SOME SELECTION THEOREMS AND PROBLEMS

R. Daniel Mauldin
Department of Mathematics
North Texas State University
Denton, Texas 76203/USA

Let I be the closed unit interval, [0,1]. Let B be a Borel subset of I×I such that for each x, B_x = {y: (x,y) ∈ B} ≠ ∅. Using the axiom of choice, we find that the Borel set B contains a uniformization (= the graph of some function f mapping I onto I). The question was raised concerning how nice or describable the function f is in the famous letters exchanged among Baire, Borel, Hadamard and Lebesgue [1]. Novikov gave the first example of a Borel subset of I×I which does not possess a Borel uniformization [2]. Kondo proved that every such Borel set B possesses a uniformization which is coanalytic [3]. Yankov [4] and von Neumann [5] proved that B contains the graph of a function f which is measurable with respect to the σ-algebra generated by the analytic subsets of I. In fact, they proved this result assuming only that B is an analytic set. Whether every Borel set B possesses a uniformization which is the difference of two coanalytic sets seems to be an unsolved problem.

Various extensions of these uniformization problems have been considered not only for their intrinsic interest but for their applications. One such problem was discussed by A.H. Stone at the 1975 Oberwolfach conference on Measure Theory [6]. The problem is essentially that of filling up a set with pairwise disjoint uniformizations so that the uniformizations are indexed in some reasonable manner. For simplicity let me formulate the problem as follows.

PARAMETRIZATION PROBLEM.
Let B be a subset of I×I so that for each x, B_x is uncountable. A parametrization of B is a map g from I×I onto B so that for each x, g(x, ·) maps I onto [x]×B_x. Given a description of B how describable can a parametrization of B be?

Stone was particularly interested in the case where B is a Borel set and he inquired about the existence of a universally measurable parametrization. Wesley proved by forcing techniques that the answer is yes. Cenzer and I proved the following theorem [7].

THEOREM 1.
Let B be an analytic subset of I×I so that for each x, B_x is uncountable.
Then there is a parametrization $g$ of $B$ such that both $g$ and $g^{-1}$ are measurable with respect to $S(I \times I)$.

Here $S(I \times I)$ denotes the family of $C$-sets of Selivanovskii, the smallest family of subsets of $I \times I$ containing the open sets and closed under operation (A). $S(I \times I)$ is a very nice family of universally measurable sets.

In the course of proving this theorem, we showed that such an analytic set $B$ possesses $2^{\aleph_0}$ pairwise disjoint uniformizations which are the graphs of functions which are measurable with respect to the $\sigma$-algebra generated by the analytic subsets of $I$. Whether such an analytic set can be filled up by pairwise disjoint uniformizations each of which is the graph of function measurable with respect to this $\sigma$-algebra is an unsolved problem. Whether the function $g$ chosen so that both $g$ and $g^{-1}$ are measurable with respect to the $\sigma$-algebra generated by the analytic subsets of $I \times I$ is also unsolved. Necessary and sufficient conditions for a Borel parametrization are given in the following theorem [8].

**THEOREM 2.**

Let $B$ be a Borel subset of $I \times I$ such that for each $x, B_x$ is uncountable. The following are equivalent.

1. $B$ has a Borel parametrization;
2. there is an atomless conditional probability distribution $\mu$ so that for each $x, \mu(x, B_x) > 0$;
3. $B$ contains a Borel set $M$ such that for each $x, M_x$ is a nonempty perfect subset of $I$.

In connection with this I showed that there is a closed uncountable subset $B$ of $I \times I$ such that for each $x, B_x$ is uncountable and yet $B$ does not have a Borel parametrization.

In view of this last result, let me pose the following problem which seems to be unsolved.

**PROBLEM.**

Let $B$ be a Borel subset of $I \times I$ so that for each $x, B_x$ is a closed uncountable set. Does $B$ have $2^{\aleph_0}$ pairwise disjoint Borel uniformizations?

That such a Borel set $B$ has a Borel uniformization was proven by Novikov [9]. That $B$ possesses $\aleph_1$ Borel uniformizations was shown by Larman [10].

We can obtain Larman's result from the following selection theorem.

**THEOREM 3.**

There are Borel measurable maps, $f_\alpha, \alpha < \omega_1$ from $2^I$ into $I$ such that for each closed set $K, f_\alpha(K) \in K$ and if $K$ is uncountable, then $f_\alpha(K) \neq f_\beta(K)$, if $\alpha \neq \beta$. 
Before proving this theorem, let us indicate how this gives \( \aleph_1 \) pairwise disjoint Borel measurable uniformizations. One simply takes the map \( \phi \) from \( I \) into \( 2^I \) defined by \( \phi(x) = B_x \). It is well known that this is a Borel measurable map. For \( \alpha < \omega_1 \), set \( g_\alpha(x) = f_\alpha(\phi(x)) \). The graphs of the functions \( g_\alpha \) are pairwise disjoint Borel uniformizations of \( B \).

**PROOF OF THEOREM 3.**

Let \( \tau \) be a Borel isomorphism of \( I \) onto \( 2^I \). For each \( \alpha < \omega_1 \), let \( D_\alpha \) be the \( \alpha \)th derived set map of \( 2^I \) onto \( 2^I \). \( D_1 \) is a Borel measurable map of class 2. Thus, for each \( \alpha < \omega_1 \), \( D_\alpha \) is a Borel measurable map. (The exact class of these maps is an unsolved problem [11].) Let \( P = \{ F \in 2^I : F \text{ is perfect} \} \). Let \( Z_\alpha = D_\alpha^{-1}(\phi) \) and let \( h : 2^I \to B \) by \( h(E) = \{ x \in E : x \text{ is an isolated point of } E \} \).

For each ordinal \( \alpha, 0 \leq \alpha < \omega_1 \), let \( B_\alpha \) be the graph of \( D_\alpha \circ \tau \) in \( I \times I \), let

\[ T_\alpha = (D_\alpha \circ \tau)^{-1}(P), \]

let

\[ K_\alpha = T_\alpha - \bigcup \{ T_\gamma : \gamma < \alpha \}; \]

and let

\[ M_\alpha = B_\alpha \cap (K_\alpha \times I). \]

Notice that for each \( \alpha \), \( M_\alpha \) is a Borel subset of \( I \times I \) and if \( x \in \pi_1(M_\alpha) = K_\alpha \times I \) then \( (M_\alpha)_x \) is a perfect set. According to Theorem 2 each \( M_\alpha \) has continuum many pairwise disjoint Borel uniformizations. So, let \( \{ Y_\beta^\alpha \}_{\beta < \omega_1} \) be a family of \( \aleph_1 \) pairwise disjoint Borel uniformizations of \( M_\alpha \).

For each \( 0 \leq \alpha < \omega_1 \), let

\[ E_\alpha = \bigcup_{\gamma \leq \alpha} \bigcup (T_\gamma^{-1}(Z_\alpha) \times I) \cap B \cup \bigcup (h(D_\alpha \circ \tau) | I - T_\alpha) \]

We note the following properties of these sets: (1) \( E_\alpha \) is a Borel subset of \( I \times I \), (2) \( E_\alpha \subset B \), (3) for each \( x \) in \( I \), \( E_x^\alpha \) is a nonempty countable set and finally (4) if \( 0 \leq \alpha < \beta < \omega_1 \) and \( Y \) is an uncountable compact subset of \( I \), then

\[ E_\alpha^{-1}(Y) \cap E_\beta^{-1}(Y) = \emptyset. \]

For each \( \alpha \), let \( g_\alpha \) be a Borel map of \( I \) into \( I \) which uniformizes \( E_\alpha \). Let \( f^\alpha(x) = g_\alpha(\phi^{-1}(x)) \) for each \( x \in 2^I \). Then \( \{ f_\alpha : \alpha < \omega_1 \} \) is a family of Borel measurable selectors for \( 2^I \) such that if \( \alpha \neq \beta \) and \( X \) is an uncountable closed subset of \( I \), \( f_\alpha(X) \neq f_\beta(X) \). Q.E.D.

The preceding proof is a variation on an argument of Larman [10]. I was simply unable to obtain Theorem 3 as a corollary of the results in [10].
There is one closely related problem which seems to bear stating.

PROBLEM.
Let \( B \) be a Borel subset of \( I \times I \) so that for each \( x, B_x \) is uncountable and is the union of countably many compact sets. Does \( B \) have \( 2^{\aleph_0} \) pairwise disjoint Borel uniformization?

Again it was shown by Larman that \( B \) has \( \aleph_1 \) Borel uniformizations provided each \( B_x \) is also a \( G_\delta \) set [10]. It may appear that this problem can be reduced to the previous problem by employing the beautiful result of Saint-Raymond [12] which states that \( B \) is the union of countably many Borel sets \( B_n \) so that for each \( n \) and each \( x, B_{nx} \) is compact. If one could insure that for some \( n \) each \( B_{nx} \) is uncountable then this last problem would be reduced to the previous problem. However, we give the following example.

EXAMPLE.
There is an \( F_\sigma \) subset \( B \) of \( I \times I \) so that for each \( x, B_x \) is uncountable and yet \( B \) does not contain a Borel set each section of which an is an uncountable compact set.

Let \( \{A_n\}_{n=1}^{\infty} \) be an increasing sequence of analytic subsets of \( I \) so that \( \bigcup A_n = I \) such that there does not exist a sequence of Borel sets \( D_n \) so that for each \( n, D_n \subseteq A_n \) and \( \bigcup D_n = I \). The existence of such sets was shown by Leo Harrington. I understand that this is a classical result although I do not know any reference.

For each \( n \), let \( M_n \) be a closed subset of \( I \times [1/2n+1, 1/2n] \) so that \( (M_n)_x \) is uncountable if and only if \( x \) is in \( A_n \). Let \( B = \bigcup M_n \). For each \( n \), let \( T_n = \bigcup \{M_p : p \leq m \} \). Suppose \( B \) contains a Borel set \( K \) so that each \( K_x \) is compact and uncountable. Notice that for each \( x \) there is some \( n \) so that \( K_x \subseteq T_{nx} \). For each \( n \), let \( D_n = \{x : K_x \subseteq T_{nx} \} \). Then each set \( D_n \) is a Borel set, \( D_n \subseteq A_n \) and \( \bigcup D_n = I \). This contradiction establishes that the \( F_\sigma \) set \( B \) has the required properties.

Let me mention that one can easily obtain the following improvement of Larman's theorem [10, p.244] by combining the results of Saint-Raymond with those of Larman.

THEOREM 4.

Let \( B \) be a Borel subset of \( I \times I \) so that for each \( x, B_x \) is uncountable and is the union of countably many compact sets. Then \( B \) has \( \aleph_1 \) pairwise disjoint Borel uniformizations.
PROOF.

First, express $B = \bigcup \{B_n : n \geq 1\}$ where each $B_n$ is a Borel set and for every $x, B_n x$ is compact. For each $n$, let $g_n$ map $I \times [1/2n + 1, 1/2n]$ onto $I \times I$ by $g_n(x, y) = (x, t_n(y))$, where $t_n(y)$ is a linear map of $[1/2n + 1, 1/2n]$ onto $[0, 1]$.

Set $B_0 = \emptyset$ and for each positive integer $n$, $M_n = B_n - B_{n-1}$. Let $K_n = g_n^{-1}(M_n)$.

Let $K = \bigcup \{K_n : n \geq 1\}$. Clearly, $K$ is a Borel subset of $I \times I$ and for each $x$, $K_x$ is both a $K_\sigma$ and a $G_\delta$ set. Also, if $(x, y) \in K$, there is a unique $n$ so that $(x, y) \in K_n$. So, define $g(x, y) = g_n(x, y) \in K$. Then $g$ is a Borel isomorphism of $K$ onto $B$ so that for each $x$, $g(x, \cdot)$ maps $[x] \times K_x$ onto $[x] \times B_x$. According to Larman's theorem, there exists $\kappa_1$ pairwise disjoint Borel uniformizations of $K$. The images of these uniformizations of $K$ are pairwise disjoint Borel uniformizations of $M.Q.E.D.$

REFERENCES


