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ON THE HAUSDORFF DIMENSION OF A SET OF COMPLEX CONTINUED FRACTIONS

BY

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1. Introduction

This note arose from some general considerations concerning geometric representations of the shift operator. Specifically, consider an infinite set T, the product space T^N, and shift operator $S : T^N \to T^N$ defined by

$$S(\langle t_1, t_2, t_3, \ldots \rangle) = \langle t_2, t_3, \ldots \rangle.$$

One can ask whether there are some natural measures on T^N with respect to which S is ergodic or mixing. From our point of view the answer depends on the geometric structure of a representation of this space. For example, if $T = N$, then there are, of course, $2^{\aleph_0}$ probability measures with respect to which S is mixing. This can be seen by noting that the permutations π of N induce distinct mixing measures $\gamma \circ h_\pi$, where γ is Gauss' measure and h_π is the natural homeomorphism of N^N induced by π. However, if one considers the extremely natural representation of N^N via the canonical continued fraction expansion of the irrational numbers in [0, 1], then there is only one ergodic measure which is connected to the geometric structure of this set, Gauss' measure. (See, for example [1, p. 40].) Gauss' measure is the only ergodic measure which is absolutely continuous with respect to Lebesgue measure; this is proved in [4, p. 114].

Let us consider $T = N \times Z$. Again, there are $2^{\aleph_0}$ measures with respect to which the shift is ergodic. There is a natural geometric representation of $(N \times Z)^N$. As is shown here, the map

$$h(\langle b_1, b_2, \ldots \rangle) = \cfrac{1}{b_1 + \cfrac{1}{b_2 + \cdots}}$$

is a homeomorphism of $(N \times Z)^N$ onto a subset of the open disc in the plane with center (1/2, 0) and radius 1/2. Our question is, is there an

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ergodic measure which is naturally connected to the geometric structure of this representation? By analogy with the representation of N^N, one could speculate that perhaps there is some ergodic measure which is absolutely continuous with respect to planar Lebesgue measure. However, we shall show that the planar measure of our representation is zero. Perhaps more to the point is the fact that the representation of N^N has Hausdorff dimension one and that Gauss' measure is absolutely continuous with respect to the corresponding Hausdorff measure, obtained from the Hausdorff measure function $h(t) = t$.

The purpose of this note is to obtain some information concerning the Hausdorff dimension of our representation of $(N \times Z)^N$. The Hausdorff dimension of the set remains an unsolved problem together with the problem of whether there is an ergodic measure which is absolutely continuous with respect to the corresponding Hausdorff measure. We show that the Hausdorff dimension of our representation is strictly between 1 and 2.

2. Preliminary Lemmas

If b_n, $n = 1, 2, \ldots$, are complex numbers, we shall use the notation

$$\frac{1}{b_1} + \frac{1}{b_2} + \cdots$$

for the continued fraction with partial numerators equal to one and partial denominators $b_1, b_2, \ldots$. Then q_n, the denominator of the n-th convergent p_n/q_n to this continued fraction, is defined by $q_0 = 1$, $q_1 = b_1$, and $q_{n+1} = b_{n+1} q_n + q_{n-1}$ for $n \geq 1$.

It is shown in this section that the convergents p_n/q_n converge provided that each $b_n \in N \times Z$. We call J the set of all limits of such continued fractions. The geometry of J is indicated in Figs. 1–4.

Basic results concerning complex continued fractions may be found in [6]. In order to prove convergence and our Hausdorff dimension estimates for J we need lower bounds for q_n, an upper bound on the distance between two continued fractions with the same first n elements, and other results. We have not attempted, however, to obtain best possible constants in Lemmas 2.3 and 2.4, as the values of these constants are not crucial to the dimension estimates. The corresponding constants in the theory of real continued fractions are 1 and $(\sqrt{5} + 1)/2$, respectively (see [5, p. 136]).

LEMMA 2.1. *Suppose* $\mathrm{Re}(z) \geq 1$. *Then* $|1/z - 1/2| \leq 1/2$.

LEMMA 2.2. *Suppose* $\mathrm{Re}(b_n) \geq 1$ *for each* n. *Then*

$$\left|\frac{q_{n-1}}{q_n} - \frac{1}{2}\right| \leq \frac{1}{2} \quad \text{for } n = 1, 2, \ldots .$$

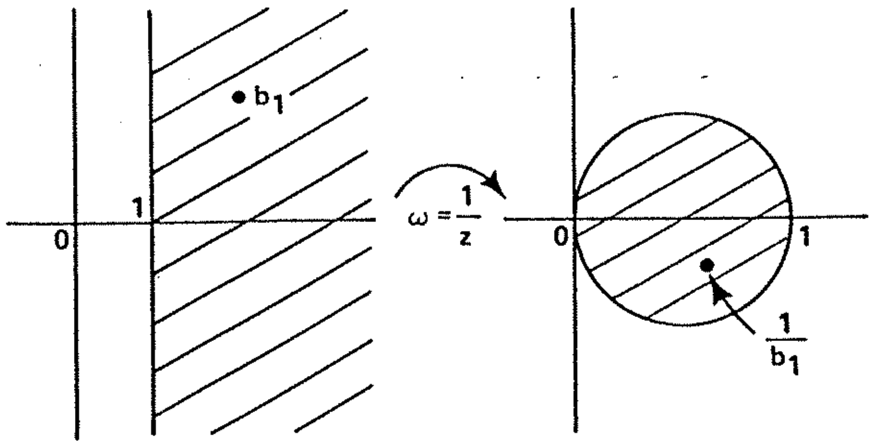

Fig. 1 Fig. 2

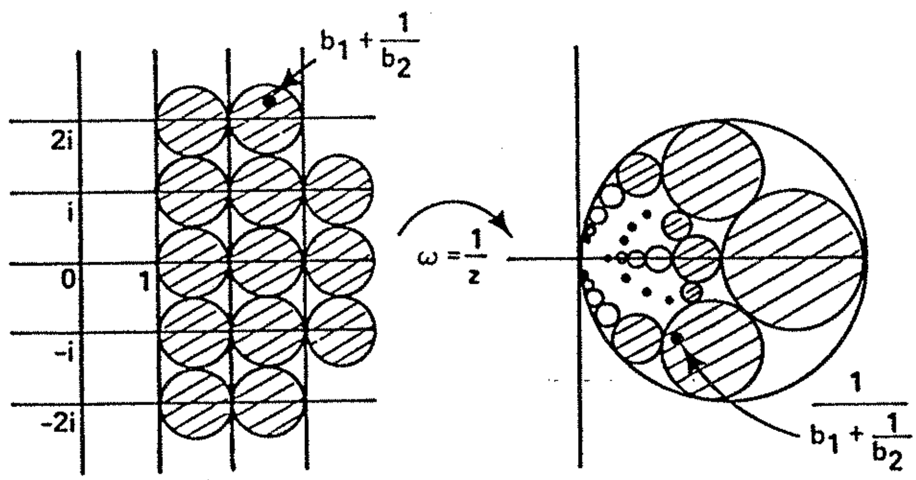

Fig. 3 Fig. 4

Proof. This is true for $n = 1$ by Lemma 2.1. Suppose it is true for $n < k$. We have

$$\frac{q_{k-1}}{q_k} = \frac{q_{k-1}}{b_k q_{k-1} + q_{k-2}} = \frac{1}{b_k + \dfrac{q_{k-2}}{q_{k-1}}}$$

As $\mathrm{Re}(b_k) \geqslant 1$, $\mathrm{Re}(b_k + q_{k-2}/q_{k-1}) \geqslant 1$ by the inductive hypothesis, so

$$\left| \frac{q_{k-1}}{q_k} - \frac{1}{2} \right| \leqslant \frac{1}{2}$$

by Lemma 2.1.

We advise the reader who wishes to follow the proofs of Lemmas 2.3, 2.4 and 2.6 that this may be easier if he draws pictures to illustrate the fairly simple geometry involved.

LEMMA 2.3. *Suppose v, w and $b_n \in \{z : \operatorname{Re}(z) \geq 1\}$, for $n = 1, 2, \ldots$. Then*

$$\left|\left(\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} + \frac{1}{v}\right) - \left(\frac{1}{b_1} + \frac{1}{b_2} + \cdots + \frac{1}{b_n} + \frac{1}{w}\right)\right| \leq \frac{\sqrt{5}}{2|q_n(q_n + q_{n-1})|} \tag{1}$$

Proof. The left-hand side of (1) is easily shown to be

$$\frac{|v - w|}{|vq_n + q_{n-1}||wq_n + q_{n-1}|} = \frac{|v - w|}{|q_n|^2\left|v - \left(\frac{-q_{n-1}}{q_n}\right)\right|\left|w - \left(\frac{-q_{n-1}}{q_n}\right)\right|} \tag{2}$$

Set $z_0 = -q_{n-1}/q_n$. Then $|z_0 + 1/2| \leq 1/2$ by Lemma 2.2.

Suppose, without loss of generality, that $\operatorname{Im}(v) \geq \operatorname{Im}(w)$. Choose v' with $\operatorname{Re}(v') = 1$, $\operatorname{Im}(v') \geq \operatorname{Im}(v)$ and

$$|v' - z_0| = |v - z_0|.$$

Similarly, choose w' with $\operatorname{Re}(w') = 1$, $\operatorname{Im}(w') \leq \operatorname{Im}(w)$ and

$$|w' - z_0| = |w - z_0|.$$

Let v'' be the point with $\operatorname{Re}(v'') = 1$ and $\operatorname{Im}(v'') = \operatorname{Im}(z_0)$. From the geometry,

$$|v'' - v| \leq |v'' - v'| \quad \text{and} \quad |v'' - w| \leq |v'' - w'|.$$

So

$$|v - w| \leq |v - v''| + |v'' - w| \leq |v' - v''| + |v'' - w'| = |v' - w'|.$$

Consequently

$$\frac{|v - w|}{|v - z_0||w - z_0|} \leq \frac{|v' - w'|}{|v' - z_0||w' - z_0|} \tag{3}$$

Let $|v' - v''| = a$, $|w' - v''| = b$, and $|z_0 - v''| = c$. Then

$$\frac{|v' - w'|}{|v' - z_0||w' - z_0|} = \frac{a + b}{\sqrt{(a^2 + c^2)(b^2 + c^2)}} \leq \frac{1}{c} = \frac{1}{|v'' - z_0|}. \tag{4}$$

Now $|z_0 + 1/2| \leq 1/2$, so putting $z_0 = x_0 + iy_0$, we have $1 - x_0 \geq 1 \geq 2|y_0|$, and thus $5(1 - x_0)^2/4 \geq (1 - x_0)^2 + y_0^2$. So

$$\frac{\sqrt{5}}{2}|v'' - z_0| \geq |1 - z_0|. \tag{5}$$

By (2), (3), (4) and (5), the left-hand side of (1) is no greater than

$$\frac{1}{|q_n|^2|v'' - z_0|} \leq \frac{\sqrt{5}}{2|q_n|^2|1 - z_0|} = \frac{\sqrt{5}}{2|q_n(q_n + q_{n-1})|}.$$

Note. A more careful argument shows that the constant $\sqrt{5}/2$ in (5) can be replaced by $3/2\sqrt{2}$ (take z_0 to be the point where the tangent through (1, 0) meets the circle $|z + 1/2| = 1/2$).

LEMMA 2.4. *Suppose $b_n \in N \times Z$ for $n = 1, 2, \ldots$. Then $|q_n| \geq n$ for $n = 1, 2$ and 3, and*

$$|q_n| \geq \left(\frac{\sqrt{13} - 1}{2}\right)^{n-1} \quad \text{for } n \geq 1.$$

Proof. Clearly, $|q_1| = |b_1| \geq 1$. Now $q_2 = b_2 b_1 + 1$; let

$$b_1 = \alpha + \gamma i, \quad b_2 = \beta + \delta i,$$

so that $\alpha \geq 1$, $\beta \geq 1$. Then

$$\begin{aligned} |q_2|^2 &= \alpha^2\beta^2 + 2\alpha\beta + (\gamma\delta - 1)^2 + \alpha^2\delta^2 + \gamma^2\beta^2 \\ &\geq 3 + (\gamma\delta - 1)^2 + \alpha^2\delta^2 + \gamma^2\beta^2. \end{aligned}$$

If $|\gamma| \geq 1$ or $|\delta| \geq 1$, we have $|q_2|^2 \geq 4$; if $\gamma = \delta = 0$,

$$|q_2|^2 \geq 3 + (\gamma\delta - 1)^2 = 4.$$

So $|q_2| \geq 2$.

It may be verified directly, in a similar manner, that $|q_3| \geq 3$.

Now suppose $n \geq 3$. We have

$$\left|\frac{q_{n+1}}{q_n}\right| = \left|b_{n+1} + \frac{q_{n-1}}{q_n}\right|.$$

Now $|q_{n-1}/q_n - 1/2| \leq 1/2$ by Lemma 2.2 and $\mathrm{Re}(b_{n+1}) \geq 1$. It is then easily verified that for $b_{n+1} \neq 1$, $|b_{n+1} + q_{n-1}/q_n| \geq d$, where d is the distance from the origin to the circle $(x - 3/2)^2 + (y - 1)^2 = 1/4$. Thus

$$\left|\frac{q_{n+1}}{q_n}\right| \geq d = \frac{\sqrt{13} - 1}{2} \quad \text{provided } b_{n+1} \neq 1. \tag{6}$$

Suppose now that $b_{n+1} = 1$. Then

$$\left|\frac{q_{n+1}}{q_{n-1}}\right| = \left|(b_n + 1) + \frac{q_{n-2}}{q_{n-1}}\right| \geq 2 > \left(\frac{\sqrt{13} - 1}{2}\right)^2 \tag{7}$$

The estimate

$$|q_n| \geq \left(\frac{\sqrt{13} - 1}{2}\right)^{n-1}$$

now follows from (6) and (7) by induction.

THEOREM 2.5. *Suppose $b_n \in N \times Z$ for $n = 1, 2, \ldots$. Then the continued fraction $\frac{1}{b_1} \, {}_{+} \, \frac{1}{b_2} \, {}_{+} \cdots$ converges to a unique complex number in the open disc $|z - 1/2| < 1/2$.*

Proof. This follows from Lemmas 2.3 and 2.4. We note however that convergence can be deduced, assuming only $\mathrm{Re}(b_n) \geq 1$ for each n, from the "parabola" theorem [6, Theorem 14.2]. To see this, note that it suffices to prove convergence for the continued fraction

$$\frac{1}{1} \, {}_{+} \, \frac{a_2}{1} \, {}_{+} \, \frac{a_3}{1} \, {}_{+} \cdots,$$

where $a_n = 1/b_n b_{n-1}$ (the *equivalence* transformation [6, p. 20]. Noting also that $|1/b_n - 1/2| \leq 1/2$, for each n, by Lemma 2.1, we see that the result follows if for all z_i, $i = 1, 2$, with $|z_i - 1/2| \leq 1/2$, $i = 1, 2$, we have

$$|z_1 z_2| - \mathrm{Re}(z_1 z_2) \leq \frac{1}{2}. \tag{8}$$

We may suppose that $|z_j - 1/2| = 1/2$, $j = 1, 2$, i.e., setting $z_j = r_j e^{i\theta_j}$, $i = 1, 2$, that $r_j = \cos\theta_j$, $i = 1, 2$.

Now it is easy to show that the point $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$ lies in the cardioid region

$$r \leq \frac{1}{2}(1 + \cos\theta), \tag{9}$$

which in turn is contained inside the parabolic region (8).

Now suppose we have

$$\frac{1}{b_1} \, {}_{+} \, \frac{1}{b_2} \cdots = \frac{1}{c_1} \, {}_{+} \, \frac{1}{c_2} \, {}_{+} \cdots .$$

Notice that $|1/z - 1/2| = 1/2$ if and only if $\mathrm{Re}(z) = 1$; so

$$\left|\left(\frac{1}{b_2} \, {}_{+} \, \frac{1}{b_3} \, {}_{+} \cdots\right) - \frac{1}{2}\right| < \frac{1}{2} \quad \text{for } \mathrm{Re}\left(b_2 + \frac{1}{b_3} \, {}_{+} \cdots\right) > \mathrm{Re}(b_2) \geq 1.$$

It follows immediately that $b_1 = c_1$, and $b_n = c_n$ can then be proved by induction. Consequently the representation is unique.

It follows that the map h defined by

$$h(\langle b_1, b_2, b_3, \ldots\rangle) = \frac{1}{b_1} \, {}_{+} \, \frac{1}{b_2} \, {}_{+} \cdots$$

is a homeomorphism of $(N \times Z)^{N}$ with the product topology onto the set J.

LEMMA 2.6. *Let J_{MN} be the set of values of continued fractions* $\frac{1}{b_1 +} \frac{1}{b_2 +} \cdots$, *where $b_j = m_j + n_j i$, $1 \leq m_j \leq M$ and $-N \leq n_j \leq N$ for $j = 1, 2, \ldots$. Then there is a constant $K > 0$, depending only on M and N, such that if z_v and z_w are points in J_{MN} with*

$$z_v = \frac{1}{b_1 +} \frac{1}{b_2 +} \cdots \frac{1}{+ b_n +} \frac{1}{v_{n+1} +} \cdots \quad \text{and} \quad z_w = \frac{1}{b_1 +} \frac{1}{b_2 +} \cdots \frac{1}{+ b_n +} \frac{1}{w_{n+1} +} \cdots,$$

and $v_{n+1} \neq w_{n+1}$, then

$$|z_v - z_w| \geq \frac{K}{|q_n(q_n + q_{n-1})|} \tag{10}$$

Proof. Set

$$v = v_{n+1} + \frac{1}{v_{n+2} +} \cdots \quad \text{and} \quad w = w_{n+1} + \frac{1}{w_{n+2} +} \cdots.$$

Then

$$\begin{aligned} |z_v - z_w| &= \frac{|v - w|}{|vq_n + q_{n-1}||wq_n + q_{n-1}|} \\ &= \frac{|v - w|}{|q_n|^2 \left|v - \left(\frac{-q_{n-1}}{q_n}\right)\right| \left|w - \left(\frac{-q_{n-1}}{q_n}\right)\right|} \end{aligned} \tag{11}$$

We first show $J_{MN} \subset F_1$, where F_1 is a closed set whose distance from the origin is positive. Note that the transformation $\omega = 1/z$ takes the line $x = c$ onto the circle $|z - 1/2c| = 1/2|c|$ and the line $y = c$ onto the circle $|z + i/2c| = 1/2|c|$.

Let

$$S_c = \left\{z : \left|z - \frac{1}{2c}\right| < \frac{1}{2|c|}\right\} \quad \text{and} \quad T_c = \left\{z : \left|z + \frac{i}{2c}\right| < \frac{1}{2|c|}\right\}.$$

Then $J_{MN} \subset F_1$, where

$$F_1 = \overline{S}_1 \setminus (S_M \cup T_N \cup T_{-N} \cup \{0\}),$$

which has positive distance from the origin, $\mathbf{0}$.

Next we need to show that J_{MN} in fact lies inside a slightly smaller closed set $F_2 \subset F_1$. The points $(1 \pm i)$ are carried by $\omega = 1/z$ to the points $(1/2 \pm i/2)$. Suppose

$$z = \frac{1}{b_1 +} \frac{1}{b_2 +} \cdots \in J_{MN}.$$

Then

$$z = \frac{1}{b_1} + \frac{1}{z'} \quad \text{where } z' \in J_{MN} \subset F_1.$$

As F_1 has positive distance from the origin, the set

$$\left\{ b_1 + \frac{1}{z'} : z' \in F_1 \right\}$$

has positive distance from $(1 \pm i)$. Consequently $J_{MN} \subset F_2$, where F_2 is a closed subset of F_1 whose distance from the points $(1/2 \pm i/2)$ is positive.

It follows that the distance between the sets $(v_{n+1} + J_{MN})$ and $(w_{n+1} + J_{MN})$ (vector addition), for admissible $v_{n+1} \neq w_{n+1}$ is bounded below by a positive constant c, which depends only on M and N. Hence

$$|v - w| > c. \tag{12}$$

Now the point $-q_{n-1}/q_n$ lies in the circle $|z + 1/2| \leq 1/2$; so we certainly have

$$\left| v - \left(\frac{-q_{n-1}}{q_n} \right) \right| \leq \sqrt{(M+2)^2 + (N+1)^2} \tag{13}$$

and

$$\begin{aligned} |w - (-q_{n-1}/q_n)| &\leq \sqrt{(M+2)^2 + (N+1)^2} \\ &\leq \sqrt{(M+2)^2 + (N+1)^2}\,|1 - (-q_{n-1}/q_n)|. \end{aligned} \tag{14}$$

Finally, combining (11), (12), (13) and (14), we obtain (10), as required.

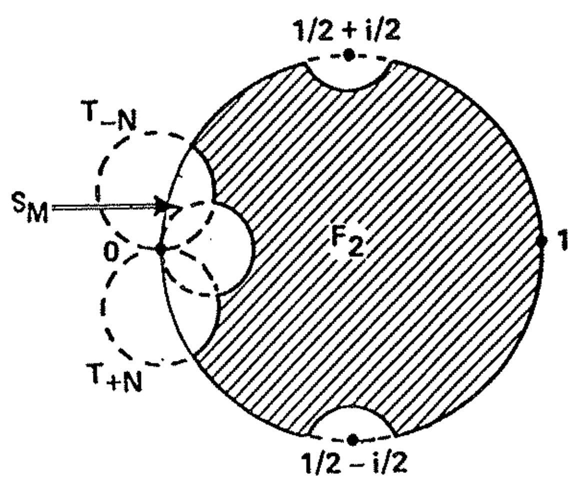

Figure 5
The Set F_2 in Lemma 2.6

3. Hausdorff Dimension

We now estimate the measure of our set of continued fractions. Our methods are not new, but follow the techniques devised by V. Jarník as exposited in [5, Chapter 3, §2]. We refer to [5] also for an introduction to Hausdorff measure.

THEOREM 3.1. *The Hausdorff dimension of J is less than* 2.

Proof. Let $I(b_1, b_2, ..., b_n)$ denote the set of values of the continued fractions whose first entries are $b_1, b_2, ..., b_n$, and let $d(b_1, b_2, ..., b_n)$ be the diameter of this set. By Lemma 2.3,

$$d(b_1, b_2, ..., b_n) \leq \frac{\sqrt{5}}{2|q_n(q_n + q_{n-1})|} = \frac{\sqrt{5}}{2|q_n|^2|\beta|}, \tag{15}$$

where $\beta = 1 + q_{n-1}/q_n$, and, in view of Lemma 2.2, $|\beta - 3/2| \leq 1/2$.
Now

$$\begin{aligned} d(b_1, b_2, ..., b_n, b_{n+1}) &\leq \frac{\sqrt{5}}{2|q_{n+1}(q_{n+1} + q_n)|} \\ &= \frac{\sqrt{5}}{2|q_n|^2|b_{n+1} + \beta - 1||b_{n+1} + \beta|}, \end{aligned} \tag{16}$$

as in [5, p. 141].

Suppose $s > 0$. By the argument of [5, Theorem 61], $\mu^{(s)}(J) = 0$ for any s for which there exist, for each n, open sets $E(b_1, b_2, ..., b_n)$, containing $I(b_1, b_2, ..., b_n)$, with

$$\sum_{b_{n+1} \in N \times Z} [d(E(b_1, b_2, ..., b_n, b_{n+1}))]^s / [d(E(b_1, b_2, ..., b_n))]^s \leq 1. \tag{17}$$

Thus, by (15), (16) and (17), it will suffice to show that for some $\varepsilon > 0$,

$$\sum_{z \in N \times Z} \frac{|\beta|^{2-\varepsilon}}{(|z + \beta - 1||z + \beta|)^{2-\varepsilon}} < 1, \text{ for } |\beta - 3/2| \leq 1/2. \tag{18}$$

Put $z = m + ni$, where $m \in N$ and $n \in Z$, and $\beta = \alpha + \gamma i$. We consider the following sum which dominates the sum in (18):

$$(\alpha^2 + \gamma^2) \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{[((m + \alpha - 1)^2 + (n + \gamma)^2)((m + \alpha)^2 + (n + \gamma)^2)]^{(2-\varepsilon)/2}}.$$

The integral test gives the convergence of this sum for $\varepsilon < 1$. Consequently the sum converges uniformly in ε in the interval $0 \leq \varepsilon < 1/2$, by the Weierstrass M test. It follows that if (18) holds for $\varepsilon = 0$, it also hold for sufficiently small $\varepsilon > 0$. Thus, we will consider the corresponding sum

$$(\alpha^2 + \gamma^2) \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{((m + \alpha - 1)^2 + (n + \gamma)^2)((m + \alpha)^2 + (n + \gamma)^2)} \tag{19}$$

We first fix m and calculate the sum over n, by integrating

$$f(z) = \frac{\pi\cot\pi z}{[(z+\gamma)^2 + (m+\alpha-1)^2][(z+\gamma)^2 + (m+\alpha)^2]}$$

along the square contour with corners $(N + 1/2)(\pm 1 \pm i)$, and applying the theory of residues. (Or, express (19) as the sum of two series by partial fractions and apply [3, p. 82, (4)].)

Then (19) becomes

$$(\alpha^2+\gamma^2)\pi \sum_{m=1}^{\infty}\left(\frac{1}{2(m+\alpha)-1}\right)$$
$$\times\left[\frac{\sinh 2\pi(m+\alpha-1)}{(\cosh 2\pi(m+\alpha-1) - \cos 2\pi\gamma)(m+\alpha-1)}\right.$$
$$\left. - \frac{\sinh 2\pi(m+\alpha)}{(\cosh 2\pi(m+\alpha) - \cos 2\pi\gamma)(m+\alpha)}\right]$$
$$= (\alpha^2+\gamma^2)\pi\left[\frac{\sinh 2\pi\alpha}{(\cosh 2\pi\alpha - \cos 2\pi\gamma)\alpha(2\alpha-1)}\right.$$
$$\left. - 4\sum_{m=0}^{\infty}\frac{\sinh 2\pi(m+\alpha)}{(\cosh 2\pi(m+\alpha) - \cos 2\pi\gamma)(2m+2\alpha-1)(2m+2\alpha)(2m+2\alpha+1)}\right]$$
$$\leq (\alpha^2+\gamma^2)\pi\left[\frac{\sinh 2\pi\alpha}{(\cosh 2\pi\alpha - 1)\alpha(2\alpha-1)}\right.$$
$$\left. - 4\sum_{m=0}^{\infty}\frac{\sinh 2\pi(m+\alpha)}{(\cosh 2\pi(m+\alpha) + 1)(2m+2\alpha-1)(2m+2\alpha)(2m+2\alpha+1)}\right]$$
$$\leq (3\alpha-2)\pi\left[\frac{\coth\pi\alpha}{\alpha(2\alpha-1)}\right.$$
$$\left. - 4\tanh\pi\alpha\sum_{m=0}^{\infty}\frac{1}{(2m+2\alpha-1)(2m+2\alpha)(2m+2\alpha+1)}\right] \qquad (20)$$

To obtain (20), we have first noted that $(\alpha - 3/2)^2 + \gamma^2 \leq 1/4$, so that $\alpha^2 + \gamma^2 \leq 3\alpha - 2$. Also,

$$\frac{\sinh 2\pi(m+\alpha)}{\cosh 2\pi(m+\alpha)+1} = \tanh\pi(m+\alpha) \geq \tanh\pi\alpha,$$

as $\tanh x$ is increasing.

By (18) it will suffice to show that

$$\sum_{m=0}^{k}\frac{1}{(2m+2\alpha-1)(2m+2\alpha)(2m+2\alpha+1)} > \frac{\coth\pi\alpha}{4}\left[\frac{\coth\pi\alpha}{\alpha(2\alpha-1)} - \frac{1}{(3\alpha-2)\pi}\right] \qquad (21)$$

for some k and all α with $1 \leq \alpha \leq 2$.

By direct calculation, e.g., for $\alpha = 2$, it can be seen that (21) is false for $k = 0$ and $k = 1$. However, with the aid of a computer it can be shown that (21) is true for $k = 2$, as follows.

With the estimate $\coth\pi\alpha \le \coth 3 < 1.01$, it is enough to prove

$$\sum_{m=0}^{2} \frac{1}{(2m + 2\alpha - 1)(2m + 2\alpha)(2m + 2\alpha + 1)} - \frac{(1.01)^2}{4\alpha(2\alpha - 1)} + \frac{1.01}{4(3\alpha - 2)(3.142)} > 0 \tag{22}$$

Simplifying the left-hand side of (22), we obtain a ninth degree polynomial $p(\alpha) = \sum_{i=0}^{9} c_i\alpha^i$ over the canonical denominator of tenth degree. (With the coefficients rounded down to the nearest integer, $p(\alpha) = 1034\alpha^9 + 4385\alpha^8 + 4345\alpha^7 - 14031\alpha^6 - 34510\alpha^5 - 5592\alpha^4 + 49092\alpha^3 + 39129\alpha^2 - 9461\alpha - 8868$.) It can now be verified that, for $1 \le \alpha \le 2$,

$$|p'(\alpha)| \le \sum_{i=1}^{9} |c_i| \cdot i \cdot 2^{i-1} < 3\cdot 10^7.$$

Thus to show $p(\alpha) > 0$ for $1 \le \alpha \le 2$, it suffices to check that $p(\alpha) > 6000$ for 5000 equally spaced values of α in the range $1 \le \alpha \le 2$.

We thank J. Neuberger for writing the necessary programs. This completes the proof.

Remark. The sum in (20) may also be estimated using [3, p. 54], (19) and (20), and the properties of the psi function $\psi(x) = d(\log\Gamma(x))/dx$ (see, for example, [2, p. 147]). However this approach seems also, eventually, to require the use of a computer.

THEOREM 3.2. *The Hausdorff dimension of J is greater than* 1.

Proof. It can actually be shown that there is an $M > 0$ and $N > 0$ such that the set J_{MN} of values of the continued fraction

$$\frac{1}{b_1} \dotplus \frac{1}{b_2} + \cdots \text{ where } b_j = m_j + n_j i,\ 1 \le m_j \le M, \text{ and } |n_j| \le N, j = 1, 2, \ldots,$$

has Hausdorff dimension greater than 1.

Again the proof follows closely that in [5, p. 141–147]. It is clear that J_{MN} is a compact, perfect set and that the estimate of Lemma 2 [5, p. 141], may be replaced by that of our Lemma 2.6. To show that $\mu^{(s)}(J_{MN})$ is positive, it then suffices to show that

$$\sum_{m=1}^{M} \sum_{n=-N}^{N} \frac{|\beta|}{[((m + \alpha - 1)^2 + (n + \gamma)^2)((m + \alpha)^2 + (n + \gamma)^2)]^{s/2}} \ge 1 \tag{23}$$

for some $s > 1$, and all $\beta = \alpha + \gamma i$ with $|\beta - 3/2| \le 1/2$.

To this end, we consider

$$\sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{|\beta|}{[(m+\alpha-1)^2+(n+\gamma)^2)((m+\alpha)^2+(n+\gamma)^2)]^{1/2}}$$

$$\geqslant \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(m+\alpha)^2+(n+\gamma)^2}$$

$$= \sum_{m=1}^{\infty} \frac{\pi}{(m+\alpha)} \frac{\sinh 2\pi(m+\alpha)}{(\cosh 2\pi(m+\alpha)-\cos 2\pi\gamma)} \tag{24}$$

again by the theory of residues (or see [3, p. 82, (4)]).

As before,

$$\frac{\sinh 2\pi(m+\alpha)}{\cosh 2\pi(m+\alpha)-\cos 2\pi\gamma} > \tanh \pi\alpha.$$

So (24) is divergent for all α with $1 \leqslant \alpha \leqslant 2$, and we may choose M and N so that

$$\sum_{m=1}^{M} \sum_{n=-N}^{N} \frac{|\beta|}{[((m+\alpha-1)^2+(n+\gamma)^2)((m+\alpha)^2+(n+\gamma)^2)]^{1/2}} > 1$$

for all $\beta = \alpha + \gamma i$ with $|\beta - 3/2| \leqslant 1/2$, and hence an $s > 1$ such that (23) holds.

The proof is now complete.

Added in proof. The proof of (22) can also be achieved by studying the changes of sign of the derivative $p'(\alpha)$ of the polynomial $p(\alpha)$. This method still requires the use of a pocket calculator, however. We thank G. Siebert for pointing out this alternative approach.

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