On the Baire system

generated by a linear lattice of functions

by

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Suppose $G$ is a linear space of real functions defined over a point set $S$ such that if $f$ is in $G$, then $[f]$ is in $G$. If $G$ is a linear lattice of functions over $S$. Also, suppose $G$ contains the constant functions over $S$. Let $B_{p}(G)$ denote the collection of all pointwise limits of sequences from the collection $\sum_{y \in S} B_{p}(G)$. Sierpiński [1] and Tucker [2] have given necessary and sufficient conditions on a function $f$ in order that it be in $B_{p}(G)$. These conditions are in terms of particular sequences of functions which converge in a uniform or a monotonic sense. Since for each ordinal $\alpha > 0$, the collection $\sum_{\gamma < \alpha} B_{p}(G)$ is a linear lattice of real functions over $S$ and it contains the constant function over $S$, these results may be extended to give necessary and sufficient conditions on a function $f$ in order that it be in $B_{p}(G)$. In this paper we characterize the collection $B_{p}(G)$, $\alpha > 0$, in terms of an associated collection of Baire sets (Theorem 7) and give some relationships between these collections and the collections described by Hausdorff in [3].

Notation. If $K$ is a lattice of functions, then $K_{\alpha}$ denotes the collection of all functions which are uniform limits of sequences from $K$, $USK$ the collection of all functions which are limits of nonincreasing sequences from $K$ and $LSK$ the collection of all functions which are limits of nondecreasing sequences from $K$. The Baire system of functions generated by $K$ is denoted by $B(K)$. If $f$ is a bounded function, $[f]$ denotes the l.u.b. norm of $f$.

**Theorem 1.** If $f$ is a bounded function in $G$, then $f^{*}$ is in $G_{\alpha}$. 

50

B. M. Schein

References


Proof. Suppose $f$ is a bounded function in $G$ and $f \neq 0$. Let $h = f(g)$, so that $h$ is in $G$. For each $n$, let

$$h_n(x) = 2^n \left( \frac{2^n}{2^n} \right) h(x) - \frac{i-1}{2^n} \left( \frac{2^n}{2^n} \right) h(x) + \frac{i}{2^n},$$

$$= \frac{2i-1}{2^n} h(x) + \frac{i(i-1)}{2^n} h(x),$$

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If $i - 1 \leq 2^n$, then

$$h_n(x) = \frac{i(2^n-1)}{2^n} h(x)$$

and if $h(x)$ is between $(i-1)2^{-n}$ and $i2^{-n}$, then $h_n(x)$ is between $(i-1)2^{-n}$ and $i2^{-n}$. So, for each $x$ in $S$,

$$|h_n(x) - h^2(x)| < |(i(2^n-1)2^{-n}) - [(i-1)2^{-n}]| = 2^{-n}(2i-1) \leq 2^{-n}(2|i+1).$$

So, $h_n(x) - h^2(x) < 2^{-n}$, $g_n+1 = 2^{-n+1} + 2^{-n}$, and the sequence $(h_n)_{n=1}^\infty$ converges uniformly to $h^2$.

For each $n$, let $g_n = \max((i-1)2^{-n}, \min(h_{n+1}, i2^{-n}))$, and let $c_n = 2^{-n}(2i-1)$, for each $i$, $2^{-n} \leq i \leq 2^n$.

Let $d_n = 1 - \sum_{i=0}^{n} c_i i 2^{-n} = 1 - \sum_{i=0}^{n} c_i (2i-1)2^{-n} 2^{-n}$.

Since $G$ is a linear lattice containing the constant functions, the function $d_n + \sum_{i=0}^{n} c_i g_i$ is in $G$, for each $n$.

Suppose $n$ is a positive integer and $(p-1)2^{-n} \leq h(x) \leq p2^{-n}$, where $-2^n+1 \leq p \leq 2^n$. Then

$$d_n + \sum_{i=0}^{n} c_i g_i(x) = \sum_{i=0}^{n} \left( \frac{2^n}{2^n} \right) (2i-1)2^{-n} + \frac{p-1}{2^n} h(x) + \sum_{i=0}^{n} \left( \frac{2^n}{2^n} \right) (i-1) \frac{i}{2^n} + \sum_{i=0}^{n} \left( \frac{2^n}{2^n} \right) \frac{i}{2^n} + d_n$$

$$= \frac{2p-1}{2^n} h(x) - \frac{p-1}{2^n} + \sum_{i=0}^{n} \left( \frac{2^n}{2^n} \right) (2i-1) \frac{i}{2^n} + \sum_{i=0}^{n} \left( \frac{2^n}{2^n} \right) \frac{i}{2^n} + d_n$$

$$= \frac{2p-1}{2^n} h(x) - \frac{p-1}{2^n} + \frac{1}{2^n} \sum_{i=0}^{n} (2i-1) + 1.$$
Theorem 6. If \( g \) is in \( LSG \), \( h \) is in \( USG \) and \( g \geq h \), then there is a function \( f \) in \( LX \cap USG \) such that \( g \geq f \geq h \) and if \( g(x) > f(x) > h(x) \), then \( g(x) > f(x) > h(x) \).

Induction of Proof. Suppose \( \{a_n\}_{n=1}^\infty \) is a non-decreasing sequence from \( G \) converging to \( g \) and \( \{h_n\}_{n=1}^\infty \) is a non-increasing sequence from \( G \) converging to \( h \). For each \( p \), let \( a_p = g_p - 2^{-p} \) and \( h_p = h_p + 2^{-p} \), so that the sequence \( \{a_n\}_{n=1}^\infty \) is an increasing sequence from \( G \) converging to \( g \) and the sequence \( \{h_n\}_{n=1}^\infty \) is a decreasing sequence from \( G \) converging to \( h \).

Let \( u_0 = a_0 \) and for each \( n \), \( u_n = \max(u_{n-1}, h_{n-1}) \) and \( u_{n+1} = \min(u_n, a_{n+1}) \). It can be shown that there is a function \( f \) such that \( u_1 \leq u_2 \leq u_3 \leq \ldots \rightarrow f \) and \( u_n \leq v_n \leq u_{n+1} \) and that \( g \geq f \geq h \). Then \( g(x) > f(x) > h(x) \), then \( g(x) > f(x) > h(x) \).

Definition. If \( R \) is a collection of subsets of \( S \), then \( W(R) \) is the collection to which \( X \) belongs if and only if \( X = \bigcup \left( \bigcap_{U \in U} U \right) \) where, for each \( n \), \( Y \in R \times \infty \) is in \( R \) and \( Y \times n \) is the complement of \( X \times n \).

The next theorem includes a characterization of \( B_1(G) \) in terms of \( W(R) \).

Theorem 7. Suppose \( f \) is a function on \( S \). Each two of the following statements are equivalent:

1. \( f \) is in \( B_1(G) \).
2. \( f \) is in \( USG : LSG \).
3. \( f \) is the uniform limit of a sequence of functions each term of which is the difference of two functions in \( USG \).
4. \( f \) is in \( B_1(USG : LSG) \), and
5. For each \( (a, b) \), the set \( (a < f < b) \) is in \( W(R) \).

Sierpiński has proved that statements 1 and 2 are equivalent in [1] and Tucker that statements 1 and 3 are equivalent in [2].

Proof that 2 implies 4 and 5. Suppose \( f \) is in \( USG : LSG \). Let \( \{a_n\}_{n=1}^\infty \) be a non-increasing sequence from \( LSG \) converging to \( f \) and let \( \{h_n\}_{n=1}^\infty \) be a non-decreasing sequence from \( USG \) converging to \( f \). For each \( n, \ p, g_p > h_p \) and it follows from Theorem 5 that there is a function \( f_p \) in \( USG : LSG \) between \( g_p \) and \( h_p \). The sequence \( \{f_p\}_{p=1}^\infty \) converges to \( f \). So, \( f \) is in \( B_1(USG : LSG) \) and 2 implies 4. Suppose \( (a, b) \) is a segment.
Hausdorff in [3] defines an ordinary function system \( E \) as a collection of functions which is a linear lattice containing the constant functions such that if \( f \) and \( g \) are in \( E \), then \( f + g \) is in \( E \) if and only if there is no \( x \) such that \( f(x) = 0 \), \( f' \) is in \( E \). A complete ordinary function system is an ordinary system which is closed under uniform limits. In [3] Hausdorff characterized \( B_1(E) \) where \( E \) is an ordinary function system in terms of associated Baire sets and proved that \( B_1(E) \) is a complete ordinary function system. Theorem 7 and the following theorem strengthen these results.

**Theorem 8.** If \( E \) is a linear lattice on \( S \) containing the constant functions, then \( B_1(E) \) is a complete ordinary function system.

*Proof.* Certainly \( B_1(E) \) is a linear lattice on \( S \) containing the constant functions. Sierpiński [1] has shown that \( USB_2(G) \cdot LSB_2(G) = B_3(G) \). So, by Theorem 5, \( B_1(E) \) is closed under uniform limits. Suppose \( f \) is in \( B_1(E) \). Let \( \{f_\alpha\}_{\alpha \in \Delta} \) be a sequence of bounded functions from \( E \) converging to \( f \). According to Theorem 2, \( f_\alpha \) is in \( G_\alpha \) for each \( \alpha \). It follows that \( f_\alpha \) is in \( B_1(E) \). If \( g \) is in \( B_1(E) \), since \( \forall \alpha \geq 0 \), \( \alpha \neq 0 \), we have \( f_\alpha \) is in \( B_1(E) \). Suppose there is no \( x \) such that \( f(x) = 0 \). Let \( \{f_\alpha\}_{\alpha \in \Delta} \) be a sequence from \( E \) converging to \( f' \) such that for each \( \alpha \), \( f_\alpha \) is bounded and \( \lim_{\alpha} f_\alpha(x) = f'(x) \) for each \( x \). Let \( \delta \), be \( \epsilon \), for each \( \alpha \). Again, according to Theorem 2, \( \delta_\alpha \) is in \( G_\alpha \). So, \( f' \) is in \( B_1(E) \) and hence \( f^{-1} = f \) is in \( B_1(E) \). So, \( B_1(E) \) is a complete ordinary function system.

**Definition.** Let \( W_\Delta(D) = D \) and for each ordinal number \( a, 0 < a < \Omega \), let \( W_a(D) = W_\Delta(D) \).

Use of transfinite induction and the fact that the collection \( \sum_{a < \alpha} B_1(G) \) is a linear lattice containing the constant functions, yields the following theorem.

**Theorem 9.** Suppose \( f \) is a function on \( S \) and \( 0 < a < \Omega \). In order that \( f \) belong to \( B_1(G) \), it is necessary and sufficient that, for each segment \((a, b)\), the set \((a < f < b)\) should be in \( W_\Delta(D) \).

*Remark.* In case \( G \) is a complete ordinary function system, we have the following relationships between the method presented here and the method given by Hausdorff [3, pp. 292-295]. The functions in \( B_1(G) \) are the functions \( f_\alpha \), if \( 0 < \xi < u \) and are the functions \( f^{\xi^+} \), if \( w < \xi < u \). Also, the sets in \( W_\Delta(D) \) are the sets \( M_\xi \), if \( 0 < \xi < u \) and are the sets \( M^{\xi^+} \), if \( w < \xi < u \).

**Theorem 10.** Suppose \( G \) is USG-LSG. If for each \( n \), \( A_n \) is in \( D \), then \( \sum_{n} A_n \) is in \( D \).

*Proof.* For each \( n \), \( A_n = (a_n < f_n < b_n) \). For each \( n \), let \( g_n = (b_n - a_n)^{-1}[\max(b_n, \max(a_n, f_n)) - a_n] \). \( g_n \) is in \( G \) and \( A_n = (0 < g_n < 1) \).

For each \( n \), let \( a_n = (1 - g_n) \cdot a_n \). It follows from Theorems 2 and 5 that \( h_n \) is in \( G \). Let \( h = \sum_{n=1}^{\infty} 2^{-n} h_n \). Again, it follows from Theorem 5 that \( h \) in \( G \) and

\[
\sum_{n=1}^{\infty} A_n = (0 < h < 1) \text{ is in } D.
\]

Theorem 10 allows us to make a simplification in describing the collection \( W_{\alpha+1}(D) \) where \( 0 < \alpha < \Omega \). Since \( USB_2(G) \cdot LSB_2(G) = B_3(G) \), it follows from Theorem 10 that the sum of countably many sets in \( W_{\alpha}(D) \) belongs to \( W_{\alpha+1}(D) \). If \( X \) is in \( W_{\alpha+1}(D) \), then \( X = \sum_{n=1}^{\infty} X_n \).

Since

\[
\sum_{n=1}^{\infty} X_n = \left( \sum_{n=1}^{\infty} X_n \right) \quad \text{and} \quad USB_2(G) \cdot LSB_2(G) = B_3(G) \quad \text{(see Sierpiński [1], p. 13)},
\]

it follows from Theorem 10, that \( \sum_{n=1}^{\infty} X_n \) is in \( W_{\alpha+1}(D) \). So, \( X \) is in \( W_{\alpha+1}(D) \) if and only if

\[
X = \sum_{n=1}^{\infty} X_n \quad \text{where, each } X_n \text{ is in } W_{\alpha}(D).
\]

The preceding results characterize the collection \( B_1(G) \) in terms of \( W_{\alpha}(D) \) for each \( a, 0 < a < \Omega \). However, \( W_{\alpha}(D) = D \) may not characterize \( B_1(G) = G \). For example, let \( G \) be the collection of all bounded functions over the interval \([0, 1]\) which are continuous except for a countable set and let \( H \) be the collection of all function on the interval \([0, 1]\) which are continuous except for a countable set. The \( D \) sets for the collection \( G \) are the \( D \) sets for the collection \( H \). The following theorem gives some conditions under which the collection \( D \) does characterize \( G \).

**Theorem 11.** Suppose \( G \) is USG-LSG. The following statements are equivalent:

1. If \( f \) is in \( G \) and \( \varphi \) is a continuous real function on the range of \( f, \varphi[f] \) is in \( G \).
2. A function \( f \) is in \( G \) if and only if for each segment \((a, b)\), \((a < f < b)\) is in \( D \).

*Proof.* By definition if \( f \) is in \( G \), then \((a < f < b)\) is in \( D \) for each segment \((a, b)\).

Suppose \( f \) is a function on \( S \) such that for each segment \((a, b)\), the set \((a < f < b)\) is in \( D \). Let \( f' = 2^{n} \tan^{-1}f \). For each \( n \), let

\[
g_n(x) = \begin{cases} 
1 & \text{if } x \in S \cap \left( b_n - \frac{i-1}{2^n} < f < b_n + \frac{i+1}{2^n} \right), \\
\frac{i}{2^n} & \text{if } x \in S \cap \left( b_n - \frac{i-1}{2^n} < f < b_n + \frac{i+1}{2^n} \right), 
\end{cases}
\]
and
\[ h_{a,i}(x) = \begin{cases} 
-1 & \text{if } x \text{ is in } S^{-\frac{i-1}{2^n}} < f' < \frac{i+1}{2^n}, \\
\frac{i-1}{2^n}, & \text{if } x \text{ is in } \left( \frac{i-1}{2^n}, \frac{i+1}{2^n} \right), 
\end{cases} \]

where \( -2^n + 1 \leq i \leq 2^n - 1 \). It follows from Theorem 4, that \( g_{a,i} \) is in \( \text{USG} \) and \( h_{a,i} \) is in \( \text{LSG} \) and we have \( g_{a,i} > f' > h_{a,i} \).

Let \( \alpha_r = \min(g_{a,1}, g_{a,2}, g_{a,3}) \) and \( \beta_r = \max(h_{a,1}, h_{a,2}, h_{a,3}) \), and for each \( n, \ a_{n+1} = \min(g_{a,1}, g_{a,2}, \ldots, g_{a,n} - \frac{1}{2^n}, a_1, \ldots, a_n) \) and \( \beta_{n+1} = \max(h_{a,1} + \frac{1}{2^n}, h_{a,2} + \frac{1}{2^n}, \ldots, h_{a,n} + \frac{1}{2^n}, \beta_1, \ldots, \beta_n) \). For each \( n, a_n \) is in \( \text{USG} \) and \( \beta_n \) is in \( \text{LSG} \) and it follows that \( a_1 \geq a_2 \geq a_3 \geq \ldots \rightarrow f' \rightarrow \infty \in \beta_3 \leq \beta_2 \leq \beta_1 \). So, \( f' \) is in \( \text{USG-LSG} = \mathcal{G} \). By assumption, \( f = \tan \left( \frac{n}{2} f' \right) \) is in \( \mathcal{G} \). So, statement 1 implies statement 2.

Now, suppose that \( f \) is in \( \mathcal{G} \) and only if for each segment \( (a, b) \), the set \( \{a < h < b\} \) is in \( D \). Suppose \( f \) is in \( \mathcal{G} \) and \( \varphi \) is a continuous real function on \( Y_f \), the range of \( f \).

Suppose \( (a, b) \) is a segment. Let \( \left\{ \langle a_p, b_p \rangle \right\}_{p=1}^{\infty} \) be a sequence of segments such that \( (a < \varphi < b) = Y_f. \frac{\sum_{p=1}^{\infty} (a_p, b_p)}{b_p - a_p} \) so that \( (a < \varphi[f] < b) \)

\[ = \sum_{p=1}^{\infty} (a_p < f < b_p). \] So, the set \( (a < \varphi[f] < b) \) is the sum of countably many sets in \( D \). Since \( \text{USG-LSG} = \mathcal{G} \), it follows from Theorem 10, that \( (a < \varphi[f] < b) \) is in \( D \). So, \( \varphi[f] \) is in \( \mathcal{G} \). Statements 1 and 2 are equivalent.

**Theorem 12.** Suppose \( \mathcal{S} \) is a countable set and \( F(S) \) denotes the collection of all real functions on \( S \). In order that \( F(S) = B(\mathcal{G}) \) it is necessary and sufficient that if \( x \) and \( y \) are in \( S \), then there be a function \( f \) in \( \mathcal{G} \) such that \( f(x) \neq f(y) \). Moreover, if \( F(S) = B(\mathcal{G}) \), then \( F(S) = B(\mathcal{G}) \).

**Proof.** Suppose that if \( x \) and \( y \) are in \( S \), then there is some \( f \) in \( \mathcal{G} \) such that \( f(x) \neq f(y) \). Let \( \langle e_p \rangle_{p=1}^{\infty} \) be a sequence of all the points of \( S \).

It follows that for each two positive integers \( p \) and \( q \), there is a function \( t_{pq} \) in \( \mathcal{G} \) such that \( (|t_{pq}| \leq 1, t_{pq}(x) = 0 \) and \( t_{pq}(e_i) = 1 \) for each \( e_p \) and each \( x \) in \( S \).

Let \( h_{aq} = \left. \frac{1}{q} \right| \right. t_{qp}, \) for each positive integer pair \( n, q \). It follows from Theorem 2 that \( h_{aq} \) is in \( \mathcal{G}_q \). If \( i \leq n \), then \( h_{aq}(e_i) = 0 \) if \( i \neq q \) and equals 1 if \( i = q \).

Suppose \( f \) is a function on \( S \). For each \( e_i \), let \( f_i = \sum_{i=1}^{q} f(e_i) h_{aq} \).

If \( i = n \), then \( f_i(e_i) = f(e_i) \). So, \( f \) is the pointwise limit of a sequence from \( \mathcal{G}_q \) and \( f \) is in \( B(\mathcal{G}) \).