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FAITHFUL EXTENSIONS OF ANALYTIC SETS TO BOREL SETS

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ABSTRACT. A faithful extension property is a property P such that for any analytic subset A of the product $X \times Y$ of two Polish spaces, X and Y, such that each Y-section of A possesses property P, there is a Borel set B including A so that each Y-section of B possesses property P. Lyapunov and others showed that various properties are faithful extension. In this paper a uniform method is given for showing that these and many other properties are faithful extension.

Introduction. In this paper we consider the following general problem: Given two Polish spaces X and Y and an analytic subset A of the product $X \times Y$ such that each section A_y (defined to be $\{x: (x,y) \in A\}$) satisfies a certain property P -- Is there a Borel set $B \supseteq A$ such that each section B_y satisfies P? Such a B will be called a *faithful extension* of A. P is called a faithful extension (or FE) property of subsets of X if for any Y and A as above, there exists a faithful extension B.

Note that for the continuum or the Baire space N^N a set is analytic *iff* it is definable in Σ^1_1 form and Borel *iff* it is Δ^1_1, that is, definable in both Σ^1_1 and Π^1_1 form. (See Rogers [5], page 373, for an explanation of definability.) This characterization follows from some general results about Polish spaces which we develop in Section one. (A space is said to be Polish if it is separable and possesses a complete metric.)

The general problem outlined above is motivated by the following classical example. Suppose that every section of a given analytic set A is a singleton; then A itself is Borel. (For N^N, we have the Π^1_1 definition: $(x,y) \in A$ *iff* (for all z) $[(z,y) \in A \rightarrow x = z]$.) The same method yields that any analytic set each section of which contains exactly n elements must be Borel. This becomes false if the condition is weakened to "contains $\leqslant$ n elements"; however, this is still an FE property.

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The intuition drawn from this example is that if P is a property of "small" sets, then P should be FE, as there will be plenty of room in which to make extensions.

In the second section, we show that Π^1_1 monotone properties are FE. This implies that the following properties are FE: finite, well-ordered, scattered, nowhere dense, and totally bounded. In the third section we show that compactness and "strongly measure zero" are FE properties.

To obtain the results mentioned above, we shall use the powerful notion of a sieve to analyze the analytic sets in question. The necessary facts concerning sieves to be used here are discussed in the first section.

Finally, at the end of the paper we state a number of problems.

1. Sieves. Let $N = \{1,2,\ldots\}$. Let S be the set of all finite sequences of positive integers;

$$S = \cup \{N^k, k = 0 \text{ to } \infty\}.$$

We consider S provided with the so-called Brouwer-Kleene ordering:

$$s = (m_1,\ldots,m_k) \leqslant (n_1,\ldots,n_\ell) = t \textit{ iff } s \supseteq t \textit{ or}$$
$$\text{(there exist i) } (m_1 = n_1 \ \& \cdots \& \ m_i = n_i \ \& \ m_{i+1} < n_{i+1}).$$

Note that S is isomorphic to the set of dyadic rationals q, $0 \leqslant q < 1$, with the usual ordering reversed.

A sieve is a map L from S into $\mathcal{F}(X)$, the closed subsets of X. For each $x \in X$, let $I_L(x) = \{q\colon x \in L(q)\}$. The set sifted by a sieve L is

$$R(L) = \{x\colon I_L(x) \text{ is not well-ordered}\}.$$

Thus, the set $C(L) = X - R(L)$ may be expressed as the union of a chain of sets:

$$C(L) = \cup \{C^\alpha(L), \alpha < \omega_1\},$$

where, for each α,

$$C^\alpha(L) = \{x\colon |I_L(x)| < \alpha\}.$$

(We use the notation $|\cdot|$ to denote order type.)

The principal facts here are that the sets $C^\alpha(L)$ are Borel and that if B is a Borel subset of X and $B \subseteq C(L)$, then there is some $\alpha < \omega_1$ such that $B \subseteq C^\alpha(L)$ - - see Kuratowski and Mostowski [8, page 389] for details. Of course, R(L) is an analytic set and every analytic subset of X is sifted by some sieve.

Notice that if L sifts R(L), then R(L) is also sifted by G, where

$$G(q) = \cap \{L(t): q \supseteq t\}.$$

The sieve G has the property that if $s \supseteq t$, then $G(s) \subseteq G(t)$. Such sieves will be said to be extension decreasing.

LEMMA 1. *Suppose* $\{q_n: n \in N\}$ *is a sequence from* S *such that for each* n, $q_n > q_{n+1}$. *Then there exists* $u \in N^N$ *such that for some infinite increasing sequence of natural numbers* p_n:

$$u|n = q_{p_n}|n,$$

where $u|n = (u(1),...,u(n))$.

The real u is just the "limit" of the sequence q_n - - see [8], pages 377-378, for details.

LEMMA 2. *Let* L *be an extension decreasing sieve and let* $x \in X$. *Then* $x \in R(L)$ *iff there is some* $u \in N^N$ *such that for every* $p \in N$, $u|p \in I_L(x)$.

PROOF. Clearly, if there is some such u, then $I_L(x)$ is not well-ordered, since it contains the decreasing sequence $u|1, u|2, u|3, \ldots$.

Conversely, if $x \in C(L)$, there exists a sequence q_n of elements of $I_L(x)$ which is decreasing. By Lemma 1, there exists some $u \in N^N$ with each $u|n = q_{p_n}|n$. Thus, for each n, $x \in L(q_{p_n}|n) \subseteq L(u|n)$ and $u|n \in I_L(x)$.

NOTE. This lemma provides a Σ^1_1 definition for the arbitrary analytic set A.

The main tool to be used here is a method of "stitching" sieves together to give a new sieve which has some useful properties. We now describe the stitching process.

Let ϕ be a one-to-one map of $N \times N$ onto N such that $\phi(i,n+1) > \phi(i,n)$ for all i and n (for example, $\phi(i,n) = 2^{i-1}(2n-1)$). The map ϕ induces maps $\sigma_i: S \to S$ by setting for each $q = (q(1),...,q(\ell))$ in S:

$$\sigma_i(q) = (q)_i = (q_i(1),...,q_i(p)) = (q(\phi(i,1)),...,q(\phi(i,p))$$

provided $\phi(i,1) \leqslant \ell$ (in which case p is the largest integer such that $\phi(i,p) \leqslant \ell$). For $u \in N^N$, let

$$u_i = \cup \{(u|p)_i: p \in N\}.$$

If $\{L_n: n \in N\}$ is a sequence of sieves, then the stitched sieve generated by this sequence is the sieve L where:

$$L(q) = \cap \{L_n((q)_n): n \in N\}$$

for each q in S.

THEOREM 3. *Let* $\{L_n: n \in N\}$ *be a sequence of extension decreasing sieves and let* L *be the generated stitched sieve. Then* L *is extension decreasing and*

$$C(L) = \cup \{C(L_n): n \in N\}.$$

Moreover, for each $\alpha < \omega_1$, $C^\alpha(L) \subseteq \cup \{C^\alpha(L_n): n \in N\}$.

Before proceeding to the proof, let us set $I(x) = I_L(x)$ and $I_n(x) = I_{L_n}(x)$.

PROOF. It can be checked that L is extension decreasing. To see that $C(L) = \cup \{C(L_n): n \in N\}$, notice that

(*) (there exists $u \in N^N$)(for all p) $(u|p \in I(x))$ *iff*

(there exists $u \in N^N$) (for all n) (for all p) $(u_n|p \in I_n(x))$.

In view of the properties of our coding function, (*) holds if and only if

(for all n) (there exists $u^n \in N^N$) (for all p) $(u^n|p \in I_n(x))$.

To see that $C^\alpha(L) \subseteq \cup \{C^\alpha(L_n): n \in N\}$, it is enough to show that if $x \in C(L)$, then

$$|I(x)| \geqslant \min\{|I_n(x)|: x \in C(L_n)\} = \rho.$$

Suppose now that $x \in C(L)$. Let $J = \{n: x \in C(L_n)\}$; notice that $J \neq \emptyset$ and let ρ be as above. For $n \in J$, $|I_n(x)| \geqslant \rho$ - - let $\{r^{n\sigma}: \sigma < \rho\}$ be a sub-ordering of $I_n(x)$ of length ρ. For each $n \notin J$, let u^n be an element of N^N such that for every p, $u^n|p \in I_n(x)$.

For each $\sigma < \rho$, let q^σ be the longest sequence such that

(†) $(q^\sigma)_n \subseteq r^{n\sigma}$ for $n \in J$

and

(††) $(q^\sigma)_n \subseteq u^n$ for $n \notin J$.

Notice that the length of q^σ is $\min\{\phi(n,\text{length}(r^{n\sigma})+1: n \in J\} - 1$ and that the terms of q^σ are completely determined by (†) and (††). Also, notice that for each $\sigma < \rho$, there is some $n \in J$ such that $(q^\sigma)_n = r^{n\sigma}$.

Finally, we have, for each $\sigma < \rho$, $q^\sigma \in I(x)$ and, for any $\sigma_1 < \sigma_2 < \rho$, $q^{\sigma_1} < q^{\sigma_2}$. This last relation can be seen from the following consideration: The assumption that

$q^{\sigma_2} \leqslant q^{\sigma_1}$ for some $\sigma_1 < \sigma_2$ would contradict the fact that $r^{n\sigma_1} < r^{n\sigma_2}$, where n is chosen with $(q^{\sigma_1})_n = r^{n\sigma_1}$. Thus, $|I(x)| \geqslant \rho$.

Before turning to the discussion of faithful extension properties, we present an unrelated application of this "stitching" theorem.

Recall that every uncountable Borel set includes a perfect subset, whereas it is possible that some uncountable co-analytic set may not. We can now give a simple proof for a known result.

PROPOSITION 4. *If, for each* n, *the co-analytic set* C_n *includes no perfect set, then* $\cup C_n$ *includes no perfect set.*

PROOF. Following the stitching theorem, let $C = \cup C^\alpha$ and each $C_n = \cup C_n^\alpha$, with each $C^\alpha \subseteq \cup C_n^\alpha$. Now each C_n^α is countable - - since Borel it would otherwise include a perfect set - - it follows that each C^α is countable. Thus no C^α can include a perfect set. But if P is a perfect subset of C (and therefore Borel), then by the boundedness principle, $P \subseteq C^\alpha$ for some α. Thus C includes no perfect set.

2. Π_1^1 **monotone properties are FE.** In this section we prove that any Π_1^1 monotone property is a faithful extension property.

DEFINITIONS. Let X be a set and P a property of subsets of X. The property P is said to be monotone provided that $K \subseteq M$ and P(M) implies P(K). If X is also a Polish space then P is said to be Π_1^1 monotone provided there is a co-analytic (Π_1^1) subset V of X^N such that a subset M of X has property P if and only if $M^N \subseteq V$. Equivalently, there is an analytic (Σ_1^1) set $T = X^N - V$ such that P(M) *iff* $M^N \cap T = \emptyset$. The set T will be called the test set for the property P. The following condition is very useful: If P is Π_1^1 monotone, then M has property P if and only if every countable subset of M has property P. Of course the converse of the preceding statement is not true. (See Cenzer [1] for further discussion of Π_1^1 monotone properties.)

Before proceeding to the main theorem of this section, let us give some examples of Π_1^1 monotone properties in a Polish space X.

EXAMPLE 1. Scattered: A subset M is said to be scattered if there is no sequence $\{x_n: n \in N\}$ of points of M which forms a dense-in-itself set. Let T be the subset of X^N consisting of all sequences which are dense-in-themselves. Then M is scattered *iff* $M^N \cap T = \emptyset$. It can be checked that T is Σ_1^1 (actually Borel) in X^N. It was announced by Lyapunov [9] that the property of scattered is faithful extension for subsets of the real line. Kozlova [4,5] gave a proof under the assumption that the Cantor-Bendixson derived set orders of the sections are uniformly bounded below ω_1, which in fact is always true [11].

EXAMPLE 2. Nowhere dense: Let $\{B_n: n \in N\}$ be a countable base for the topology of X. A subset M of X fails to be nowhere dense if and only if there is a sequence $\{x_n: n \in N\}$ of points of M whose closure includes some B_n. Thus, let T be the subset of X^N consisting of all sequences whose closure includes some B_n. Again, it can be checked that T is Borel.

EXAMPLE 3. Totally bounded: Let $\{B_n: n \in N\}$ be a sequence of balls which forms a base for the topology of X. Now M is not totally bounded if and only if there is a countable subset K of M such that $C\ell(K)$ is not compact -- equivalently, there is a sequence $\{x_n: n \in N\}$ of points of M and a positive integer p such that no finite collection $\{B_{n_1},...,B_{n_k}\}$ of balls of diameter $< 1/p$ from the base covers $C\ell(K)$. Let T_p be the set of all sequences from X with this property and let $T = \cup T_p$. Then M is totally bounded if and only if $T \cap M^N = \emptyset$. It can be checked that T is a Borel set. We note that the property of total boundedness is faithful extension also follows from the fact that compactness is faithful extension. Of course, compactness is not a Π^1_1 monotone property (see Section 3).

EXAMPLE 4. Let X be the Euclidean plane and P the property of having no vertical line segment included in the closure. It can be checked that this is a Π^1_1 monotone property.

EXAMPLE 5. Cardinality: The property of containing $\leqslant$ n points is a Π^1_1 monotone property. As a corollary, if A is analytic and every section of A consists of exactly n points, then A is a Borel set. Finiteness is also Π^1_1 and monotone. However, countability is not Π^1_1, although it is monotone and turns out to be FE. Novikov [12] showed that the property of containing $\leqslant$ n points is faithful extension. It was shown by Lyapunov [9] that finiteness is a faithful extension property. Luzin [10] showed that countability is FE.

EXAMPLE 6. Let X have a Borel linear order $<$. This means that $\{(x_1,x_2): x_1 < x_2\}$ is a Borel subset of X^2. The property of being well ordered with respect to this order is a Π^1_1 monotone property. This follows from the fact that a subset A of X is well ordered with respect to $<$ if and only if there does not exist a sequence $\{a_n\}_{n=1}^{\infty}$ of elements of A such that $a_{n+1} < a_n$, for each n.

For the usual order on the real line, Lyapunov [9] announced that being well ordered is faithful extension. Kozlova [4,5] gave a proof under the assumption that the order types of the sections are uniformly bounded below ω_1, which is always true [10].

THEOREM 5. *Every* Π^1_1 *monotone property is a faithful extension property.*

PROOF. Let A be an analytic subset of $X \times Y$ each y-section of which satisfies the Π^1_1 monotone property P; let $C = X \times Y - A$. Let $T \subseteq X^N$ be a Σ^1_1 test set for the property P. Let K be an extension decreasing sieve which sifts A and, for each n, let L_n be given by:

$$(\{x_i\colon i \in N\}, y) \in L_n(q) \textit{ iff } (x_n, y) \in K(q).$$

Then L_n sifts $\{(\{x_i\colon i \in N\}, y)\colon (x_n, y) \in A\} = X \times Y - D_n$. Also, L_n is extension decreasing and $C^\alpha(L_n) = \{(\{x_i\colon i \in N\}, y)\colon (x_n, y) \in C^\alpha(K)\}$.

Let L be the stitched sieve generated by the L_n's. Then

$$C(L) = \cup\{D_n\colon n \in N\}.$$

Notice that $T \times Y$ is analytic and $T \times Y \subseteq C(L)$. Thus, there is an ordinal $\alpha < \omega_1$ such that

(*) $T \times Y \subseteq C(L) \subseteq \{C^\alpha(L_n)\colon n \in N\}$.

Consider $B = X \times Y - C^\alpha(K)$. Certainly B is a Borel set containing A. Suppose that, for some y, B_y does not have property P. Then there is some $t = \{t_i\colon i \in N\} \in T$ such that, for all i, $(t_i, y) \in B$; thus $(t_i, y) \notin C^\alpha(K)$.

On the other hand, $t \in T$ implies by (*) that, for some n, $(t, y) \in C^\alpha(L_n)$, which means that $(t_n, y) \in C^\alpha(K)$. This contradiction establishes that P is a faithful extension property.

3. **Other results and open questions.** Some properties which are not Π^1_1 monotone are nonetheless faithful extension properties. Luzin [10, page 247] showed that countability, which is monotone but not Π^1_1, is FE (another argument for this is given in [11]).

Most of the FE properties previously considered imply countability; the exceptions are totally bounded and nowhere dense. This seems to confirm our original intuition that FE properties are properties of small sets. On the other hand, the properties of compactness and σ-compactness, of having measure 0, and of being first category all are properties of small sets, but do not imply countability and are not Π^1_1 monotone.

Since all countable sets are first category, of measure 0, and σ-compact, these cannot be Π^1_1 monotone properties, although the first two are monotone.

Compactness is neither Π^1_1 nor monotone, although it satisfies one direction of our condition: If M is not compact, then there is a countable $K \subseteq M$ which is not compact. (Given a cover $\{G_n: n \in N\}$ of M which has no finite subcover, just let K contain one point from each non-empty $G_{n+1} - (G_1 \cup G_2 \cup \cdots \cup G_n)$.)

Any non-monotone property P, such as compactness, can be extended to a monotone property Q given by Q(K) *iff* (there exists $M \supseteq K$)P(M). Now a set M is compact *iff* it is both closed and totally bounded. Thus every subset K of M is also totally bounded. On the other hand, the closure of a totally bounded set is still totally bounded and is therefore compact. Thus a set K is totally bounded *iff* it is included in some compact set. In other words, total boundedness is the smallest monotone extension of compactness.

We can use these facts to show that compactness is FE. Let $A \subseteq X \times Y$ be analytic with each section A_y compact and therefore totally bounded. Let $\{d_n: n \in N\}$ be a countable dense subset of X and let

$$(d_n,y) \in D \textit{ iff } (\text{there exists } x)\ (\text{there exists } \epsilon)\ [(x,y) \in A \ \&\ |x-d_n| < \epsilon$$

$$\&\ (\text{for all } m < n)|x-d_m| \geqslant \epsilon].$$

D is clearly analytic and we claim the following:

(1) For each $y \in Y$, $A_y \subseteq \overline{D_y}$;

(2) For each $y \in Y$, D_y is totally bounded.

Statement (1) is obvious, since for any $x \in A_y$ and any $\epsilon > 0$, D_y contains a point within ϵ of x.

Here is a proof of statement (2): Fixing y and ϵ, let $\{U_1,\ldots,U_n\}$ be a cover of A_y by $\epsilon/3$-balls. For each $j < n$, let i_j be the least i such that $d_i \in U_i$, and let N be the maximum of $\{i_1,\ldots,i_n\}$. Expand each U_i to an ϵ-ball V_i. Then any point not in $V_1 \cup \cdots \cup V_n$ is at least $2\epsilon/3$ distant from any point in A_y; on the other hand, any point in A_y is within $2\epsilon/3$ of one of $\{d_1,\ldots,d_N\}$. It follows that $D_y \subseteq \{d_1,\ldots,d_N\} \cup V_1 \cup \cdots \cup V_n$ so that D_y has a finite cover of ϵ-balls. Thus D is totally bounded.

Applying Proposition 6 for totally bounded, we can now extend D to a Borel set $E \subseteq \{d_n: n \in N\}$, each section E_y of which is totally bounded. Let

$$B = \cup\{\overline{E_y}: y \in Y\} = \{(x,y): (\text{for all } k)\ (\text{there exists } n)\ [(d_n,y) \in E \\ \& \ |d_n - x| < 1/k]\}.$$

Then B is a Borel set from its description. Each section of B is the closure of a totally bounded set and is therefore compact. Finally, for each y, $A_y \subseteq \overline{D_y} \subseteq \overline{E_y} = B_y$, so that $A \subseteq B$. This completes the proof of the following.

PROPOSITION 7. *Compactness is an* FE *property.*

Kozlova [6] showed that compactness is a faithful extension property of subsets of the real line.

The property of having measure 0 can also be approximated by a Π^1_1 monotone property, although not in the same fashion. It turns out Jordan content zero is a faithful extension property. Recall that M has content zero provided that for each positive ϵ, there is a finite cover of M by open intervals with the sum of the lengths being less than ϵ. It is not difficult to see that this is a Π^1_1 monotone property. Any compact set of measure zero is of content zero. In fact, a set K has content zero *iff* it is included in some compact set of measure zero.

Of course, this depends on the definition of measure in terms of the lengths of intervals as in the real line.

PROPOSITION 8. *If every section of an analytic* $A \subseteq R \times R$ *has content zero, then* A *can be extended to a Borel set* B *each section of which has content zero.*

COROLLARY 9. *If every section of an analytic* $A \subseteq R \times R$ *is compact and has measure* 0, *then* A *can be extended to a Borel set* B *each section of which has measure* 0. (These results hold when R is either the irrationals or the complete real line.)

Here is one final observation. The class of relations P over subsets of a given topological space X is of course a Boolean algebra. The class of Π^1_1 monotone properties is of course a sub-ring of this algebra. More generally, the class of all monotone FE properties is closed under conjunction; this can be seen by the following. Given FE properties P and Q and an analytic set A each section of which satisfies both P and Q, there will be Borel extensions B and B* of A such that each section of B satisfies P and each section of B* satisfies Q. Since P and Q are monotone, it follows that each section of B B* satisfies both P and Q. Thus B B* is a faithful extension for the property P & Q.

Thus any two of the monotone properties shown above to be FE can be combined to yield a new FE property, such as being both countable and nowhere dense.

We leave the reader with the following open questions.

PROBLEM 1. Is the class of monotone FE properties closed under disjunction? Is the class of FE properties closed under conjunction and/or disjunction?

PROBLEM 2. Which of the following properties are FE - - first category, σ-compact (in a Polish space which is not σ-compact), measure zero (with respect to some regular, σ-finite, complete Borel measure)?

PROBLEM 3. If P is FE and Q is defined by: Q(M) *iff* M is the countable union of sets with property P, does it follow that Q is FE?

Since the paper was first written during the spring of 1976 several developments have taken place. It was pointed out to us by John Burgess [16] that our results could be viewed as "faithful separation" or "fully faithful extension" theorems rather than "faithful extension" theorems. A property P is a faithful separation property provided that if E and A are disjoint analytic subsets of the product X X Y of two Polish spaces X and Y and for each x, E_x has property P, then there is a Borel set B which separates E from A; $E \subset B$, $B \cap A = \phi$ and for each x, B_x has property P. Regarding Problem 2, Saint-Raymond [6] proved that σ-compactness is a faithful separation property and more. The present authors have shown that first category and measure zero are faithful separation properties [7]. Also, G. Hillard has independently demonstrated that first category is a faithful separation property. This is discussed by Dellacherie [18]. Finally, some of the ideas presented have been employed by Louveau [17].

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