$, $f(b)$ and $f(c)$. Now, if $d \in \operatorname{sp}\{a, b\}$, $d \neq a, b$, there is a circle W passing through d and intersecting each of U and V in two points. Thus, $f(d) \in \operatorname{sp}\{f(a), f(b), f(c)\}$. It follows that $\operatorname{sp}\{a, b\}$ is mapped into a three-dimensional subspace. By Lemma 17, $f(\operatorname{sp}\{a, b\}) = S \setminus \{n\}$, where S is a sphere and $n \in S$. Let $f(z) = n$. Let X be the circle passing through a, b and z. Then $f(X) \subseteq S$. Since $X \cap \operatorname{sp}\{a, b\} = \{a, b\}$, $f(X)$ can meet S in only three points. This contradiction establishes that f maps planes to planes.

Now, by composing f with a similarity of H, we may assume that $f(0) = 0$ and there are linearly independent points a, b such that $f|_{\operatorname{sp}\{a,b\}}$ maps the unit circle onto itself in this two-dimensional space. So, f is an isometry on this plane.

Next notice that if $x \notin \operatorname{sp}\{a, b\}$, then $f(\operatorname{sp}\{a, b, x\}) = \operatorname{sp}\{a, b, f(x)\}$. This follows from the fact that if $z \in \operatorname{sp}\{a, b, x\}$ and $z \notin \operatorname{sp}\{a, b\}$, then there is a circle T such that $x \in T$, $z \in T$ and T contains two points of $\operatorname{sp}\{a, b\}$. Thus $f(z) \in f(T)$ and $f(T)$ contains three points in $\operatorname{sp}\{a, b, f(x)\}$. So, $f(z) \in \operatorname{sp}\{a, b, f(x)\}$. Now, by induction it follows that f maps subspaces of dimension n onto subspaces of dimension n.

Finally, let x and y be points of H and consider $E = \operatorname{sp}\{a, b, x, y\}$. There is an isometry of H which maps E onto $\operatorname{sp}\{a, b, f(x), f(y)\}$ and which is an isometry of $\operatorname{sp}\{a, b\}$ onto itself. Thus, by composing f with some isometry of H, we can assume that f maps E onto itself and $f|_{\operatorname{sp}\{a,b\}}$ is an isometry. But by Corollary 2, $f|_E$ is a similarity. Since f must expand the unit sphere of E by some constant c, we see that $c = 1$ and f is an isometry of E. This implies that the modified f is an isometry of H and therefore our original map is a similarity.

We remark that, in [15], the fundamental theorem of chronogeometry is generalized to a Hilbert space setting.

Problems

1. Are there bijections f of $\mathbb{R}^n$ which map each circle onto a finite union of line segments? Or even polygonal arcs?
2. Is CH necessary for Example 9 to hold? Does such an example exist for $n = 2$ (even with CH)?
3. Let f be a bijection of $\mathbb{R}^2$ onto itself which maps polygons onto polygons. Is f piecewise affine on every bounded set?
4. Let f be a bijection of $\mathbb{R}^2$ onto itself taking polygons with n sides onto polygons with n sides. Must f be affine?
5. Let f be a bijection of $\mathbb{R}^3$ onto itself which takes each circle into a plane. Must f be affine?
6. Is there a bijection of $\mathbb{R}^2$ onto itself taking each circle into a triangle (or into a square)?
7. Is there a measurable bijection of $\mathbb{R}^2$ onto itself taking each circle into a quadrilateral?

REFERENCES

1. Aczél, J. and McKiernan, M. A., 'On the Characterization of Plane Projective and Complex Möbius-Transformations', *Math. Nachr.* **33** (1967), 315–337.
2. Alexandrov, A. D., 'A Contribution to Chronogeometry', *Canad. J. Math.* **19** (1967), 1119–1128.
3. Aleksandrov, A. D., 'Mappings of Families of Sets', *Soviet Math. Dokl.* **11** (1970), 376–380.
4. Beckman, F. S. and Quarles, D. A., 'On Isometries of Euclidean Spaces', *Proc. Amer. Math. Soc.* **4** (1953), 810–815.
5. Benz, W., 'A Characterization of Plane Lorentz Transformations', *J. Geom.* **10** (1977), 45–56.
6. Borchers, H. J. and Hegerfeldt, G. C., 'The Structure of Space-Time Transformations', *Comm. Math. Phys.* **28** (1972), 259–266.
7. Cacciafesta, F., 'An Observation about a Theorem of A. D. Alexandrov Concerning Lorentz Transformations', *J. Geom.* **18** (1982), 5–8.
8. Carathéodory, C., 'The Most General Transformations of Plane Regions which Transform Circles into Circles', *Bull. Amer. Math. Soc.* **43** (1937), 573–579.
9. Darboux, G., 'Sur le théorème fondamental de la géométrie projective', *Math. Ann.* **17** (1880), 55–61.
10. Gibbons, J. and Webb, C., 'Circle-Preserving Functions of Spheres', *Trans. Amer. Math. Soc.* **248** (1979), 67–83.
11. Guc, A. K., 'On Mappings of Families of Sets', *Soviet Math. Dokl.* **14** (1973), 506–508.
12. Kuz'minykh, A. V., 'On the Characterization of Isometric and Similarity Mappings', *Soviet Math. Dokl.* **20** (1979), 82–84.
13. Kuz'minykh, A. V., 'Affineness of Convex-Invariant Mappings', *Siberian Math. J.* **16** (1975), 918–922.