σ-ideals and related Baire systems

by

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Suppose \( S \) is a metric space with metric \( d \), \( E \) is a proper \( \sigma \)-ideal of subsets of \( S \) and \( \mathcal{G} \) is the collection of all real functions defined on \( S \) which are continuous almost everywhere with respect to \( E \). Let \( \mathcal{B}_\alpha(\mathcal{G}) \) be \( \mathcal{G} \) and for each ordinal number \( \alpha < \omega_1 \), let \( \mathcal{B}_\alpha(\mathcal{G}) \) be the collection of all pointwise limits or sequences taken from the collection \( \sum_{\gamma \leq \alpha} \mathcal{B}_\gamma(\mathcal{G}) \).

In this paper, the collections \( \mathcal{B}_\alpha(\mathcal{G}) \), the analytic representable functions or Baire functions of class \( \alpha \) generated by \( \mathcal{G} \), are characterized in terms of an associated collection of Baire type sets (Theorem 1). These Baire type sets are characterized by a relation to the classical Baire sets (Theorems 2a, b, and c). In Theorem 3, the collections \( \mathcal{B}_\alpha(\mathcal{G}) \), \( \alpha > 0 \) are characterized by a relation to Baire’s class \( \alpha \). Finally, in the case the space \( S \) is separable, a theorem of T. Traczyk is used to give another characterization of the collections \( \mathcal{B}_\alpha(\mathcal{G}) \), \( \alpha > 0 \), (Theorem 4).

Notation. The collection of all sets of the form \( (a, b) \), where \( (a, b) \) is a number segment and \( f \) is in \( \mathcal{G} \), is denoted by \( \mathcal{D} \). If \( L \) is a collection of subsets of \( S \), then \( W_\alpha(L) \) denotes \( L \) and for each ordinal number \( \alpha \), \( 0 < \alpha < \omega_1 \), \( W_\alpha(L) \) denotes the collection to which \( X \) belongs if and only if \( X = \sum_{\gamma = 1}^{\omega_1} X_\gamma \), where, for each \( \gamma \), there is some \( \varepsilon_\gamma < \alpha \) such that \( X_\gamma \) is in \( W_{\varepsilon_\gamma}(L) \) and \( X_\gamma \) is the complement of \( X_\gamma \).

**THEOREM 1.** Suppose \( \alpha \) is an ordinal number, \( 0 < \alpha < \omega_1 \). A real function \( f \) on \( S \) is in \( \mathcal{B}_\alpha(\mathcal{G}) \) if and only if for each number segment \( (a, b) \), the set \( (a < f < b) \) is in \( W_\alpha(L) \).

Indication. It is true that \( \mathcal{G} \) is a linear lattice of real functions on \( S \) containing the constant real functions on \( S \). Also, if \( f \) is in \( \mathcal{G} \) and \( U \) is a continuous real function on the range of \( f \), then \( U(f) \) is in \( \mathcal{G} \). Also, it is true that \( \mathcal{G} \) is also containing \( \mathcal{G} \) or \( L(\mathcal{G}) \), where \( U(\mathcal{G}) \) is the collection of all limits of nonincreasing (nondecreasing) sequences from \( \mathcal{G} \). Using these facts, Theorem 1 for the case \( \alpha = 0 \) follows from Theorem 11 of [6] and the cases \( 0 < \alpha < \omega_1 \) follow from Theorem 9 of [5].
As was pointed out in [5], the collection \( G \) is a complete ordinary function system as defined by F. Hausdorff [1, Chapter 9] and we have the following relationships between the method presented here and the method of F. Hausdorff. The functions in \( B_d(G) \) are the functions \( f^1 \), if \( 0 \leq \xi < \omega \) and are the function \( f^{\xi+1} \), if \( \omega \leq \xi < \Omega \). Also, the sets in \( W_d(D) \) are the sets \( M^n \), if \( 0 \leq \xi < \omega \) and the sets \( M^{\xi+1} \), if \( \omega \leq \xi < \Omega \).

In case \( G \) is \( C \), the continuous functions on \( S \), then \( B_d(C) \) is \( \Phi_n \), the \( a \) Bien functions or analytic representable functions of class \( a \) as described by K. Kuratowski in [2, p. 392] and the collection \( D \) is \( G_n \), the collection of all open sets. For each \( n \geq 0 \), let \( B_n \) be \( B_d(C) \) and let \( W_n = W_d(G_n) \).

It can be shown by transfinite induction that we have the following relationship between the collections \( W_n \), the analytic representable sets of class \( a \) and the Borel sets of class \( a \) as defined in [2, p. 312]:

\[
W_n = \begin{cases}
G_{a,}\quad a \text{ is even and finite}, \\
F_{a,}\quad a \text{ is odd and finite}, \\
F_{a+1,}\quad a \text{ is even and infinite}, \\
G_{a+1},\quad a \text{ is odd and infinite}.
\end{cases}
\]

Theorem 2 characterizes the collections \( W_d(D) \) in the general case, in terms of the collections \( W_n \).

**Theorem 2a.** A subset \( X \) of \( S \) is in the collection \( W_d(D) \) if and only if \( X \) is a subset of an \( F_n \) set in the \( \sigma \)-ideal \( R \), \( X \) is in \( W_n \), or \( X \) is the sum of a set in \( W_n \) and a subset of an \( F_n \) set in \( R \).

**Proof.** Suppose \( X \) is in \( D = W_d(D) \). Let \( f \) be a function in \( G = B_d(G) \) and \( (a, b) \) a segment such that \( (a < f < b) \) is \( X \). For each \( n \), let \( H_n \) be the set of all points \( p \) such that the discontinuity of \( f \) at \( p \) is \( >1/n; H_n \) is a closed set in the \( \sigma \)-ideal \( R \).

Suppose \( f \) is continuous at some point \( a < f < b \). For each point \( p \) of continuity of \( f \) in \( (a < f < b) \), let \( S_p \) be an open set containing \( p \) such that \( S_p \) is a subset of \( (a < f < b) \). Let \( K \) be the sum of all the \( S_p \)’s. The set \( K \) is an open set and is a subset of \( (a < f < b) \). The set \( X = (a < f < b) \) is \( K \) or \( X = (a < f < b) = K + (a < f < b) = \sum_{n=1}^{\infty} H_n \). So, if \( f \) is continuous at some point \( a < f < b \), then \( X \) is an open set or \( X \) is the sum of an open set and a subset of an \( F_n \) set in the \( \sigma \)-ideal \( R \).

If \( f \) is not continuous at any point \( a < f < b \), then \( X \) is a subset of an \( F_n \) set in \( R \).

Now, suppose that \( X \) is an open set. Let \( f \) be the function defined as follows:

\[
f(p) = \begin{cases}
1, & \text{if } d(p, S-X) \geq 1, \\
d(p, S-X), & \text{if } d(p, S-X) < 1,
\end{cases}
\]

where \( d(p, S-X) \) means the distance from the point \( p \) to the set \( S-X \). The function \( f \) is continuous on \( S \) and the set \( X \) is \( (0 < f < 2) \) and so \( X \) is in \( W_d(D) = D \).

Suppose \( X = K + H \), where \( K \) is an open set and \( H \) is a subset of \( \sum_{n=1}^{\infty} H_n \), where each \( H_n \) is a closed set in the \( \sigma \)-ideal \( R \). For each \( p \), let \( M_p = (S-K) 
M_p \) is closed in \( R \) and \( X = K + (X-K) \cdot H \).

Let \( f \) be the function on \( S \), defined as follows:

\[
f(p) = \begin{cases}
1, & \text{if } p \text{ is in } K \text{ and } d(p, S-K) \geq 1, \\
d(p, S-K), & \text{if } p \text{ is in } K \text{ and } d(p, S-K) < 1, \\
1/n, & \text{if } p \text{ is in } (X-K) \cdot H \text{ and } M_p \text{ is the first term of the sequence } M_1, M_2, ..., \text{ which contains } p, \\
-1/n, & \text{if } p \text{ is in } \sum_{n=1}^{\infty} M_p - (X-K) \cdot H \text{ and } M_p \text{ is the first term of the sequence } M_1, M_2, ..., \text{ which contains } p, \\
0, & \text{if } p \text{ is in } S-(K+\sum_{n=1}^{\infty} M_n).
\end{cases}
\]

The function \( f \) is continuous at each point of \( \sum_{n=1}^{\infty} M_p \), \( f \) is in \( B_d(G) = G \) and \( \sigma \)-ideal \( R \) is \( X \), the set \( X \) is in \( D = W_d(D) \). There is a similar argument to show that \( X \) is a subset of an \( F_n \) set in \( R \), then \( X \) is in \( W_d(D) \).

This completes Theorem 2a.

**Theorem 2b.** A subset \( X \) of \( S \) is in the collection \( W_d(D) \) if and only if there is an \( F_n \) set, \( X \) in \( R \), a set \( A \) in \( W_n \), and a subset \( B \) of \( K \) such that \( X = A \cdot K + B \).

**Proof.** Suppose \( X \) is in \( W_d(D) \). Then \( X = \sum_{n=1}^{\infty} \bigcap_{X_{n,p}} X_{n,p} \), where for each \( n \), \( p \), \( X_{n,p} \) is in \( D = W_d(D) \). Then \( X = \sum_{n=1}^{\infty} \bigcap_{X_{n,p}} X_{n,p} \cdot B \). But, since the collection \( W_d(D) \) is countably additive, \( X = \sum_{n=1}^{\infty} X_{n,p} \), where for each \( n \), \( X_{n} \) is in \( W_d(D) \). For each \( n \), \( X_{n} = A_{n} + B_{n} \), where \( A_{n} \) is an open set and \( B_{n} \) is a subset of an \( F_n \) set in the \( \sigma \)-ideal \( R \). \( X = \sum_{n=1}^{\infty} X_{n} = \bigcap_{n=1}^{\infty} (A_{n} + B_{n}) \).

So, \( X = (\bigcap_{n=1}^{\infty} A_{n}) + C \), where \( C \) is a subset of an \( F_n \) set, \( K \text{ in } R \).
So, $X = \left( \prod_{n=1}^{\infty} A_n \right) \cdot C = \left( \prod_{n=1}^{\infty} A_n \right) \cdot (K' + (K - C))$. Letting $A = \left( \prod_{n=1}^{\infty} A_n \right)$ and $B = A \cdot (K - C)$ we have $X = A \cdot K' + B$, where $A$ is an $F_0$ set and $B$ is a subset of $K$, an $F_0$ set in the $\sigma$-ideal $E$. Now, suppose $X = A \cdot K' + B$, where $A$ is in $W_1$, $K$ is an $F_0$ set in the $\sigma$-ideal $E$ and $B$ is a subset of $K$. Since $W_1$ is an additive class, $E = A + K$ is in $W_1$ and $E' = \left( \prod_{n=1}^{\infty} A_n \right)$, where for each $n$, $A_n$ is open. So, $X = A \cdot K' + B = A \cdot K' + E \cdot B = E \cdot (K' \cdot B)$. Let $C = K \cdot B$. Then $X = E' \cdot C' = (E' + C')$. $X = \left( \prod_{n=1}^{\infty} A_n \right) \cdot (C' + \prod_{n=1}^{\infty} A_n) = \left( \prod_{n=1}^{\infty} (A_n \cdot C) \right) = \sum_{n=1}^{\infty} (A_n \cdot C)$. It follows from Theorem 2a, that for each $n$, $A_n + C$ is in $W_1(D)$ and it follows from the definition of $W_1(D)$ that $X$ is in $W_1(D)$. This completes Theorem 2b.

Theorem 2c. Suppose $1 < a < \infty$. A subset $X$ of $S$ is in $W_1(D)$ if and only if $X = A + B$, where $A$ is in $W_1$ and $B$ is a subset of an $F_0$ set in the $\sigma$-ideal $E$.

Proof for $a = 2$. Suppose $X$ is in $W_1(D)$. Then

(1) $X = \sum_{n=1}^{\infty} \prod_{p=1}^{\infty} X_{n,p}$, where for each $n, p, X_{n,p}$ is in $W_1(D)$. Noting Theorem 2b, for each $n, p$, let $X_{n,p} = A_{n,p} \cdot K_{n,p} + B_{n,p}$, where $A_{n,p}$ is in $W_1$ and $B_{n,p}$ is a subset of an $K_{n,p}$, an $F_0$ set in $R$. For each $n, p$:

$X_{n,p} = (A_{n,p} \cdot K_{n,p} + B_{n,p})' = (A_{n,p} + K_{n,p}) \cdot B_{n,p}$

and

$B_{n,p} = K_{n,p} + (K_{n,p} - B_{n,p})$.

So, $X_{n,p} = (A_{n,p} + K_{n,p}) \cdot (K_{n,p} - B_{n,p})$.

(2) $X_{n,p} = A_{n,p} \cdot K_{n,p} + (A_{n,p} + K_{n,p}) \cdot (K_{n,p} - B_{n,p})$. Using (2) in (1) we have $X = \sum_{n=1}^{\infty} \prod_{p=1}^{\infty} (A_{n,p} \cdot K_{n,p} + (A_{n,p} + K_{n,p}) \cdot (K_{n,p} - B_{n,p}))$ and expanding this we have that $X = \sum_{n=1}^{\infty} \prod_{p=1}^{\infty} (A_{n,p} \cdot K_{n,p} + B_{n,p})$, where $B$ is a subset of an $F_0$ set in $R$.

For each $n, p$ let $T_{n,p} = A_{n,p} \cdot K_{n,p}$. The set $A_{n,p}$ is in $W_1$ and $K_{n,p}$ is an $F_0$ set. So, $K_{n,p}$ is in $W_1$ and since $W_1$ is an additive class, $T_{n,p}$ is in $W_1$.

Since $T_{n,p} = A_{n,p} \cdot K_{n,p}$ for each $n, p$, we have $X = \sum_{n=1}^{\infty} \prod_{p=1}^{\infty} T_{n,p} + B$. The set $A = \sum_{n=1}^{\infty} \prod_{p=1}^{\infty} T_{n,p}$ is in $W_1$.

Now, suppose $X = A + B$, where $A$ is in $W_1$ and $B$ is a subset of $\sum_{n=1}^{\infty} M_n$, each $M_n$ is a closed set in $R$. Since $A$ is in $W_1$, $A = \sum_{n=1}^{\infty} \prod_{p=1}^{\infty} A_{n,p}$, where for each $n, p, A_{n,p}$ is in $W_1$. So,

$X = \sum_{n=1}^{\infty} \prod_{p=1}^{\infty} A_{n,p} + \sum_{n=1}^{\infty} B_n \cdot M_n$.

However, for each $n$,

$B_n \cdot M_n = (M_n + (M_n - B_n))^\prime$.

so that for each $p$,

$A_{n,p} + B_n \cdot M_n = A_{n,p} + (M_n + (M_n - B_n))^\prime$.

Since for each $n$, $M_n$ is closed, $M_n$ is in $W_1$ and since $W_1$ contains $W_1$ and $W_1$ is finitely multiplicative we have that for each $p$, $A_{n,p} \cdot M_n$ is in $W_1$. It follows from Theorem 2b that $X_{n,p} = A_{n,p} \cdot M_n + A_{n,p} \cdot (M_n - B_n)$ is in $W_1(D)$. So, $X = \sum_{n=1}^{\infty} \prod_{p=1}^{\infty} X_{n,p}$, is in the collection $W_1(D)$. This completes the argument for Theorem 2c for the case $a = 2$.

There are arguments for the cases $a > 2$ similar to the argument given here for the case $a = 2$.

Theorems 2a, b, and c give a characterization of each collection $W_1(D)$ in terms of $W_1$. From these theorems, we see that if $a$ is a countable ordinal number, other than 1, then $X$ is in $W_1(D)$ if and only if there is a set $A$ in $W_1$ and a set $B$, which is a subset of an $F_0$ set in the $\sigma$-ideal $E$ such that $X = A + B$.

Theorem 3 characterizes each collection $B_1(D)$, the analytic representable functions or Baire functions of class $\alpha$ generated by $G$, in terms of $B_1$, the Baire function of class $\alpha$.

Theorem 3. Suppose $f$ is a function on $S$ and $0 < a < \infty$. The function $f$ is in $B_1(G)$ if and only if there is a function $g$ in $B_1$ and an inner limiting set $E$ such that $f = g$ on $E$ and $E$ belongs to the $\sigma$-ideal $F$.

Proof. Suppose $f$ is in $B_1(G)$. Let $f_1, f_2, f_3, \ldots$ be a sequence from $B_1(G) = G$ converging to $f$. For each $n$, let $H_n$ be the set of all points of discontinuity of $f_n$ and let $H = \sum_{n=1}^{\infty} H_n$. $H$ is in $E$ and $H$ is an $F_0$ set. Let $E = S - H$; $E$ is an inner limiting set. For each $n, f_\delta$, the partial function of $f_n$ over $E$ is in $C(E)$, the collection of all continuous functions over $E$. 
So, \( f_S \) is in \( B_\alpha(C(E)) \). It follows by a theorem of K. Kuratowski [2, p. 434] that \( f_S \) can be extended to \( S \) without changing its class. So, there is a function \( g \) in \( B_\alpha \) such that \( g_S = f_S \).

Now, suppose \( E \) is an inner limiting set, \( S - E \) is in \( B_\alpha \), \( f \) is a function of \( S \) and there is a function \( g \) in \( B_\alpha \) such that \( f_S = g_S \). Let \( g_1, g_2, g_3, \ldots \) be a sequence from \( B_\alpha = C \) converging to \( g \) and let \( S - E = \sum_{\beta=1}^\infty K_\beta \), where each \( K_\beta \) is closed.

For each \( \alpha \), let

\[
\begin{cases}
  g_\alpha(x) = \begin{cases}
    g(x), & \text{if } x \text{ is in } S - (K_1 + \ldots + K_\alpha), \\
    f(x), & \text{if } x \text{ is in } K_1 + K_2 + \ldots + K_\alpha.
  \end{cases}
\end{cases}
\]

For each \( \alpha \), \( f_\alpha \) is continuous at each point of \( S - (K_1 + \ldots + K_\alpha) \). For each \( \alpha \), \( f_\alpha \) is in \( B_\alpha \). Let \( f_1, f_2, f_3, \ldots \) be a sequence converging to \( f \) such that for each \( \alpha \), \( f_\alpha \) belongs to \( B_\alpha \). Suppose \( f \) is in \( B_\alpha \). Let \( f_1, f_2, f_3, \ldots \) be a sequence converging to \( f \) such that for each \( \alpha \), \( f_\alpha \) is in \( B_\alpha \). Let \( f_1, f_2, f_3, \ldots \) be a sequence converging to \( f \) such that for each \( \alpha \), \( f_\alpha \) is in \( B_\alpha \), where \( \gamma_\alpha < \alpha \), and let \( S - E = \sum_{\beta=1}^\infty K_\beta \), where each \( K_\beta \) is closed.

For each \( \alpha \), let

\[
\begin{cases}
  g_\alpha(x) = \begin{cases}
    g(x), & \text{if } x \text{ is in } S - (K_1 + K_2 + \ldots + K_\alpha), \\
    f(x), & \text{if } x \text{ is in } K_1 + K_2 + \ldots + K_\alpha.
  \end{cases}
\end{cases}
\]

For each \( \alpha \), \( f_\alpha \) is in \( B_\alpha \). Let \( f_1, f_2, f_3, \ldots \) be a sequence converging to \( f \) such that for each \( \alpha \), \( f_\alpha \) belongs to \( B_\alpha \). Let \( f_1, f_2, f_3, \ldots \) be a sequence converging to \( f \) such that for each \( \alpha \), \( f_\alpha \) is in \( B_\alpha \). Then Theorem 3 follows by transfinite induction.

In case \( S \) is a separable metric space we can obtain another characterization of the collections \( B_\alpha \), \( \alpha > 0 \) from a theorem of T. Traczyk [7]. In [7], Traczyk makes use of the following definition.

DEFINITION. Suppose \( S \) is a metric space, \( D \) is a metric space and \( f \) is a mapping from \( S \) into \( D \). The function \( f \) has property \( A_\alpha \) at the point \( x_0 \) of \( S \) if for every \( \epsilon > 0 \), there is a neighborhood \( E \) of \( x_0 \), a mapping \( g \) of \( S \) into \( \alpha \), and a set \( A \) in \( I \) such that

\[
|f(x) - g(x)| < \epsilon, \quad \text{for every } x \text{ in } A \cdot E.
\]

Traczyk gives the following theorem in [7]:

Suppose \( S \) is a separable metric space, \( D \) is a separable and complete metric space and \( f \) is a mapping from \( S \) into \( D \). If \( \alpha > 0 \) and for each closed subset \( E \) of \( S \), the mapping \( f_E \) has property \( A_\alpha \) with respect to \( F \) at some point of \( E \), then there is a mapping \( g \) in \( \alpha \), a set \( A \) in \( I \) such that \( f(x) = g(x) \).

This theorem is a generalization of some earlier results of G. Lepper and later Lepper generalized this result [4, Theorem III].

Before using this theorem of Traczyk, consider the following situation. The space \( S \) is the real numbers and \( \alpha \) is the collection of all sets of Lebesgue measure 0. Suppose we let \( E \) be \( I \), the \( \alpha \)-ideal of Traczyk’s definition. As can be seen from Theorem 3, if \( f \) is in \( B_\alpha \) and \( \alpha > 0 \) then \( f \) satisfies the hypothesis of Traczyk’s theorem. However, every measurable function satisfies the hypothesis of Traczyk’s theorem for \( \alpha = 1 \), but the Baire system generated by \( G \), the collection of all functions continuous almost everywhere does not contain all the measurable functions see [6, Theorem 3]. So, it does not suffice to let \( R = I \).

In order to get a characterization of \( B_\alpha \) using Traczyk’s theorem we do the following. Noting that if \( f \) is continuous almost everywhere (with respect to \( R \)), then it is continuous except for an \( F_\alpha \) set in \( R \), let \( R \) be the collection of all sets in \( R \) which are subsets of \( F_\alpha \) sets in \( R \). \( R \) is \( \alpha \)-ideal. Let \( L \) be the \( \alpha \)-ideal of Traczyk’s definition stated above. Then Theorem 1 follows easily from Traczyk’s theorem.

THEOREM 4. Suppose the metric space \( S \) is separable and \( 0 < \alpha \). A function \( f \) is in \( B_\alpha \) if and only if for each closed subset \( E \) of \( S \), the mapping \( f_E \) has property \( D_\alpha \) (where the \( \alpha \)-ideal \( I \) is \( R \)) with respect to \( F \) at some point of \( E \).

References


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