REPRESENTATIONS OF WELL-FOUNDED PREFERENCE ORDERS

DOUGLAS CENZER AND R. DANIEL MAULDIN

A preference order, or linear preorder, on a set $X$ is a binary relation $\leq$ which is transitive, reflexive and total. This preorder partitions the set $X$ into equivalence classes of the form $[x] = \{y : x \leq y \text{ and } y \leq x\}$. The natural relation induced by $\leq$ on the set of equivalence classes is a linear order. A well-founded preference order, or prewellordering, will similarly induce a well-ordering. A representation or Paretian utility function of a preference order is an order-preserving map $f$ from $X$ into the $\mathbb{R}$ of real numbers (provided with the standard ordering). Mathematicians and economists have studied the problem of obtaining continuous or measurable representations of suitably defined preference orders $[4, 7]$. Parametrized versions of this problem have also been studied $[1, 7, 8]$. Given a continuum of preference orders which vary in some reasonable sense with a parameter $t$, one would like to obtain a continuum of representations which similarly vary with $t$.

Specifically, let $T$ and $X$ be Polish (that is, complete separable metric) spaces. For each $t$ in $T$, let $B_t$ be a nonempty subset of $X$, a preference order on $B_t$ and let $E_t = \{ (x, y) : x \leq_t y \}$. Finally, set

$$E = \{ (t, x, y) : x \leq_t y \},$$

and set

$$B = \{ (t, x) : x \in B_t \}.$$

Suppose that $E$ (and therefore $B$) is a Borel measurable set. This will be the general setting throughout the paper.

We will say that $E$ is section-wise closed if, for each $t$, $E_t$ is closed with respect to $B_t \times B_t$; in this case, each preference order $\leq_t$ will possess a continuous representation by a result of Debreu $[4]$. The second author

Received February 3, 1982 and in revised form November 24, 1982. This research was supported by National Science Foundation grant MCS-81-01581 and by a Faculty Research Grant from North Texas State University.

496
showed in [7] that if $E$ is section-wise closed, then there is an $S(T \times X)$-measurable map $f$ of $B$ into $\mathbb{R}$ such that, for each $t$, $f(t, \cdot)$ is a continuous representation of $\preceq$. (Here $S(T \times X)$ form the C-sets of Selivanovskii or the husin heirarchy [6, p. 468].) Under the further assumption that each $B_t$ is $\alpha$-compact, it was also shown in [7] that the map $f$ may be taken to be Borel measurable.

In this paper we obtain significant improvements in the above results in the case that each preference order $\preceq_t$ is well-founded.

**Theorem 3.3.** Let $E$ be a Borel subject of the product $T \times X \times X$ of Polish spaces such that, for each $t$, $E_t = \{(x, y) : (t, x, y) \in E\}$ is a well-founded preference order on $B_t = \{x : (t, x, x) \in E\}$. Then there is a Borel measurable map $f$ from $B = \{(x, t) : x \in B_t\}$ into $\mathbb{R}$ such that each $f(t, \cdot)$ is a representation of $E_t$.

If $E$ is section-wise closed, then we show that the map $f$ constructed in the above theorem can be modified so as to be continuous on each section.

**Theorem 4.2.** Suppose that $E$ satisfies the hypothesis of Theorem 3.1 and that, for each $t$, $E_t$ is closed with respect to $B_t \times B_t$. Then there is a Borel measurable map $f$ from $B$ into $\mathbb{R}$ such that each $f(t, \cdot)$ is a continuous representation of $E_t$.

This answers Question (2) of [7] in the affirmative.

We note that the methods of this paper are quite different from those of [7]. The construction of the map $f$ in Theorem 3.3 does not require that $E$ be section-wise closed and does not depend on any selection principles.

**1. Ordinal representations.** In this section, we introduce the notion of an ordinal representation of a preference order and of a continuum of preference orders. We show the existence of ordinal representations for individual well-founded preference orders and give a sufficient condition for the continuity of such a representation. Finally, we show that the existence of an ordinal representation (with range bounded to some countable ordinal) for a continuum of preference orders implies the existence of a representation into the real line.

An ordinal representation of a preference order $\preceq$ on a set $B$ is simply an order-preserving map $\phi$ from $B$ into the class of ordinal numbers. Suppose now that $\preceq$ is a well-founded Borel preference order on a Borel subset $B$ of a Polish space $X$. Then $\preceq$ possesses a natural ordinal representation, which we will now describe. Let $x \sim y$ denote the equivalence relation ($x \preceq y$ and $y \preceq x$) and let $x \preceq y$ denote ($x \preceq y$ and
not \((y \leq x)\). Let the ordinal \(o(\leq) = \kappa\) be the order type of the induced well-ordering on the equivalence classes of \(\sim\); it follows from Theorem 3.1 that \(\kappa\) is countable. For \(X \in B\), let \(o(x)\) be the order type of \(\leq\) restricted to the predecessors of \([x]\). Note that

\[
o(\leq) = \sup \{o(x) + 1 : x \in B\}.
\]

The map \(o: B \rightarrow (\leq)\) is clearly an ordinal representation.

Furthermore, since each equivalence class \([x]\) is a Borel subset of \(X\), \(o^{-1}(A)\) will be Borel for any set \(A\) of ordinals. This will clearly apply to any representation of a Borel preference order.

Let the class of ordinals be given the usual order topology with a subbase of open sets of the two forms \(\{\alpha: \alpha < \beta\}\) and \(\{\alpha: \alpha > \beta\}\). If \(\leq\) has a continuous representation \(\phi\), then, for each \(y \in B\), both \(\{x: x \leq y\} = \{x: \phi(x) \leq \phi(y)\}\) and \(\{x \geq y\}\) must be relatively closed subsets of \(B\). A preference order satisfying the above condition was said to be continuous in [7]. This condition is easily seen to be equivalent to the following: that the set \(E = \{(x, y): x \leq y\}\) is a relatively closed subset of \(B \times B\).

**Lemma 1.1.** An ordinal representation \(\phi\) of a continuous preference order on \(B\) is continuous if and only if, for each ordinal \(\beta\), \(\{x: \phi(x) > \beta\}\) is a relatively open subset of \(B\).

**Proof.** For any ordinal \(\beta\), \(\{x: \phi(x) < \beta\}\) equals either \(B\) or \(\{x: x \leq y\}\), where \(\phi(y)\) is the least ordinal in the range of \(\phi\) which is greater than or equal to \(\beta\).

For a continuous preference order \(\leq\), the map \(o\) defined above is a continuous ordinal representation, since, for each ordinal \(\beta\), \(\{x: o(x) > \beta\}\) equals either \(\emptyset\) or \(\{x: x > y\}\), where \(o(y) = \beta\).

We will next indicate how (continuous) ordinal representations may be used to obtain (continuous) representations into the real line. It is a classical result of Cantor that any countable linear ordering can be imbedded into the real line. For a well-ordering, the image can be taken to be a closed set. This fact is a straightforward consequence of the countable axiom of choice.

**Lemma 1.2.** For any countable ordinal \(\kappa\), there exists a bicontinuous order isomorphism \(i\) of \(\kappa = \{\alpha: \alpha < \kappa\}\) onto a closed subset \(K\) of the real line.

It should be remarked that any order isomorphism \(i\) from an initial segment \(\kappa\) of the ordinals onto a closed set of reals must be bicontinuous. This can be seen as follows. For any real \(r\), \(\{\alpha: i(\alpha) < r\} = \{\alpha: \alpha < \beta\}\), where \(\beta\) is either \(\kappa\) or the least such that \(i(\beta) \geq r\); also, \(\{\alpha: i(\alpha) \geq r\}\) is either empty or equals \(\{\alpha: \alpha \geq \beta\}\), where \(i(\beta)\) is the least upper bound of \(K\).
\(\cap (-\infty, r].\) The inverse map from \(K\) onto \(\kappa\) is just the natural representation of the standard order on \(K\) and is therefore continuous as shown above.

Now if \(\preceq\) is a continuous preference order on \(B\), let \(o\) be the natural ordinal representation mapping \(B\) onto \(o(\preceq) = \kappa\) and let \(i\) be a continuous order isomorphism of \(\kappa\) onto a closed subset \(K\) of the real line. Then the composition of \(f: B \to K\), defined by \(f(x) = i(o(x))\) is clearly a continuous representation of \(B\) into the real line.

The problem is more interesting when we are given a continuum of preference orders. Therefore, let the Borel subset \(E\) of the product \(T \times X \times X\) of Polish spaces define a continuum of preference orders \(\preceq_i\) on the sets \(B_i\) as described in the introduction. An ordinal representation of \(E\) is a map \(\phi\) from \(B\) into the class of ordinals such that, if \(x\) and \(y\) belong to \(B_i\), then \(x \preceq_i y\) if and only if \(\phi(t, x) \leq \phi(t, y)\). It is important to note that the natural map \(\phi\), defined by letting \(\phi(t, \cdot)\) be the natural ordinal representation of \(B_i\), is not necessarily a Borel map, even assuming that \(E\) is section-wise closed. An example will be given in Section two. The failure of this natural first guess for a Borel representation of a continuum of preference orders necessitates the inductive construction given in this paper.

However, once we construct a continuous or Borel ordinal representation for \(E\) which maps \(B\) into some countable ordinal \(\kappa\), Lemma 1.2 can be used to obtain a continuous or Borel representation mapping \(B\) into the real line.

2. Reduction, separation and boundedness. The classical Separation Theorem of Lusin states that disjoint analytic subsets \(A_1\) and \(A_2\) of a Polish space \(Y\) may be separated by a Borel set \(D\) so that \(A_1 \subset D\) and \(A_2 \cap D = \emptyset\). Now suppose that \(\sim\) is an analytic equivalence relation on \(Y\), that is, \(\{ (x, y) : x \sim y \}\) is an analytic subset of \(Y \times Y\); in fact, we have in mind the equivalence relation on the product space \(T \times X\) induced by a Borel continuum of preference orders \(\preceq_i\). Define the saturation \(S(A)\) of a subset \(A\) of \(Y\) by

\[ S(A) = \{ x : (\exists y \in A) x \sim y \}. \]

Of course, the saturation of an analytic set is also analytic. We will need the "invariant" separation theorem first obtained by Ryll-Nardzewski and a "downward closed invariant" reduction theorem.

**Theorem 2.1.** (Invariant Separation) Let \(\sim\) be an analytic equivalence relation on a Polish space \(Y\). Then any two disjoint saturated analytic subsets \(A_1\) and \(A_2\) of \(Y\) may be separated by a saturated Borel set \(D\).
The Reduction Theorem of Kuratowski [6, p. 508] for a infinite sequence \( \{C_1, C_2, \ldots \} \) of coanalytic sets whose union is Borel states that there exists a sequence \( \{D_1, D_2, \ldots \} \) of pairwise disjoint Borel sets such that \( D_n \subset C_n \) for each \( n \) and \( \bigcup C_n = \bigcup D_n \). Now, if each \( C_n \) is saturated, then \( S(D_n) \) and \( Y \cap C_n \) are disjoint, saturated analytic sets. Thus, by the Invariant Separation Theorem above, there exists a Borel \( B_n \) such that \( D_n \subset S(D_n) \subset B_n \subset C_n \). This gives the first part of Theorem 2.2.

**Theorem 2.2.** (Invariant Reduction) Let \( \sim \) be an analytic equivalence relation on a Polish space \( Y \) and let \( \{C_0, C_1, C_2, \ldots \} \) be a sequence of saturated coanalytic subsets of \( Y \) such that \( \bigcup C_n = D \) is Borel. Then there exists a sequence \( \{B_n : n < \omega \} \) of saturated Borel sets such that \( B_n \subset C_n \) for all \( n \) and such that \( \bigcup B_n = D \). Furthermore, if \( \leq \) is a Borel linear ordering on the equivalence classes of \( \sim \) and each \( C_n \) is closed downward, then each \( B_n \) may be taken to be closed downwards.

**Proof.** The proof of the first part was given above. Now fix \( n \) and suppose that \( C_n = C \) is closed downward. Let the saturated Borel subset \( B_n = B^0 \) of \( C \) be given by the above and let

\[
A^0 = \{ y : (\exists x \in B^0)(y \leq x) \}.
\]

Then \( A^0 \) is a saturated analytic subset of \( C \), so by Theorem 2.1, there is a saturated Borel set \( B^1 \) with \( A^0 \subset B^1 \subset C \). Proceeding inductively, we obtain a sequence \( B^0 \subset A^0 \subset B^1 \subset A^1 \subset \ldots \) of saturated subsets of \( C \) such that each \( B^i \) is Borel and each \( A^i \) is analytic and closed downwards. Then \( \bigcup_i B^i \) will be Borel, saturated and closed downwards.

The invariant separation and reduction theorems are both subsumed under the main result of [2].

The classical Boundedness Principle of Lusin and Sierpinski states that any analytic subset of the family of countable well-ordering must be bounded in length by some countable ordinal. This can be used to see that a Borel continuum of well-founded preference orders is similarly bounded in length.

We will use the Boundedness Principle as incorporated in the Inductive Definability Theorem of [3]. We recall that a monotone operator over the Polish space \( Y \) is a map \( \Gamma \) from the power set \( 2^Y \) into \( 2^Y \) such that, whenever \( K \subset M \subset Y \), \( \Gamma(K) \subset \Gamma(M) \). \( \Gamma \) constructs a transfinite sequence \( \{\Gamma^\alpha : \alpha \text{ an ordinal}\} \) by letting \( \Gamma^0 = \emptyset \), \( \Gamma^{\alpha+1} = \Gamma(\Gamma^\alpha) \) for all \( \alpha \) and \( \Gamma^\lambda = \bigcup_{\alpha < \lambda} \Gamma^\alpha \) for limit \( \lambda \).

The closure \( \text{Cl}(\Gamma) = \Gamma^\infty \) of \( \Gamma \) is \( \bigcup \alpha \Gamma^\alpha \); the closure ordinal \( |\Gamma| \) is the least such that \( \Gamma^\alpha = \Gamma^\infty \). The following theorem is given in [3, p. 58].
THEOREM 2.3. (Inductive Definability) Let $\Gamma$ be a coanalytic monotone operator on a Polish space $Y$. Then

(a) For each countable ordinal $\alpha$, $\Gamma^\alpha$ is a coanalytic subset of $Y$.
(b) $\Gamma^\omega$ is a coanalytic subset of $Y$.
(c) $|\Gamma| \leq \omega_1$.
(d) For any analytic subset $A$ of $\Gamma^\omega$, there is a countable ordinal $\alpha$ such that $A \subset \Gamma^\alpha$.

Part (d) can be viewed as a generalization of the Boundedness Principle.

3. Borel representations. Let $E$ be a Borel continuum of well-founded preference orders $\preceq$, on the Borel subset $B$ of the product $T \times X$ of Polish spaces, as described in the introduction. For each $t$, let $o(t)$ be the order type of the induced well-ordering on the equivalence classes of $\sim_t$; for each $x$, let $o(x, t)$ be the order type of $\preceq_t$ restricted to the $\preceq_t$-predecessors of $[x]$. Let $o(E) = \sup_t o(t)$.

THEOREM 3.1. Let $E$ be a Borel subset of the product $T \times X \times X$ of Polish spaces such that, for each $t$, $E_t = \{(x, y); (t, x, y) \in E\}$ is a well-founded preference order on $B_t$. Then $o(E)$ is countable and each of the following sets are coanalytic:

\[
\{ (t, x) : o(t, x) < \alpha \}, \quad \{ (t, x) : o(t, x) \equiv \alpha \},
\{ t : o(t) \equiv \alpha \}, \text{ and } \{ t : o(t) < \alpha \}.
\]

Proof: Define the $\prod_1$ monotone operator $\Gamma$ over $B$ by:

\[
(t, x) \in \Gamma(K) \iff (\forall y) \left( (y \preceq_t x) \rightarrow (t, y) \in K \right).
\]

It is easily seen by induction on $\alpha$ that

\[
\Gamma^\alpha = \{ (t, x) : o(t, x) < \alpha \};
\]

in addition, $C1(\Gamma) = B$ and $|\Gamma| = o(E)$.

$\Gamma^\alpha$ is $\prod_1$ by Theorem 2.3(a). Also,

\[
o(t, x) \equiv \alpha \iff o(t, x) < \alpha + 1
\]

\[
o(t) \equiv \alpha \iff (\forall x) \ o(i, x) < \alpha;
\]

\[
o(t) < \alpha \iff (\exists \beta < \alpha) \ o(t) \equiv \beta.
\]
Now by Theorem 2.3(d), $B = \mathcal{C}_1(\Gamma) = \Gamma^\omega$ for some countable ordinal $\alpha$; it follows that $o(E)$ is countable.

We are now ready for the first of our two main theorems.

**Theorem 3.2.** Let $E$ be a Borel continuum of well-founded preference orders on a subset $B$ of $T \times X \times X$ as described in Theorem 3.1. Then $E$ possesses a Borel ordinal representation $\phi: B \rightarrow o(E)$.

**Proof.** The proof is by induction on $\alpha = o(E)$.

($\alpha = 1$). Just let $\phi(t, x) = 0$ for all $(t, x)$ in $B$.

($\alpha + 1$). Suppose the theorem holds for $o(E) = \alpha$ and let $B, E$ be given with $o(E) = \alpha + 1$.

Let

$$U = \{ (t, x) \in B : o(t, x) \geq \alpha \} \quad \text{and} \quad L = \{ (t, x) \in B : (\exists y) x <_{t} y \}. \]

Then $U$ and $L$ are disjoint saturated analytic subsets of $B$. By the Invariant Separation Theorem (2.1), there exist disjoint saturated Borel sets $B_L \supset L$ and $B_U \supset U$ such that $B_L \cup B_U = B$. Define a Borel continuum $E_L$ of well-founded preference orders on $B_L$ by

$$E_L = E \cap \{ (t, x, y) : (t, x) \in B_L \quad \text{and} \quad (t, y) \in B_L \}. \]

Now $o(E_L) = \alpha$, so by the induction hypothesis, $E_L$ possesses a Borel ordinal representation $\phi_L: B_L \rightarrow \alpha$. Define the representation $\phi$ of $E$ by:

$$\phi(t, x) = \begin{cases} 
\alpha & \text{if } (t, x) \in B_U, \\
\phi_L(t, x) & \text{if } (t, x) \in B_L.
\end{cases} \]

Each $\phi^{-1}(\{\beta\})$ is either $B_U$, $\emptyset$, or $\phi_L^{-1}(\{\beta\})$ and is therefore a Borel subset of $B$. If $(t, x)$ and $(t, y)$ are both in $B_U$, then

$$x \leq_{t} y \iff \phi_L(t, x) \leq \phi_L(t, y) \iff \phi(t, x) \leq \phi(t, y). \]

If $(t, x)$ and $(t, y)$ are both in $B_U$, then $x \sim_{t} y$ and $\phi(t, x) = \phi(t, y) = \alpha$. Finally, if $(t, x) \in B_L$ and $(t, y) \in B_U$, then $(t, y) \not\in L$, so for all $z \in B$, $z \leq_{t} y$; it follows that $x \leq_{t} y$. Since $(t, x) \not\in B_U$ and $B_U$ is saturated, we must have $x \leq_{t} y$. Of course

$$\phi(t, x) = \phi_L(t, x) < \alpha = \phi(t, y). \]

Thus $\phi$ is an ordinal representation.

($\lambda = \text{limit}$). Let $\lambda = \lim_n(\alpha_n)$, where $\{\alpha_n : n < \omega\}$ is an increasing sequence and the theorem holds for each ordinal $\alpha < \lambda$. Suppose that $o(E) = \lambda$. For each $n$, let
\[ C_n = \{ (t, x): o(t, x) < \alpha_n \}. \]

Then each \( C_n \) is \( \prod_1 \) and saturated (in fact, closed downwards). Furthermore, each \( C_n \subset C_{n+1} \) and \( \bigcup_n C_n = B \). By the Invariant Reduction Theorem (2.2), there is a sequence \( \{ B_n; n < \omega \} \) of saturated Borel sets such that \( \bigcup B_n = B \) and, for each \( n \), \( B_n \subset C_n \) and \( B_n \) is closed downwards.

Let \( E_n = E \cap \{ (t, x, y); (t, x) \in B_n \text{ and } (t, y) \in B_n \} \).

Let \( o_n(t, x) \) be the order of \( (t, x) \) in \( E_n \), \( o_n(t) \) the order of \( B_n \), and \( \tau_n = o(E_n) \). Note that \( \tau_n \leq \alpha_n \). By the induction hypothesis, each \( E_n \) possesses a Borel ordinal representation \( \phi_n: B_n \rightarrow \tau_n \). Define the map \( \phi: E \rightarrow \lambda \) by

\[ \phi(t, x) = \min \{ \phi_n(t, x); (t, x) \in B_n \}. \]

Then for each ordinal \( \beta \), we have

\[ \phi(t, x) > \beta \iff (\forall n)( (t, x) \in B_n \rightarrow \phi_n(t, x) > \beta) \]

and

\[ \phi(t, x) < \beta \iff (\exists n)( (t, x) \in B_n \text{ and } \phi_n(t, x) < \beta). \]

It follows that \( \phi \) is Borel measurable. Now, given \( (t, x) \) and \( (t, y) \) in \( B \), such that \( x \leq_y \), choose \( n \) so that \( (y, t) \in B_n \) and \( \phi_n(y, t) = \phi(y, t) \). Since \( B_n \) is closed downward, \( (t, x) \in B_n \) and since \( \phi_n \) is a representation,

\[ \phi_n(t, x) \leq \phi_n(t, y) = \phi(t, y). \]

But this implies that \( \phi(t, x) \leq \phi(t, y) \) since \( \phi(t, x) \) is the minimum of the \( \phi_n(t, x) \). Similarly, if \( x \leq_y \), then \( \phi(t, x) < \phi(t, y) \). This completes the proof of Theorem 3.2.

**Theorem 3.3.** Let \( E \) be a Borel continuum of well-founded preference orders on \( B \) as described in Theorem 3.1. Then \( E \) possesses a Borel representation \( f: B \rightarrow R \).

**Proof.** Let \( \phi: B \rightarrow o(E) = \kappa \) be given by Theorem 3.2 and let \( i: \kappa \rightarrow K \) be given by Lemma 1.2. Define \( f \) by \( f(x) = i(\phi(x)) \).

One may wonder why we don't dispense with reduction and separation and just let \( \phi(t, x) = o(t, x) \). The following example indicates that this may not be possible even when \( o(E) = 2 \) and each \( E_t \) is continuous. Let \( T \) and \( X \) be the space of irrational numbers, let \( S \) be an analytic non-Borel subset of \( T \), let \( A = S \times \{ 0 \} \cup T \times \{ 1 \} \) and let \( f \) be a continuous map of \( X \) onto \( A \). Now \( f(x) = (f_1(x), f_2(x)) \), where both \( f_1 \) and \( f_2 \) are continuous. Define the closed subset \( B \) of \( T \times X \) by
\[ B = \{ (t, x) : f_1(x) = t \} \]
and the closed subset \( E \) of \( T \times X \times X \) by
\[ E = \{ (t, x, y) : f_1(x) = f_1(y) = t \quad \text{and} \quad f_2(x) \leq f_2(y) \} \].

Also define the closed sets
\[ B_i = \{ (t, x) \in B : f_2(t, x) = i \} \quad \text{for} \quad i = 0 \text{ or } 1. \]

Of course, the map \( f_2 \) is a continuous representation of \( E \), but it does not always agree with the order map \( o(t, x) \). In fact, let
\[ C_0 = \{ (t, x) : o(t, x) = 0 \}. \]

Then
\[ C_0 = B_0 \cup [B_1 \cap ((T - S) \times X)]. \]

If \( C_0 \) were Borel, then \( C_0 \cap B_1 = (T - S) \times X \) would also be Borel, whereas it is clearly a coanalytic non-Borel set by our choice of \( S \).

4. Continuous representations. Suppose that we have a Borel representation \( \phi : B \to o(E) \) for a continuum \( E \) of continuous well-founded preference orders. We will now systematically repair any discontinuities of \( \phi \) and thus obtain a section-wise continuous representation of \( E \).

**Theorem 4.1.** Suppose that \( E \) is section-wise closed and that \( \phi : B \to o(E) \) is a Borel representation of \( E \). Then \( E \) possesses a section-wise continuous Borel representation \( \bar{\phi} : B \to o(E) = \kappa \).

**Proof.** We will construct a decreasing sequence \( \{ \phi_\alpha : \alpha \leq \kappa \} \) of Borel representations of \( E \) such that \( \phi_0 = \phi \) and, for all \( \alpha \leq \kappa \),

(1) for all \( t \in T \) and all \( \sigma < \alpha \):
\[ \{ x : \phi_\alpha(t, x) > \sigma \} \text{ is open in } B_t. \]

(2) for all \( (t, x) \in B \) and all \( \sigma < \beta < \alpha \):
\[ \phi_\beta(t, x) > \sigma \iff \phi_\alpha(t, x) > \sigma. \]

The map \( \bar{\phi} = \phi_\kappa \) will be a Borel ordinal representation which is section-wise continuous by (1) and Lemma 1.1. The construction of the maps \( \phi_\alpha \) is by induction and as usual, there are two cases to consider: successor and limit.
(Case 1: \( \alpha + 1 \)) Suppose that \( \phi_{\beta} \) has been constructed, satisfying (1) and (2), for all \( \beta \leq \alpha \). Define the saturated coanalytic subset \( C \) of \( B \) by

\[
C = \{ (t, x) : \sup \{ \phi_{\alpha}(t, y) + 1 : y <_t x \} \leq \alpha < \phi_{\alpha}(t, x) \}
\]

\[
= \{ (t, x) : \phi_{\alpha}(t, x) > \alpha \text{ and } (\forall y)(\exists x)(y <_t x \Rightarrow \phi_{\alpha}(t, y) < \alpha) \}.
\]

Define the analytic set \( A \) which is a subset of \( C \) by

\[
A = \{ (t, x) : \phi_{\alpha}(t, x) > \alpha \text{ and } (\forall n)(\exists y)(d(x, y) < \frac{1}{n} \text{ and } \phi_{\alpha}(t, y) \leq \alpha) \},
\]

where \( d \) is the metric on \( X \).

Now \( A \) contains precisely those points of \( C \) at which \( \phi_{\alpha} \) is discontinuous because of the indicated gap: \( \sup \{ \phi_{\alpha}(t, y) : y <_t x \} \leq \alpha \) whereas \( \phi_{\alpha}(t, x) \rightarrow \alpha \). Notice that in fact if \( (t, x) \in A \), then \( \sup \{ \phi_{\alpha}(t, y) : y <_t x \} \) must equal \( \alpha \). To see this, suppose

\[
\sup \{ \phi_{\alpha}(t, y) : y <_t x \} = \beta < \alpha.
\]

Since \( (t, x) \in A \), there is a sequence \( \{ y_n : n < \omega \} \) converging to \( x \) such that \( \phi_{\alpha}(t, y_n) < \alpha \) for each \( n \). Since \( \phi_{\alpha}(t, x) > \alpha \) and \( \phi_{\alpha} \) is a representation, \( y_n <_t x \) for each \( n \). Now, according to (1), \( U = \{ y : \phi_{\alpha}(t, y) > \beta \} \) is open in \( B_t \). Since \( x \in U \), it follows that for some \( n \), \( y_n \in U \) and therefore \( \phi_{\alpha}(t, y_n) > \beta \). This is a contradiction.

Thus we can repair \( \phi_{\alpha} \) for \( (t, x) \in A \) letting \( \phi_{\alpha+1}(t, x) = \alpha \). For \( (t, x) \in C - A \),

\[
\{ y : x \leq_t y \} = \{ y : \phi_{\alpha}(t, y) > \alpha \}
\]

is already open and we can let \( \phi_{\alpha+1}(t, x) = \alpha \) anyway.

Now the saturated analytic set \( S(A) \) is included in the saturated coanalytic set \( C \), so by the Invariant Separation Theorem (2.1) there is a saturated Borel set \( D \) with \( A \subset S(A) \subset D \subset C \). Notice that if \( D_t \neq \emptyset \), then \( D_t \) consists of exactly one \( \sim_t \) equivalence class, since \( C \) has this property. Define the map \( \phi_{\alpha+1} \) by

\[
\phi_{\alpha+1}(t, x) = \begin{cases} 
\alpha, & \text{if } (t, x) \in D, \text{ and } \\
\phi_{\alpha}(t, x), & \text{otherwise.}
\end{cases}
\]

Since \( (t, x) \in D \) implies \( \phi_{\alpha}(t, x) > \alpha \), we have

\[
\phi_{\alpha+1}(t, x) \leq \phi_{\alpha}(t, x) \quad \text{for all } (t, x) \in B.
\]

The map \( \phi_{\alpha+1} \) is Borel measurable since both \( D \) and \( \phi_{\alpha} \) are Borel.
We next show that $\phi_{\alpha+1}$ is a representation. Certainly, $\phi_{\alpha+1}$ is invariant on $\sim \tau$ equivalence classes. All we need to show is that if $x \prec_\tau y$, then $\phi_{\alpha+1}(t, x) < \phi_{\alpha+1}(t, y)$. Suppose $x \prec_\tau y$ and $y \in D_{\tau}$; then

$\phi_{\alpha+1}(t, y) = \alpha$, \quad $(t, x) \notin C$ and
$\phi_{\alpha+1}(t, x) = \phi_{\alpha}(t, x) < \alpha$.

Suppose $x \prec_\tau y$ and $y \notin D_{\tau}$; then

$\phi_{\alpha+1}(t, y) = \phi_{\alpha}(t, y) > \phi_{\alpha+1}(t, x)$.

Thus $\phi_{\alpha+1}$ is an ordinal representation.

It remains to show that (1) and (2) hold for $\alpha + 1$. Given $\sigma < \alpha$, we have, for all $(t, x) \in B$:

$$\tag{3} \phi_{\alpha+1}(t, x) > \sigma \leftrightarrow \phi_{\alpha}(t, x) > \sigma.$$

It follows that $\{x : \phi_{\alpha+1}(t, x) > \sigma\}$ is open in $B_{\tau}$.

Now suppose $\phi_{\alpha+1}(t, x) > \alpha$. There are two sub-cases. First, suppose that $D_{\tau} \neq \emptyset$ and choose $y_0 \in D_{\tau}$. Then $\phi_{\alpha+1}(t, y_0) = \alpha$ and $y_0 \prec_\tau x$. Thus

$x \in \{y : y_0 \prec_\tau y\} \subset \{y : \phi_{\alpha+1}(t, y) > \alpha\}$.

Second, suppose that $D_{\tau} = \emptyset$; in this case, $\phi_{\alpha+1} = \phi_{\alpha}$. Since $A \subset D$, $A_{\tau}$ is also empty and $x \notin A_{\tau}$. Thus by the definition of $A$, there is some $n$ such that

$x \in \left(B_{\tau} \cap \{y : d(x, y) < \frac{1}{n}\}\right) = \{y : \phi_{\alpha+1}(t, y) > \alpha\}$.

In either case, it follows that $\{x : \phi_{\alpha+1} > \alpha\}$ is open in $B_{\tau}$. This establishes (1).

Given $\sigma < \beta < \alpha + 1$, it follows that $\sigma < \alpha$. Thus by (3):

$$\phi_{\alpha+1}(t, x) > \sigma \leftrightarrow \phi_{\alpha}(t, x) > \sigma \leftrightarrow \sigma_{\beta}(t, x) > \sigma.$$

This establishes (2) and completes the proof of Case I.

(Case II: $\lambda$ = limit). Suppose that $\phi_{\alpha}$ has been constructed satisfying (1) and (2) for all $\alpha < \lambda$. Define the map $\phi_{\lambda} : B \rightarrow \kappa$ by

$$\phi_{\lambda}(t, x) = \min \{\phi_{\alpha}(t, x) : \alpha < \lambda\}.$$

Clearly $\phi_{\lambda}$ is less than or equal to $\phi_{\alpha}$ for all $\alpha < \lambda$.

Since $\{\phi_{\alpha} : \alpha < \lambda\}$ is a decreasing sequence of ordinal representations, it follows that $\phi_{\lambda}$ is a representation. For each $\sigma < \kappa$, we have
\[ \phi_\lambda(t, x) > \sigma \iff (\forall \alpha < \lambda) \phi_\alpha(t, x) > \sigma \quad \text{and} \]
\[ \phi_\lambda(t, x) < \sigma \iff (\exists \alpha < \lambda) \phi_\alpha(t, x) < \sigma. \]

It follows that \( \phi_\lambda \) is Borel measurable.

For any \( \sigma < \beta < \lambda \), any \( t \) and any \( x \), it follows from the definition of \( \phi_\lambda \) that if \( \phi_\lambda(t, x) > \sigma \), then \( \phi_\beta(t, x) > \sigma \). Now if \( \phi_\beta(t, x) > \sigma \), then by (2) of the hypothesis, \( \phi_\alpha(t, x) > \sigma \) for all \( \beta \leq \alpha < \lambda \) and, since the maps \( \{ \phi_\alpha : \alpha < \lambda \} \) are decreasing, \( \phi_\alpha(t, x) \geq \phi_\beta(t, x) > \sigma \), if \( \alpha < \beta \).

So, if \( \phi_\beta(t, x) > \sigma \), then \( \phi_\lambda(t, x) > \sigma \). This establishes (2). In particular, if \( \sigma < \lambda \), then
\[ \phi_\lambda(t, x) > \sigma \iff \phi_{\sigma+1}(t, x) > \sigma. \]

In other words,
\[ \{ x : \phi_\lambda(t, x) > \sigma \} = \{ x : \phi_{\sigma+1}(t, x) > \sigma \} \]
and is open in \( B \), by the (1) of the induction hypothesis. This establishes (1) and completes the proof of Theorem 4.1.

**THEOREM 4.2.** Let \( E \) be a Borel subset of the product \( T \times X \times X \) of Polish spaces such that, for each \( t, E_t \) is a continuous well-founded preference order on \( B_t \). Then \( E \) possesses a section-wise continuous Borel ordinal representation \( \phi : B \to o(E) \) and a section-wise continuous Borel representation \( f : B \to K \) of \( B \) onto a closed subset \( K \) of the real line.

**Proof.** The first part is immediate from Theorem 4.1 and Theorem 3.2. As in the proof of Theorem 3.3, let \( i : o(E) \to K \) be a continuous order isomorphism of \( o(E) \) onto a closed subset \( K \) of the real line (given by Lemma 1.2). Let \( \phi \) be the section-wise continuous ordinal representation as in the first part. Finally, let \( f = i \circ \phi \).

It should be pointed out that the general Question (1) of [7] remains open: whether every section-wise continuous Borel preference order has a section-wise continuous Borel (or even \( \mathcal{B} \) \( \mathcal{B} \)-measurable) econ. 71980), 165-173.

**Added in proof.** Some results similar to those in [7] were obtained by A. Wieczorek, J. Math. Econ. 7 (1980), 165-173.

**REFERENCES**


University of Florida,
Gainsville, Florida;
North Texas State University,
Denton, Texas