Some additive properties of sets of real numbers

by

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Abstract. Some problems concerning the additive properties of subsets of $\mathbb{R}$ are investigated. From a result of G. G. Lorentz in additive number theory, we show that if $P$ is a nonempty perfect subset of $\mathbb{R}$, then there is a perfect set $M$ with Lebesgue measure zero so that $P+M = \mathbb{R}$. In contrast to this, it is shown that (1) if $S$ is a subset of $\mathbb{R}$ is concentrated about a countable set $C$, then $\lambda(S+R) = 0$, for every closed set $P$ with $\lambda(P) = 0$; (2) there are subsets $G_1$ and $G_2$ of $\mathbb{R}$ both of which are subspaces of $\mathbb{R}$ over the field of rationals such that $G_1 \cap G_2 = \emptyset$, $G_1 + G_2 = \mathbb{R}$, and $\lambda(G_1) = \lambda(G_2) = 0$. Some other results are obtained under various set theoretical conditions. If $2^{\aleph_0} = \aleph_1$, then there is an uncountable subset $X$ of $\mathbb{R}$ concentrated about the rationals such that if $\lambda(G) = 0$, then $\lambda(G+X) = 0$; if $V = L$, then $X$ may be taken to be coanalytic.

P. Erdős and E. Straus conjectured and G. G. Lorentz proved that if $1 \leq a_1 < a_2 < \ldots$ is an infinite sequence of integers, then there always is an infinite sequence of integers $1 \leq b_1 < b_2 < \ldots$ of density zero so that all but finitely many positive integers are of the form $a_i + b_j$ [1]. In this note we investigate the measure theoretic analogues of this result.

Throughout this paper, the real line will be denoted by $R$. If $A$ and $B$ are subsets of $R$, then $A + B = \{a + b : a \in A, b \in B\}$.

**Theorem 1.** Let $P$ be a nonempty perfect subset of $\mathbb{R}$. Then there is a perfect set $M$ with Lebesgue measure zero so that $P + M = \mathbb{R}$.

Let us note that it suffices to prove the theorem under the additional assumption that $P \subseteq [0, 1]$. Let us also note that under this assumption it suffices to prove the existence of a closed set $M$ so that $P + M$ contains some closed interval. With this in mind, for each $n$ and $i$, set $I(i, n) = [(2i, 2i+1); 2^n]$. For each $n$, set

$$A_n = (\{ i : \text{int}(I(i, n)) \cap P \neq \emptyset \})$$

and

$$P_n = \bigcup \{ I(i, n) : i \in A_n \}.$$  

Clearly, $P_1 \supseteq P_2 \supseteq \ldots$ and $\bigcap P_n = P$.

We will prove the following lemma.

**Lemma 2.** There is a sequence of positive integers $m_1 < m_2 < m_3 < \ldots$ and a sequence $(B_p)_{p=1}^{\infty}$ of sets of nonnegative integers so that

1) for each $p$, $B_p \subseteq [1, 2^{m_p}]$,
Let $B_2 = \bigcup \{B(i,j) : 1 \leq i \leq i_1, 1 \leq j \leq t_1\}$. Let $K_2(i,j) = \bigcup \{(p,m) : p \in B(i,j)\}$.

Then
\[
\lambda(K_2(i,j)) \leq 2^{\nu_m} \log^2(i(i)/l(i)) \leq 2^{\nu_m} \log^2(i(i)/l(i)) < 8/3i_1 t_1
\]
and
\[
P_{m_1} + K_2(i,j) = [a_1 + b_1 + 1/2m', q_1 + b_1 + 1/2m'].
\]

Set $K_2 = \bigcup \{K_2(i,j) : 1 \leq i \leq i_1, 1 \leq j \leq t_1\}$. Then
\[
\lambda(K_2) < 1/3 \cdot 2^{-3}
\]
and
\[
K_2 = M_1 \cup (M_1 + 1/2m').
\]

Set $M_2 = K_2 \cup (K_2 - 1/2m')$. Then
\[
\lambda(M_2) < 2^{-3}
\]
and
\[
P_{m_1} + M_2 = [1 + 1/2m', 2].
\]

Let us remark that Theorem 1 has the following corollaries.

**Corollary (Talagrand [2]).** Let $\alpha$ be an analytic subset of $R$ such that if $X$ is a closed subset of $R$ of measure zero, then $X + \alpha$ has measure zero. Then $\alpha$ is countable.

Talagrand proved this result for arbitrary abelian locally compact groups. We will show later in this paper that this result cannot be extended to coanalytic sets.

We give another corollary of Theorem 1 which implies a theorem of S. J. Taylor [4].

**Corollary.** Let $P$ be a perfect subset of $R$. There is a perfect subset $M$ of $R$ with Lebesgue measure zero such that the linear measure of the planar set $M \times P$ is infinite.

Proof. Let $M$ be a perfect subset of $R$ so that $M + P = R$ and such that $\lambda(M) = 0$. Consider the shear transformation $T : R \to R$ defined by $T(x, y) = (x, x + y)$. Since, $\nu_2(T(M \times P))$, the projection of $T(M \times P)$ into the second coordinate, is all of $R$ and the Lebesgue measure of $\nu_2(T(M \times P))$ is no more than the linear measure of $T(M \times P)$, $T(M \times P)$ has infinite linear measure.

Notice that if $E \subseteq R^2$, the linear measure of $T(E)$ is no more than three times the linear measure of $E$, it follows that the linear measure of $M \times P$ must be infinite. Q.E.D.

We note that our proof of the preceding corollary shows that if $A$ is a subset of $R$ such that for every subset $G$ of $R$ with Lebesgue measure zero, $A \times G$ has linear measure zero, then $A + E$ has Lebesgue measure zero, for every set $E$ with Lebesgue measure zero.

**Question.** Is the converse of this result also true?

**Theorem 3.** Let $P$ be a nonempty perfect subset of $R$. There is a subset $M$ of $R$ with Lebesgue measure 0 so that if $F = \bigcup_{i=1}^{m} X_i$, then there is some $i$ so that $X_i + M = F$.

Proof. Let $\{p_i\}_{i=1}^{m}$ be a countable dense subset of $P$. For each $n$ and $m$, let $M(n, m)$ be a perfect subset of $R$ with Lebesgue measure zero so that

\[
(P \cap [m + 1, m + 1/|m|]) + M(n, m) = R.
\]
Let $G$ be a $G_2$ subset of $R$ with Lebesgue measure 0 which contains $\bigcup M(n, m)$.

Suppose $P = \bigcup X_i$ and for each $i$, $X_i + G \not\subset R$.

For each $i$, let $r_i \in R - (X_i + G)$. Thus,

$$X_i \cap (G - r_i) = \emptyset$$

and

$$\bigcup_{i=1}^{m} X_i \cap \bigcap_{i=1}^{m} (G - r_i) = \emptyset.$$

But by construction each $G - r_i$ is a dense $G_2$ set with respect to $P$. Q.E.D.

Let us remark that Theorem 3 contrasts with several results in the opposite direction. The remainder of this paper is devoted to these contrasts.

Recall that a subset $M$ of $R$ is concentrated about a countable set $C$ provided every open set which includes $C$ contains all but countably many points of $M$.

**Theorem 4.** If $S$ is a subset of $R$ which is concentrated about a countable subset $C$, then $\lambda(S + P) = 0$, for every closed set $P$ with Lebesgue measure zero.

**Proof.** It is enough to prove this for compact closed sets $P$ with $\lambda(P) = 0$.

Let $C = \{x_n: n \in N\}$. Let $\varepsilon > 0$ and let $V'$ be an open set with $\lambda(V') < \varepsilon$ and

$$V' \supseteq \bigcup (P + x).$$

Let $T = \{x \in S: (P + x) \cap (R - V') \not\subseteq \emptyset\}$. It can be checked that $T$ is closed with respect to $S$. Thus, $S - T$ is open with respect to $S$ and contains $C$. Therefore, $S - T$ contains all but countably many points of $S$. This implies that $\lambda(S + P) < \varepsilon$. Q.E.D.

One may think that if $S$ is concentrated then $\lambda(S + P) = 0$, for every set $P$ of measure zero. However, we have the following theorems.

**Theorem 5.** There are subsets $G_1$ and $G_2$ of $R$ both of which are subspaces of $R$ over the field of rationals such that $G_1 \cap G_2 = \{0\}$, $G_1 + G_2 = R$ and both $G_1$ and $G_2$ have Lebesgue measure zero.

The proof of this theorem will be based on the next lemma. Let us set some notation first. Let $K_1$ be the set of all $x$ which can be expressed in the form

$$x = \sum_{i=1}^{m} a_i/(2i)!,$$

where $0 \leq a_i < 2i$, $i = 1, 2, \ldots$

Let $K_2$ be the set of all $x$ which can be expressed in the form

$$x = \sum_{i=1}^{m} a_i/(2i+1)!,$$

where $0 \leq a_i < 2i+1$, $i = 1, 2, \ldots$

**Lemma 6.** Let $H_i$ be the subgroup of $R$ generated by $K_i$, $i = 1, 2$. Then $H_1 + H_2 = R$ and $\lambda(H_1) = \lambda(H_2) = 0$.

**Proof.** Since every $x$ in $[0, 1]$ can be written in the form

$$x = \sum_{i=1}^{m} a_i/2^i,$$

where, $0 \leq a_i < 2i$, $i = 1, 2, \ldots$, it follows that $H_1 + H_2 = R$.

The subgroup $H_1$ can be expressed as

$$H_1 = \bigcup_{\sigma} \{p_1 K_1 + \ldots + p_r K_r\},$$

where the union is taken over all finite sequences of integers. Thus, in order to show that $H_1$ has measure zero, it suffices to fix $\{p_1, \ldots, p_r\}$ and show that $L = p_1 K_1 + \ldots + p_r K_r$ has measure zero. If $x \in L$, then $x$ can be written

$$x = \sum_{i=1}^{m} \left(\sum_{i=1}^{r} p_i a_i^{(2i)}\right)/2^i!$$

with $0 \leq a_i^{(2i)} < 2i$, $i = 1, 2, \ldots, r = 1, 2, \ldots$. There is a positive integer $m$ so that

$$x = \sum_{i=1}^{m} c_i/2^i!,$$

where $|c_i| \leq m(2k)$, $k = 1, 2, \ldots$. For each $k$, $c_k = 2k(2k)! + d$, where $|d| \leq m$ and $0 \leq d < 2k$. So, $x$ can be expressed as

$$x = \sum_{i=1}^{m} b_i/2^i!,$$

where $|b_i| \leq m(2i)$, $i = 1, 2, \ldots$. Let $E(m)$ be the set of all such $x$.

It suffices to show that $E(m)$ has measure zero. For each tuple $(a_1, \ldots, a_m)$ such that $|a_1|, |a_2|, \ldots, |a_m| \leq m$ and $0 \leq a_i < 2i$, let $H(a_1, \ldots, a_m)$ be the closed interval with center $\sum a_i/2^i!$ and radius $4m/(2m - 1)!$. For each $n$, $E(m) \subseteq \bigcup H(a_1, \ldots, a_m)$, where the union is taken over all appropriate $2n$-tuples. But,

$$\lambda(H(a_1, \ldots, a_m)) = \sum_{i=1}^{m} \lambda(H(a_1, \ldots, a_m)) \leq \frac{8m}{(2m - 1)!} \left(2^{m+1} \cdot 4 \cdot \ldots \cdot 2n\right).$$

Since this last expression goes to zero as $n$ increases, $E(m)$ has measure zero.

A similar argument shows that $H_2$ also has measure zero.

**Proof of Theorem 5.** Let $F_1$ be the subspace of $R$ over the rationals which is generated by the additive subgroup $H_i$, $i = 1, 2$. Thus,

$$F_1 = \bigcup (r_1 H_1 + \ldots + r_r H_r).$$
where the union is taken over all tuples of rationals. But,

$$\frac{q_1}{p_1} H_1 + \ldots + \frac{q_n}{p_n} H_1 = \left( \frac{1}{p_1 \ldots p_n} \right) \left( \sum_{i=1}^{n} q_i p_1 \ldots p_{i-1} H_i \right).$$

Therefore, each set $r_1 H_1 + \ldots + r_n H_1$ has measure zero. Thus, $V_1$ and $V_2$ have measure zero.

Set $G_1 = V_1$. The set $G_2$ will be constructed by transfinite recursion.

We now consider $R - G_1 = \{ x_0 : x_0 = g_{10} + v_{20} \}$ where $g_{10} \in G_1$ and $v_{20} \in V_2$. Set $G_{20} = \{ q v_{20} : q \in Q \}$. Then $G_{20} \subseteq T_{20}$. $G_{20}$ is a subspace of $R$ over $Q$, the rationals, $G_{20} \cap G_1 = \{ 0 \}$ and $x_0 \in G_1 + G_{20}$.

Suppose $0 < \alpha < \delta$ and for every $\beta$, $0 < \beta < \alpha$, subsets $G_{2\beta}$ of $R$ have been determined so that $G_{2\beta} \subseteq F_2$, $G_{2\beta}$ is a subspace of $R$ over $G_1$, $G_{2\beta} \cap G_1 = \{ 0 \}$, $x_\beta \in G_1 + G_{2\beta}$ and if $0 < \alpha < \beta < \delta$, then $G_{2\beta} \subseteq G_{2\alpha}$. Set $T_{2\beta} = \{ (g_{10} : \beta < \alpha) \}$. If $x_\beta \notin G_1 + T_{2\beta}$, set $G_{2\beta} = F_{2\beta}$. If $x_\beta \in G_1 + T_{2\beta}$, write $x_\beta = g_{10} + v_{2\beta}$, where $g_{10} \in G_1$ and $v_{2\beta} \in V_2$. Set $G_{2\beta} = \{ t + q v_{2\beta} : t \in T_{2\beta} \}$. In either case $G_{2\beta}$ still satisfies the defining conditions. Finally set $G_\delta = \bigcup \{ G_{2\alpha} : \alpha < \delta \}$. Q.E.D.

Next, we note that under some set theoretic assumption an even stronger example along the lines of Theorem 5 can be given.

**Theorem 7**: Suppose that the union of less than continuously many meager subsets of $R$ is meager. There are subsets $G_1$ and $G_2$ of $R$ both of which are subspaces of $R$ over the field of rationals, both of which meet every meager set in a set of cardinality less than $2^{|\delta|}$ and such that $G_1 \cap G_2 = \{ 0 \}$ and $G_1 + G_2 = R$. (Of course, if every subset of $R$ with cardinality $< 2^{|\delta|}$ has measure zero, then $G_1$ and $G_2$ both have measure zero. If CH holds, then $G_1$ and $G_2$ are both Luzin sets.)

Proof. Let $a_\alpha$ be the first ordinal with cardinality $2^{|\delta|}$. Well-order the closed nowhere dense subsets of $R$ into type $a_\alpha$: $F_0, F_1, \ldots, F_\alpha, \ldots, a < a_\alpha$. Also, well-order $R - \{ 0 \}$ into type $a_\alpha$: $x_0, x_1, \ldots, x_\alpha, \ldots, a < a_\alpha$. We denote the rational numbers by $Q$ and if $\alpha \subseteq R$, then $\langle S \rangle_0 \alpha = \langle S \rangle_0 \alpha$. It will be shown by transfinite recursion that there are elements $x_\alpha, y_\alpha, a \subseteq a_\alpha$ of $R$ which have the following properties for each $a < a_\alpha$. (Notation: for $t < a_\alpha$,

1. $\langle S \rangle_0 + (\langle T \rangle_0, \gamma \therefore) \subseteq [x_\alpha, y_\alpha]$
2. $\langle S \rangle_0 \cap (\langle T \rangle_0, \gamma \therefore) = \{ 0 \}$
3. if $\gamma \subseteq a_\alpha$, then

$$\langle S \rangle_0 \cap (\langle T \rangle_0, \gamma) \subseteq \langle S \rangle_0 \cap (\langle T \rangle_0, \gamma)$$

and

$$\langle T \rangle_0 \cap (\langle F \rangle_0, \beta \therefore) \subseteq \langle T \rangle_0 \cap (\langle F \rangle_0, \beta \therefore)$$

Let us note that once this construction has been carried out, then the conclusion of the theorem follows immediately upon setting $G_1 = \langle S \rangle_0$ and $G_2 = \langle T \rangle_0$. 

Construct $t_0$ and $t_0$ as follows. Set

$$B = (\{ \langle q F \rangle_0 : q \in Q \} \cup \{ \langle q_0 - q F \rangle_0 : q \in Q \}) \cup (\{ \langle x \rangle_0 : x \in Q \})$$

Choose $x_0$ to be an element of the residual set $R - B$ and set $t_0 = x_0 - t_0$

Clearly,

$$\langle x_0 \rangle_0 + \langle \langle x_0 \rangle_0 \rangle \subseteq [x_0, q x_0]

\langle x_0 \rangle_0 \cap \langle \langle x_0 \rangle_0 \rangle = \{ 0 \}

\langle \langle x_0 \rangle_0 \rangle \cap \langle x_0 \rangle_0 = \{ 0 \}

$$

Suppose $0 < \alpha < \beta$ and elements $x_0, a_0, a_1$ have been determined so that if $\alpha < \beta$, then conditions 1, 2, and 3 all hold.

Define $s_\alpha$ and $t_\alpha$ as follows. First, set $S_\alpha = \{ x_\alpha : \alpha < \beta \}$ and $T_\alpha = \{ x_\alpha : \alpha > \beta \}$ and $W_\alpha = \bigcup \{ F_\alpha : \alpha < \beta \}$. If $x_\alpha \in \langle S \rangle_0 \cap (\langle T \rangle_0, \gamma)$, set $t_\alpha = \gamma = 0$. If $x_\alpha \notin \langle S \rangle_0 + (\langle T \rangle_0, \gamma)$, then choose $s_\alpha$ to be an element of $R$ which is not in any of the following meager sets:

$$\langle S \rangle_0 + (\langle T \rangle_0, \gamma) \subseteq [y, \gamma]$$

or

$$\bigcup \{ \langle q W_\beta \rangle : q \in Q \} \cup \langle S \rangle_0 \alpha$$

or

$$\{ x_\alpha : x_\alpha \therefore \in \langle x_\alpha \rangle_0 \}$$

Finally, set $t_\alpha = y - t_\alpha$

Setting $S_\alpha = \{ x_\alpha \} \cup \{ s_\alpha \}$ and $T_\alpha = T_\alpha \cup \{ t_\alpha \}$, we have

$$\langle S \rangle_0 \cap (\langle T \rangle_0, \gamma) \subseteq [x_\alpha, y_\alpha]$$

Suppose $w \in \langle S \rangle_0 \cap (\langle T \rangle_0, \gamma)$. There are rationals $\alpha$ and $\beta \in \langle S \rangle_0$, and $v \in (\langle T \rangle_0, \gamma)$ to that

$w = \alpha + \beta \in \alpha + \beta \therefore$

If $x_\alpha \in \langle S \rangle_0 \cap (\langle T \rangle_0, \gamma)$, then $t_\alpha = 0$, and $w \in \langle S \rangle_0 \cap (\langle T \rangle_0, \gamma) = \{ 0 \}$. If $x_\alpha \notin \langle S \rangle_0 + (\langle T \rangle_0, \gamma)$, then

$\alpha + \beta \in \alpha + \beta \therefore$

Or,

$(x_\alpha + \alpha) \beta \therefore = \mu + \beta \therefore$

We consider two cases.

Case 1. $\alpha + \beta = 0$. Then $x_\beta = \alpha + \beta$. Since $x_\alpha \notin \langle S \rangle_0 + (\langle T \rangle_0, \gamma)$, $\alpha = 0$. Then $\alpha = 0$ and $\mu = \beta = \alpha$ is an element of $\langle S \rangle_0 \cap (\langle T \rangle_0, \gamma)$. Thus, $\alpha = 0$. 


Case 2. \(a + b \neq 0\). Then 
\[ x_r = \frac{1}{a + b} \frac{1}{a + b} + \frac{b}{a + b} x_r. \]
But, this is prohibited by the choice of \(x_r\). Thus,
\[ \langle S_{\gamma}, q \cap (T_{\xi} q) = \{0\} \rangle. \]

Next, suppose \(y \leq \tau\) and \(w \in \langle S_{\gamma} q \cap (\cup \{F_{\beta} : \beta \leq \gamma\}\rangle).\) There is an element \(\mu \in \langle S_{\gamma} q \rangle\) and a rational \(r\) so that \(w = \mu + r\xi,\) if \(x_r \in \langle S_{\gamma} q \cap (T_{\xi} q)\rangle\) then \(x_r = 0\) and it follows that \(w \in \langle S_{\gamma} q \rangle\) if \(x_r \notin \langle S_{\gamma} q \rangle\) and \(r \neq 0\), then \(x_r = (1/r)w - \mu.\) But this is prohibited by the choice of \(x_r\). Therefore, \(r = 0, w = \mu\) and it follows that \(w \in \langle S_{\gamma} q \rangle\).

Thus,
\[ \langle S_{\gamma} q \cap (\cup \{F_{\beta} : \beta \leq \gamma\}) \rangle \in \langle S_{\gamma} q \rangle. \]

It can be shown in a similar fashion that
\[ \langle T_{\xi} q \cap (\cup \{F_{\beta} : \beta \leq \gamma\}) \rangle \in \langle T_{\xi} q \rangle. \]
Q.E.D.

Our final goal is to show that under certain conditions there is an uncountable concentrated set such that the sum of this with every set of Lebesgue measure zero still has measure zero. This is the content of Theorems 12 and 13. First, we prove some lemmas which hold outright.

Let us make the following conventions.
Define \(\Psi : 2^S \rightarrow [0, 1]\) by
\[ \Psi((x) = \sum_{s \in S} x_s 2^{-s}, \]
If \(S \subset N,\) let
\[ P_S = \{x \in \Psi : \exists y \in S (x_r = 0)\} \]
and let
\[ Q_S = P(P_S) = [0, 1]. \]

Notice that if \(S \subset T \subset N,\) then \(P_S \subset P_T\) and \(Q_S \subset Q_T.\) Also, if \(S\) is infinite, then \(P_S\) and \(Q_S\) are perfect sets.

For each \(S \subset N,\) set
\[ R_S = \{Q_{S'}, S' : S' \subset S, |S' T| < \omega\} \}
\[ \cup \{Q_{S'} : S' \subset S, T = S' S\} \}

Of course, if \(T = S\), then \(R_T = R_S.\)

Finally, if \(f \in N^N, P_f, Q_f, R_f\) denote \(P_{\text{null}(f)}, Q_{\text{null}(f)}\), and \(R_{\text{null}(f)},\) respectively.

**Lemma 6.** If \(I\) and \(J\) are subintervals of \(R,\) then \(\lambda(I + J) = \lambda(I) + \lambda(J).\)

**Lemma 7.** Let \(I\) be a subinterval of \(R\) and let \(f \in N^N\) be strictly increasing. For each \(n \in N,\)
\[ \lambda(I + Q_f) \leq 2^n \lambda(I) + 2^{-n}. \]

**Proof**. Notice that for each \(n \in N, Q_f\) is a subset of a union of \(2^n\) intervals each of length \(2^{-n} + \phi.\) Thus, this lemma follows immediately from the preceding lemma. Q.E.D.

Let us make the following convention, if \(f, g \in N^N\), say \(f \leq g\) past \(n\) provided \(\forall m > n(f(m) < g(m))\) and \(f \leq g\) if and only if \(\exists n(f \leq g\) past \(n).\)

**Lemma 8.** Let \(U\) be an open subset of \(R\) with \(\sigma(U) < +\infty.\) For each \(s > 0\) and \(n \in N,\) there is a strictly increasing \(f \in N^N\) such that \(f \leq g\) is strictly increasing and \(f \leq g\) past \(n,\) then
\[ \lambda(U + Q_f) \leq 2^n \lambda(U) + s. \]

**Proof.** By blocking the components of \(U,\) there is a sequence \(\{V_n\}_{n=1}^\infty\) of pairwise disjoint sets such that \(U = \cup V_n,\) each \(V_n\) is a union of finitely many open intervals and for each \(k > 1,\)
\[ \lambda(U - \cup V_{<k}) \leq 2^{2k-2} \]

Consequently, for each \(k > 1, \lambda(V_k) \leq 2^{-2k-2} \mu\) with
\[ V_k = \cup \{I : l < r\} \]
where the sets \(I\) are disjoint open intervals.

Now, choose a strictly increasing \(f\) so that \(2^{-a} + r_f \leq 2^{-2} + 1\) and so that for \(k > 1,\)
\[ 2^{-2k-2} + r_f \leq 2^{-2} + 1 \]

Assume \(g \in N^N, g\) is strictly increasing and \(g \geq f\) past \(n.\) Then
\[ \lambda(V_k Q_f) \leq \sum_{l : r_l < r_f} \lambda(I_l + Q_f) \leq \sum_{l : r_l < r_f} \lambda(I_l) + 2^{-2k-2} \]
\[ \leq 2^k \lambda(U) + r_f \leq 2^{-2k-2} \leq 2^{n} \lambda(U) + s. \]

Also, for \(k > 1,\)
\[ \lambda(V_k Q_f) \leq \sum_{l : r_l < r_f} \lambda(I_l + Q_f) \leq \sum_{l : r_l < r_f} \lambda(I_l) + 2^{-2k-2} \]
\[ \leq 2^{2k} \lambda(V_k) + r_f \leq 2^{-2k-2} + r_f \leq 2^{-2k-2} + s. \]
Thus,
\[ \lambda(U + Q_f) \leq 2^n \lambda(U) + \sum_{k=1}^\infty 2^{-2k-2} = 2^n \lambda(U) + s. \]
Q.E.D.

**Lemma 9.** If \(\lambda(G) = 0,\) then there is a strictly increasing element \(g\) of \(N^N\) such that \(\lambda(G + Q_f) = 0,\) whenever \(h\) is strictly increasing and \(g \leq h.\)

**Proof.** Let \(\{U_n\}_{n=1}^\infty\) be a sequence of open sets with \(U_n \supseteq U_{n+1},\) \(\lambda(U_n) \leq 2^{-n}\) and \(G \subseteq \cup U_n.\) For each \(n,\) let \(f_n\) be a strictly increasing element of \(N^N\) such that if \(h\) is strictly increasing and \(f_n \leq h\) past \(n,\) then
\[ \lambda(U_n + Q_f) \leq 2^n \lambda(U_n) + 1/n. \]
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Proof. Let \( \langle h_1 : \alpha < \omega_1 \rangle \) be a family of strictly increasing elements of \( N^n \) such that

(a) if \( \alpha < \beta \), then \( h_\alpha \leq h_\beta \),
(b) if \( \alpha < \beta \), then \( \text{ran}(h_\alpha) \subseteq \text{ran}(h_\beta) \),
(c) \( \forall \gamma \in N^n, \exists \delta (g(\delta) = h_\gamma) \). 

Choose \( x_0 \in R_q \) and for each \( \alpha, 0 < \alpha < \omega_1 \), \( x_\alpha \in R_q - \{x_0 : \alpha < \omega_1 \} \). Let \( X = \{x_\alpha : \alpha < \omega_1 \} \). It follows from Lemmas 9, 10, and 11 that \( X \) satisfies (1), (2) and (3) of the conclusion. Q.E.D.

Let us remark that Theorem 12 cannot be proved as it stands under MA + \( \neg \text{CH} \), since MA + \( \neg \text{CH} \) implies that no uncountable set can be concentrated on the rationals. Of course, under MA + \( \neg \text{CH} \) it is true that \( \lambda(G + E) = 0 \) for any set \( G \) with \( \lambda(G) = 0 \) and \( |E| < 2^{\omega_1} \). Finally, our proof of Theorem 12 can be easily modified under the assumption of MA + \( \neg \text{CH} \) to yield a subset \( X \) of \( R \) of size \( c \) so that if \( U \) is an open set containing the rationals then \( |X - U| < c \) and such that if \( \lambda(G) = 0 \), then \( \lambda(G + E) = 0 \).

Added in proof. Friedman and Talagrand [6] have done this.

Open Question. Can one prove in ZFC that there is an \( X \) satisfying (1) and (2) of Theorem 12?

Added in proof. C. Becerra has shown that the answer is no.

Finally, we comment on where an \( X \) satisfying the conditions of Theorem 12 can lie in the projective hierarchy. From Theorem 1, \( X \) cannot contain a perfect set, so \( X \) cannot be analytic. Also, one cannot even produce a projective \( X \) in ZFC + \( \text{CH} \), since Solovay has shown that it is consistent with \( \text{CH} \) that every uncountable projective set contains a perfect subset. If we assume \( V = L \), then a standard argument, due to Gödel will produce an \( X \) which is \( \Delta_1^1 \) (PCA + CPACA).

A somewhat more careful argument yields:

Theorem 13. If \( \mathbf{V} = \mathbf{L} \), then there is an \( X \) satisfying the conditions of Theorem 12 which is coanalytic (\( = \pi_1^1 \)).

Proof. It is sufficient to show that there is a subset \( H = \{h_\alpha : \alpha < \omega_1 \} \) of \( N^n \) satisfying conditions (a), (b) and (c) listed in the proof of Theorem 12 so that \( H \) is coanalytic and such that \( h_\alpha \neq h_\beta \), if \( \alpha \neq \beta \). We may then define \( z_\alpha \in \mathcal{P}(\lambda) \), where \( z_\alpha(\alpha) = 1 \) if and only if \( \alpha \in \text{ran}(h_\alpha) \). The set \( X = \{x_\alpha : \alpha < \omega_1 \} \) will then be coanalytic, since \( X = \mathcal{P}(\text{ran}(g(h))) \) where \( g : N^n \to N^n \) by \( g(h) = \text{ran}(h) \). The map \( g \) is Borel measurable and when restricted to the Borel set \( D \), of strictly increasing elements of \( N^n \) it is also one-to-one. Thus, \( g(D) \) is a Borel isomorphism of \( D \) onto \( g(D) \). Since \( H \subseteq D \), \( g(H) \) will also be coanalytic.

To construct such an \( H \), let

\[ A = \{g < \omega_1 : L_q + ZF = P \text{ and } L_q \text{ is point-definable} \} \]

Since \( A \) is unbounded in \( \omega_1 \), let \( \langle h_\alpha : \alpha < \omega_1 \rangle \) be an increasing enumeration of \( A \).

If \( g = g_\beta \), define \( h_\beta \) to be the \( \leq \text{L-first} \) element of \( N^n \) such that

\[ \beta \]
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Theorem 16. If $V = L$, then there is an $X$ satisfying the conditions of Theorem 15 which is coanalytic.

The proofs of these theorems are similar to those given for Theorems 12 and 13. These proofs use Lemma 11 as it stands and the following two lemmas which are analogous to Lemmas 9 and 10.

Lemma 17. If $\lambda(G) = 0$, then there is a strictly increasing element $y$ of $[^m\omega]^1$ such that the linear measure of the planar set $G \times Q_2$ is zero, whenever $h$ is strictly increasing and $g \leq^* h$.

We indicate how this lemma follows immediately from Theorem 1 of Taylor's paper. One only need note the following connections. Let $G \subseteq R$ with $\lambda(G) = 0$. Let $\{a_n\}$ be a sequence with $a_n > 0$ for each $n$ such that if $\langle a_n \rangle$ (Taylor's notation), then $G \times \omega^\delta(0, a_n)$ (Taylor's notation) has zero linear measure. Let $g$ be a strictly increasing element of $[^m\omega]^1$ so that for each $n$, $2^{-2n} < a_n$. Now, if $h$ is strictly increasing and $g \leq^* h$, then $(0) \times Q_2 = \omega^\delta(0, b_n)$, where $b_n = 2^{-2n}$.

Lemma 18. If $S \subseteq N$ and $G \times Q_2$ has zero linear measure, then $G \times R$ has zero linear measure.

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References


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