Some examples of $\sigma$-ideals and related Baire systems

by

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In [4], the author gave some characterizations of the Baire system of functions generated by the collection of all functions continuous almost everywhere with respect to a $\sigma$-ideal $R$ of a metric space $S$.

In this paper are given some examples of $\sigma$-ideals, and some theorems which are connected with them [6]. The first example is the $\sigma$-ideal of all first category sets. Kuratowski has analysed the Baire system related to this $\sigma$-ideal [2]. In case $S$ is complete, separable and uncountable, this Baire system does not generate all the real functions on $S$. Theorem 1 gives necessary and sufficient conditions on a $\sigma$-ideal $R$ so that the related Baire system is the set of all real functions on $S$. In Theorem 2 a characterization of scattered or dispersed subsets of complete and separable metric spaces is given as a consequence of Theorem 1.

A second example is the $\sigma$-ideal of all countable subsets of $R$. The final example is the $\sigma$-ideal of all sets of measure 0 where $\mu$ is a complete, regular, $\sigma$-finite measure on $S$, $\mu(S) \neq 0$. In Theorem 3, it is shown that the related Baire system should be the set of all $\mu$-measurable functions is equivalent to each of the following three conditions: 1) the Baire process of taking pointwise limits should end in one step, 2) there be a scattered (dispersed) subset $M$ of $S$ such that $\mu(S - M) = 0$ and 3) the Baire system be the collection of all real functions on $S$.

The notation of this paper is the same as in [4].

Example 1. Suppose $S$ is a separable and complete metric spaces and $R$ is the $\sigma$-ideal of all first category subsets of $S$. Then $B_2(G) = G$ is the collection of all functions continuous almost everywhere with respect to $E$. Kuratowski has shown that a function $f$ is in $B_2(G)$ if and only if there is a subset $E$ of $S$ such that $S - E$ is of the first category and $f_E$ is continuous [2]. Moreover, Kuratowski showed that in this case $B(G) = B_2(G)$. This means that the Baire system connected with sets of the first category is generated by taking limits one time. This contrasts with
the fact that in case $S$ contains a perfect set, the Baire system, generated by the continuous functions is not generated by countably many iterations of the process of taking limits.

Even though $G$ is a more numerous collection than $C$, the continuous functions on $S$, in general $B(G)$ is still a proper subset of $P(S)$, the set of all real functions on $S$. In fact if $S$ is uncountable, then since $S$ is complete and separable, there is a subset $A$ of $S$ such that if $M$ is a perfect subset of $S$, then $A \cdot M$ and $A \cdot \overline{M}$ are dense subsets of $M$. The characteristic function of $A$ would not belong to $B(G)$, since it is not continuous on any perfect set and every second category set contains a perfect set. The following theorem characterizes the $\sigma$-ideals $E$ in a complete and separable metric space such that $B(E)$ should be $P(S)$.

**Theorem 1.** Suppose $S$ is a complete and separable metric space and $R$ is a $\sigma$-ideal of $S$. In order that $B(G)$ should be $P(S)$ it is necessary and sufficient that there be a scattered (or dispersed) subset $M$ of $S$ such that $M'$ is in $R$. Moreover, if $F(S) = B(G)$, then $F(S) = B(G)$.

Proof. Suppose $B(G) = F(S)$ and $S$ is not scattered. Since $S$ is complete, $S$ contains a perfect set.

Let $A$ be a subset of $S$ such that if $F$ is a perfect subset of $S$, then $F \cap A$ and $F \setminus A$ are dense subsets of $S$ and let $h$ be the characteristic function of $A$. If $K$ is a perfect subset of $S$, then $h_S$ is totally discontinuous; $h$ belongs to $F(S) = B(G)$. It can be shown by transfinite induction that if $f$ is in $B(G)$, then there is a countable subcollection $G$ of $G$ such that $f$ belongs to the Baire system generated by $G$. So, there is a countable subcollection $G$ of $S$ such that $h$ belongs to $B(G)$. Let $N$ be the subset of $S$ to which $h$ belongs if and only if some function in $G$ is discontinuous if $x$. The set $N$ is the $\sigma$-ideal $R$.

Let $M = S - N$ and suppose $M$ is not scattered. Let $H$ be a subset of $M$ which is dense in itself; $H$ is perfect. If $f$ is in $G$, then $f$ is continuous at each point of $H$. Since $H$ is a dense subset of $S$, $M$ is continuous at each point of $H$. Since $H$ is a dense subset of $M$, $M$ is continuous except for a set of the first category with respect to $H$. If $f$ is in $G$, then $f$ is in $G$, the collection of all functions over $H$ which are continuous except for a first category set with respect to $H$. So, $h_S$ belongs to $B(G)$. Therefore, by Kuratowski's theorem there is a subset $N$ of $S$ such that $A$ is the first category with respect to $H$ and $h_S$ is continuous. Let $L$ be a perfect subset of $M$ dense in itself; $M$ is scattered and $N = M$ in $E$.

Now, suppose $M$ is a scattered set of $S$. If $S$ is $M$, then $S$ is countable and it follows from Theorem 12 of $[3]$ that $F(S) = B(M)$. So, suppose $M$ is not $S$ and $M = M$ is the $\sigma$-ideal $R$.

The set $M$ is an inner limit set and since $S$ is separable, $M$ is countable; $M: \eta_1, \eta_2, \eta_3, \ldots$. Let $E_1, E_2, E_3, \ldots$ be a monotonic sequence of open sets whose intersection is $M$.

Let $E_1$ be a spherical open set containing $\eta_1$ such that (1) $E_1$ is a subset of $E_2$, and (2) no boundary point of $E_1$ belongs to $M$. For each $n$, and for each $n$, $E_n \subseteq M+1$, let $E_{n+1}$ be a spherical open set containing $\eta_n$ such that (1) $E_{n+1}$ is a subset of $E_n$, (2) no boundary point of $E_{n+1}$ belongs to $M$, (3) if $1 \leq p \leq n$, $E_{n+1}$ is a subset of $E_p$, and (4) $E_{n+1}$ is an $\epsilon$-neighborhood of $E_n$ and $E_{n+1}$ are mutually exclusive if $i \neq j$.

For each $n$, let $D_n = \sum_0^n k + n \prod_{1}^{n} (n+1)$; $D_1, D_2, D_3, \ldots$ is a sequence of open sets whose common point is $M$.

Suppose $f$ belongs to $P(S)$. For each $n$, let

$$f(x) = \begin{cases} f(x), & x \in D_n, \\ 1, & x \in E_1 \cap \bigcup_{i=n}^{\infty} \bigcap_{n}^{\infty} \bigcap_{k}^{\infty} (n+1) \\ f(i), & x \in $n+1$, 1 \leq x \leq n. 

For each $n$, $f$ is continuous at each point of $M$. The sequence $f_1, f_2, f_3, \ldots$ is a sequence of functions on $G$ which converge to $f$. So, $F(S) = B(G)$. This completes the proof of Theorem 1.

Theorem 1 easily yields a characterization of scattered subsets of complete and separable metric spaces.

**Theorem 2.** Suppose $M$ is a subset of a complete and separable metric space $S$. The set $M$ is scattered if and only if every real function on $S$ is the limit of a sequence of functions each continuous at each point of $M$.

Theorem 2 follows from Theorem 1 by taking the $\sigma$-ideal $R$ to be the class of all subsets of $S$ which do not intersect $M$.

**Example 2.** Suppose $E$ is the class of all countable subsets of a metric space $S$. It follows from Theorem 3 of $[1]$ that a function $f$ on $S$ is in $B(E)$ if and only if there is a function $g$ in $B(E)$ such that the set $(f \neq g)$ is countable. G. Tucker $[5]$ has shown that this is true in a more general setting. Certainly, $B_1(E)$ is a subset of $B(E)$. In general $B_1(E)$ is a proper subset of $B_1(E)$. For example, if $S$ is $K$, $1$, and $f$ is on the rational numbers and $1$ on the irrational numbers, then $f$ is the limit of a sequence of step functions. So, $f$ is in $B_1(E)$ but $f$ is not in $B_1(E)$.

However, if $h$ is in $B_1(E)$, then by Theorem 3 of $[4]$, there is a function $g$ in $B_1(E)$ such that the set $(g \neq h)$ is countable. The function $h - g$ is $0$ except for a countable set. It follows that $h - g$ is in $B_1(E)$. So, $h = (h - g) + g$ is in $B_1(E)$. Therefore, $B_1(E)$ is a subset of $B_1(E)$. So, if $h$ is the $\sigma$-ideal of all countable subsets of $S$, then $B_1(E)$ may be a proper subset of $B_1(E)$, but if $a > 1$, then $B_1(E) = B_1(E)$.
The final theorem and example are connected with measures.

In what follows, suppose \( S \) is a complete and separable metric space and \( \mu \) is a complete, regular, \( \sigma \)-finite measure on \( S \) and \( \mu(S) > 0 \) and \( B \) is the \( \sigma \)-ideal of all clients of measure 0. If \( f \) is a function in \( B = B_0(G) \), then \( f \) is continuous. Theorem 2 of \( [4] \) and the fact that \( \mu \) is a topological measure that \( f \) is a measurable function. So, \( B_0(G) \) is a collection of measurable functions. Hence, \( B(G) \), the Baire system generated by \( G \) is a collection of measurable functions. If there is a scattered subset \( S \) of \( \mathbb{R} \) such that \( \mu(S - M) = 0 \), then it follows from Theorem 1, that \( S \) is the class of all measureable functions. Theorem 3 characterizes the measures \( \mu \) such that \( B(G) \) is the collection of all measurable functions.

**Theorem 3.** Suppose \( \mu \) is a complete, regular, \( \sigma \)-finite measure on the complete and separable metric space \( S \) and \( \mu(S) > 0 \) and \( B \) is the \( \sigma \)-ideal of all clients of measure 0. Each two of the following statements are equivalent:

1. \( B(G) \) is the collection of all measurable functions,
2. there is a scattered subset \( S \) of \( \mathbb{R} \) such that \( S = M \) or \( \mu(S - M) = 0 \).
3. \( B(G) = \mathcal{F}(S) \), and
4. \( B(G) = B_0(G) \).

**Proof.** Suppose \( B(G) \) is the collection of all measureable functions and \( S \) is not scattered. Let \( S_p \) be the subset of \( S \) to which \( p \) belongs if and only if \( \mu(p) = 0 \). There are not uncountably many points of \( S \) which do not belong to \( S_p \).

Suppose \( \mu(S_p) > 0 \). Since \( \mu \) is regular, there is a closed subset of \( S_p \) having positive measure. Let \( S_q \) be a perfect subset of \( S_p \) having positive measure. If there is no point \( p \) of \( S_q \) and open set \( D \) containing \( p \) such that \( \mu(S_q - D) = 0 \), then let \( S_r = S_q \). If there is such a point, let \( T \) be the set of all such points \( p \) and for each point \( p \) in \( T \), let \( B_p \) be an open set containing \( p \) such that \( \mu(B_p) < \epsilon \). Let \( E \) be a countable subcollection of the set of all \( B_p \)'s covering \( T \). The set \( E^c \), the sum of the members of \( E \), is an open set and \( \mu(E^c) = 0 \). Let \( S_t = S_r - E^c \). \( S_t \) is a closed subset of \( S_q \). Suppose there is a point \( x \) of \( S_t \) and an open set \( B_s \) containing \( x \) such that \( \mu(B_s) = 0 \) and \( B_s \) is an open set containing \( x \) and \( \mu(B_s + E^c) = \mu(B_s) + \mu(E^c) = 0 \). This is a contradiction. So, if \( E \) is an open set intersecting \( S_t \), then \( \mu(R - S_t) > 0 \). It follows that \( S_t \) is a perfect subset.

Let \( K \) be a countable dense subset of \( S_t \). Suppose \( \mu(K) = 0 \). Let \( A \) be an inner limiting set containing \( K \) such that \( \mu(A) = 0 \). \( S_t - A \) is an inner limiting set with respect to \( S_t \). \( S_t - A \) is of the first category with respect to \( S_t \).

Let \( A \) be a subset of \( A \) such that if \( H \) is a perfect subset of \( A \), then \( H \cdot (A - A) \) is a dense subset of \( H \) and let \( h \) be the characteristic function of \( A \). If \( L \) is a perfect subset of \( A \), then \( h_L \) is not continuous. Since the measure \( \mu \) is complete, the function \( h \) is a measurable function; \( h \) belongs to \( B(G) \). It follows from Theorem 3 of \( [4] \) that there is a function \( g \) in \( B(G) \) and an inner limiting set \( L \) such that \( \mu(S - L) = 0 \) and \( g_L = h_L \).

Let \( S = L - \sum_{p=1} F_p \), where each \( F_p \) is closed. For each \( p \), if \( F_p \) intersects \( S_1 \), then \( \mu(F_p - S_1) = 0 \) and so \( F_p - S_1 \) is a closed nowhere dense subset of \( S_1 \). So, \( (S - L) - S_1 \) is of the first category in \( S_1 \). Let \( K \) be a perfect subset of \( S_1 \) and \( K \) is of the first category in \( S_1 \). \( K \) is a perfect subset of \( A \) and \( \mu(K) = 0 \). The function \( g_K \) belongs to \( B(C(K)) \) and it follows from Theorem 1 that \( \mu(K) = 0 \). So, \( \mu(S_1) = 0 \).

Let \( M = \mathbb{R} - S_1 \). \( M \) is countable and every function on \( S \) is a measurable function. Also, a function belongs to \( G \) if and only if \( f \) is continuous at each point of \( M \). It follows from Theorem 1 that \( M \) is scattered. So, statement 1 implies statement 2.

Certainly, statement 2 implies statement 1 and it follows from Theorem 2 that statement 2 and 3 are equivalent and that statement 3 implies statement 4.

Suppose \( B(G) = B_0(G) \) and \( S \) is not scattered. Let \( S_p \) be the set of all points \( p \) such that \( \mu(p) = 0 \). Suppose \( \mu(S_p) > 0 \). Let \( S_q \) be a perfect subset of \( S_p \) such that if \( D \) is an open set intersecting \( S_q \), then \( \mu(D - S_p) > 0 \).

Let \( A \) be an inner limiting set such that \( S_p - S_q \cdot A \) is of the first category in \( S_p \) and \( \mu(A) = 0 \). Let \( h \) be the characteristic function of \( A \). \( h \) is in \( B(G) \). So, \( h \) is in \( B(G) \) and \( h \) is in \( B_0(G) \).

By Theorem 3 of \( [4] \), there is an inner limiting set \( A_p \) and a function \( g_p \) in \( B_0(G) \) such that \( \mu(S - A_p) = 0 \) and \( g_p = h \).

The set \( A \) is a subset of \( S_p \), since \( S_p \) has positive measure; \( A \cdot S_q \) is an inner limiting set with respect to \( S_p \) and \( \mu(S_q - A_p - S_q) = 0 \). \( S_p - S_q \cdot A \) is not continuous at \( S_p \). Let \( F_p \) be closed for each \( p \), \( F_p \) is closed and \( \mu(F_p) = 0 \). Since \( S_p \) is not continuous at \( S_p \), \( S_p \) is dense in \( S_p \). So, \( S_p - A_p \cdot S_q \) is of the first category with respect to \( S_p \).

The function \( g_{S_p} \) is in \( B_0(G(S_p)) \). So, the set \( (g_{S_p})/2 \) is the sum of countably many closed sets. Also, since \( g_p \) and \( h \) agree almost everywhere, \( \mu((g_{S_p} + h)/2) = 0 \). So, \( (g_{S_p} + h)/2 \) is of the first category in \( S_p \).

Since each set on the right hand side is of the first category in \( S_p \), \( S_p \cdot A \) is of the first category in \( S_p \). This is a contradiction. So, \( \mu(S_p) = 0 \).
A corrected correction

by

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The result of this note is that there exist an algebra $A$ of real-valued functions on a set, closed under uniform convergence and all continuous finitary operations, and a homomorphism of $A$ onto a non-archimedean ordered field which has a countable cofinal subset.

I denoted this in [2] (omitting proof). In [1] we correctly deduced this from a claimed example of a completely regular space $X$, an increasing sequence of zero sets $Z_t$ of $\beta X$ disjoint from $X$, and a point $x$ each of whose neighborhoods meets every difference $Z_t - Z_{t+1}$. It remains to repair that example, and to thank Eleanor Aron for pointing out the error. (The algebra $A$ consists of the continuous functions on $X$ having continuous real-valued extensions over the complement of some $Z_t$; the maximal ideal at $x$ provides the ordered field, and the verification [1] is straightforward.)

Let $X$ be the product of $n$ copies of the positive integers. Let $p_t$ be the $t$th coordinate function, $q_t = p_1 + \ldots + p_t$, $r_t$ the reciprocal of $q_t$, $s_t$ the continuous extension of $r_t$ over $\beta X$. Let $Z_t$ be the zero set of $s_t$. Evidently the $Z_t$ are increasing and disjoint from $X$. Consider all closed subsets of $X$ defined by conditions of the form $p_n > \varphi(p_1, \ldots, p_{n-1})$. No finite family of such conditions is inconsistent; so the closure in $\beta X$ of all these sets has a common point $x$. For any neighborhood $U$ of $x$, for any index $n$, there are natural numbers $a_1, \ldots, a_n$ such that on the points of $U \cap X$ where $p_1 = a_1, \ldots, p_{n-1} = a_{n-1}$, $p_n$ is unbounded; for a bound $\varphi(a_1, \ldots, a_n)$ will yield a contradiction. So $(U \cap X)^-$ meets $Z_n - Z_{n+1}$, and every neighborhood of $x$ contains one of these.

References