A CROSS SECTION THEOREM
AND AN APPLICATION TO $C^*$-ALGEBRAS

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ABSTRACT. The purpose of this note is to prove a cross section theorem for certain equivalence relations on Borel subsets of a Polish space. This theorem is then applied to show that cross sections always exist on countably separated Borel subsets of the dual of a separable $C^*$-algebra.

See Auslander-Moore [2], Bourbaki [3], Kuratowski [9], and Mackey [12] for the main results and notation in Polish set theory used in this paper.

The main result of this note is the following theorem.

THEOREM 1. Let $B$ be a Borel subset of the Polish space $X$. Let $R$ be an equivalence relation on $B$ such that each $R$-equivalence class is both a $G_δ$ and an $F_σ$ in $X$, and such that the $R$-saturation of each relatively open subset of $B$ is Borel. Then the quotient Borel space $B/R$ is standard, and there is a Borel cross section $f: B/R \to B$ for $R$.

Notice that if the $R$-saturation of each relatively closed subset of $B$ is Borel, then the $R$-saturation of each relatively open subset of $B$ is Borel, for each relatively open subset of $B$ is the countable union of relatively closed sets.

A number of preliminary lemmas are proved first.

LEMMA 2. Let $(Y, d)$ be a separable metric space and let $R$ be an equivalence relation on $Y$ such that the $R$-saturation of each open set is Borel. Then there is a Borel set $S$ whose intersection with each $R$-equivalence class is complete with respect to $d$ is nonempty, and whose intersection with each $R$-equivalence class is at most one point.

PROOF. By the proofs (but not the statements) of Theorem 4, p. 206, Bourbaki [3] and Lemme 2, p. 279, Dixmier [4], there exists a decreasing sequence of Borel subsets of $Y$, say $S_n$, so that $S_n \cap R(y) \neq \emptyset$, diameter$(S_n \cap R(y)) \to 0$, and $\cap_{n \geq 1} (S_n \cap R(y)) = \cap_{n \geq 1} (((S_n \cap R(y)) \cap R(y))$ for each $y$ in $Y$. Let $S = \cap_{n \geq 1} S_n$. $S$ is a Borel subset of $Y$, the intersection of $S$ with each complete $R$-equivalence class is nonempty, and the intersection of $S$ with each $R$-equivalence class is at most one point. Q.E.D.

LEMMA 3. Let $Y$ be a Polish space and $D$ a subset of $Y$ which is both a $G_δ$ and
an $F_\sigma$. Then there is an open set $V$ in $Y$ so that $D \cap V$ is nonempty and $D \cap V$ is closed in $V$.

Proof. We may assume that $\overline{D} = Y$. Since $D$ is a $G_\delta$, $Y - D$ is a countable union of closed sets. None of these closed sets has an interior, for $D$ is dense in $Y$. But $D$ is a countable union of closed sets. The Baire category theorem implies that one of these closed sets has an interior. Hence, there exists an open set $V$ so that $V$ is contained in $D$. Q.E.D.

Lemma 4. Let $X$ and $R$ be as in Theorem 1. Then $X/R$ is countably separated.

Proof. Let $V_m$ ($m \geq 1$) be a basis for the topology of $X$. Then the $R(V_m)$ ($m \geq 1$) are Borel sets which separate the $R$-equivalence classes. To see this, let $a$ and $b$ be elements of $X$ so that $R(a)$ and $R(b)$ are disjoint. If $R(b)$ is not contained in $\overline{R(a)}$, there exists a $c$ in $R(b)$ and a positive integer $m$ so that $c$ is in $V_m$ and $V_m \cap \overline{R(a)} = \emptyset$. But then $V_m \cap R(a) = \emptyset$, and so $R(V_m) \cap R(a)$ is empty. Hence, $R(b)$ is contained in $R(V_m)$ and $R(a)$ is contained in $X - R(V_m)$. So we may assume that $R(b)$ is contained in $\overline{R(a)}$. It follows from Lemma 3 that there is an integer $m$ so that $R(a) \cap V_m$ is nonempty and $R(a) \cap V_m = \overline{R(a)} \cap V_m$. But then $R(V_m) \cap R(b)$ is empty. If not, there exists a $c$ in $R(b) \cap V_m \subset \overline{R(a)} \cap V_m = R(a) \cap V_m$. Hence, $R(b) = R(c) = R(a)$. Contradiction. Hence, $R(a)$ is contained in $R(V_m)$ and $R(b)$ is contained in $X - R(V_m)$. Q.E.D.

Proof of Theorem 1. If $V$ is an open subset of $X$ and $U = B \cap V$, $R$ defines an equivalence relation $R_U$ on $U$ by $R_U(b) = R(b) \cap U$. The $R_U$-saturation of any open set is Borel. Now $V$ itself is a Polish space, and each $R_U$-equivalence class is both a $G_\delta$ and an $F_\sigma$ in $V$. Hence, by Lemma 2, there is a Borel set $S$ which intersects each $R_U$-equivalence class at at most one point, and which intersects each $R_U$-equivalence class which is closed in $V$ in exactly one point. Let $V_m$ ($m \geq 1$) be a basis for the topology of $X$. For each $V_m$, let $S_m$ be a corresponding $S_m$ and let $S' = \bigcup_{m \geq 1} S_m$. $S'$ is a Borel subset of $X$. $S'$ intersects each $R$-equivalence class in at most countably many points. Furthermore, $S'$ intersects each $R$-equivalence class in at least one point by Lemma 3. $X/R$ is countably separated, and therefore is Borel isomorphic to an analytic subset of $[0, 1]$ by Proposition 2.9, p. 8, of Auslander-Moore [2]. Let $g: S' \to X/R$ be the natural surjective Borel mapping. The graph of $g$, say $C$, is a Borel subset of $S' \times [0, 1]$. Horizontal sections of $C$ are at most countable. Hence, theorems of S. Braun and N. Luzin (see 42.4.5, p. 378, and 42.5.3, p. 381, of Hahn [8]) show that the horizontal projection of $C$, namely $X/R$, is standard and that there exists a Borel subset $S''$ of $S'$ so that $g|S''$ is a bijection onto $X/R$. Let $f = (g|S'')^{-1}$. $f$ is a Borel mapping by Souslin's theorem. Q.E.D.

See Dixmier [5] for most of the notation and results on $C^*$-algebras used in this note. The Borel structure on the dual of a $C^*$-algebra is that generated by the hull-kernel topology. The following corollary might be a useful tool in
proving local versions of known theorems in $C^*$-algebras and group representations (see, for instance, Moore's appendix to Auslander and Kostant [1]).

**Corollary 5.** Let $\hat{\mathcal{A}}$ be a separable $C^*$-algebra and let $B$ be a Borel subset of $\hat{\mathcal{A}}$ whose relative Borel structure separates points. Then $B$ is standard, and there is a Borel cross section $f: B \to \text{Irr}(\hat{\mathcal{A}})$.

**Proof.** Let $p: \hat{\mathcal{A}} \to \text{kernel}(\hat{\mathcal{A}})$, $\hat{\mathcal{A}} \to \text{Prim}(\mathcal{A})$, be the natural open mapping. From the definition of the topology of $\hat{\mathcal{A}}$, $U$ is open in $\hat{\mathcal{A}}$ if and only if $p(U)$ is open in $\text{Prim}(\mathcal{A})$, in which case $U = p^{-1}(p(U))$. Now consider the set $\mathcal{S}$ of all subsets $B$ of $\hat{\mathcal{A}}$ such that $B = p^{-1}(p(B))$. $\mathcal{S}$ is clearly closed under countable unions, and $\hat{\mathcal{A}}$ is closed under complements since $p$ is surjective. Since $\mathcal{S}$ contains the open subsets of $\hat{\mathcal{A}}$, $\mathcal{S}$ therefore contains all Borel subsets of $\hat{\mathcal{A}}$. Therefore, $B$ is a Borel subset of $\hat{\mathcal{A}}$ if and only if $p(B)$ is a Borel subset of $\text{Prim}(\mathcal{A})$. Hence, if $B$ is a Borel subset of $\hat{\mathcal{A}}$, and if the relative Borel structure on $B$ separates points, then $p$ is one-to-one on $B$, and $B$ and $p(B)$ are Borel isomorphic. But $\text{Prim}(\mathcal{A})$ is a standard Borel space by Theorem 2.4 of Effros [7]. Hence, $p(B)$, and therefore $B$, are standard Borel spaces.

Let $q: \text{Irr}(\mathcal{A}) \to \hat{\mathcal{A}}$ be the natural continuous open mapping. As $\text{Prim}(\mathcal{A})$ is $T_0$ with a countable basis for its topology, each point of $\text{Prim}(\mathcal{A})$ is the intersection of a closed set and a $G_\delta$. Hence, $q^{-1}(b) = (p \circ q)^{-1}(p(b))$ is a $G_\delta$ in $\text{Irr}(\mathcal{A})$ for all $b$ in $B$. Each $q^{-1}(b)$ is also an $F_\sigma$ by Lemma 2.7 and Lemma 4.1 of Effros [6]. Let $R$ be the equivalence relation on $q^{-1}(B)$ given by point inverses under $q$. Each $R$-equivalence class is both a $G_\delta$ and an $F_\sigma$ in $\text{Irr}(\mathcal{A})$. The $R$-saturation of a relatively open subset of $q^{-1}(B)$ is again relatively open, and therefore Borel, since $q|q^{-1}(B)$ is open onto $B$. Hence, Theorem 1 and Soussin's theorem show that $q^{-1}(B)/R$ and $B$ are Borel isomorphic, and there is a cross section $f: B \to \text{Irr}(\mathcal{A})$. Q.E.D.

The following corollary has some applications. Consider the following setup. Let $X$ be a standard Borel space, $Y$ a Polish space, and $R$ an equivalence relation on $Y$ such that the $R$-saturation of open sets is Borel. $R$ gives rise to an equivalence relation $R'$ on $X \times Y$ by $R'(x, y) = \{x\} \times R(y)$.

**Corollary 6.** Let $B$ be a Borel subset of $X \times Y$ which is saturated with respect to $R'$. Suppose that each $R'$-equivalence class contained in $B$ is, viewed as a subset of $Y$, both a $G_\delta$ and an $F_\sigma$. Then $B/R'$ is standard, and there exists a Borel cross section $f: B/R' \to B$ for $R'$.

**Proof.** There exists a Polish space $Z$ and a one-to-one Borel mapping $p: Z \to X$. Let $g: (z, y) \to (p(z), y)$, $Z \times Y \to X \times Y$. Let $R'' = g^{-1}(R')$ and $B'' = g^{-1}(B)$. Each $R''$-equivalence class contained in $B''$ is both a $G_\delta$ and an $F_\sigma$ since each $R'$-equivalence class in $B$ is, viewed as a subset of $Y$, a $G_\delta$ and an $F_\sigma$, and since the vertical sections of $Z \times Y$ are closed. The $R''$-saturation of an open set in $Z \times Y$ is Borel. It suffices to prove this for open rectangles. Let $U \times V$ be open in $Z \times Y$, where $U$ is open in $Z$ and $V$ is open in $Y$. But
\[ R''(U \times V) = g^{-1}(R'(p(U) \times V)) = g^{-1}(p(U) \times R(V)), \]

which certainly is Borel in \( Z \times Y \). Hence, by Theorem 1, \( B'/R'' \) is standard, and there exists a Borel cross section \( f': B'/R'' \to B' \). Choose a sequence \( B'_n \) \((n \geq 1)\) of \( R''\)-saturated Borel subsets of \( B' \) which separate the \( R''\)-equivalence classes. Then the \( g(B'_n) \) \((n \geq 1)\) are \( R'\)-saturated Borel subsets of \( B \) which separate the \( R'\)-equivalence classes. Hence, \( B/R' \) is countably separated. Furthermore, \( g(f'(B'/R'')) \) is a Borel transversal for the \( R'\)-equivalence classes of \( B \). Let \( h: g(f'(B'/R'')) \to B/R' \) be the natural one-to-one Borel mapping. Then \( B/R' \) is standard by Souslin’s theorem, and \( f = h^{-1}: B/R' \to B \) is a Borel cross section for \( R' \). Q.E.D.

The following examples help to clarify the hypotheses of Theorem 1.

**Example 7.** Note that Theorem 1 may fail if each \( R \)-equivalence class is only required to be an \( F_\sigma \) set, even if the \( R \)-saturation of each open set is open and \( B/R \) is metrizable. This follows from the fact that if \( A \) is an analytic nonborelian subset of \( J \), the irrational numbers, then there is a Borel subset \( B \) of \( J \times J \) such that the projection map restricted to \( B \) is open and projects \( B \) onto \( A \). Also, each vertical section of \( B \) may be taken to be an \( F_\sigma \) subset of \( J \times J \) (see Taimanov [11]).

**Example 8.** There is a Borel subset \( B \) of \( J \times J \) such that each vertical section of \( B \) is an \( F_\sigma \) subset of \( J \), the projection \( \pi \) onto the first axis, restricted to \( B \), is open, \( \pi(B) = J \), and yet there is no Borel cross section (in this case, there is no Borel uniformization). Recall that if \( E \) is a subset of \( X \times Y \), then a uniformization of \( E \) is a subset \( F \) of \( E \) such that \( E_x \neq \emptyset \) if and only if \( F \) consists of exactly one point, where \( E_x = \{ y | (x, y) \in E \} \).

First, let \( M \) be a Borel subset of \( J \times J \) such that \( \pi(M) = J \), \( M \) has no Borel uniformization, and each vertical section of \( M \) is closed. The existence of such an \( M \) can be seen as follows. Let \( C_1 \) and \( C_2 \) be disjoint coanalytic subsets of \( J \) which are not Borel separable (see Sierpinski [10] for the existence of these \( C \)'s). Let \( A_1 = J - C_1 \) and \( A_2 = J - C_2 \). \( A_1 \) and \( A_2 \) are analytic sets whose union is \( J \). Let \( M_i \) be a closed subset of \( J \times J \) which projects onto \( A_i \) \((i = 1, 2)\). Let \( M \) be the Borel set which is the union of \( M_1 \) and \( M_2 \). If \( \Gamma \) were a Borel uniformization of \( M \), then \( D = \pi(\Gamma \cap (M_1 - M_2)) \) would be a Borel subset of \( J \) which contains \( C_2 \) and has empty intersection with \( C_1 \). Thus, \( M \) has no Borel uniformization. This argument for the existence of \( M \) is due to D. Blackwell.

Identify \( J \) with \( N^N \). Let \( h_{n_1 \ldots n_k} \) be a homeomorphism of \( J \) onto \( J(n_1, \ldots, n_k) = \{ z | z \text{ is in } J \text{ and } z_i = n_i \text{ (1} \leq i \leq k) \} \), and let \( T_{n_1 \ldots n_k}(x, z) \) \(\to (x, h_{n_1 \ldots n_k}(z)) \), \( J \times J \to J \times J \). Let \( B = \bigcup T_{n_1 \ldots n_k}(M) \). Then \( B \) is a Borel subset of \( J \times J \), \( \pi | B \) is open, \( \pi(B) = J \), and each vertical section of \( B \) is an \( F_\sigma \). If \( \Gamma \) were a Borel uniformization for \( B \), then \( C = \bigcup T_{n}^{-1}((\Gamma \cap T_{n}(M)) - \bigcup_{k<n} T_{k}(M)) \) would be a Borel uniformization of \( M \). Here \( k \) and \( n \) denote finite multi-indices and \( \prec \) is the usual lexicographic order.

Suppose that \( B \) is a Borel subset of a Polish space \( X \), \( R \) is an equivalence relation on \( B \) such that each equivalence class is a \( G_\delta \) in \( X \), and such that the
saturation of relatively open sets is Borel. D. Miller has pointed out to the authors that Lemmas 3 and 4 may be altered slightly to prove that $B/R$ is countably separated. Let $V_m$ ($m \geq 1$) be as in Lemma 4. We claim that the $R(V_m)$ ($m \geq 1$) separate the $R$-equivalence classes. Let $a$ and $b$ be in $X$ so that $R(a)$ and $R(b)$ are disjoint. If $R(b)$ is not contained in $R(a)$, proceed as in Lemma 4. So suppose that $R(b)$ is contained in $R(a)$. By a symmetric argument we may assume that $R(a)$ is contained in $R(b)$. Thus, we may assume that $R(a) = R(b)$. But $R(a)$, being a $G_D$, is comeager in $R(a)$, and $R(b)$, being a $G_D$, is comeager in $R(b)$. Hence, $R(a) \cap R(b)$ is nonempty, a contradiction. The following questions remain. Is $B/R$ standard? Even if $B/R$ is standard, is there a cross section? The authors do not know the answers to these questions even if $R$ is an open equivalence relation and $B/R$ is metrizable. Note that if the last question has an affirmative answer, then there is a natural Borel cross section from $\text{Prim}(\mathcal{C}) \to \text{Irr}(\mathcal{C})$.

BIBLIOGRAPHY


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