On the Borel Subspaces of Algebraic Structures

R. DANIEL MAULDIN

Introduction. Let G be a connected abelian Polish group which admits an element of infinite order. Theorem 1; for each ordinal α, $1 \leq \alpha < \omega_1$, there is a subgroup of G which is of exactly additive class α in G. For each ordinal, α, $2 \leq \alpha < \omega_1$, there is a subgroup of G which is of exactly multiplicative (ambiguous) class α in G. Theorem 2; if X is an infinite dimensional separable Banach space, then there are subspaces of X of exactly the same Borel classes as in the previous theorem.

Let us recall that the sets of multiplicative (additive) class 0 in a metric space are the closed (open) sets; the sets of multiplicative (additive) class α, $\alpha > 0$ are those sets which can be expressed as the intersection (union) of countably many sets of additive (multiplicative) class less than α; the sets of ambiguous class α are the sets which are of both multiplicative and additive class α; a set is of exactly multiplicative (additive) class α provided it is of that class but is not of additive (multiplicative) class α; a set is of exactly ambiguous class α provided it is of ambiguous class α, but is not of any lower class [3].

Let us recall that a subset H of an abelian group G is said to be independent provided that if $h_1, \ldots, h_n$ are elements of H, $b_1, \ldots, b_n$ are integers and

$$\sum_{i=1}^{n} b_i h_i = e,$$

then $b_1 = b_2 = \ldots = b_n = 0$.

The idea of the proof of Theorem 1 is first to obtain a compact, perfect, totally disconnected subset M of G which is independent (Theorem 0) and then to simply take a subset H of M of a given Borel class and show that $\langle H \rangle$, the subgroup of G generated by H, is of the same class.

Theorem 0. *Let G be a connected, abelian Polish group which has an element of infinite order. Then there is an independent compact, perfect, totally disconnected subset M of G.*

Proof. For each integer a, set

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$$T(a) = \{y : ay = e\}.$$

Clearly, each set $T(a)$ is closed. Also, each set $T(a)$ is nowhere dense. Otherwise, $T(a)$ would then be a clopen subgroup of the connected group G and G would not have an element of infinite order.

Suppose n is a positive integer and for each sequence $a_1, \ldots, a_n$ of integers, the set

$$T(a_1,\ldots,a_n) = \left\{(x_1,\ldots,x_n) \in G \times \ldots \times G : \sum_{i=1}^{n} a_i x_i = e\right\}$$

is a nowhere dense subset of G^n. Let $b_1, \ldots, b_n, b_{n+1}$ be positive integers and consider

$$T(b_1,\ldots,b_n,b_{n+1}) = \{(x_1,\ldots,x_{n+1}) : \sum_{i=1}^{n} b_i x_i = e\}.$$

Clearly, $T(b_1,\ldots,b_{n+1})$ is a closed set. If this set is not nowhere dense, then there are nonempty open subsets $V_1, \ldots, V_n, V_{n+1}$ of G so that

$$V_1 \times V_2 \times \ldots \times V_{n+1} \subseteq T(b_1,\ldots,b_{n+1}).$$

Fix $x_{n+1} \in V_{n+1}$. If $z_i = x_i - x'_i$, where $x_i, x'_i \in V_i$, for $i = 1, \ldots, n$, then

$$\sum_{i=1}^{n} b_i x_i + b_{n+1} x_{n+1} = e,$$

$$\sum_{i=1}^{n} b_i x'_i + b_{n+1} x_{n+1} = e.$$

Thus,

$$\sum_{i=1}^{n} b_i z_i = e,$$

for all $z_i \in V_i - V_i$. This means

$$(V_1 - V_1) \ldots (V_n - V_n) \subseteq T(b_1,\ldots,b_n).$$

This contradiction establishes that each of the "hyperplanes" $a_1 x_1 + \ldots + a_n x_n = e$ is meager. The existence of a Cantor set of independent elements now follows from a result of J. Mycielski [4]. □

Let us remark that if G is a connected locally compact abelian group, then G certainly has an element of infinite order [1, page 389].

Theorem 1. *For each ordinal α, $1 \leq \alpha < \omega_1$, there is a subgroup of G which is of exactly additive class α in G. For each ordinal α, $2 \leq \alpha < \omega_1$, there is a subgroup of G which is of exactly multiplicative (ambiguous) class α in G.*

Proof. In order to proceed with the proof of Theorem 1, let us make the following conventions and notations.

Let $\leq$ be a linear ordering on $M \cup (-M)$ so that $\{(x,y) \in (M \cup M^{-1})^2 : x \leq y$ is closed$\}$. If $E \subseteq G$, then $-E = \{-t : t \in E\}$ and $E^n = \{(t_1,\ldots,t_n) : t_i \in E, 1 \leq i \leq n\}$. For each n, let

$$W_n = \{(x_1,\ldots,x_n) \in (M \cup (-M))^n : x_1 \leq x_2 \leq \ldots \leq x_n\}$$

and

$$T_n = \{(x_1,\ldots,x_n) \in (M \cup (-M))^n : x_i \neq -x_j, \text{ for } 1 \leq i \leq j \leq n\}.$$

Clearly, W_n is closed with respect to $(M \cup (-M))^n$ and T_n is open with respect to $(M \cup (-M))^n$. For each n, let $f_n : G^n \to G$ by $f_n(x_1,\ldots,x_n) = x_1 + \ldots + x_n$.

Notice that each element $x \in \langle M \rangle$, $x \neq e$, has a unique representation as

$$x = x_1 + \ldots + x_n,$$

where each $x_i \in M \cup (-M)$, $x_1 \leq x_2 \leq \ldots \leq x_n$ and $x_i \neq -x_j$, if $1 \leq i < j \leq n$.

For each subset H of M, set

$$S_n(H) = W_n \cap T_n \cap (H \cup (-H))^n$$

and

$$Z_n(H) = f_n(S_n(H)).$$

Notice that: (1) $f_n | S_n(H)$ is a homeomorphism of $S_n(H)$ onto $Z_n(H)$, (2) $Z_n(H) \cap Z_m(H) = \phi$, if $n \neq m$, (3) $\cup Z_n(H) = \langle H \rangle - \{e\}$, and (4) $\langle H \rangle \cap M = H$.

Let H be a subset of M of exactly additive class α in M for some α, $1 \leq \alpha \leq \omega_1$. Of course, H is of the same class in G. It can be checked that each of the sets $S_n(H)$ is also of additive class α. If $\alpha = 1$, then the sets $S_n(H)$ are actually K_σ sets, the union of countably many compact sets. Thus, each of sets $Z_n(H)$ is a K_σ set. Therefore, $\langle H \rangle$ is a K_σ set and since $\langle H \rangle \cap M = H$, $\langle H \rangle$ cannot be of any lower class or a G_δ set in G. If $\alpha > 1$, then the sets $Z_n(H)$ are of additive class α in G since this property is an intrinsic invariant [2, page 432]. Thus, $\langle H \rangle$ is of additive class α in G. Also, since $\langle H \rangle \cap M = H$, $\langle H \rangle$ cannot be of any lower class in G or of multiplicative class α in G. Thus, for each $\alpha > 1$, G has a subgroup of exactly additive class α.

Now, let $1 < \alpha < \omega_1$ and take H to be a subset of M of exactly multiplicative class α in M (and therefore in G). As before, each of the sets $Z_n(H)$ is of multiplicative class α in G. Thus, $\langle H \rangle$ is a Borel subgroup of G and since $\langle H \rangle \cap M = H$, $\langle H \rangle$ cannot be of lower class than α and cannot be of additive class α. Thus, all that need be shown is that the class of

$\langle H \rangle$ does not go up. To see this, notice the following. The sets $V_n = Z_n(M)$ are pairwise disjoint F_σ subsets of G and $Z_n(H) \subset V_n$.

If α is not a limit ordinal, let

$$Z_n(H) = \bigcap_{p=1}^{\infty} A_{n_p},$$

where each A_{n_p} is of additive class $\alpha - 1$. For each p, set $B_p = \bigcup_{n=1}^{\infty} (A_{n_p} \cap V_n)$. Each set B_p is of additive class $\alpha - 1$ and $\bigcap_{p=1}^{\infty} B_p = \bigcup_{n=1}^{\infty} Z_n(H) = \langle H \rangle$. Thus, $\langle H \rangle$ is of exactly multiplicative class α.

If α is a limit ordinal, let $\{\alpha_p\}_{p=1}$ be an increasing sequence of ordinals converging to α such that $\alpha_p > 1$ for each p. For each n, let $Z_n(H) = \bigcap_{p=1}^{\infty} A_{n_p}$, where A_{n_p} is of additive class α_p. As before, the sets B_p are of additive class α_p. Thus, $\langle H \rangle$ is of exactly multiplicative class α.

The existence of subgroups of G of exactly ambiguous class α, $1 < \alpha < \omega_1$ can be shown in exactly the same manner.

Thus, the only real restriction on the Borel subgroups of the groups under consideration is that every open or G_δ subgroup is closed. It seems that the same results should hold for nonabelian groups.

Our second theorem is a simple extension of the result of Klee [2]. Klee proved that each separable infinite-dimensional Banach space has a subspace of exactly additive class α, $1 \leq \alpha < \omega_1$. Klee left open the problem of subspaces of other classes [2, page 198]. However, we will indicate how Klee's procedure settles the other cases also.

The essential ingredient is the existence of a special lineraly independent arc.

Theorem 2. *Let X be an infinite-dimensional separable Banach space. For each ordinal α, $1 \leq \alpha < \omega_1$, there is a dense subspace A_α of X which is exactly of additive class α in X. For each ordinal α, $2 \leq \alpha < \omega_1$, there is a subspace $M_\alpha(E_\alpha)$ of X which is exactly of multiplicative (ambiguous) class α in X.*

Proof. Let ϕ be a continuous one-to-one map of the closed interval $[0,1]$ into X so that $A = \phi([0,1])$ is a linearly independent subset of X and if Z is an infinite subset of A, then sp(Z), the linear span of Z, is dense in X [2].

For each n, let

$$W_n = \{(\phi(t_1),\phi(t_2),\ldots,\phi(t_n)): 0 \leq t_1 < t_2 < \ldots < t_n \leq 1\}$$

and set

$$T_n = (\mathbf{R} - \{0\})^n.$$

For each n, let $f_n : \mathbf{R}^n \times X^n \to X$ by

$$f_n((r_1,\ldots,r_n), (x_1,\ldots,x_n)) = \sum_{i=1}^{n} r_i x_i.$$

For each subset H of A, set

$$S_n(H) = T_n \times (W_n \cap H^n)$$

and

$$Z_n(H) = f_n(S_n(H)).$$

Notice that: (1) $f_n \mid S_n(H)$ is a homeomorphism of $S_n(H)$ onto $Z_n(H)$, (2) $Z_n(H) \cap Z_m(H) = \phi$, if $n \neq m$, (3) $\cup Z_n(H) = \text{sp}(H) - \{0\}$ and (4) $\text{sp}(H) \cap A = H$.

The remainder of the argument proceeds exactly as in the argument for the preceding theorem. □

Finally, we note that if G is a connected abelian Polish group which admits elements of arbitrarily high finite order, then it admits an element of infinite order. Also, R. R. Kallman points out that there is a connected abelian Polish group which does not admit an element of infinite order. In fact, every element has order two. One such group consists of the measure algebra on the unit interval. Let $\mathscr{M}$ be the family of Lebesgue measurable subsets of [0,1] and let $\mathscr{N}$ be the family of null sets. Consider the space $G = \mathscr{M}/\mathscr{N}$ provided with the distance function $p([A], [B]) = \mu(A \Delta B)$. Then G is complete and separable under the metric p. Define $[A] \circ [B] = [A \Delta B]$. Clearly, G is a group under this operation and every element has order two. It can be checked that the map $([A],[B]) \to [A] \circ [B]$ is continuous. Thus, G is a Polish group. Finally, G is connected since if $[B] \in G$, then the map $f(x) = [B \cap [0,x]]$ is a continuous map from [0,1] into G with $f(0) = e$, the identity in G and $f(1) = [B]$. It is also true that G is an exotic group in the sense of Christensen and Herer [5], *i.e.,* G has no strongly continuous unitary representations in $L(H)H$ being some Hilbert space. This is true since G is abelian and therefore, if ϕ were such a representation, then ϕ would be a continuous character. But G has no continuous characters.

References

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North Texas State University, Denton, TX 76203

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