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Inductive Definability: Measure and Category

DOUGLAS CENZER

Department of Mathematics, University of Florida, Gainesville, Florida 32611

AND

R. DANIEL MAULDIN

Department of Mathematics, North Texas State University, Denton, Texas 76203

INTRODUCTION

The purpose of this paper is threefold. Our first purpose is to exposit a major concept from descriptive set theory, inductive definability, and to present some of the major results concerning inductive operators. The classical version of this theory is carried out in the first two sections and the effective version in the fifth section.

Our second purpose is to demonstrate a powerful unity of viewpoint provided by inductive definitions. This is shown by deriving several well-known results from this viewpoint which had been previously proved by diverse methods. This point is demonstrated in the first section by deriving several well-known examples of analytic (and coanalytic) sets which are not Borel sets; in the third and fourth sections by the proofs of some "faithful extension" and reflection theorems. Again, this point is demonstrated in the fifth section in the presentation of several results from effective descriptive set theory.

Our third purpose is to present some new results. Our first new result is given in the third section where it is shown (Theorem 3.3) that several classical "definability" results may be unified with the use of inductive definitions. New results are given in Theorems 4.1 and 4.2 where we demonstrate a definability and reflection principle with respect to conditional probability distributions. In the fifth section, we present new proofs of several known results and give a new characterization of $\rho(x)$, the least ordinal not recursive in x.

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In the sixth section, we present some more new results. These concern uniformizations of coanalytic sets and the relatively new concept of parametrizations of coanalytic sets. We demonstrate the fact that if C is a coanalytic subset of the product of two Polish spaces and each X-section of C is large, then C has $2^{\aleph_0}$ Borel uniformizations and a particularly nice parametrization, a parametrization which is measurable with respect to the σ-algebra generated by the analytic sets. The proofs of these theorems are made possible through the employment of the results presented earlier in this paper concerning inductive operators and some recent work of the second author.

1. Inductive Definability

Let X be some fixed set. An inductive operator Γ over X is a map from the power set 2^X to 2^X such that $K \subseteq \Gamma(K)$ for all $K \subseteq X$: in this paper most of the operators considered are monotone, that is, for any $K \subseteq M \subseteq X$, $\Gamma(K) \subseteq \Gamma(M)$. We shall identify a subset K of X with its characteristic function, so $K(x) = 1$ if $x \in K$ and $K(x) = 0$ if $x \notin K$.

Let $A \subseteq X$ be given. The operator Γ constructs from A a transfinite sequence $\{\Gamma^\alpha(A)\colon \alpha \in On\}$ as follows:

$$\begin{aligned} \Gamma^0(A) &= A, \\ \Gamma^{\alpha+1}(A) &= \Gamma(\Gamma^\alpha(A)) && \text{for all ordinals } \alpha, \\ \Gamma^\lambda(A) &= \bigcup \{\Gamma^\alpha(A)\colon \alpha < \lambda\} && \text{for limit ordinals } \lambda. \end{aligned} \tag{1.1}$$

The set C inductively defined from A by Γ is $\mathrm{Cl}(\Gamma; A) = \bigcup \{\Gamma^\alpha(A)\colon \alpha \in On\}$; C is called the closure of Γ on A. For some ordinal $\alpha < \mathrm{card}(X)^+$, $\Gamma^{\alpha+1}(A) = \Gamma^\alpha(A) = \mathrm{Cl}(\Gamma; A)$; the least such ordinal is $|\Gamma; A|$, the closure ordinal of Γ on A. Also, $|\Gamma|$ means $|\Gamma; \varnothing|$ and $\mathrm{Cl}(\Gamma)$ means $\mathrm{Cl}(\Gamma; \varnothing)$. The operator Γ is said to inductively define the set $\mathrm{Cl}(\Gamma)$.

A subset K of X is said to be a fixed point of Γ if $\Gamma(K) = K$. The following fact is often useful.

THEOREM 1.1. *If Γ is a monotone inductive operator over a set X and $A \subseteq X$, then $\mathrm{Cl}(\Gamma; A)$ is the intersection of the family of fixed points of Γ which include A.*

Proof. Let $C = \mathrm{Cl}(\Gamma; A) = \bigcup \{\Gamma^\alpha(A)\colon \alpha \in On\}$ and let $D = \bigcap \{K\colon K \supseteq A$ and $\Gamma(K) = K\}$. Since $C \supseteq A$ and $\Gamma(C) = C$, $D \subseteq C$.

Let $K \supseteq A$ and $\Gamma(K) = K$. Then $K \supseteq \Gamma^0(A) = A$ and it can be seen by transfinite induction on $\alpha \in On$, that $K \supseteq \Gamma^\alpha(A)$ for all $\alpha \in On$, since if $\Gamma^\alpha(A) \subseteq K$, then $\Gamma^{\alpha+1}(A) = \Gamma(\Gamma^\alpha(A)) \subseteq \Gamma(K) = K$. ∎

We now present some examples of monotone definitions.

EXAMPLES 1.2. Let $(G, \cdot)$ be a group and let Γ be defined by

$$\Gamma(K) = K \cup \{a^{-1}b \colon a, b \in K\}.$$

Then of course, for any subset A of G, $\mathrm{Cl}(\Gamma; A) = \langle A \rangle$, the subgroup of G generated by A and $|\Gamma; A|$ can vary from 0 to ω. It will follow from our results below that if G is a Polish group and A an analytic subset of G, then $\langle A \rangle$ is also analytic.

EXAMPLE 1.3. Let Y be a set and let $X = 2^Y$. For $\mathscr{K} \subseteq X$, define the following operators.

$$\mathscr{C}(\mathscr{K}) = \{Y - K \colon K \in \mathscr{K}\},$$

$$\Sigma(\mathscr{K}) = \{\text{countable unions } \bigcup K_n \text{ with each } K_n \text{ in } \mathscr{K}\},$$

and

$$\Pi(\mathscr{K}) = \{\text{countable intersections } \bigcap K_n \text{ with each } K_n \text{ in } \mathscr{K}\}.$$

These are all monotone operators.

Suppose now that Y is a topological space with family τ of open sets and let $\mathscr{B}$ be the family of Borel subsets of Y, that is, the smallest family including τ and closed under complementation and countable unions. A number of schemes for generating the Borel sets from τ have been given: see, for example, [17]. However, the Borel sets may also be given as the closure on τ of the monotone operator Δ defined by

$$\Delta(\mathscr{K}) = \Sigma(\mathscr{K}) \cup \mathscr{C}(\mathscr{K}).$$

In case every open set is an F_σ set we have $\Delta(\tau) = \{U \colon U \text{ is open or closed}\}$, $\Delta^2(\tau) = \tau \cup F_\sigma$ and $\Delta^3(\tau) = F_\sigma \cup G_\delta$. Of course, $\mathrm{Cl}(\Delta; \tau) = \mathscr{B}$.

Depending on the space Y, this operator Δ has $|\Delta; \tau|$ between 0 and ω_1. In fact, no matter what family, τ, of subsets of Y one starts with, $\mathrm{Cl}(\Delta; \tau)$ is the Borel family generated by τ. Recently, Miller [19] has shown that it is consistent that for each $\alpha < \omega_1$, there be a family τ of subsets of the reals such that $|\Delta; \tau| = \alpha$.

Now let X be a Polish space—that is, a topological space such that there is a metric which generates the same topology under which the space is separable and complete. The interval $I = [0, 1]$ and the space J of irrationals in I (not with the usual metric though) are typical Polish spaces. A subset A of X is analytic (or Σ^1_1), if there is a Borel subset B of $X \times J$ such that $A = \pi_1(B) = \{x \colon (\exists y \in J)(x, y) \in B\}$; a subset C of X is coanalytic (CA or Π^1_1), if $X - A$ is analytic. In fact, an arbitrary uncountable Polish space can be used here in place of J. The Souslin–Kleene theorem states that a set is Borel if and only if it is both $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$ (that is, $\mathbf{\Delta}_1^1$). Of course, there are $\mathbf{\Pi}_1^1$ sets which are not Borel (some examples are given below).

The primary topic in this section concerns the families of Borel, analytic, and coanalytic operators over a Polish space X. Roughly speaking, an operator Γ is Borel (open, analytic, etc.) if $\Gamma(K)$ is always Borel (open, analytic, etc.) relative to K and is defined uniformly on 2^X.

An operator Δ over a Polish space X is said to be Borel (or $\mathbf{\Delta}_1^1$) if it is defined in one of the following ways:

(a) $\Delta(K) = B$, where B is a fixed Borel subset of X;

(b) $\Delta(K) = f^{-1}(K)$, where f is a fixed Borel map from X to X;

(c) $\Delta(K) = X - K$; (1.4)

(d) $\Delta(K) = \Delta_1(\Delta_2(K))$, where Δ_1 and Δ_2 are previously defined Borel operators;

(e) $\Delta(K) = \bigcup \{\Delta_n(K) : n \in N\}$, where the Δ_n are previously defined Borel operators for $n \in N = \{0, 1, 2, \dots\}$.

Thus, a map from 2^X into 2^X is a Borel operator if and only if it is in the smallest family which contains the operators described in parts a, b, and c and which is closed with respect to the operations described in parts d and e. The operators of parts a and b are monotone, but not inductive and the operator of part c is neither monotone nor inductive.

An operator Γ over the Polish space X is analytic or $\mathbf{\Sigma}_1^1$ (resp. coanalytic or $\mathbf{\Pi}_1^1$) if there is a Polish space Y and a Borel operator Δ over $X \times Y$ such that for all x and K:

$$
\begin{aligned}
x \in \Gamma(K) \text{ IFF } \quad & (\exists y)(x, y) \in \Delta(K \times Y), \\
\text{(resp.)} \quad & (\forall y)(x, y) \in \Delta(K \times Y).
\end{aligned}
\tag{1.5}
$$

Of course, both classes are closed under countable unions and intersections; every Borel operator is both $\mathbf{\Sigma}_1^1$ and $\mathbf{\Pi}_1^1$.

We hope to convince the reader that the classes of inductive and particularly monotone inductive operators in these families have an interesting and useful theory. The main tools of this theory are given in the following theorem. Parts of this theorem will be proved in this section and parts will be proved in Section 2. We will interject some examples and application of this theorem along the way.

THEOREM 1.6 (Inductive definability). *Let X be a Polish space.*

(a) *If Δ is a Borel operator over X, then Δ^α is also a Borel operator for each $\alpha < \omega_1$ and $\Delta(B)$ is a Borel subset of X, if B is.*

(b) *If the monotone operator Γ and the set A are both analytic (resp. coanalytic), then for each countable ordinal α, $\Gamma^{\alpha}(A)$ is analytic (resp. coanalytic).*

(c) *If the monotone operator Γ and the set A are both coanalytic, then* $\mathrm{Cl}(\Gamma; A)$ *is coanalytic.*

(d) *For any coanalytic subset C of X, there is a monotone Borel operator Δ over $X \times J$ and a real $r \in J$ such that $C = \{x: (x, r) \in \mathrm{Cl}(\Gamma)\}$.*

(e) *If Γ is a coanalytic monotone operator with closure C, on the coanalytic subset P of X, then for any analytic subset A of X with $A \subseteq C$, there is a countable ordinal α such that $A \subseteq \Gamma^{\alpha}(P)$.*

(f) *If the inductive operator Γ is either* (1) *Borel or* (2) *monotone and either analytic or coanalytic, then $|\Gamma| \leqslant \omega_1$.*

Let us remark that there is a useful and natural method of associating an inductive operator with any given operator. Given $\Phi: 2^X \to 2^X$, define Γ by

$$\Gamma(K) = K \cup \Phi(K).$$

Clearly, Γ is inductive and if Φ is respectively Borel, analytic or coanalytic, then so is Γ. If Φ is monotone, then Γ is monotone inductive and it can be shown by transfinite induction that Γ and Φ will then produce the same sets inductively from ϕ: $\Phi^{\alpha}(\phi) = \Gamma^{\alpha}(\phi)$, for all ordinals α.

One proves Theorem 1.6a by noting that the family of all operators which satisfy Theorem 1.6a contains the operators described in (1.4) parts a, b, and c and is closed under the operations described in part d and e.

Before proceeding further with the proof of Theorem 1.6, we give some examples of inductive definitions and some applications of Theorem 1.6.

As our next example, we will give a Borel monotone inductive definition of a (actually coanalytic) subset C of $J \times J$ which is *universal* for the Borel subsets of J. This means that $\{C_x: x \in J\}$ is precisely the family of Borel subsets of J, where $C_x = \{y: (x, y) \in C\}$. Of course, C cannot be Borel itself by a simple diagonal argument: C is coanalytic by Theorem 1.6(c). Such a set was first constructed by Sierpinski [26].

Our definition of the set C depends on the fact that the Borel subsets of J can be generated from the open sets by taking countable unions and countable intersections, but not complements—call the sets generated in this way the positive Borel sets. The family of positive Borel subsets of a given topological space X is always included in the family of Borel sets and will be the entire family if it is closed under complementation. This will be the case in any metric space X, by the following argument: Any positive Borel set B is either open or the countable union or intersection of previously generated sets $\{B_n: n \in N\}$. If B is open, then $X - B = \bigcap \{M_n: n \in N\}$, where $M_n = \{x$: the distance from x to $X - B$ is less than $1/n\}$; each M_n is open, so $X - B$ is positive Borel. If $B = \bigcup B_n$ or $\bigcap B_n$, then $X - B = \bigcap (X - B_n)$ or $\bigcup (X - B_n)$; by induction, each $X - B_n$ is positive Borel so, again $X - B$ is positive Borel.

The space J of irrationals is of course a metric space, but it is also homeomorphic to the product N^N of countably many countable discrete spaces; henceforth J will refer to this product space. For $u = (u(0), u(1),...) \in J$ and $i \in N$, let $\pi_i(u) = (u(p_i), u(p_i^2), u(p_i^3),...)$, where p_i is the ith prime number. The function which takes u to $(\pi_0(u), \pi_1(u),...)$ maps J onto J^N. Now the topology of J has a subbase consisting of the sets $V(m, n) = \{u: u(m) = n\}$ for fixed m and n in N. The Borel sets are generated from these by countable union and intersection.

Each Borel set receives an index (in fact, infinitely many) in the following manner: $V(m, n)$ gets any u with $u(0) = 0$, $u(1) = m$ and $u(2) = n$. If $B = \bigcup \{A_i : i \in N\}$ (resp. $\bigcap A_i$), and for each i, u_i is a code for A_i, then B gets as an index any u with $u(0) = 1$ (resp. $u(0) = 2$) and, for each i, $\pi_i(u) = u_i$. It can be checked that (1) each Borel set has continuumly many indices, (2) no sequence is an index for two distinct Borel sets and (3) there are continuumly many sequences which are not indices for any Borel set.

Let C be the set of pairs (u, v) such that v belongs to the Borel set with index u. C is clearly universal for the Borel subsets of J. Now C is the closure of the monotone Borel inductive operator Δ, defined by:

$$(u, v) \in \Delta(K) \text{ IFF } \begin{cases} (u, v) \in K \\ \text{OR } u(0) = 0 \text{ AND } v(u(1)) = u(2) \\ \text{OR } u(0) = 1 \text{ AND } (\exists i)(\pi_i(u), v) \in K \\ \text{OR } u(0) = 2 \text{ AND } (\forall i)(\pi_i(u), v) \in K. \end{cases} \tag{1.7}$$

It follows from Theorem 1.6(c) that C is in fact a coanalytic subset of $J \times J$. Note that, for $\alpha \geqslant \omega$, Δ^α is universal for the Borel sets of class $<\alpha$.

We next give some examples of Σ_1^1 and Π_1^1 operators and some applications of Theorem 1.6(b, c, e).

Recall the operator Γ over the group G which has on A the closure $\langle A \rangle$, the subgroup of G generated by A. If G is a Polish group (that is, a Polish space with continuous multiplication map from $G \times G$ to G making G a group), then Γ is a Σ_1^1 monotone inductive operator. Since $|\Gamma; A| \leqslant \omega$, it follows from Theorem 1.6(b) that whenever A is analytic, $\langle A \rangle = \mathrm{Cl}(\Gamma; A)$ is also an analytic subset of G.

The closure operator on a topological space with metric d is defined by

$$\Gamma(M) = \bar{M} = \{x: (\forall k)(\exists y \in M)\, d(x, y) < 1/k\}$$

is a Σ_1^1 monotone and inductive operator. We will indicate that Γ is Σ_1^1.

Similar arguments can be given for the other examples in this paper. Let $Y = X^N$ and let Δ be the Borel operator over $X \times Y$ defined by

$$\Delta(A) = \bigcap_{k=1}^{\infty} (B_k \cap f_k^{-1}(A)),$$

where $B_k = \{(x, (y_n)): d(x, y_k) < 1/k\}$ and $f_k(x, (y_n)) = (y_k, (y_n))$. Notice that

$$x \in \Gamma(K) \leftrightarrow \exists (y_n) \qquad (x, (y_n)) \in \Delta(K \times Y).$$

Thus, Γ is $\boldsymbol{\Sigma}_1^1$.

The (Cantor–Bendixson) derived set operator, defined by

$$M' = \{x: x \in \overline{M - \{x\} \cap M}\}$$

is also $\boldsymbol{\Sigma}_1^1$. However, it is not inductive, since $M' \subseteq M$ rather than $M \subseteq M'$. The dual operator Ψ defined by

$$\Psi(K) = X - (X - K)'$$

will be inductive, monotone and $\boldsymbol{\Pi}_1^1$.

Now let A be a fixed analytic subset of a Polish space X, and apply Ψ to the $\boldsymbol{\Pi}_1^1$ set $X - A$. It is easily seen that, for each countable ordinal α, $X - \Psi^{\alpha}(X - A)$ is the αth derived set of A and that $X - \mathrm{Cl}(\Psi; X - A)$ is the largest subset of A which is dense-in-itself. It follows from Theorem 1.6(c) that the largest dense-in-itself subset of an analytic set is also analytic.

For an analytic subset A of a product space $X \times Y$, a similar $\boldsymbol{\Pi}_1^1$ monotone operator can be defined with $\boldsymbol{\Pi}_1^1$ closure C so that, for each $x \in X$, $Y - C_x$ is the largest dense-in-itself subset of A_x. This leads to our first definability result. Recall that a set is said to be *scattered* if it includes no non-empty dense-in-itself subset.

THEOREM 1.8 (Luzin [5]). *Let A be an analytic subset of the product $X \times Y$ of Polish spaces. Then $D = \{x \in X: A_x$ is scattered$\}$ is a coanalytic subset of X.*

Proof. Let Γ be the monotone, inductive, $\boldsymbol{\Pi}_1^1$ operator described above with closure C, i.e., $\Gamma(K) = \bigcup \{\{x\} \times \Psi(K_x): \quad x \in X\}$. Thus, $C = \mathrm{Cl}(\Gamma; (X \times Y) - A)$. So, by Theorem 1.6(c), C is $\boldsymbol{\Pi}_1^1$. Now,

$$\begin{aligned} A_x \text{ is scattered} \quad & \text{IFF } C_x = Y \\ & \text{IFF } (\forall y)(x, y) \in C. \end{aligned}$$

Since C is $\boldsymbol{\Pi}_1^1$, D is also $\boldsymbol{\Pi}_1^1$. ∎

A generalized version of this result will be given in Section 3.

As an application of the boundedness principle in Theorem 1.6 (part e), we give the following related theorem (proved by second author in *Pacific J. Math.* **74** (1978), 169–177).

THEOREM 1.9. *Let X and Y be Polish spaces and A an analytic subset of $X \times Y$. If A_x is scattered for each x in X, then there is an ordinal α, $\alpha < \omega_1$, such that the αth derived set of A_x is empty, for all x in X.*

Proof. Using the notation from proof of Theorem 1.8, we have $C_x = Y$, for each x in X. So C, the closure of Γ, is $X \times Y$. Since C is a Borel set, it follows from Theorem 1.6e that there is a countable ordinal α such that $X \times Y = C \subseteq \Gamma^\alpha$. This means that the αth derived set of each section is empty. ∎

The next examples require some definitions.

Let S be the set $\bigcup \{N^k : k \in N\}$ be the set of all finite sequences of non-negative integers with the usual Brouwer–Kleene ordering:

$$s = (m_0, \dots, m_{k-1}) \leqslant (n_0, \dots, n_{l-1}) = t \text{ IFF}$$
$$[s \text{ extends } t\ (s \supseteq t) \text{ OR } (\exists j)(m_0 = n_0 \,\&\, \cdots \,\&\, m_{j-1} = n_{j-1} \,\&\, m_j < n_j)].$$

Note that $(S, \leqslant)$ is isomorphic to the set of dyadic rationals q, $0 \leqslant q < 1$, with the usual ordering reversed; $(m_0, \dots, m_k)$ corresponds to $(2^{-m_0}) + (2^{-m_0-m_1}) + \cdots + (2^{-m_0-m_1-\cdots-m_k})$, and the empty sequence $\varnothing$ corresponds to 0. For any $s = (m_0, \dots, m_{k-1})$ and any $i \in N$, let $s * i = (m_0, \dots, m_{k-1}, i)$. For any $u \in J$ and $n \in N$, let $u \mid n = (u(0), \dots, u(n-1))$; also, write $s \subseteq u$ for $(\exists n)$ $(s = u \mid n)$.

For $R \subseteq S$, let $x_R \in 2^S$ be defined by $x_R(s) = 1$ IFF $s \in R$. The coanalytic set $W \subseteq 2^S$ is defined by:

$$\begin{aligned} W &= \{x_R : R \text{ is well-ordered}\} \\ &= \{x : \forall (y_0, y_1, \dots)(\exists n)[x(y_n) = 0 \text{ OR } y_{n+1} \geqslant y_n]\}. \end{aligned} \tag{1.10}$$

For x_R in W, let $\sigma(x_R)$ be the order type of R. For $p \in S$ and $R \subseteq S$, let $R \restriction p = \{s \in R : s < p\}$; for $x \in 2^S$ and $s \in S$, $x \restriction p(s) = 1$ IFF $x(s) = 1 \,\&\, s < p$.

W can be defined by the following simple closed operator:

$$\begin{aligned} x \in \Delta(K) \quad & \text{IFF } x \in K \\ & \text{OR } (\forall s)\, x(s) = 0 \\ & \text{OR } (\forall p)\, x \restriction p \in K. \end{aligned} \tag{1.11}$$

In fact, for each ordinal α,

$$\Delta^{\alpha} = \{x: \sigma(x) < \alpha\}.$$

Thus $W = \mathrm{Cl}(\Delta)$, $|\Delta| = \omega_1$, and by Theorem 1.6(c), W is coanalytic.

Closely related is the set W^*, defined by:

$$W^* = \{x: (\forall u)(\exists n)\, x(u \mid n) = 0\}. \tag{1.12}$$

Now x in 2^S is said to be regular if, whenever $s \subseteq t$, $x(s) \geqslant x(t)$.

LEMMA 1.13. *For regular* x, $x \in W$ IFF $x \in W^*$.

Proof. The direction ($\rightarrow$) is immediate, even for x not regular, since $u \mid 0 > u \mid 1 > \cdots$ is a descending chain. For the other direction, suppose that $s_0 > s_1 > \cdots$ is a descending chain with each $x(s_i) = 1$ and let $u = \lim(s_i)$. This limit exists even if x is not regular, but if x is regular then each $x(u \mid n) = 1$. ∎

A subset A of a space X is said to be reducible to a subset B of a space Y provided there is a continuous map f of X into Y so that $A = f^{-1}(B)$.

W^* is an example of a set which is reducible to the closure of a Borel monotone inductive operator. Define the operator Δ on $2^S \times S$ by:

$$\begin{aligned} x \in \Delta(K) \quad & \text{IFF } (x, q) \in K \\ & \text{OR } q \not\subseteq x \\ & \text{OR } (\forall i)(x, q^*i) \in K. \end{aligned} \tag{1.14}$$

It is easily seen that, for all x and q:

$$(x, q) \in \mathrm{Cl}(\Delta) \qquad \text{IFF } (\forall u)[q \subseteq u \rightarrow (\exists n)\, x(u \mid n) = 0], \tag{1.15}$$

so that $x \in W^*$ if and only if $(x, \varnothing) \in \mathrm{Cl}(\Delta)$.

Another such example, due to Luzin [15], is the set $D \subseteq J$, defined by:

$$D = \{x: \neg(\exists y)(\forall n)\, x(y(n)) \propto x(y(n+1)), \tag{1.16}$$

where $i \propto j$ means i is less than j and divides j.

D is reducible to the closure of the Borel monotone inductive operator Δ on $J \times N$, defined by:

$$\begin{aligned} (x, n) \in \Delta(K) \qquad & \text{IFF } (x, n) \in K \\ & \text{OR } (\forall i) \neg (n \propto x(i)) \\ & \text{OR } (\forall j > 1)(x, n \cdot j) \in K. \end{aligned} \tag{1.17}$$

It is easily seen that $(x, n) \notin \mathrm{Cl}(\Delta)$ if and only if there is a y such that $n \propto x(y(0))$ and, for each i, $x(y(i))$ is less than and divides $x(y(i+1))$. Thus $D = \{x: (x, 1) \in \mathrm{Cl}(\Delta)\}$.

It is easy to construct a continuous map ϕ: $2^S \to J$ such that $W^* = \phi^{-1}(D)$. Let $s_0, s_1, \ldots,$ enumerate S and let $p_0, p_1, \ldots,$ enumerate the prime numbers, both without repetition. For $s = (m_0, \ldots, m_{k-1})$, let $f(s) = p_{m_0}^{1+m_0} \cdot p_{1+m_0+m_1}^{1+m_1} \cdot \cdots \cdot p_{k-1+m_0+\cdots+m_{k-1}}^{1+m_{k-1}}$, so that $s \subseteq t$ if and only if $f(s) \mid f(t)$. Define the map ϕ by

$$\begin{aligned} \phi(x)(n) &= p_n &&\text{if} \quad x(s_n) = 0 \\ &= f(s_n) &&\text{if} \quad x(s_n) = 1. \end{aligned} \tag{1.18}$$

We next consider the definition of an analytic set by means of a sieve. The reader is referred to Kuratowski and Mostowski [14] for historical background.

Let X be a Polish space. For our purposes, a sieve is a map L from S into the space of closed subsets of X such that if $s \subseteq t$, then $L(s) \supseteq L(t)$. (A more general definition is possible but reduces to this situation [5, p. 21].) For each x in X, let $I_L(x) = \{q: x \in L(q)\}$; each of these sets is a regular subset of S. The analytic set $A(L)$ defined by the sieve is

$$A(L) = \{x: (\exists u)(\forall n)\, x \in L(u \mid n)\}. \tag{1.19}$$

The coanalytic set $C(L)$ defined by L is $X - A(L)$, that is,

$$\begin{aligned} C(L) &= \{x: (\forall u)(\exists n)\, x \notin L(u \mid n)\} \\ &= \{x: I_L(x) \text{ is well-ordered by } \leqslant\}. \end{aligned} \tag{1.20}$$

The second equality is of course a consequence of Lemma 1.12.

For example, let $L(q) = \{x: 0 \subseteq q \to x(0) = 1\}$ define a sieve on the space 2^S. Then $C(L)$ is the set W^* defined in (1.12).

For a second example, for each $q = (m_0, \ldots, m_{k-1}) \in S$, let $L((m_0, \ldots, m_{k-1})) = \{x \in J: (\forall i \leqslant k-1)(\exists j) \quad x(j) = (2+m_0) \cdot \ldots \cdot (2+m_i)\}$. Then $C(L)$ is the set D defined in (1.16).

The fundamental result, due to Sierpinski [27], is that any coanalytic subset of a Polish space can be given by a sieve in the above manner. Thus the set W of well-ordering of S can be thought of as a "universal" coanalytic set in the following sense.

PROPOSITION 1.21. *If C is a Π_1^1 subset of a Polish space X, then there is a Borel map ψ: $X \to 2^S$ such that $C = \psi^{-1}(W)$.*

Proof. Just let $\psi(x)(s) = 1$ IFF $x \in L(s)$. ∎

If X is the space J of irrationals, then the sets $L(q)$ can be taken to be clopen and the map ψ can be taken to be continuous. As a consequence of (1.13) and (1.18), the sets W^* and D are also "universal" $\mathbf{\Pi}_1^1$ sets.

Now for any sieve L, the set $C(L)$ can be written as the union of the $\aleph_1$ Borel sets

$$C^\alpha = \{x: I_L(x) \text{ is a well-ordering of type } <\alpha\}, \tag{1.22}$$

for $\alpha < \omega_1$. The important Boundedness Principle of Lusin and Sierpinski states that if B is a $\mathbf{\Sigma}_1^1$ subset of X and $B \subseteq C(L)$, then B is included in some C^α.

The expression of $C(L)$ as the increasing union of the Borel sets C^α is similar in form to an inductive definition of $C(L)$, but does not in general correspond to any inductive definition. (For example, whenever $C^\alpha = C^{\alpha+1}$ for some $\alpha < \omega_1$). In the remainder of this section, we give a Borel monotone inductive definition for the coanalytic set $C(L)$, proving Theorem 1.6(d), and established a general "Boundedness Principles" for such inductive definitions.

Given a sieve L, define the inductive Borel monotone operator Δ_L over $X \times S$ as follows: (Compare with 1.14.)

$$\begin{aligned} (x, q) \in \Delta_L(K) \qquad & \text{IFF (i) } (x, q) \in K \\ & \text{OR (ii) } x \notin L(q) \\ & \text{OR (iii) } (\forall i)(x, q * i) \in K. \end{aligned} \tag{1.23}$$

It is easily seen that, for all x and q:

$$(x, q) \in \mathrm{Cl}(\Delta_L) \qquad \text{IFF } (\forall u)[q \subseteq u \rightarrow (\exists n)(x \notin L(u \mid n))], \tag{1.24}$$

so that

$$x \in C(L) \qquad \text{IFF } (x, 0) \in \mathrm{Cl}(\Delta_L). \tag{1.25}$$

Theorem 1.6(d) follows from these considerations, since any $\mathbf{\Pi}_1^1$ set can be given by a sieve L and therefore by the corresponding monotone Borel operator Δ over $X \times S$ and since S can be embedded in J.

If $(x, q) \in \Delta^1 = \Delta^1(\varnothing)$, then $x \notin L(q)$. So, $(\forall i) x \notin L(q * i)$ and $(\forall i)$ $(x, q * i) \in \Delta^1$. It can be seen by induction on α that

$$(x, q) \in \Delta^\alpha \rightarrow (\forall i)\,(x, q * i) \in \Delta^\alpha. \tag{1.26}$$

It should be noted that the levels of the inductive definition of the set $C(L)$ using the operator Δ_L do not correspond exactly to the levels C^α. For each countable ordinal α, let $B^\alpha = \{x: (x, 0) \in \Delta_L^\alpha\}$. For $x \in C$ and $q \in I_L(x)$, let

$|q|_x$ be the image of q under the natural isomorphism of the well-ordered set $I_L(x)$ with an ordinal. It can be seen by induction on $|q|_x$ that, for any x and q, $(x, q) \in \Delta_L^{|q|+1}$. It follows that, for each countable ordinal α, $C^\alpha \subseteq B^\alpha$. It is not true in general that $B^\alpha \subseteq C^\alpha$.

On the other hand, there is a Borel monotone operator Γ over $X \times S$ such that for each ordinal α

$$C^\alpha = \{x: (x, 0) \in \Gamma^\alpha\}$$

The operator Γ is defined as follows

$$(x, q) \in \Gamma(K) \leftrightarrow \text{(i) } (x, q) \in K \quad \text{OR}$$

$$\text{(ii) } x \notin L(q) \text{ and } \forall s[(s \in L(q) \vee s < q) \Rightarrow (x, s) \in K].$$

It can be seen by transfinite induction that $\Gamma^\alpha(\varnothing) = \{(x, q): \sigma(I_L(x) \upharpoonright q) < \alpha\}$. The Boundedness Principle of Lusin and Sierpinski can now be obtained as a corollary of Theorem 1.6(e) as follows.

Let A be an analytic subset of X. If $A \subseteq C(L)$, then $A \times \{0\} \subseteq \mathrm{Cl}\,\Gamma$. By Theorem 1.6(e) there is some $\alpha < \omega_1$ so that $A \times \{0\} \subseteq \Gamma^\alpha$ which means $A \subseteq C^\alpha$.

2. Proof of the Inductive Definability Theorem

Parts (a) and (d) of Theorem 1.6 were proven in Section 1. In this section, the remainder of the theorem is demonstrated.

The key fact is that, for any Borel operator Δ over a Polish space X and any fixed $x \in X$ and $K \subseteq X$, the determination of $\Delta(K)$ at the point x depends on only countably much information about K. More precisely:

Lemma 2.1. (a) *If Δ is a Borel operator over the Polish space X, then for any $x \in X$ and $K \subseteq X$, there are countable sets $U \subseteq K$ and $V \subseteq X - K$ such that, for any set M with $U \subseteq M$ and $V \subseteq X - M$, $x \in \Delta(K)$ if and only if $x \in \Delta(M)$.*

(b) *If Γ is a Σ_1^1 monotone operator over X, then for any $x \in X$ and $K \subseteq X$, $x \in \Gamma(K)$ if and only if (for some countable $U \subseteq K$) $x \in \Gamma(U)$.*

(c) *If Γ is a Π_1^1 monotone operator over X, then for any $x \in X$ and $K \subseteq X$, $x \in \Gamma(K)$ if and only if (for all countable $V \subseteq X - K$) $x \in \Gamma(X - V)$.*

Proof. Part (a) can be seen by induction on the class of Borel operators. We now prove part (b); the proof of (c) is similar. Let the Σ_1^1 monotone operator Γ be given by $x \in \Gamma(K)$ IFF $(\exists y)(x, y) \in \Delta(K \times Y)$, where Δ is Borel. Let x and K be given with $x \in \Gamma(K)$; choose y so that $(x, y) \in$

$\Delta(K\times Y)$. By part (a), there exist countable $U\subseteq K\times Y$ and $V\subseteq(X-K)\times Y$ such that whenever $M\supseteq U$ and $(X\times Y)-M\supseteq V$, then $(x,y)\in\Delta(M)$. Let $T=\mathrm{proj}_x(U)$; T is a countable subset of K. Now $T\times Y\supseteq U$ and $(X-T)\times Y\supseteq V$, so that $(x,y)\in\Delta(T\times Y)$ and, therefore, $x\in\Gamma(T)$. This proves the direction ($\rightarrow$) of part (b). The other direction of (b) follows from the monotonicity of Γ. ∎

This lemma has several applications.

First of all, let Γ be an inductive Borel operator. Let $K=\Gamma^{\omega_1}(\phi)$ and suppose that $x\in\Gamma(K)$; let U and V be given by part (a) of Lemma 2.1. Since U is countable, it is included in some countable level of K. Let α be the least countable ordinal such that $U\subseteq\Gamma^{\alpha}(\phi)$ and let $M=\Gamma^{\alpha}(\phi)$; of course, $V\subseteq X-K\subseteq X-M$. It follows that $x\in\Gamma(M)=\Gamma^{\alpha+1}(\phi)\subseteq K$. Thus $|\Gamma|\leqslant\omega_1$ as claimed. The argument when Γ is Σ^1_1 and monotone is similar. This establishes two-thirds of Theorem 1.6(f). The proof for Π^1_1 monotone operators will be given later.

The second application of Lemma 2.1 is the following.

PROPOSITION 2.2. *The family of Σ^1_1 (resp. Π^1_1) subsets of a Polish space is closed under Σ^1_1 (resp. Π^1_1) monotone operators.*

Proof. Let Y be the Polish space X^N. If Γ is Σ^1_1 monotone, let $M\subseteq X\times Y$ be $\{(x,(y_n)): x\in\Gamma(\{y_0,y_1,\ldots\})\}$. Let Δ be a Borel operator over $X\times Z$ so that $x\in\Gamma(K)\leftrightarrow\exists z(x,z)\in\Delta(K\times Z)$. Then $M=\pi_{12}(B)$, where $B=\{(x,(y_n),z): (x,z)\in\Delta(\{y_0,y_1,\ldots\}\times Z)\}$. It can be checked that B is a Borel set for each Borel operator Δ. Thus, M is Σ^1_1 and for any $\Pi^1_1 A\subseteq X$, Σ $\Gamma(A)=\{x: (\exists y)[(x,y)\in M\ \&\ (\forall n)\ (y_n\in A)]\}$, so that $\Gamma(A)$ is also Σ^1_1.

If Γ is Π^1_1, let M be $\{(x,y_0,y_1,\ldots): x\in\Gamma(X-\{y_0,y_1,\ldots\})\}$. Then M is Π^1_1 and for any $\Pi^1_1 C\subseteq X$, $\Gamma(C)=\{x: (\forall y)[x,y)\in M$ OR $(\exists n)(y_n\in C)]\}$, so that $\Gamma(C)$ is also Π^1_1. ∎

Part (b) of Theorem 1.6 is of course an immediate corollary of this proposition.

The third application of Lemma 2.1 involves the family of fixed points of an operator. A set K is said to be a fixed point of the operator Γ if $\Gamma(K)=K$. Now let Γ be a fixed Π^1_1 monotone operator on a Polish space X. Our goal is to show that the closure of Γ is coanalytic, which will yield part (c) of Theorem 1.6.

LEMMA 2.3. $\mathrm{Cl}(\Gamma)=\bigcap\{K$: *$K$ is a co-countable fixed point of Γ*$\}$.

Proof. Let $C=\mathrm{Cl}(\Gamma)$ and suppose $x\notin C$. We will construct a co-countable fixed point K with $x\notin K$. By Lemma 2.1(c), there is a co-countable $B_0\supseteq C$ such that $x\notin\Gamma(B_0)$. If $\Gamma(B_0)=B_0$, then we are finished; if not, let $y_0,y_1,\ldots$ enumerate $\Gamma(B_0)-B_0$. Since $B_0\supseteq C=\Gamma(C)$, each

$y_i \notin \Gamma(C)$. As above, for each i there is a co-countable $D_i \supseteq C$ such that $y_i \notin \Gamma(D_i)$. Let $B_1 = B_0 \cap \bigcap \{D_i : i \in N\}$; then B_1 is co-countable, $B_0 \supseteq B_1 \supseteq C$ and each $y_i \notin \Gamma(B_1)$, that is, $\Gamma(B_1) \subseteq B_0$. Continuing in this manner, we obtain a descending sequence $\{B_n : n \in N\}$ of co-countable sets such that, for all n, $C \subseteq \Gamma(B_{n+1}) \subseteq B_n$ and $x \notin B_n$. Finally, let $K = \bigcap \{B_n : n \in N\}$; then $x \notin K$, K is co-countable, and $\Gamma(K) \subseteq \bigcap \{\Gamma(B_{n+1}) : n \in N\} \subseteq \bigcap \{B_n : n \in N\} = K$, so K is a fixed point. ∎

Now let $Y = X^N$ as above and let $M \subseteq Y$ be

$$\begin{aligned} &\{(y_0, y_1, \ldots) : \Gamma(X - \{y_n : n \in N\} \subseteq X - \{y_n : n \in N\})\} \\ &= \{(y_0, y_1, \ldots) : (\forall n)\, y_n \notin \Gamma(X - \{y_n : n \in N\})\}. \end{aligned} \tag{2.4}$$

Then M is Σ_1^1 and it is clear from Lemma 2.3 that, for any x,

$$x \in \mathrm{Cl}(\Gamma) \qquad \text{IFF } (\forall y)[\, y \in M \rightarrow (\forall n)(y_n \neq x)]. \tag{2.5}$$

This completes the proof of Theorem 1.6(c) in case $A = \varnothing$ by demonstrating that $\mathrm{Cl}(\Gamma)$ is a Π_1^1 set. To complete the proof of (c), let A be a Π_1^1 subset of Y and define a coanalytic monotone operator Φ by setting $\Phi(K) = \Gamma(A \cup K)$. Since $\Phi^\alpha(\varnothing) = \Gamma^\alpha(A)$, for all $\alpha > 0$, $\mathrm{Cl}(\Gamma; A) = \mathrm{Cl}(\Phi)$. Thus, $\mathrm{Cl}(\Gamma; A)$ is Π_1^1.

Recall the derived set operator from Section 1. The implication of Lemma 2.3 is that the largest dense-in-itself subset of any set K is the union of the family of countable subsets of K which are dense-in-themselves.

Finally, we demonstrate the Boundedness Principle for inductive definitions, Theorem 1.6(e). The proof of this Principle requires some discussion of the notion of a pre-well-ordering associated with an inductive operator Γ over the space X:

$$\begin{aligned} |x|_\Gamma &= (\text{least } \alpha)\, x \in \Gamma^{\alpha+1}, && \text{if } x \in \mathrm{Cl}(\Gamma), \\ &= \infty, && \text{otherwise.} \end{aligned} \tag{2.6}$$

This induces a pre-well-ordering on X.

Suppose now that Γ inductively defines a subset of X and Δ inductively defines a subset of Y. Define

$$\begin{aligned} &R(x, y) && \text{IFF } |x|_\Gamma \leqslant |y|_\Delta \;\&\; x \in \mathrm{Cl}(\Gamma); \\ &S(x, y) && \text{IFF } |x|_\Gamma < |y|_\Delta \;\&\; x \in \mathrm{Cl}(\Gamma). \end{aligned} \tag{2.7}$$

It was discovered by Kunen that a simultaneous inductive definition can be given for R and S using the following identities: (See Moschovakis [20, p. 27])

$$\begin{aligned} &R(x, y) && \text{IFF } x \in \Gamma(\{x' : S(x', y)\}); \\ &S(x, y) && \text{IFF } y \notin \Delta(\{y' : \neg R(x, y')\}). \end{aligned} \tag{2.8}$$

Suppose further that X and Y are Polish spaces, that Γ is $\mathbf{\Pi}^1_1$ monotone and Δ is $\mathbf{\Delta}^1_1$ monotone. In view of our remark following Theorem 1.6, we can assume that Γ and Δ are also inductive. Keeping (2.8) in mind, define a $\mathbf{\Pi}^1_1$ monotone operator Λ over $\{0, 1\} \times X \times Y$ by:

$$\begin{aligned} (0, x, y) \in \Lambda(K) \qquad & \text{IFF } x \in \Gamma(\{x' : (1, x', y) \in K\}); \\ (1, x, y) \in \Lambda(K) \qquad & \text{IFF } y \notin \Delta(\{y' : (0, x, y') \notin K\}). \end{aligned} \tag{2.9}$$

Now, for any x and y, $R(x, y)$ if and only if $(0, x, y) \in \mathrm{Cl}(\Lambda)$ and $S(x, y)$ if and only if $(1, x, y) \in \mathrm{Cl}(\Lambda)$, so that both R and S are $\mathbf{\Pi}^1_1$. This is a generalization of a result of Lusin and Sierpinski. (See [13].)

Now suppose A is $\mathbf{\Sigma}^1_1$ in X, $A \subseteq \mathrm{Cl}(\Gamma)$, and let $\alpha = \sup\{|x|_\Gamma + 1 : x \in A\}$. Then

$$y \in \Delta^\alpha \qquad \text{IFF } (\exists x)(x \in A \;\&\; \neg S(x, y)), \tag{2.10}$$

so that Δ^α is also analytic. But if Δ is the inductive definition given in (1.10) of the set W, it follows that α is countable. Since $A \subseteq \Gamma^\alpha$, this completes the proof of part (e) of Theorem 1.6, when $P = \varnothing$.

Given an arbitrary coanalytic subset P of X, define a coanalytic monotone operator Φ by $\Phi(K) = \Gamma(P \cup K)$. Then Φ is $\mathbf{\Pi}^1_1$ monotone inductive and it can be shown that $\Phi^\alpha(\phi) = \Gamma^\alpha(P)$, for all $\alpha > 0$. Now if A is an analytic subset of X and $A \subseteq \mathrm{Cl}(\Gamma; P)$, then $A \subset \mathrm{Cl}(\Phi)$, which implies by the above that $A \subset \Phi^\alpha$ for some $\alpha < \omega_1$. Thus, $A \subset \Gamma^\alpha(P)$ as desired.

Now for any x in $\mathrm{Cl}(\Gamma)$, $\{x\}$ is an analytic subset of $\mathrm{Cl}(\Gamma)$ and is therefore included in some countable level. It follows that, for any $\mathbf{\Pi}^1_1$ monotone operator Γ, $|\Gamma| \leqslant \omega_1$. Together with the first application of Lemma 2.1 above, this completes the proof of the final part of the Inductive Definability Theorem.

The following is an immediate consequence of Theorem 1.6

COROLLARY 2.11. *If Γ is a $\mathbf{\Delta}^1_1$ monotone inductive operator, then* $\mathrm{Cl}(\Gamma)$ *is Borel if and only if $|\Gamma| < \omega_1$; if Γ is $\mathbf{\Pi}^1_1$ and monotone, then $|\Gamma| < \omega_1$ if* $\mathrm{Cl}(\Gamma)$ *is Borel.*

In particular, a coanalytic set C is Borel if and only if its "sieve" inductive definition (1.16) closes at a countable level. Also, this gives an alternate proof of the fact that the set W defined in (1.9) is not Borel.

3. THE FAITHFUL EXTENSION PRINCIPLE

In this section, we consider the application of the Inductive Definability Theorem to the "faithful extension" problem of [5]. This is in preparation for some new results of this type which are demonstrated in Section 4.

Let Y be a Polish space and P a property of subsets of Y, P is said to be monotone decreasing if $K \subseteq M$ and $P(M)$ imply $P(K)$ and monotone increasing if $K \subseteq M$ and $P(K)$ imply $P(M)$. For example, the property of being countable is monotone decreasing; the property of being dense in Y is monotone increasing. P is monotone increasing if and only if the dual property P^*, defined by $P^*(K)$ IFF $P(Y-K)$ is monotone decreasing; P is monotone decreasing IFF P^* is monotone increasing.

An obviously equivalent formulation can be given in terms of monotone operators: P is monotone increasing if and only if there is a monotone operator θ such that, for any K, $P(K)$ IFF $\theta(K) = Y$; P is monotone decreasing if and only if there is a monotone (but not inductive) operator Λ such that $P(K)$ IFF $\Lambda(K) = \varnothing$. If the operator $\theta(\Lambda)$ is Σ^1_1, Π^1_1, etc., then the property P is said to be Σ^1_1 (Π^1_1, etc.) monotone increasing or decreasing. Notice that if P is Π^1_1 monotone increasing and given by the operator θ, then P^* is Σ^1_1 monotone decreasing and is given by the dual operator θ^*, defined by $\theta^*(K) = Y - \theta(Y-K)$, and conversely.

The following is an immediate corollary of Lemma 2.1.

Proposition 3.1. *If the property P is Σ^1_1 monotone decreasing, then, for any K, $P(K)$ IFF (for every countable $M \subseteq K$) $P(M)$; if P is Π^1_1 monotone increasing, then $P(K)$ IFF (for every co-countable $M \supseteq K$) $P(M)$.*

The property of being scattered is Σ^1_1 monotone decreasing; the operator Λ is defined by $\Lambda(K) =$ the largest subset of k which is dense-in-itself. The property of being nowhere dense is also Σ^1_1 monotone decreasing; the operator Λ is defined by $\Lambda(K) =$ the interior of the closure of k. Other examples are the properties of being totally bounded and of being well-ordered (given a Borel linear ordering of the space Y).

These properties were studied in [5], where they were defined by the following alternate characterization.

Proposition 3.2. *A property P is Σ^1_1 monotone decreasing if and only if there is some Π^1_1 subset V (the test set) of Y^N such that, for any $K \subseteq Y$, $P(K)$ IFF $K^N \subseteq V$.*

Proof. ($\rightarrow$) Let the test set V be $\{(y_0, y_1, \ldots): P(\{y_n : n \in N\})\}$ and apply Proposition 3.1. ($\leftarrow$) Define a Σ^1_1 monotone operator by $\Lambda(K) =$ the union of the sets $\{y_n : n \in N\}$ such that $(y_0, y_1, \ldots) \notin V$: it can be checked that $\Lambda(K) = \varnothing$ if and only $K^N \subseteq V$. ∎

For example, K is finite if and only if $K^N \subseteq V = \{(y_0, y_1, \ldots): \{y_n : n \in N\}$ is finite$\} = \{(y_0, y_1, \ldots): (\exists n)(\forall m)(\exists k < n)\, y_m = y_k\}$; here V is actually an F_σ subset of Y^N.

We now present the two basic results concerning monotone properties.

The first is a generalization of Theorem 1.8 and second is a generalization of Theorem 1.9.

THEOREM 3.3 (Definability). *Let A be an analytic subset of the product $X \times Y$ of Polish spaces and let P be a Σ_1^1 monotone decreasing property of subsets of Y. Then $D = \{x \in X: P(A_x)\}$ is a coanalytic subset of X. Equivalently, if C is a coanalytic subset of $X \times Y$ and P is a Π_1^1 monotone increasing property, then $\{x \in X: P(C_x)\}$ is coanalytic.*

Proof. Let Θ be a Π_1^1 monotone operator such that $P(K)$ IFF $\Theta(K) = Y$ and let Δ be a Borel monotone operator over $X \times Y \times S$ such that $(x, y) \in C$ IFF $(x, y, 0) \in \mathrm{Cl}(\Delta)$ as in (1.23). Define the Π_1^1 monotone operator Γ over $(X \times Y \times S) \cup X$ by:

$$\Gamma(K) = \Delta(K \cap (X \times Y \times S)) \cup \{x: \Theta(\{y: (x, y, 0) \in K\}) = Y\}. \quad (3.4)$$

Then, for each ordinal α,

$$\Gamma^{\alpha+1} = \Delta^{\alpha+1} \cup \{x: P(\{y: (x, y, 0) \in \Delta^{\alpha}\})\} \quad (3.5)$$

so that

$$\mathrm{Cl}(\Gamma) = \mathrm{Cl}(\Delta) \cup D. \quad (3.6)$$

Now $\mathrm{Cl}(\Gamma)$ is coanalytic by Theorem 1.6(c); since $D = \mathrm{Cl}(\Gamma) \cap X$, it follows that D is also coanalytic. ∎

THEOREM 3.4 (Faithful extension/reflection). *Let A be an analytic subset of the product $X \times Y$ of Polish spaces and let P be a Σ_1^1 monotone decreasing property of subsets of Y such that $P(A_x)$ for all x in X. Then there is a Borel $B \supseteq A$ such that $P(B_x)$ for all x in X. (Equivalently, let C be a coanalytic subset of $X \times Y$ and let P be Π_1^1 monotone increasing property such that $P(C_x)$ for all x. Then there is a Borel $B \subseteq C$ such that $P(B_x)$ for all x in X.*

Proof. Let C, P, Θ, Δ and Γ be as above in the proof of Theorem 3.3 and assume that $P(C_x)$ for all x in X. Then the corresponding set $D = \{x: P(C_x)\} = X$, so by (3.6), $X \subseteq \mathrm{Cl}(\Gamma)$. By Theorem 1.6(e), there is a countable ordinal α such that $X \subseteq \Gamma^{\alpha}$. This implies, by (3.5) that for all x $P(\{y: (x, y, 0) \in \Delta^{\alpha}\})$. Now let

$$B = \{(x, y): (x, y, 0) \in \Delta^{\alpha}\}. \quad (3.8)$$

It is clear that the set B has the desired properties. ∎

4. Measure and Category

In this section, we obtain results analogous to Theorem 3.3 and 3.4 for the properties of "largeness" in the sense of measure and category, which are monotone but not in general Π_1^1. For example, all co-countable subsets of $I=[0, 1]$ have Lebesgue measure 1, so this property cannot be not by Π_1^1 monotone by our previous results. Let us fix the setting and make some definitions.

Let X and Y be Polish spaces. There are two versions of largeness connected with the geometric notion of category; that of being non-meager and of being co-meager. Recall that a subset K of Y is said to be of the first category, or meager, if it is the union of countably many nowhere dense sets: K is co-meager if $Y-K$ is meager.

A subset of Y can also be said to be large provided there is a probability measure defined on the Borel subsets of Y which gives this set positive measure or measure 1. We shall deal with a more general notion than being large with respect to one fixed measure. We shall allow the measure to vary in a measurable fashion.

A conditional probability distribution on $X \times Y$ is a map μ from $X \times \mathscr{B}(Y)$, where $\mathscr{B}(Y)$ is the family of Borel subsets of Y such that for each x in X, $\mu_x = \mu(x, \cdot)$ is a countably additive measure on $\mathscr{B}(Y)$ such that $\mu_x(Y)=1$ and, for any fixed Borel subset B of Y, the function $\mu(\cdot, B)$ is Borel measurable (equivalently, $\{(x, r)\colon \mu_x(B)=\mu(x, B) \geqslant r\}$ is a Borel subset of $X \times I$). For each x, the measure μ_x has a unique extension to the family of μ_x-measurable sets, where μ_x-measurable is taken in the usual Caratheodory sense. In what follows, we shall also use $\mu_x(E)$ to denote the measure of a μ_x-measurable set E under this extension.

Fix a conditional probability distribution μ on $X \times Y$. The following theorems are proved in this section.

THEOREM 4.1 (Definability). *If C is a coanalytic subset of $X \times Y$, then the following sets are also coanalytic*:

(a) $\{(x, r)\colon \mu(x, C_x) > r\} \subseteq X \times I$,

(b) $\{x\colon X_x$ *is non-meager*$\}$,

(c) $\{x\colon C_x$ *is co-meager*$\}$.

THEOREM 4.2 (Reflection). *If C is a coanalytic subset of $X \times Y$ such that each section C_x, respectively,*

(a) *has the property that $\mu(x, C_x) > r$ (for some fixed r),*

(b) *is non-meager,*

(c) *is co-meager,*

then there is a Borel subset B of C such that each section B_x, respectively,

(a) *has the property that $\mu(x, B_x) > r$,*
(b) *is non-meager,*
(c) *is co-meager.*

In each of these theorems, the condition "measure $>r$" can be replaced by the condition "measure $\geqslant r$."

Let C now be a fixed coanalytic subset of $X \times Y$ and let Δ be the Borel monotone inductive operator over $X \times Y \times S$ with $(x, y) \in C$ if and only if $(x, y, 0) \in \mathrm{Cl}(\Delta)$ as defined in (1.23).

Let $C(x, q) = \{y: (x, y, q) \in \mathrm{Cl}(\Delta)\}$ and, for each α, let $C^\alpha(x, q) = \{y: (x, y, q) \in \Delta^\alpha\}$. Thus, $C^1(x, q) = \{y: (x, y) \notin L(q)\} = (X \times Y) - L(q)$. Clearly, $C^\beta(x, q) \subseteq C^\alpha(x, q)$, if $\beta < \alpha$ and $C(x, q) = \bigcup C^\alpha(x, q)$.

Now, for $\alpha > 0$, if $(x, y, q) \in \Delta^{\alpha+1}$, then according to (1.23) either $(x, y, q) \in \Delta^\alpha$, in which case, according to (1.26), $(\forall i)(x, y, q * i) \in \Delta^\alpha$, or $(x, y) \notin L(q)$, in which case $(\forall i)(x, y) \notin L(q * i)$ and $(\forall i)(x, y, q * i) \in \Delta^\alpha$ or $(\forall i)(x, y, q * i) \in \Delta^\alpha$. Thus we have

LEMMA 4.3. *For any ordinal $\alpha > 0$, any $x \in X$ and $q \in S$:*

$$C^{\alpha+1}(x, q) = \bigcap_{i=0}^{\infty} C^\alpha(x, q * i).$$

The plan for proving Theorem 4.1 is to start with the obvious definability of C^1 and proceed inductively using Lemma 4.3 and standard properties of measure and category. It will be very helpful if the intersection of Lemma 4.3 is decreasing, that is, for $i < j$,

$$C^\alpha(x, q * i) \supseteq C^\alpha(x, q * j). \tag{4.4}$$

This is not true in general; however, it is always possible to construct a similar inductive definition of C for which it is true. For example, define the monotone, inductive, Borel operator Δ_0 over $X \times Y \times \bigcup_{k=1}^\infty S_k$ by $(x, y, (q_1, \ldots, q_n)) \in \Delta_0(K) \leftrightarrow (x, y, (q_1, \ldots, q_n)) \in K$ or $(\forall i \leqslant n)$ $[(x, y) \notin L(q_i)$ or $(\forall j)((x, y, (q * j)) \in K)]$. Then $(x, y) \in C(L) = C$ IFF $(x, y, (0)) \in \mathrm{Cl}(\Delta_0)$. If one sets $C_0^\alpha(x, (q_1, \ldots, q_n)) = \{y: (x, y, (q_1, \ldots, q_n)) \in \Delta_0^\alpha\}$, then

$$C_0^{\alpha+1}(x, (q_1, \ldots, q_n)) = \bigcap_{m=0}^{\infty} [C_0^\alpha(x, (q_1 * 0, \ldots, q_1 * m, \ldots, q_n * 0, \ldots, q_n * m))].$$

This is a decreasing sequence. For simplicity, we assume that the sets $C^\alpha(x, q)$ described above satisfy condition (4.4).

Now given the sets C and $C^\alpha(x, q)$ as described above, define

$$D = \{(x, q, r): \mu_x(C(x, q)) > r\}$$

and, for each α,

$$D^\alpha = \{(x, q, r): \mu_x(C^\alpha(x, q)) > r\}.$$

Since $C_x = C(x, 0)$ for all x, it suffices for Theorem 4.1(a) to prove that D is Π^1_1.

Now $D^1 = \bigcup \{D(q): q \in S\}$, where $D(q) = \{(x, q, r): \mu_x((X \times Y) - L(q)) > r\}$. Since each $(X \times Y) - L(q)$ is open and μ is a conditional measure, each $D(q)$ is also Borel and therefore D^1 is Borel.

It follows from (4.3) and (4.4) that, for each $\alpha > 0$,

$$\mu_x(C^{\alpha+1}(x, q)) > r$$

if and only if

$$(\exists t > r)(\forall i)\, \mu_x(C^\alpha(x, q * i)) > t,$$

so that

$$(x, q, r) \in D^{\alpha+1} \text{ if and only if } (\exists t > r)(\forall i)(x, q * i, t) \in D^\alpha.$$

Of course, for limit λ, $\mu(C^\lambda(x, q)) > r$ if and only if

$$(\exists \alpha < \lambda) \qquad (\mu_x(C^\alpha(x, q)) > r).$$

With this in mind, define a Borel monotone operator Γ over $X \times S \times I$ by:

$$(x, q, r) \in \Gamma(K)$$

if and only if

$$\text{(i)}\ \ (x, q, r) \in D^1$$

or

$$\text{(ii)}\ \ (\exists t > r)(\forall i) \qquad (x, q * i, t) \in K$$

It can be seen by induction on α that

$$\Gamma^\alpha = D^\alpha = \{(x, q, r): \mu_x(C^\alpha(x, q)) > r\} \text{ for all ordinals } \alpha. \tag{4.5}$$

Thus, $D = \mathrm{Cl}(\Gamma)$.

It follows that, for any x and r:

$$\mu_x(C_x) > r \text{ if and only if } (x, 0, r) \in \mathrm{Cl}(\Gamma). \tag{4.6}$$

Since $\mathrm{Cl}(\Gamma) = D$ is coanalytic by Theorem 1.6(d), $\{(x, r): \mu_x(C_x) > r\}$ is also coanalytic.

Of course, $\mu_x(C_x) \geqslant r$ if and only if $(\forall t < r)\mu_x(C_x) > t$, therefore $\{(x, r): \mu_x(C_x) \geqslant r\}$ is also coanalytic. This completes the proof of Theorem 4.1(a).

Now suppose that each section C_x has measure $>$ some fixed real r. Then, by (4.6), the Borel set $X \times \{0\} \times \{r\}$ is included in $\mathrm{Cl}(\Gamma)$. It follows from the Boundedness Principle (Theorem 1.6(e)) that, for some countable α, $X \times \{0\} \times \{r\} \subseteq \Gamma^\alpha$. By (4.5), $\mu(C^\alpha(x, 0)) > r$ for all $x \in X$. Let $B = \{(x, y): y \in C^\alpha(x, 0)\} = \{(x, y): (x, y, 0) \in \Delta^\alpha\}$. B is a Borel subset of C by Theorem 1.6(a). For each $x \in X$, $B_x = C^\alpha(x, 0)$, so that $\mu_x(B_x) > r$ by the choice of α.

If each $\mu_x(C_x) \geqslant$ some fixed r, then for rational $t < r$, each $\mu_x(C_x) > t$. By the above, there are Borel sets $B(t) \subseteq C$ with each $\mu_x(B(t)_x) > t$. Let $B = \bigcup B(t)$; it is clear that B is a Borel subset of C and that each $\mu_x(B_x) \geqslant r$. This completes the proof of Theorem 4.2(a).

We now turn to the category portions of Theorems 4.1 and 4.2. The following is obvious

$$\bigcup \{A_i : i \in N\} \text{ is meager if and only if } (\forall i) A_i \text{ is meager.} \tag{4.7}$$

Let $\{G_n\}_{n=1}$ enumerate a basis of non-empty open balls for the space Y. The following lemma is a consequence of the Baire category theorem and the fact that in a Polish space all analytic and coanalytic sets have the property of Baire.

Lemma 4.8. *Let K be a coanalytic or analytic subset of the Polish space Y. Then* (a) *K is co-meager if and only if $(\forall m)(G_m \cap K$ is non-meager$)$*; (b) *K is non-meager if and only if $(\exists n)(G_n - K$ is meager$)$.*

If $s = (m_0, m_1, \ldots, m_{l-1})$, let $G_s = G_{m_0} \cap \cdots \cap G_{m_{l-1}}$; also, set $G_0 = Y$. As our basis notion of largeness in category, we take the property "$G_s \cap K$ is non-meager."

As the inductive definition of a coanalytic set is based on countable intersections and unions, we need to determine the largeness of $\bigcap_{k=0}^\infty A_k$ and $\bigcup_{k=0}^\infty A_k$ in terms of the largeness of the sets A_k. One of these is trivial. We assume the sets A_k have the property of Baire.

$$G_s \cap \left(\bigcup_k A_k\right) \text{ is non-meager} \tag{4.9}$$

if and only if $(\exists k)[G_s \cap A_k$ is non-meager$]$.

Now $G_s \cap (\bigcap A_k) = \bigcap (G_s \cap A_k)$ is non-meager if and only if $(\exists n)[G_n - \bigcap (G_s \cap A_k)$ is meager], by 4.8(b). But

$$G_n - \bigcap (G_s \cap A_k) = \bigcup (G_n - (G_s \cap A_k)),$$

so by 4.7, $G_s \cap \bigcap A_k$ is non-meager if and only if

$$(\exists n)(\forall k)[G_n - (G_s \cap A_k) \text{ is meager}]$$

if and only if

$$(\exists n)(\forall k)[(Y - G_n) \cup (G_s \cap A_k) \text{ is co-meager}].$$

Now by 4.8(a), $(Y - G_n) \cup (G_s \cap A_k)$ is co-meager if and only if

$$(\forall m)[(G_m - G_n) \cup (G_{s*m} \cap A_k) \text{ is non-meager}].$$

Finally, we have

$$G_s \cap \bigcap_{k=0}^{\infty} A_k \text{ is non-meager} \tag{4.10}$$

if and only if

$$(\exists n)(\forall k)(\forall m)[G_m - G_n \text{ is non-meager or } G_{s*m} \cap A_k \text{ in non-meager}].$$

Now let the sets C and $C^\alpha(x, q)$ be as previously described. Define

$$D = \{(x, q, s) \colon G_s \cap C(x, q) \text{ is non-meager}\}$$

and

$$D^\alpha = \{(x, q, s) \colon G_s \cap C^\alpha(x, q) \text{ is non-meager}\}.$$

As for the measure case, it will be shown that D is the closure of a monotone Borel operator Γ and is therefore a $\mathbf{\Pi}^1_1$ subset of $X \times S \times S$.

Recall that each $C^1(x, q)$ is open. Now, in a Polish space, an open set is meager if and only if it is empty. Let T be a countable dense subset of Y. Then, for any x and q, $G_s \cap C^1(x, q)$ is non-meager if and only if $(\exists t \in T)[t \in G_s \cap C^1(x, q)]$. Thus D^1 is an open subset of $X \times S \times S$, where S has the discrete topology.

It follows from 4.3 and 4.10 that, for each $a > 0$, $G_s \cap C^{a+1}(x, q))$ is non-meager if and only if $(\exists n)(\forall k)(\forall m)[G_m - G_n$ is non-meager or $G_{s*m} \cap C^a(x, q * k)$ is nonmeager], so that $(x, q, S) \in D^{a+1}$ if and only if

$$(\exists n)(\forall k)(\forall m)[G_m - G_n \text{ is non-meager or } (x, q * k, s * m) \in D^\alpha].$$

Of course, for limit λ, $G_s \cap C^\lambda(x, q)$ is non-meager if and only if

$$(\exists \alpha < \lambda)[G_s \cap C^\alpha(x, q) \text{ is non-meager}].$$

Now define the operator Γ over $X \times S \times S$ by:

$$(x, q, s) \in \Gamma(K)$$

if and only if

(i) $(x, q, s) \in D^1$ or
(ii) $(\exists n)(\forall k)(\forall m)[G_m - G_n$ is non-meager or $(x, q * k, s * m) \in K]$.

It can be seen by induction on α that

$$\Gamma^\alpha = D^\alpha = \{(x, q, s) \colon G_s \cap C^\alpha(x, q) \text{ is non-meager}\} \tag{4.11}$$

It follows that, for any x and s:
Thus, $D = \mathrm{Cl}(\Gamma)$ and by Theorem 1.6(d) is Π^1_1.

$$G_s \cap C_x \text{ is non-meager if and only if } (x, 0, s) \in \mathrm{Cl}(\Gamma). \tag{4.12}$$

Now C_x is non-meager if and only if $G_0 \cap C_x$ is non-meager and C_x is co-meager if and only if $(\forall s)$ $[G_s \cap C_x$ is non-meager]. Both of these relations must therefore be coanalytic. This completes the proof of Theorem 4.1. ∎

Now suppose that each section C_x is non-meager. Then, by (4.12), the Borel set $X \times \{0\} \times \{0\}$ is included in $\mathrm{Cl}(\Gamma)$. Choose a countable ordinal α, using Boundedness, such that $X \times \{0\} \times \{0\} \subseteq \Gamma^\alpha$ and let $B = \{(x, y) \colon (x, y, 0) \in \Delta^\alpha\}$. It is clear that B is a Borel subset of C each section of which is non-meager.

If each C_x is co-meager, then $X \times \{0\} \times S \subseteq \mathrm{Cl}(\Gamma)$. The rest of the argument is the same.

This completes the proof of Theorem 4.2. ∎

Let us remark that it follows from Theorem 4.2 that if each section of a coanalytic subset C of $X \times Y$ has measure r, for some fixed $r > 0$, then C includes a Borel set B such that each section of B has measure r. Thus each B_x is the same size as C_x. However, if we only assume that each C_x has positive measure, then there may not be a Borel set $B \subseteq C$ so that for each x, $\mu_x(B_x) = \mu_x(C_x)$. For example, let E be a coanalytic non-Borel subset of I and let $C = (E \times I) \cup ((I - E) \times]0, \frac{1}{2}])$. Then C is a coanalytic subset of $I \times I$. If B were a Borel set, $B \subseteq C$ and for each x, $\lambda(B_x) = \lambda(C_x)$, then according to Theorem 4.1, $\{x \colon \lambda(B_x) = \frac{1}{2}\} = I - E$ would be a Borel set. Here λ is Lebesgue measure.

5. Effective Definability

In this section we obtain *effective* versions of the results in Sections 1 and 2. These are applied to give a new characterization of the least ordinal $\rho(x)$ not recursive in x and several theorems, due to Sacks [22], Tanaka [29, 30] and others, mostly concerning $\rho(x)$.

We begin with a brief review of the field of effective descriptive set theory, which is a blend of topology and logic first developed by Kleene [10] and Addison [1].

The topological notions of open, closed, Borel and analytic sets have effective analogues. The fundamental concept here is the recursive function, which is the analogue of the continuous function.

A mapping f from a subset of $N^k \times J^l$ to N is said to be *partial recursive* (p.r.) if it is defined in one of the following ways, where k, l, and n vary over N, $i < j$, $k < 1$, $m = (m_0, \ldots, m_{k-1})$ varies over N^k and $\mathbf{x} = (x_0, \ldots, x_{l-1})$ varies over J^l. (For any $\mathbf{m}, \mathbf{x}, f$, and g, $f(\mathbf{m}, \mathbf{x}) \simeq g(\mathbf{m}, \mathbf{x})$ means either $f(\mathbf{m}, \mathbf{x}) = g(\mathbf{m}, \mathbf{x})$ or both are undefined).

(i) $f(\mathbf{m}, \mathbf{x}) \simeq 0$.

(ii) $f(\mathbf{m}, \mathbf{x}) \simeq m_i$.

(iii) $f(\mathbf{m}, \mathbf{x}) \simeq m_i + 1$.

(iv) $f(\mathbf{m}, \mathbf{x}) \simeq x_j(m_i)$.

(v) $f(\mathbf{m}, \mathbf{x}) \simeq g(g_0(\mathbf{m}, \mathbf{x}), \ldots, h_{n-1}(\mathbf{m}, \mathbf{x}), x)$, where $g, h_0, \ldots, h_{n-1}$ are previously defined p.r. functions. (5.1)

(vi) $f(0, \mathbf{m}, \mathbf{x}) \simeq g(\mathbf{m}, \mathbf{x})$
$f(p+1, \mathbf{m},, \mathbf{x}) \simeq h(p, \mathbf{m}, f(p, \mathbf{m}, x), \mathbf{x})$, where g and h are previously defined p.r. functions.

(vii) $$f(m, x) \simeq \begin{cases} (\text{least } p)[\, g(p, \mathbf{m}, \mathbf{x}) = 0 \text{ and } (\forall t < p)\, g(t, \mathbf{m}, \mathbf{x}) > 0] \\ \qquad \text{if such } p \text{ exists;} \\ \text{undefined otherwise.} \end{cases}$$

(Note that there are only countably many p.r. functions).

A p.r. map f from $D \subseteq N^k \times J^l$ to N is said to be recursive if $D = N^k \times J^l$.

For a fixed real z, f is said to be p.r. in z if there is a p.r. map g such that $f(m, x) \simeq g(m, x, z)$ for all m and x.

A subset of $N^k \times J^l$ is said to be recursive (in z) if its characteristic function is p.r. (in z).

Clauses (i) through (vi) generate the primitive recursive functions. For example, if $f(0, m) = 0$ and $f(p+1, m) = f(p, m) + 1$ for all p, then f is primitive recursive. Of course f is just the addition function.

Also primitive recursive are the coding functions $\#$, defined by $\#(\phi) = 1$

and $\#(s_0,\dots,s_k)=2^{1+s_0}3^{1+s_1}1\cdots p_k^{1+s_k}$ (where p_k is the kth prime), and the set $Seq=\{\#(s): s\in S\}$.

Clause (vii) introduces partial functions. For example, let $g(m,n)=m+n$ for all m—then g is primitive recursive. Now let $f(n)=$ (least m) $g(m,n)=0$. Then f is p.r. and has domain and range of $\{0\}$. Also partial recursive are the functions π_i and lh, defined by $lh(s)=k$ and $\pi_i(s)=s_i$ when $s=\#(s_0,\dots,s_{k-1})$.

A real number $x\in J$ is a function from N to N; countably many are recursive. In particular, for any

$$s=(s(0),s(1),\dots,s(k-1)\in S,$$

the real $(s(0),s(1),\dots,s(k-1),0,0,0,\dots)$ is recursive. Thus the recursive real numbers are dense in J. In fact, all algebraic numbers are recursive; the standard transcendental numbers such as e and π are also recursive.

The partial recursive functions can be enumerated effectively as $f_0,f_1,f_2,\dots$, to satisfy the following.

THEOREM 5.2. *The function f, defined by $f(n,\mathbf{m},\mathbf{x})=f_n(\mathbf{m},\mathbf{x})$ is partial recursive.*

The reader is referred to Hinman [8] or Rogers [21] for further details.

Now a map ϕ: $J^l\to J$ is said to be recursive (in z) if the map f_ϕ defined by $f_\phi(m,\mathbf{x})=\phi(\mathbf{x})(m)$ is recursive (in z). For example, let ϕ: $J^2\to J$ be defined by $x\circ y=\phi(x,y)=(x(0),y(0),x(1),y(1),\dots)$. Then

$$\begin{aligned} f_\phi(m,x,y) &= y(k) \qquad \text{if} \quad m=2k;\\ &= x(k) \qquad \text{if} \quad m=2k-1. \end{aligned}$$

It can be checked that f_ϕ is recursive. Such functions as multiplication and the "less than" relation can also be shown to be recursive. An example of a non-recursive function is the map E: $J\times J\to N$, defined by

$$\begin{aligned} E(x,y) &= 1 \qquad \text{if} \quad x=y;\\ &= 0 \qquad \text{if} \quad x\neq y. \end{aligned}$$

The recursive functions are the effective analogue of the continuous functions.

Recall that a map ϕ: $J^l\to J$ is continuous if and only if whenever $\phi(x)(m)=n$ there are finite sequences $s_i\subset x_i$ such that for any y with each $s_i\subset y_i$, $\phi(y)(m)=n$. The following lemma is proved by straightforward induction on the class of partial recursive functions.

LEMMA 5.3. *Let f be a partial recursive map from $N^k\times J^l$ into N such* *that* $f(\mathbf{m}, \mathbf{x}) \simeq n$. *Then there are* $s_i \subset x_i$ *such that for any* $(\mathbf{y}) \in J^l$, $f(\mathbf{m}, \mathbf{y}) \simeq n$ *whenever each* $s_i \subset y_i$.

It follows immediately that if ϕ: $J^l \to J$ is recursive in some fixed $z \in J$, then ϕ is continuous. The converse is now demonstrated. For simplicity, let ϕ be a continuous map from J to J. Let A be the set of $\#(s * m * n)$ such that, for any x, $s \subset x$ implies $\phi(x)(m) = n$. Define z by:

$$\begin{aligned} z(s * m) &= \#(0, n) \qquad \text{if} \quad (s, m, n) \in A; \\ &= \#(1, 0) \qquad \text{if} \quad (\forall n)(s, m, n) \notin A. \end{aligned}$$

Now, let

$$f(x, m) = (\text{least } k)\, \pi_0(z(x * k * m)) = 0.$$

In other words, $f(x, m)$ is least k so that the value $\phi(x)(m)$ is determined by $(x(0), x(1), \ldots, x(k-1))$.

Let

$$g(x, m) = \#(xf(x, m)) * m.$$

This codes the information needed by z to determine $\phi(x)(m)$. In fact,

$$\phi(x)(m) = \pi_1(z(g(x, m)).$$

This establishes that ϕ is recursive in the real z which yields the following proposition.

PROPOSITION 5.4. *A function* ϕ: $J^l \to J$ *is continuous if and only if there is some* $z \in J$ *such that* ϕ *is recursive in* z.

Recall that the basic open subsets of J are the intervals $J[s] = \{u: s \subset u\}$ for $s \in S$. Of course, the map Ψ is recursive, where

$$\begin{aligned} \Psi(s, u) &= 1 \qquad \text{if } s \subset u; \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Now any open set G is the countable union of these basic sets, so that, for some real z: $u \in G$ if and only if $(\exists p)\, \Psi(z(p), u) = 1$. A subset P of $N^k \times J^l$ is said to be Σ_1^0 (resp. Π_1^0) in z provided there is an $R \subseteq N^{k+1} \times J^l$ recursive in z such that, for all $\mathbf{m}$ and $\mathbf{x}$:

$$P(\mathbf{m}, \mathbf{x}) \qquad \text{if and only if } (\exists p)\, R(p, \mathbf{m}, \mathbf{x})$$
$$(\text{resp. } (\forall p)\, R(p, \mathbf{m}, \mathbf{x})).$$

PROPOSITION 5.5. *A subset G of J^l is open (resp. closed) if and only if there is some $z \in J$ such that G is Σ_1^0 (resp. Π_1^0) in z.*

Proof. This follows immediately from Proposition 5.4.

Similar comparisons can be made between F_σ and Σ_2^0 sets and so forth. More important here is the effective analogue of the analytic set. Recall that any analytic subset A of J^l can be defined by a Suslin scheme L of closed sets so that $A = \{\mathbf{x}: (\exists u)(\forall n)\mathbf{x} \in L(u \mid n)\}$. A is said to be Σ_1^1 in z provided there is a subset R of $N \times J^l$ recursive in z so that

$$A = \{x: (\exists u)(\forall n) R(u \mid n, x)\}$$

A is Π_1^1 in z provided $J^l - A$ is Σ_1^1 in z, and Δ_1^1 in z provided A is both Π_1^1 in z and Σ_1^1 in z.

PROPOSITION 5.6. *A subset A of J^l is analytic (resp. coanalytic, Borel) if and only if there is some $z \in J$ such that A is Σ_1^1 (resp. Π_1^1, Δ_1^1) in z.*

Remark. This correspondence can be extended throughout the projective hierarchy. The reader is referred to Hinman [8] for further information on effective descriptive set theory.

We can similarly define the class of Δ_1^1, Π_1^1, Σ_1^1 monotone operators over $N^k \times J^l$. Briefly, the Borel operators on $N^k \times J^l$ can be assigned indices in the manner of (1.7)—the operator Δ is Δ_1^1-in-z provided some index of Δ is recursive in z. An analytic (coanalytic) operator defined as in (1.5) is Σ_1^1-in-z (Π_1^1-in-z) if the corresponding operator Δ is.

PROPOSITION 5.7. *A monotone operator Δ over $N^k \times J^l$ is Borel (resp. analytic, coanalytic) if and only if there is some $z \in J$ such that Δ is Δ_1^1 (resp. Σ_1^1, Π_1^1) in z.*

Remark. Theorem 3.2 directly implies the existence of universal $\Sigma_1^0, \Pi_1^0, \Sigma_1^1, \Pi_1^1$ and also universal open, closed, analytic and coanalytic sets. This correspondence has given rise to the notation employed in descriptive set theory. Thus, for example, analytic sets are said to be $\mathbf{\Pi}_1^1$ (or written in boldface type) corresponding with the effective Π_1^1 (lightface).

As with the Borel sets, there will be a Π_1^1 operator which is universal for the class of Borel operators. If Δ_τ denotes the Borel operator over X with index τ, then we have $\Phi(K) = \{(\tau, y) \in J \times X: y \in \Delta_\tau(K_\tau)\}$. There will also be a universal $\Pi_1^1(\Sigma_1^1)$ operator for the class of coanalytic (analytic) operators. These last two universal operators can be transformed into monotone $\Pi_1^1(\Sigma_1^1)$ operators which are universal for the class of monotone coanalytic (analytic) operators, by applying Lemma 2.1. For example, let Φ be a universal Π_1^1 operator, then let $\Omega(K) = \{(\tau, y)$: for all countable

$Z \subset X - K_\tau$, $y \in \Phi_\tau(X - Z)\}$. It can be checked that Ω is a monotone Π_1^1 operator which is universal for the monotone coanalytic operators.

There are only countably many Π_1^1 monotone operators; for each $x \in J$, let $\alpha(x) = \sup\{(\text{least } \alpha)(x \in \Gamma^\alpha)\colon \Gamma$ is Π_1^1 monotone and $x \in \mathrm{Cl}(\Gamma)\}$. Thus each $\alpha(x)$ is a countable ordinal. Those ordinals α for which some $\alpha(x) = \alpha$ are called admissible. It is clear that they form a cofinal subset of ω_1. There are numerous other characterizations of the admissible ordinals. Some of these are given below.

LEMMA 5.8. *For any* $x \in J$, $\alpha(x) = \sup\{(\textit{least } \alpha)(1 \in \Gamma^\alpha)\colon \Gamma$ *is* Π_1^1 *monotone in x and* $1 \in \mathrm{Cl}(\Gamma)\}$.

Proof. Recall that $1 = (0, 0, 0,\ldots)$. For any $n \in N$ and $u \in J$, let $n * u = (n, u(0), u(1),\ldots)$ and let $u^+ = (u(1), u(2),\ldots)$; for $K \subseteq J$, let $K(n) = \{x\colon n * x \in K\}$. Now given Π_1^1 monotone Δ with $x \in \Gamma^{\alpha+1} - \Gamma^\alpha$, define Γ, Π_1^1 monotone in x, by

$$u \in \Gamma(K) \qquad \text{if and only if } (u(0) = 1 \text{ and } u^+ \in \Gamma(K(1)) \text{ or}$$
$$(u = 1 \text{ and } x^+ \in K(1)).$$

It is clear that for $\beta \leqslant \alpha$, $\Gamma^\beta = 1 * \Delta^\beta$ and that $1 \in \Gamma^{\alpha+2} - \Gamma^{\alpha+1}$. Now suppose we are given Γ, Π_1^1 monotone in x, such that $1 \in \Gamma^{\alpha+1} - \Gamma^\alpha$. For $u, v \in J$ recall that $u \circ v = (u(0), v(0), u(1), v(1),\ldots)$ for $K \subseteq J$, let $\Pi_x(K) = \{u\colon u \circ x \in K\}$. Let Ω be a universal monotone Π_1^1 operator over $J \times J$ defined above.. We may assume that $\Gamma = \Omega_x$. Let $\hat{\Gamma}(K) = \{z \circ y\colon y \in \Omega_z(\Pi_z(K))\}$. Thus,

$$x \circ y \in \hat{\Gamma}(K) \qquad \text{if and only if } y \in \Gamma(\Pi_x(K)).$$

As in (1.25), $x \in C$ IFF $(x, 0) \in \mathrm{Cl}(\Delta)$. It can be assumed that $\hat{\Gamma}$ is monotone. Clearly $x \circ 1 \in \hat{\Gamma}^{\alpha+1} - \hat{\Gamma}^\alpha$. Suppose, without loss of generality, that $x(0) = 0$ and define Π_1^1 monotone Δ by

$$u \in \Delta(K) \qquad \text{if and only if } (u(0) = 1 \text{ and } u^+ \in \hat{\Gamma}(K(0))) \text{ or}$$
$$(u(0) = 0 \text{ and } 0 * (u \circ I) \in K).$$

Then, for all β, $\Delta^\beta(1) = \hat{\Gamma}^\beta$ and thus $x \in \Gamma^{\alpha+2} - \Gamma^{\alpha+1}$.

Of course the "1" in the statement of this theorem could be replaced by any recursive real, natural number, or finite sequence. As a corollary to the proof, each $\alpha(x)$ must be a limit ordinal. Another corollary is the following.

PROPOSITION 5.9. *For any* $x, y \in J$, *if x is* Δ_1^1 *in y, then* $\alpha(x) \leqslant \alpha(y)$.

Thus, for any Δ_1^1 real x, $\alpha(x)$ is the least admissible ordinal $\alpha(1)$. The effective version of Theorem 1.3 can now be stated.

THEOREM 5.10. (a) *If Δ is a monotone operator over J which is Π^1_1 in z and $A \subseteq J$ is Π^1_1 in z, then* $\mathrm{Cl}(\Gamma; A)$ *is Π^1_1 in z.*

(b) *For any $C \subseteq J$ which is Π^1_1 in z, there is a monotone operator Δ over $S \times J$ which is Δ^1_1 in z such that*

$$C = \{x: (x, 0) \in \mathrm{Cl}(\Delta)\};$$

(c) *If Γ is a monotone operator over J which is Π^1_1 in z, then for any subset A of* $\mathrm{Cl}(\Gamma)$ *which is Σ^1_1 in z, $A \subseteq \Gamma^{\alpha(z)}$;*

(d) *If Γ is a monotone operator over J which is Δ^1_1 (resp. Π^1_1) in z, then for each ordinal $\alpha < \alpha(z)$ (resp. $\leqslant \alpha(z)$), Γ^α is Δ^1_1 (resp. Π^1_1) in z.*

Proof. The proof is simply a refinement of that of Theorem 1.6. Some remarks are necessary.

(a) $\mathrm{Cl}(\Gamma)$ is once again the intersection of all the co-countable fixed points of Γ; if Γ is Π^1_1 in z, this gives a Π^1_1 in z definition for $\mathrm{Cl}(\Gamma)$.

(b) Suppose $C = \{x: (\forall u)(\exists n)\, R(u \mid n, x)$ with R recursive in z. Then $C = C(L)$ where, for all $x \in J$, $I_L(x) = \{q: R(q, x)\}$ and is recursive in x and z. For any $y \in J$, let $\rho(y)$ be the supremum of the ordinals γ for which there is a well-ordering of S of length γ which is recursive in y – $\rho(y)$ is usually called ω_1^y by recursion theorists. (These recursive orderings were first studied by Spector [28].) Recall the canonical Δ^1_1 monotone inductive definition Δ of C, given in (1.23):

$$\begin{aligned} (x, q) \in (K) &\text{ if and only if } R(q, x) \\ &\text{or } (\forall i)(x, q * i) \in K. \end{aligned} \tag{5.11}$$

As in (1.25), $x \in C \leftrightarrow (x, 0) \in \mathrm{Cl}(\Delta)$. If $x \in C$, then $I_L(x)$ is a well-ordering of S recursive in z and x; it follows that $(x, 0) \in \Gamma^{(z \circ x)}$. If $1 \in C$, then $(1, 0) \in \Gamma^{\rho(z)}$.

(c) Given z, A and Γ as described, let α be the least such that $A \subseteq \Gamma^\alpha$ and define a monotone operator Δ which is Π^1_1 in z by

$$\begin{aligned} u \in \Delta(K) \qquad &\text{if and only if } (u(0) = 1 \text{ and } u^+ \in \Gamma(K(1))) \text{ or} \\ &(u = 1 \text{ and } A \subseteq K(1)). \end{aligned}$$

Then $1 \in \Delta^{\alpha+1} - \Delta^\alpha$ and therefore $\alpha < \alpha(z)$.

(d) Fix z and let C be $\{n: f_n(-, z) \in W\}$, where W represents the set of all well-orderings on N as described in (1.4). Of course, C is Π^1_1 in z, but not Σ^1_1 in z. Let the Δ^1_1-in-z monotone operator Δ over N have closure C as in (1.11). Note that $|\Delta| \leqslant \alpha(z))$ and $|\Delta| \leqslant \rho(z)$.

Now suppose that Γ is any monotone operator Π_1^1 in z and that $1 \in \Gamma^{\beta+1} - \Gamma^{\beta}$. We will show that $\beta < |\Delta|$. Recall from Section 1 the pre-well-orderings R and S defined by

$$R(x, y) \quad \text{if and only if } |x|_\Gamma \leqslant |y|_\Delta \text{ and } x \in \mathrm{Cl}(\Gamma);$$

$$S(x, y) \quad \text{if and only if } |x|_\Gamma < |y|_\Delta \text{ and } x \in \mathrm{Cl}(\Gamma).$$

Both R and S are Π_1^1 in z and if $\beta \geqslant |\Delta|$, then, for any $n \in N$:

$$n \in \mathrm{Cl}(\Delta) \quad \text{if and only if } \neg S(1, n).$$

This provides a $\Sigma_1^1 - in - z$ definition of C, which is a contradiction; thus $\beta < |\Delta|$. It follows that $\alpha(z) \leqslant |\Delta| \leqslant \rho(z)$; thus $|\Delta| = \alpha(z)$.

Now suppose $\alpha < \alpha(z)$; choose some $n \in \Delta^{\alpha+1} - \Delta^{\alpha}$. Then

$$\Gamma^{\alpha} = \{x: x \in \mathrm{Cl}(\Gamma) \text{ and } S(x, n)\}$$

and is therefore Π_1^1 in z.

$$\Gamma^{\alpha(z)} = \{x: x \in \mathrm{Cl}(\Gamma) \text{ and } (\exists n)(n \in \mathrm{Cl}(\Delta) \text{ and } S(x, n)\}$$

and is also Π_1^1 in z.

If Γ is actually Δ_1^1, then the pre-well-orderings R' and S', defined by

$$R'(x, y) \quad \text{if and only if } |x|_\Delta \leqslant |y|_\Gamma \text{ and } x \in \mathrm{Cl}(\Delta),$$

$$S'(x, y) \quad \text{if and only if } |x|_\Delta < |y|_\Gamma \text{ and } y \in \mathrm{Cl}(\Delta)$$

are also Π_1^1 in z. Thus, given $\alpha < \alpha(z)$ and $n \in \Delta^{\alpha+1} - \Delta^{\alpha}$,

$$\Gamma^{\alpha} = \{y: \neg R'(n, y)\}$$

and is therefore Σ_1^1 in z as well as Π_1^1 in z. ∎

We saw during the above that, for any z, $\alpha(z) \leqslant \rho(z)$. On the other hand, let R be a well-ordering of a subset of N which is recursive in z and has order type ρ. For $n \in N$, let $\sigma(n)$ be the order type of $R \restriction n$. Define a monotone operator Δ over N which is Δ_1^1 in z by

$$n \in \Delta(K) \quad \text{if and only if } (\forall m)[R(m, n) \to m \in K].$$

Then for any n, $|n|_\Delta = \sigma(n)$ so that $|\Delta| \geqslant \rho$. Thus $\alpha(z) \geqslant \rho(z)$.

PROPOSITION 5.12. *For all $z \in J$, $\alpha(z) = \rho(z)$; that is, the supremum of the ordinals α such that, for some Π_1^1 monotone Γ, $z \in \Gamma^{\alpha+1} - \Gamma^{\alpha}$ equals the least ordinal which is not recursive in z.*

Remark. Parts (c) and (d) of Theorem 5.10 were combined by A. Blass and the first author in [3] to yield the following.

THEOREM 5.13. *If the monotone inductive operator Γ is Δ^1_1 in z, then $\Gamma^{\alpha(z)}$ is the union of the Δ^1_1-in-z subsets of* $\mathrm{Cl}(\Gamma)$ *and is also the union of the Σ^1_1-in-z subsets of* $\mathrm{Cl}(\Gamma)$. *(Thus any other Γ' with the same closure must agree with Γ at level $\alpha(z)$.)*

The remainder of this section is devoted to some effective theorems concerning measure and category.

Suppose that μ is a countably additive probability measure on J. Then $\{(s,q): \mu(J[s]) = q\}$ is a countable subset of $S \times J$ and is therefore Borel. If this set is actually Δ^1_1 in z, then μ is said to be Δ^1_1 in z. For example, Lebesgue measure is Δ^1_1. We now present some simple refinements of Theorems 4.1(a) and 4.2(a).

PROPOSITION 5.14. *If $C \subseteq J$ is Π^1_1 (resp. Δ^1_1) in z and the measure μ is Δ^1_1 in z, then $\{r: \mu(C) > r\}$ is Π^1_1 (resp. Δ^1_1) in z.*

Proof. Suppose that $C = \{x: (\forall u)(\exists n) R(u \mid n, x)$, where R is recursive in z, and that Δ is defined from R as in (1.23). As in the proof of Theorem 4.1(a), we can define a monotone operator Δ^1_1 in z such that for any α, q and r,

$$\begin{aligned} (q,r) \in \Gamma^{\alpha} \qquad &\text{if and only if} \\ &\mu(\{x: (x,q) \in \Delta^{\alpha}) > r. \end{aligned} \tag{5.14}$$

It follows that $\mu(C) > r$ if and only if $(0,r) \in \mathrm{Cl}(\Gamma)$; thus

$$\{r: \mu(C) > r\} \text{ is } \Pi^1_1 \text{ in } z$$

that is, $\mu(C)$ is Π^1_1-in-z, as defined by Tanaka [29]. ∎

Now $A = \{(0,r): \mu(C) \leqslant r\}$ is a Σ^1_1-in-z subset of $\mathrm{Cl}(\Gamma)$. By Theorem 5.10(c), there is some $\alpha < \alpha(z)$ such that $A \subseteq \Gamma^{\alpha}$.

Let $B = \{x: (0,x) \in \Delta^{\alpha}\}$; by choice, $\mu(B) = \mu(C)$; by Theorem 5.10(d), B is Δ^1_1 in z. This completes the proof of the following refinement of Theorem 4.2(a).

THEOREM 5.15. *If $C \subseteq J$ is Π^1_1 in z and the measure μ is Δ^1_1 in z, then there is a Δ^1_1-in-z subset B of C with $\mu(B) = \mu(C)$.*

Now suppose that an inductive definition Γ which is Π^1_1-in-z is given with $\mathrm{Cl}(\Gamma) = C$, a subset of J. Of course, C is Π^1_1-in-z, so by Theorem 5.15, there is some $B \subseteq C$ such that B is Δ^1_1 in z and $\mu(B) = \mu(C)$. Now by Theorem 5.10(c), $B \subseteq \Gamma^{\alpha(z)}$. Thus $\mu(C) = \mu(\Gamma^{\alpha(z)})$.

THEOREM 5.16. *If the monotone inductive operator Γ is Π^1_1 in z and the measure μ is Δ^1_1 in z, then* $\mu(\mathrm{Cl}(\Gamma)) = \mu(\Gamma^{\alpha(z)})$.

Now if $\alpha(x) > \alpha(z)$, then there is some Π^1_1 monotone operator Γ such that $x \in \mathrm{Cl}(\Gamma) - \Gamma^{\alpha(z)}$. For each fixed Γ, it follows from Theorem 5.16 that $\mu(\mathrm{Cl}(\Gamma) - \Gamma^{\alpha(z)}) = 0$. But there are only countably many such operators Γ. This proves the following.

THEOREM 5.17. *If the probability measure μ is Δ^1_1 in z, then* $\mu(\{x: \alpha(x) \leqslant \alpha(z)\}) = 1$.

Note that the ordinal $\alpha(1)$ is the least admissible ordinal. Since Lebesgue measure is Δ^1_1 (in 1), we have the following theorem of Sacks [22].

COROLLARY 5.18. $\{x: \alpha(x) = \alpha(1)\}$ *has Lebesgue measure* 1.

Another corollary of 5.17 follows directly from Proposition 5.6.

THEOREM 5.19. *If μ is a Borel probability measure, then for some countable ordinal α,* $\mu(\{x: \alpha(x) \leqslant \alpha\}) = 1$.

It should be noted that results 15, 16, 17, and 18 are essentially due to Sacks [22] and Tanaka [30]. The analoguous results for category are due to Hinman [7] and Thomason [31]. They are proved as were the above. We consolidate them into the following.

THEOREM 5.20. (a) *If $C \subseteq J$ is Π^1_1 in z, then there is a Δ^1_1-in-z subset B of C with the same category as C*; (b) *if the monotone inductive operator Γ is Π^1_1-in-z, then $\Gamma^{\alpha(z)}$ has the same category as* $\mathrm{Cl}(\Gamma)$; (c) $\{x: \alpha(x) = \alpha(1)\}$ *is comeager.*

6. UNIFORMIZATIONS AND PARAMETRIZATIONS

Throughout this section, X and Y will be uncountable Polish spaces and C will be a coanalytic subset of $X \times Y$.

A uniformization of a subset E of $X \times Y$ is a subset F of E such that $E_x \neq \varnothing$ if and only if F_x consists of exactly one point. The Kondo–Addison–Novikov theorem [11] asserts that C has a coanalytic uniformization. We shall show that if each section of C is large, then C has $2^{\aleph_0}$ disjoint Borel uniformizations. We shall also show that C has a universally (absolutely) measurable parametrization.

A parametrization of C is a one-to-one map, g, of $X \times Y$ onto C such that for each x, $g(x, \cdot)$ maps Y onto C_x. Such a parametrization is said to be universally measurable provided that both g and g^{-1} are measurable with respect to the σ-algebra of all universally measurable sets. This σ-algebra is generated as follows. For each finite measure μ defined on the Borel subsets of $X \times Y$, let $\mathcal{M}(\mu)$ consist of all the subsets of $X \times Y$ which are measurable in Caratheodory's sense with respect to the outer measure generated by μ. The intersection of all the families $\mathcal{M}(\mu)$ forms the family of universally measurable sets.

In [6], the authors show that if A is an analytic subset of $X \times Y$ such that for each x, A_x is uncountable, then A has an $S(X \times Y)$ measurable parametrization. By $S(X \times Y)$ is meant the smallest σ-algebra of subsets of $X \times Y$ containing the open sets which is also closed under operation A. These sets are the "C sets" introduced by Selivanowski [24]. It is well known that $S(X \times Y)$ is a proper subfamily of $\Delta_2^1(X \times Y) = PCA(X \times Y) \cap CPCA(X \times Y)$ [12] and that $S(X \times Y)$ is a proper subfamily of the universally measurable sets. Our parametrization theorem for the coanalytic side is slightly better. We show that if each C_x is large, then C has a $\mathcal{B}\mathcal{A}(X \times Y)$ measurable parametrization, where $\mathcal{B}\mathcal{A}(X \times Y)$ is the σ-algebra of subsets of $X \times Y$ generated by the analytic sets. Of course, $\mathcal{B}\mathcal{A}(X \times Y)$ is a proper subfamily of $S(X \times Y)$. We do not know whether C has a $\mathcal{B}\mathcal{A}(X \times Y)$ measurable parametrization if it is only assumed that each C_x contains a perfect subset. Let us note that it is not necessarily true that such a set C contains a Borel set each section of which is uncountable. We also do not know whether every analytic subset A of $X \times Y$ such that each A_x is uncountable has a $\mathcal{B}\mathcal{A}(X \times Y)$ measurable parametrization.

THEOREM 6.1. *Assume that Y is dense-in-itself and for each x, C_x is not meager. Then C has $2^{\aleph_0}$ disjoint Borel uniformizations and C has a $\mathcal{B}\mathcal{A}(X \times Y)$ measurable parametrization.*

Proof. According to Theorem 4.2, C includes a Borel set B such that each B_x is not meager in Y. According to a theorem proved in [18], there is a Borel parametrization k of $X \times Y$ onto B. Also, according to a theorem proved in [18], it follows that B, and therefore C, has $2^{\aleph_0}$ disjoint Borel uniformizations.

The proof is completed by a Schröder–Bernstein type argument as used by the authors in [6]. Let $S_0 = C - B$ and $T_0 = (X \times Y) - C$. Thus,

$$\begin{aligned} X \times Y &= B \cup S_0 \cup T_0 \\ &= T_0 \cup S_0 \cup (T_1 \cup S_1) \cup \cdots \cup (T_n \cup S_n) \cup \cdots \cup D, \end{aligned}$$

where $T_n = k^n(T_0)$, $S_n = k^n(S_0)$ and $D = \bigcap_{p=1}^{\infty} k^p(B)$. Also,

$$\begin{aligned} C &= B \cup S_0 \\ &= S_0 \cup (T_1 \cup S_1) \cup \cdots \cup (T_n \cup S_n) \cup \cdots \cup D. \end{aligned}$$

Set $H = D \cup \bigcup_{n=0}^{\infty} S_n$ and $G = \bigcup_{n=0}^{\infty} T_n$ and define

$$g(z) = z, \qquad \text{if } z \in H;$$
$$= k(z), \qquad \text{if } z \in G.$$

It can be easily checked that g is a one-to-one map of $X \times Y$ onto C and that for each x, $g(x, \cdot)$ is a one-to-one map of Y onto C_x.

If U is an open subset of $X \times Y$, then

$$g^{-1}(U) = g^{-1}(U \cap H) \cup g^{-1}(U \cap G)$$
$$= (U \cap H) \cup k^{-1}(U \cap G).$$

Since k is a Borel isomorphism, the sets S_n, T_n, H and G are in the family $\mathscr{B}\mathscr{A}(X \times Y)$. Also, $k^{-1}(M)$ is in $\mathscr{B}\mathscr{A}(X \times Y)$ if and only if M is. Thus, $g^{-1}(U)$ is in $\mathscr{B}\mathscr{A}(X \times Y)$. Similarly, $(g^{-1})^{-1}(U) = g(U) = g(U \cap H) \cup g(U \cap G) = (U \cap H) \cup k(U \cap G)$, so g^{-1} is also $\mathscr{B}\mathscr{A}(X \times Y)$ measurable.

Let us note that the methods of Theorem 6.1 may be used to generalize a result of Sarabadhikari, who shows in [23] that if B is a Borel set in $X \times Y$ such that each B_x is not meager, then B has a Borel uniformization. (It is not assumed that Y is dense-in-itself.) Clearly, from what has been said here, this same result holds when B is only assumed to be coanalytic.

We now turn to another method of stating that a set is "large."

THEOREM 6.2. *Let μ be a conditional probability distribution on $X \times Y$ such that for each x, μ_x is nonatomic and $\mu(x, C_x) > 0$. Then C has $2^{\aleph_0}$ disjoint Borel uniformizations and C has a $\mathscr{B}\mathscr{A}(X \times Y)$ measurable parametrization.*

Proof. According to Theorem 4.2, there is a Borel set B lying in C such that for each x, $\mu(x, B_x) > 0$. According to a theorem proven in [18], there is a Borel parametrization of B. The remainder on the proof is the same as the proof of the preceding theorem. ∎

Let us note that the methods of Theorem 6.2 may be used to generalize a result of Blackwell and Ryll-Nardzewski, who show in [2] that if μ is a conditional distribution on $X \times Y$ and B is a Borel subset of $X \times Y$ such that for each x, $\mu(x, B_x) > 0$, then B has a Borel uniformization. Clearly, the same result holds when B is only assumed to be coanalytic.

The two theorems presented in this section led to the following problem.

PROBLEM. Assume that for each x, C_x contains a nonempty perfect set. Does C have a $\mathscr{B}\mathscr{A}(X \times Y)$ measurable parametrization? What about an $S(X \times Y)$ measurable or universally measurable parametrization?

We do know one line of attack for a positive solution to this problem which fails. If one could show that C contains a Borel set each section of which is uncountable, then it would follow from the results of [6], that C has an $S(X\times Y)$ measurable parametrization. Consider, however, the following example.

EXAMPLE 6.3. Let D be a coanalytic subset of J such that there is some sieve sifting D for which every constituent of D with respect to this sieve is uncountable [13]. Now, let

$$C=\{(x,y)\in J\times D: o(y)>\omega_1^x\},$$

where $o(y)$ is the order type of the constituent to which y belongs. Clearly C is coanalytic and each C_x contains a perfect set.

Let us assume that there is a Borel set B lying in C such that for each x, B_x is uncountable. This implies that there is a countable ordinal γ such that $B\subset C_\gamma$, where

$$C_\gamma=\{(x,y)\in D: o(y)\leqslant\gamma\}.$$

But, there is some x such that $\gamma<\omega_1^x$. For this x, B_x must be empty, which is a contradiction.

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