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PROJECTIVE WELL-ORDERINGS AND EXTENSIONS OF LEBESGUE MEASURE

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The problem considered here is one raised by Ulam in the Scottish Book [8]:

Can one define a countably additive measure on the algebra of all projective subsets of the unit interval which, for Borel sets, coincides with Lebesgue measure?

If ZFC and if the existence of an inaccessible cardinal is consistent, then it is consistent that all projective sets are Lebesgue measurable [7]. Also, the axiom of projective determinancy implies that all projective sets are Lebesgue measurable [4]. However, it is not known if the axiom of projective determinancy is consistent with ZFC. Kakutani and Oxtoby [2] showed that Lebesgue measure has an extension to a very large family of subsets of the unit interval. Hulanicki [1] also obtained results pertaining to the extension of Lebesgue measure. The purpose of this note is to show that if there is a projective well-ordering of $I = [0, 1]$ into type ω_1, then the answer to Ulam's question is negative.

Let us set the following notation. If X is a Polish space (complete separable metric space), then $\mathscr{P}(X)$ denotes the family of all projective subsets of X and $\mathscr{B}(X)$ denotes the family of all Borel subsets of X. We will simply write $\mathscr{P}$ or $\mathscr{B}$ if X is understood. We will denote by (H) the following proposition:

There is a well-ordering $\leqslant$ of the interval I into type ω_1 such that $W = \{(x, y): y \leqslant x\}$ is a projective subset of $I \times I$.

Of course, if Gödel's axiom of constructibility holds, then there is a well-ordering such that W is a projective set of class $\Delta_2^1 = (PCA) \cap (CPCA)$.

THEOREM 1. *Suppose* (H) *holds, X and Y are Polish spaces, and K is a projective subset of $X \times Y$. Then there is a projective subset G of $X \times Y$ which uniformizes K.*

Proof. Without loss of generality, we may assume Y is uncountable. Since Y is Borel isomorphic to $[0, 1]$, there is a projective subset V of

$Y \times Y$ which defines the well-ordering $\leqslant$ of Y onto type ω_1. To obtain a uniformization we simply choose the first point in each nonempty section of K.

Let $G = \{(x, y) \in K: (x, y') \in K \Rightarrow y \leqslant y'\}$. We have

$$(X \times Y) - G = \big((X \times Y) - K\big) \cup \pi_{12}(S),$$

where

$$S = \{(x, y, y'): (x, y) \in K, \ (x, y') \in K \text{ and } y' < y\}.$$

Clearly, the set S is projective in $X \times Y \times Y$. Therefore, G is a projective set. Also, G uniformizes K.

Note. The proof of Theorem 1 is well known. It is simply one used by Addison under the assumption that $V = L$ ([4], Theorem 5.2, p. 805). For our purposes, the fact that every projective set has a projective uniformization is important, not the "sharp" estimate on the class of the uniformization in terms of the class of the set.

THEOREM 2. *Suppose* (H) *holds and M is a projective subset of $I \times I$ such that M_x is countable for each x. Then there is a projective subset K of $I \times I^N$ such that $(x, \langle y_n \rangle) \in K$ if and only if $\{y_n: n \in N\} = M_x$.*

Proof. Let $K_1 = \big\{(x, \langle y_n \rangle) \in I \times I^N: \{y_n: n \in N\} \supset M_x\big\}$. Now,

$$(I \times I^N) - K_1 = \pi_{12}(S),$$

where

$$S = \{(x, \langle y_n \rangle, y) \in I \times I^N \times I: \ (x, y) \in M \ \text{ and } \ y_n \neq y \ \text{ for every } \ n \in N\}.$$

Clearly, S is a projective subset of $I \times I^N \times I$. Thus, K_1 is a projective subset of $I \times I^N$.

Let $K_2 = \{(x, \langle y_n \rangle) \in I \times I^N: (x, y_n) \in M \text{ for every } n \in N\}$. To see that K_2 is a projective subset of $I \times I^N$, let $\varphi: (I \times I)^N \to I \times I^N$ be defined by $\varphi(\langle x_n, y_n \rangle) = (x_1, \langle y_n \rangle)$. Clearly, $\varphi|D$ is a homeomorphism onto $I \times I^N$, where $D = \{\langle x_n, y_n \rangle: x_1 = x_2 = \ldots\}$. Now, $L = D \cap (M \times M \times \ldots)$, is a projective subset of $(I \times I)^N$ of the same projective class as M ([3], p. 454). Since $\varphi(L) = K_2$, K_2 is a projective subset of $I \times I^N$. Of course, the set $K = K_1 \cap K_2$ has the required properties.

THEOREM 3. *Suppose* (H) *holds and M is a projective subset of $I \times I$ such that M_x is countable for each x. Then*

$$M = \bigcup_{n=1}^{\infty} G_n,$$

where, for each n, G_n is a projective set which uniformizes M.

Proof. Let K be a projective subset of $I \times I^N$ satisfying the conclusion of Theorem 2. Let V be a projective subset of $I \times I^N$ which uniformizes K. For each n, let $G_n = \varphi_n(V)$, where $\varphi_n\colon I\times I^N \to I\times I$ is defined by $\varphi_n(x, \langle y_p\rangle) = (x, y_n)$. Since φ_n is a continuous map, G_n is a projective subset of $I\times I$. Also, each G_n uniformizes M and $\bigcup G_n = M$.

THEOREM 4. *If* (H) *holds, then there is no countably additive measure, defined on all the projective subsets of the interval* $[0, 1]$, *which coincides with Lebesgue measure on the Borel sets.*

Proof. Let $\leqslant$ be a well-ordering of $[0, 1]$ into type ω_1 such that $W = \{(x, y)\colon y \leqslant x\}$ is a projective subset of I. According to Theorem 3, there is a sequence of functions f_n such that $\{f_n(x)\colon n \in N\} = \{y\colon y \leqslant x\} = W_x$ for each x and the graph G_n of f_n is a projective subset of $I\times I$ for each n.

Let us note that, for each n, G_n is actually in the σ-algebra $\mathscr{P}(I)\otimes\mathscr{B}(I)$ (the σ-algebra generated by all sets of the form $P\times B$, where $P\in\mathscr{P}(I)$ and $B\in\mathscr{B}(I)$). To see this, for each n and k set

$$T_{nk} = \bigcup_{i=0} f_n^{-1}\big([i/k,\ (i+1)/k]\big)\times[i/k,\ (i+1)/k].$$

These sets are in $\mathscr{P}\otimes\mathscr{B}$ and, for each n,

$$G_n = \bigcap_{k=1} T_{nk}.$$

Now, suppose there is a countably additive measure $\tilde{\lambda}$ defined on $\mathscr{P}(I)$ which extends the Lebesgue measure λ defined on $\mathscr{B}(I)$. There is a unique countably additive measure $\tilde{\lambda}\times\lambda$ defined on $\mathscr{P}\otimes\mathscr{B}$. We calculate the $(\tilde{\lambda}\times\lambda)$-measure of W by Fubini's theorem:

$$\tilde{\lambda}\times\lambda(W) = \int_0^1\int_0^1 \chi_W(x, y)\,d\lambda(y)\,d\tilde{\lambda}(x) = 0.$$

Also,

$$\tilde{\lambda}\times\lambda(W) = \int_0^1\int_0^1 \chi_W(x, y)\,d\tilde{\lambda}(x)\,d\lambda(y) = 1.$$

This contradiction establishes the theorem.

Let us note that perhaps Sierpiński [6] first used Fubini's theorem in this manner. Further references are given in [5].

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