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PROBLEM 24 OF THE "SCOTTISH BOOK" CONCERNING ADDITIVE FUNCTIONALS

BY

IWO LABUDA (VANCOUVER) AND R. DANIEL MAULDIN (DENTON, TEXAS)

About 1935 S. Mazur asked ([6], Problem 24) the following question: In a Banach space E an additive functional f is given such that, for any continuous functions $x\colon [0, 1] \to E$, the composed function fx is measurable. Is f continuous?

The answer to Mazur's problem is contained in the following more general

THEOREM. *Let $E = (E, |\cdot|)$ be a Banach space, F a Hausdorff topological vector space, and $f\colon E \to F$ an additive operator. If fx is Lebesgue measurable* (1) *for any continuous function $x\colon [0, 1] \to E$, then f is continuous.*

Let $(e_n)_{n=1}^{\infty}$ be a sequence of elements in E such that $e_n \to 0$. It is sufficient to show that $\big(f(e_n)\big)_{n=1}^{\infty}$ is a bounded set in F. We may assume that

$$\sum_{n=1}^{\infty} |e_n| < \infty .$$

Consider the map $w\colon C = \{0, 1\}^N \to E$ defined by

$$w(1_\alpha) = \sum_{n\in\alpha} e_n,$$

where $\alpha \subset N = \{1, 2, \ldots\}$ and 1_α is the characteristic function of α on N. Let λ be the Haar probability measure on the Cantor group C, that is to say, the product of (1/2, 1/2)-measure on coordinate groups $\{0, 1\}$ (with addition modulo 2). We first show the following

LEMMA. *$fw\colon C \to F$ is λ-measurable.*

(1) It is known that for a function $g\colon [0, 1] \to X$, where X is a metric space, all usual definitions of measurability of g with respect to Lebesgue measure coincide. It will be sufficient to adopt here the (apparently) weakest one: g is *Lebesgue measurable* if for any Borel subset of X its inverse image by g is Lebesgue measurable.

Proof. Denote by m the Lebesgue measure on $[0, 1]$. There exists a (perfect nowhere dense) subset of $[0, 1]$, K say, such that there is a homeomorphism h of C onto K such that

(1) $h^{-1}(B)$ is λ-measurable if B is an m-measurable subset of K.

The existence of such a K is classical; we indicate its construction after [2], Chapter 8, Ex. 4.

Remove from $[0, 1]$ the open interval with center $1/2$ and length $1/4$. Denote by P_0 the left closed interval, and by P_1 the right one, obtained in such a way. Suppose we have already defined closed intervals $P_{\alpha_1 \ldots \alpha_n}$, where $(\alpha_i)_{i=1}^n \in \{0, 1\}^n$. Removing from $P_{\alpha_1 \ldots \alpha_n}$ the open interval with center in the center of $P_{\alpha_1 \ldots \alpha_n}$ and with length $(1/4)^{n+1}$, we obtain two closed intervals: $P_{\alpha_1 \ldots \alpha_n, 0}$ – the left one, and $P_{\alpha_1 \ldots \alpha_n, 1}$ – the right one.

Put

$$K = \bigcap_{n=1}^{\infty} \bigcup_{(\alpha_j) \in \{0,1\}^n} P_{\alpha_1 \ldots \alpha_n} = \bigcup_{(\alpha_j) \in C} \bigcap_{n=1}^{\infty} P_{\alpha_1 \ldots \alpha_n}$$

and

$$\{h((\alpha_j))\} = \bigcap_{j=1}^{\infty} P_{\alpha_1 \ldots \alpha_j}.$$

It can be checked that h and K constructed in this way have the following property:

(2) $\lambda(h^{-1}(B)) = 2m(B)$ for any Borel subset B of K.

The latter implies (1).

Now, let $x\colon K \to E$ be given by $x = wh^{-1}$. As w is continuous, so is x. By a theorem of Dugundji ([1], 4.1), x has a continuous extension, $\tilde{x}$ say, to the whole of $[0, 1]$. By the assumption in our Theorem, $f\tilde{x}$ is m-measurable. Thus $fx = f\tilde{x}|_K$ is m-measurable. Since $w^{-1}(D) = h^{-1}x^{-1}(D)$ for $D \subset E$, it follows from (1) that fw is λ-measurable.

In order to complete the proof of the Theorem we use Lemma 2.1 of [4] with fw as an additive map on C (cf. also [5] and [3]).

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INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
POZNAŃ

DEPARTMENT OF MATHEMATICS
TEXAS STATE UNIVERSITY
DENTON, TEXAS

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