SEVERAL CONSTRUCTIONS IN THE EREMENKO-LYUBICH CLASS

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Abstract. We use a theorem of Bishop in [Bis15] to construct several functions in the Eremenko-Lyubich class $\mathcal{B}$. First it is verified, that in Bishop's initial construction [Bis15] of a wandering domain in $\mathcal{B}$, all wandering Fatou components must be bounded. Next we modify this construction to produce a function in $\mathcal{B}$ with wandering domain and uncountable singular set. Finally we construct a function in $\mathcal{B}$ with unbounded wandering Fatou components. It is shown that these constructions answer two questions posed in [OS16].

Contents

1. Introduction

From the dynamical viewpoint, an entire function $f : \mathbb{C} \to \mathbb{C}$ partitions the plane into two sets. There is the *Fatou set*, denoted $\mathcal{F}(f)$, that consists of the points where the family $(f^n)_{n\geq 1}$ is normal. And there is the *Julia set* - the complement of the Fatou set, denoted $\mathcal{J}(f)$. The Fatou set is open, and its components are called the *Fatou components* of f. The Fatou components are the regions of the plane where the dynamics of f are non-chaotic.

It is not difficult to see that $\mathcal{F}(f)$ is invariant under iteration by f. If we denote U as a component of the Fatou set, it is natural to study the behavior of the forward iterates $f^n(U)$. We use the following definition: U is called *periodic* if $f^n(U) \subseteq U$ for some n, and *preperiodic* if U is eventually mapped into a periodic component. On the other hand U is called a *wandering domain* if $f^n(U) \cap f^m(U) = \emptyset$ over all $n \neq m$.

Dennis Sullivan proved in [Sul85] that wandering domains do not occur for polynomials. On the other hand for more general entire functions, wandering domains are known to exist. We call a function $f : \mathbb{C} \to \mathbb{C}$ *transcendental* if f is entire but is not a polynomial. The first example of a transcendental function with a wandering domain was in fact produced before Sullivans' result - this was given by Baker in [Bak76].

One can prove no-wandering domain theorems also for certain subclasses of transcendental functions. We introduce several new terms to define such subclasses. Given an entire function $f : \mathbb{C} \to \mathbb{C}$, the *singular values* of f are defined as the set of critical values together with the *asymptotic values*: $w \in \mathbb{C}$ is called an asymptotic value if there is a curve $\gamma : [0, \infty) \to \mathbb{C}$ approaching ∞ so that $f(\gamma(t)) \to w$ as $t \to \infty$. The singular set of f, denoted $\mathcal{S}(f)$ is the closure of the set of singular values of f.

Now we define the subclass of transcendental functions as promised: the Speiser class, denoted $\mathcal{S}$, is the collection of transcendental functions with finite singular set. This class is supposed to mimic in some sense the polynomials. Indeed, shortly after Sullivan's theorem, two groups of mathematicians (Goldberg and Keen [GK86], Eremenko and Lyubich [EL92]) proved the no-wandering domain theorem in the Speiser class.

A slightly more general subclass of transcendental functions is defined to be the collection of transcendental functions where the singular set is bounded, but not necessarily finite. This is called the Eremenko-Lyubich class and is denoted $\mathcal{B}$. The question of whether wandering domains existed in $\mathcal{B}$ was open until recently. In [Bis15] Bishop constructs functions in $\mathcal{B}$ with wandering domain.

Much of the theory for the class $\mathcal{B}$ was developed in [EL92]. There it was proven that in class $\mathcal{B}$, Fatou components can not escape uniformly to infinity. In particular this implies that in order for a wandering domain to occur in class $\mathcal{B}$ it would have to *oscillate* - i.e. return to some compact subset of the plane infinitely often. The first example of such an oscillating wandering domain was indeed given in [EL87], however this function was not in class $\mathcal{B}$. [EL87] also contains other interesting constructions of transcendental functions with specified dynamics that are built using approximation theory. Similar work is done in [Her84] where an entire function is built with a simply connected wandering domain.

Bishop's construction [Bis15], on the other hand, relies on the folding theorem, proven in the same paper, that we would like to spend some time discussing.

A *Shabat* polynomial is a complex polynomial $p(z)$ with two critical values, normalized to be ± 1. If we look at the preimage $p^{-1}[-1, 1]$ of any such polynomial, we see a tree in the complex plane - with vertices corresponding to preimages of ± 1. For example the preimage under $p(z) = \frac{3z^4}{8} - \frac{z^3}{2} - \frac{3z^2}{4} + \frac{3z}{2} + \frac{3}{8}$ with $p'(z) = \frac{3}{2}(z-1)^2(z+1)$ is shown in Figure 1.

On the other hand, if we specify any tree T in the plane, we can produce a Shabat polynomial so that $p^{-1}[-1, 1]$ is equivalent to T, that is, there is a homeomorphism of $\mathbb{C}$ taking $p^{-1}[-1, 1]$ onto T. This was first proven by Grothendieck - see for example [SZ93].

This establishes a correspondence between finite trees in the plane and Shabat polynomials. The theorem of Bishop we would like to discuss is a sort of generalization of this correspondence to the 'infinite case'. Namely, this theorem establishes a similar correspondence between a certain subclass of infinite trees and a subclass of transcendental functions.

An illustrative example is the transcendental function $\cosh(z)$. The hyperbolic cosine has two critical values ± 1, and one may verify that $\cosh^{-1}[-1, 1]$ is the imaginary axis, with vertices corresponding to multiples of πi. This is illustrated in Figure 2.

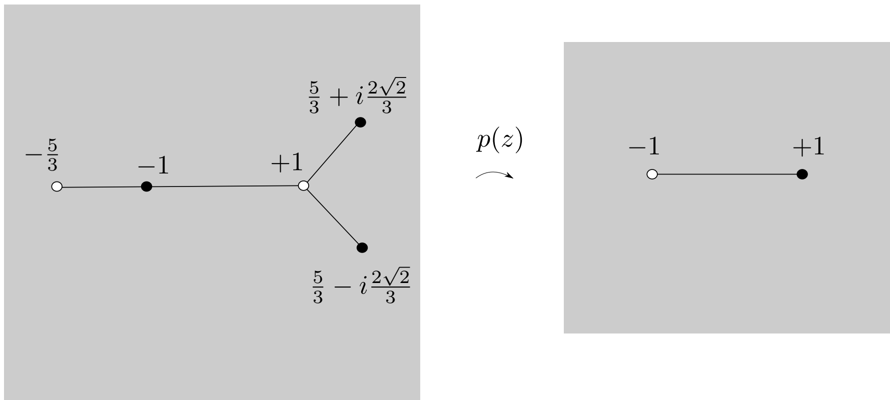

FIGURE 1. The preimage $p^{-1}[-1,1]$ of the shabat polynomial given by $p(z) = 3z^4/8 - z^3/2 - 3z^2/4 + 3z/2 + 3/8$.

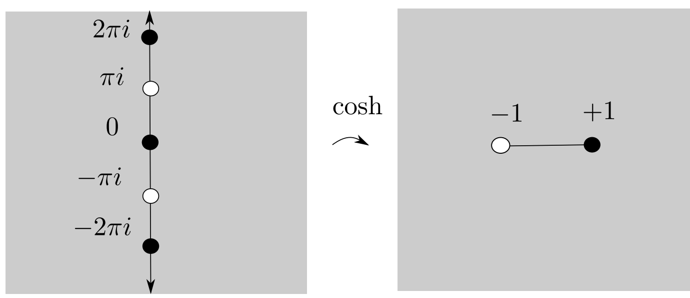

FIGURE 2. The preimage of $[-1,1]$ under the hyperbolic cosine.

Indeed, if one takes any transcendental function f with two critical values ± 1 and no asymptotic values, $f^{-1}[-1,1]$ is an infinite tree in $\mathbb{C}$. The collection of such functions with two critical values and no asymptotic values is a subclass of the Speiser class $\mathcal{S}$ and is denoted $\mathcal{S}_{2,0}$.

Bishop's theorem, on the other hand, starts with an infinite tree T satisfying certain geometric properties, and then produces an $f \in \mathcal{S}_{2,0}$ so that $f^{-1}[-1,1]$ is a quasiconformal perturbation of T^*, where T^* is T with some vertices and branches added. Before giving a rigorous formulation of the theorem's statement, it is instructive to outline some of the strategy in its proof.

If we start with an infinite tree T with alternate vertices labeled ± 1, we can denote the components of $\mathbb{C} \setminus T$ by Ω_j. We would like to keep in mind our goal is to produce $f \in \mathcal{S}_{2,0}$ so that $f^{-1}[-1,1]$ approximates T. Well each Ω_j can be mapped conformally to the right half-plane $\mathbb{H}_r$ by a map τ_j. We define τ on $\cup\Omega_j$ to be τ_j in each Ω_j. In turn, $\mathbb{H}_r$ can be mapped holomorphically onto $\mathbb{C} \setminus [-1,1]$ by cosh. This is illustrated in Figure 3.

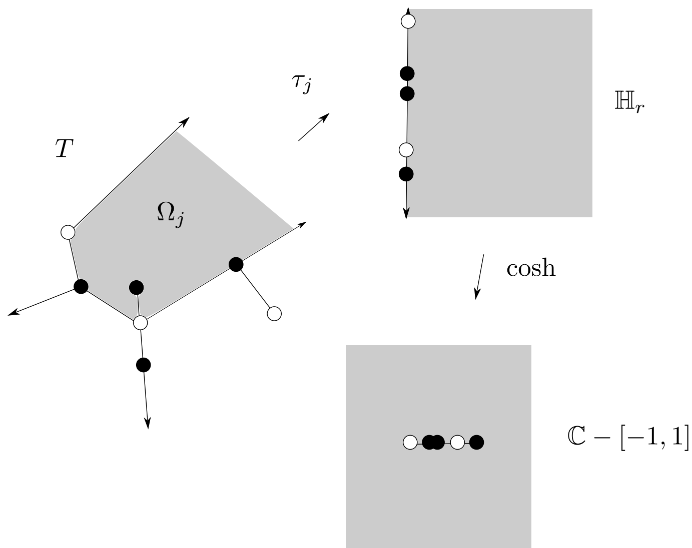

FIGURE 3. A definition of f on a component Ω_j of $\mathbb{C} \setminus T$.

There are several problems to be addressed - first of all, under this composition the vertices of T are not necessarily sent to ± 1. Moreover, one would like to define f globally on $\mathbb{C}$ as $\cosh \circ \tau_j$ on each region Ω_j - but there is no reason to expect that the continuous extensions of $\cosh \circ \tau_j$ on either side of a branch of T match up. The technical work done in [Bis15] is in replacing the map τ by a map η that agrees with τ outside a small neighborhood of T, so that $\cosh \circ \eta$ is continuous across T and maps vertices of T to ± 1. Moreover, in the course of this construction the tree T is modified by adding extra branches and vertices, and we call this new tree T^*. Once this is achieved, one has a quasiregular map $\cosh \circ \eta$ whose preimage of $[-1, 1]$ is the tree T^*. Subsequently, one applies the measurable Riemann mapping theorem to obtain a quasiconformal ϕ so that $\cosh \circ \eta \circ \phi$ is holomorphic. We take $f = \cosh \circ \eta \circ \phi$ as shown in Figure 4.

Notice that it is no longer the case that $f^{-1}[-1, 1]$ is T or even T^*, but rather a quasiconformal image of T^*. But in fact, the support of the dilatation of the quasiconformal maps may be taken so as to be concentrated in small areas of the plane. Consequently, this quasiconformal image of T^* is in fact a good geometric approximation of T^* and hence also T.

Moreover, it is not essential that T be a tree - many of the arguments will still hold as long as T is a bipartite graph and no two bounded components of $\mathbb{C} \setminus T$ share a boundary edge. We call such a bounded component of $\mathbb{C} \setminus T$ a *D-component* and an unbounded component (mapping first to $\mathbb{H}_r$ then to $\mathbb{C} \setminus [-1, 1]$) an *R-component*. In the case of a D-component we take τ to map conformally to the unit disc $\mathbb{D}$ rather

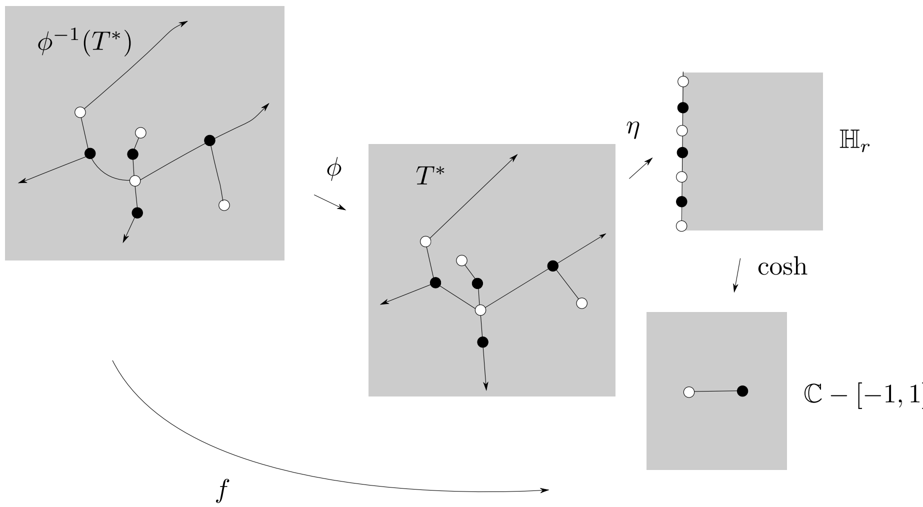

FIGURE 4. Bishop's strategy in constructing an entire f.

than $\mathbb{H}_r$. Then rather than following τ by cosh, we post-compose with $z \to z^d$, followed by a quasiconformal map ρ which is the identity in a neighborhood of the boundary of $\mathbb{D}$ and perturbs the critical value 0. This is illustrated in Figure 5.

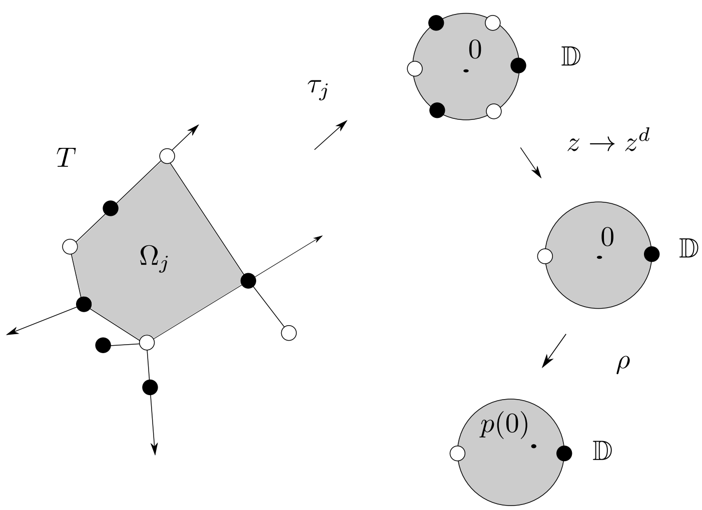

FIGURE 5. The map f on a D-component.

For such a graph T we define $\sigma \circ \tau$ as above for D-components and as $\cosh \circ \tau$ for R-components.

It is in this slightly more general setting that we first present a statement of the theorem more carefully (we will present an alternate version in Section 4). We start by introducing a restriction on the infinite trees we will look at. We say that T has *uniformly bounded geometry* if:

(1) The edges of T are C^2 with uniform bounds.
(2) The angles between adjacent edges are bounded uniformly away from zero.
(3) Adjacent edges have uniformly comparable lengths.
(4) For non-adjacent edges e and f, $\text{diam}(e)/\text{dist}(e, f)$ is uniformly bounded from above.

We also need to place some restrictions on the conformal maps τ we described above. For any edge e in the graph T, there are two τ images of e corresponding to the two sides of e. The τ-size of e is defined as the minimum of the two lengths of the τ images of e.

Lastly, we define for any $r > 0$ a neighborhood of T in which the definition of τ is adjusted, so that we may obtain continuity across the edges of T:

$$T(r) := \bigcup_{e \text{ an edge of } T} \{z \in \mathbb{C} | \text{dist}(z, e) < r\text{diam}(e)\}$$

It is only in this neighborhood of T (for a choice of r) that new branches and vertices are added in the construction to yield T^*.

Now we are ready to state Theorem 7.1 from [Bis15]:

Theorem 1.1. *Let T be an unbounded connected graph and let τ be a conformal map defined on each complementary domain $\mathbb{C} \setminus T$ as above. Assume that:*

(i) No two D-components of $\mathbb{C} \setminus T$ share a common edge.
(ii) T is bipartite with uniformly bounded geometry.
(iii) The map τ on a D-component with $2n$ edges maps the vertices to $2n^{th}$ roots of unity.
(iv) On R-components the τ-sizes of all edges is uniformly bounded from below.

Then there is an $r_0 > 0$, a transcendental f, and a K-quasiconformal map ϕ of the plane, with K depending only on the uniformly bounded geometry constants, so that $f = \sigma \circ \tau \circ \phi$ off $T(r_0)$. Moreover the only critical values of f are ± 1 - corresponding to the vertices of T, and those critical values assigned by the D-components.

Indeed Bishop's example of an $f \in \mathcal{B}$ with wandering domain is built by applying the above theorem to a particular graph shown in Figure 8 on page 10 whose vertices and other features we will discuss in the next section. Our discussion will mimic closely [FGJ15] which contains a lucid exposition of this construction. The singular set in this example is ± 1, $1/2$, and a sequence of complex numbers converging to $1/2$. In section two we will present this construction following the exposition in [FGJ15], modifying some details so as to provide an example of $f \in \mathcal{B}$ with wandering domain and an *uncountable* singular set.

In the third section of this paper we will verify that the Fatou components of Bishop's original construction are indeed bounded. The last section of this paper is dedicated to modifying this graph in order to construct an $f \in \mathcal{B}$ whose wandering Fatou components are *unbounded*. This last example is also interesting because this f has only two critical values - the other singular values are all asymptotic.

We conclude the introduction by describing a motivation for these constructions coming from the paper [OS16]. In [OS16] the authors study an alternative partition of the plane. Rather than partitioning $\mathbb{C} = \mathcal{J}(f) \cup \mathcal{F}(f)$ for a given entire function f, one partitions the plane into those points which stay bounded, escape to infinity, or do neither. More precisely define the escaping set as $I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}$, the set $K(f)$ as those points which stay bounded under iteration by f, and

$$BU(f) = \mathbb{C} \setminus [I(f) \cup K(f)]$$

In other words for any $z \in BU(f)$, the orbit $(f^n(z))_{n \geq 1}$ contains both bounded and unbounded subsequences. [OS16] contains several theorems about the set $BU(f)$ for general transcendental functions. They also ask the following (called question 3 in [OS16]):

Is there a transcendental entire function with an unbounded wandering domain in $BU(f)$, all of whose iterates are unbounded?

The answer is yes, and we provide such a construction in section 4 of this paper. [OS16] also studies the *ω-limit set* $\Lambda(z, f)$ of a point z in a wandering domain U. $\Lambda(z, f)$ is defined as the accumulation set of the orbit $(f^n(z))_{n \geq 1}$. Indeed it turns out that for any $z_1, z_2 \in U$ it is the case that $\Lambda(z_1, f) = \Lambda(z_2, f)$ [Fat20]. Thus we can write $\Lambda(U, f)$ unambiguously. The authors of [OS16] ask the following (called question 2 in [OS16])

Is there a transcendental entire function f with a wandering domain U so that $\Lambda(U, f)$ is uncountable?

We will detail the construction of such a function in Section 2. Indeed, $\Lambda(U, f)$ will contain the uncountable singular set of the function f.

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2. A Transcendental Function with Uncountable Singular Set

We will now detail Bishop's original procedure to produce $f \in \mathcal{B}$ with wandering domain - but we will introduce some slight modifications to ensure that the singular set $\mathcal{S}(f)$ is uncountable, answering a question in [OS16]. As already mentioned - [FGJ15] is an excellent resource that contains an exposition of Bishop's original construction. We follow [FGJ15] quite closely.

We first build the graph to which we will apply Bishop's theorem. There are no adjustments to be made here - we merely summarize the material contained in [FGJ15]. Consider the half strip:

$$S^+ := \{x + iy \in \mathbb{C} : x > 0, |y| < \pi/2\}$$

For any $\lambda \in \pi\mathbb{N}^+$, we have a Riemann map from S^+ to the right half plane $\mathbb{H}_r$ given by $z \to \lambda \cdot \sinh(z)$. λ is a parameter we will fix later. We may follow this Riemann map by cosh so that the composition $\sigma \circ \tau(z) := \cosh(\lambda \sinh(z))$ is holomorphic and maps ∂S^+ onto $[-1, 1]$. This is illustrated in Figure 6.

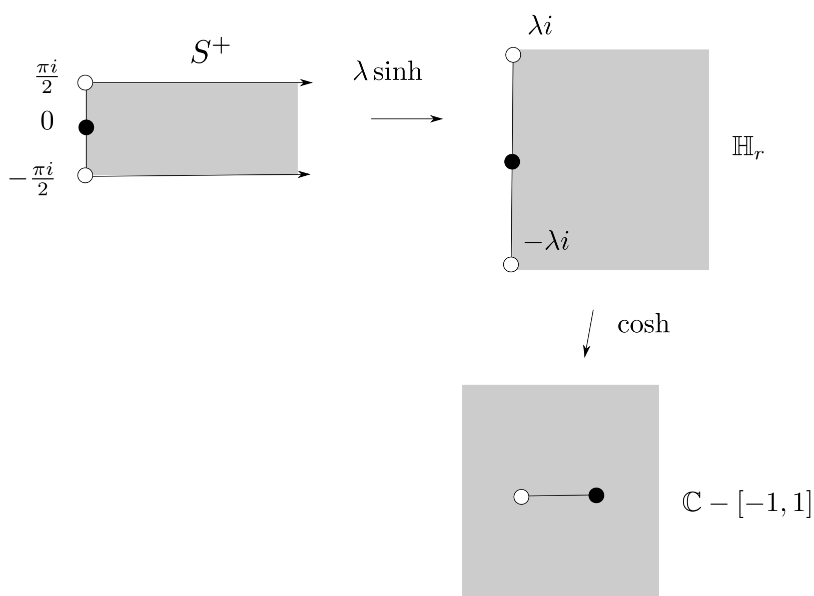


Figure 6. The definition of f in S^+.

Recall, we would like the alternating vertices of our graph to be sent to ± 1. So we will choose $\pm i\pi/2$ as two vertices, and we will choose some special vertices $(a_n \pm i\pi/2)$ along the lines $y = \pm\pi/2$. The real parts a_n ($n \geq 1$) are defined as

$$a_n := \cosh^{-1}\left(\frac{\pi}{\lambda}\left\lceil \frac{\lambda}{\pi}\cosh(n\pi)\right\rceil\right)$$

where $\lceil x \rceil$ denotes the integer part of the real number x. One may verify that $n\pi - 10^{-1} < a_n \le n\pi$. In fact we will add all preimages of ± 1 on S^+ under the map $\sigma \circ \tau$:

$$\left\{ i \sin^{-1}\left(\frac{\pi}{\lambda}k\right) : k \in \mathbb{Z} \text{ and } \frac{-\lambda}{\pi} \le k \le \frac{\lambda}{\pi} \right\} \bigcup \left\{ \cosh^{-1}\left(\frac{\pi}{\lambda}k\right) \pm i\frac{\pi}{2} : k \in \mathbb{Z} \text{ and } k \ge \frac{\pi}{\lambda} \right\}$$

Next we build the D-components of our graph. We consider for all $n \ge 1$:

$$D_n := \{z \in \mathbb{C} : |z - z_n| < 1\} \text{ where } z_n := a_n + i\pi$$

Each such D_n is mapped conformally to $\mathbb{D}$ by $z \to z - z_n$. As explained in our exposition of Bishop's theorem, in D_n we define $\sigma \circ \tau(z) := \rho_n((z - z_n)^{d_n})$ where $d_n \in 2\mathbb{N}^*$ is a parameter to be fixed later, and ρ_n is a quasiconformal map. This is illustrated in Figure 7.

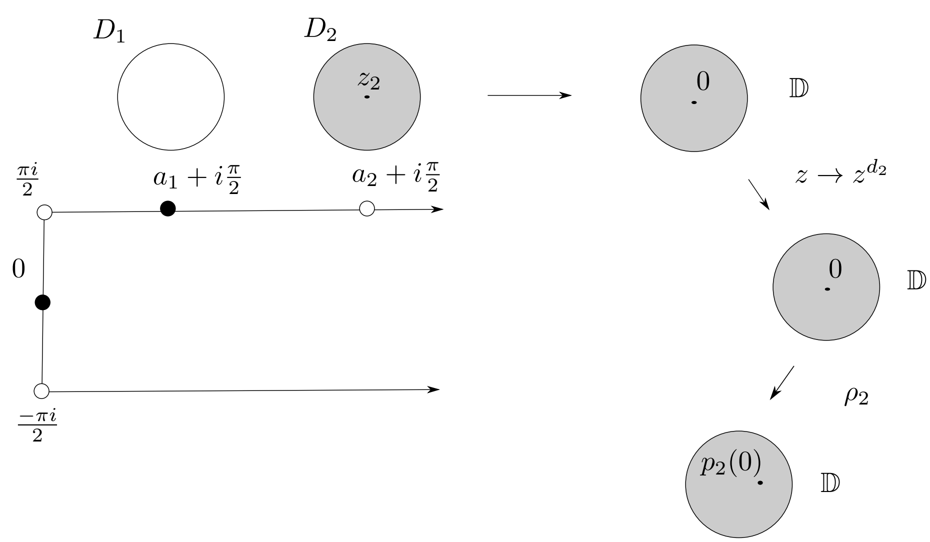


FIGURE 7. The definition of f on D-components.

We fix these quasiconformal maps ρ_n so that they fix the boundary of $\mathbb{D}$ and $\rho_n(0) = w_n$ where w_n is a parameter to be fixed later in a small neighborhood $\mathcal{N}_{1/2}$ of $1/2$. Furthermore we ensure ρ_n is conformal in $\frac{3}{4}\mathbb{D}$ and ρ_n is K_ρ-quasiconformal where K_ρ does not depend on n. The precise definition of ρ_n is given in [FGJ15]. Notice that the dilatation of ρ_n is supported on

$$\left\{ z \in \mathbb{C} : \left(\frac{3}{4}\right)^{1/d_n} < |z - z_n| < 1 \right\}$$

which shrinks in area exponentially as $d_n \to \infty$. The vertices on ∂D_n are defined to be the preimages of ± 1 under $\sigma \circ \tau$, namely the translated $(2d_n)^{\text{th}}$ roots of unity.

Next we define vertical segments on our graph connecting each $a_n + i\pi/2$ to $z_n - i\pi$, and from $z_n + i\pi$ to infinity. We also define a vertical segment connecting $i\pi/2$ to infinity. We refer the reader to [FGJ15] for the definitions of vertices along these vertical segments - we will not need them. Finally, we reflect this construction along the real and imaginary axes to obtain our final graph pictured in Figure 8.

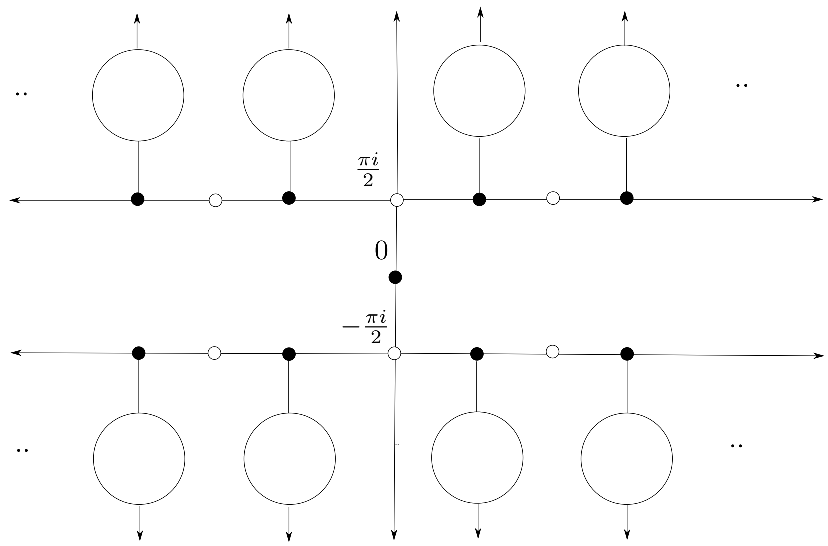


FIGURE 8. The graph to which we will apply Bishop's folding theorem.

We omit the argument that this graph satisfies the bounded geometry conditions needed in order to apply Bishop's theorem - this is contained in the proof of Theorem 3.1 from [FGJ15]:

Theorem 2.1. *For every choice of the parameters* $(\lambda, (d_n)_{n\geq 1}, (w_n)_{n\geq 1})$ *so that* $\lambda \in \pi\mathbb{N}^*, d_n \in 2\mathbb{N}^*$, *and* $w_n \in \mathcal{N}_{1/2}$ *for all* $n \geq 1$, *there exists a transcendental entire function* f *and a quasiconformal map* $\phi : \mathbb{C} \to \mathbb{C}$ *so that:*

(a) for every $z \in \mathbb{C}$, $f(\overline{z}) = \overline{f(z)}$ *and* $f(-z) = f(z)$*;*

(b) $f \circ \phi^{-1}$ *extends the maps* $(\sigma \circ \tau)|_{S^+}$ *and* $(\sigma \circ \tau)|_{D_n}$ *for every* $n \geq 1$*:*

$$f(z) = \begin{cases} \cosh(\lambda \sinh(\phi(z))) & \text{if } \phi(z) \in S^+ \\ \rho_n((\phi(z) - z_n)^{d_n}) & \text{if } \phi(z) \in D_n \end{cases} \tag{2.1}$$

(c) f *has no asymptotic values; and its set of critical values is* $\{\pm 1\} \cup \{w_n : n \geq 1\}$ *(hence* f *is in class* $\mathcal{B}$*).*

(d) $\phi(0) = 0, \phi(\mathbb{R}) = \mathbb{R}$, ϕ *is conformal in* S^+ *and its dilatation is uniformly bounded above by a universal constant* $K > 1$ *which does not depend on the parameters.*

In the original construction of [Bis15] exposited in [FGJ15], it is shown how the parameters in the above theorem may be chosen so as to produce $f \in \mathcal{B}$ with wandering domain. In the original construction $\{w_n : n \geq 1\}$ accumulates only on $1/2$, so that the singular set $\mathcal{S}(f)$ is countable (this is clear from Lemma 3.2 of [FGJ15]). We proceed in modifying the construction so as to ensure that $\{w_n : n \geq 1\}$ accumulates on an uncountable set so that $\mathcal{S}(f)$ is uncountable. We still continue however to follow the line of argument in [FGJ15], making adjustments as needed.

Let's give a sketch of our argument before we begin with details. One should keep in mind that the map $f(z)$ inside S^+ is roughly $\exp(\exp(z))$, expanding out to infinity quickly, whereas f returns the discs D_n back to $\mathbb{D}$. If we look at two discs we call D_{p_1}, D_{p_2}, we can see that f maps D_{p_1} near the origin, but also there is a preimage $f^{-2}(D_{p_2})$ of D_{p_2} near the origin. This is illustrated in Figure 9.

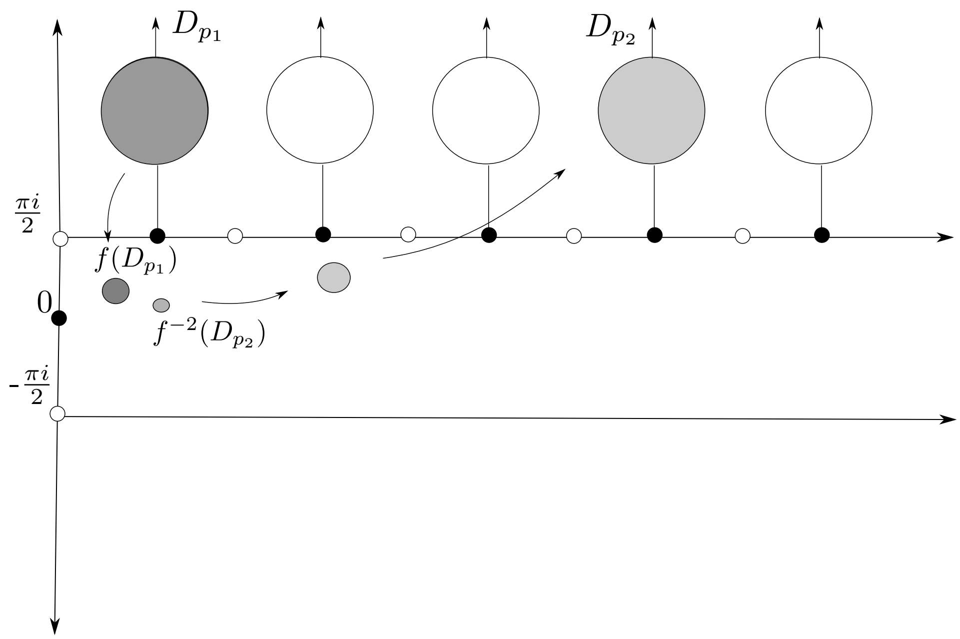


FIGURE 9. The map f before carefully choosing the parameters.

We will do two things:

(a) shrink $f(D_{p_1})$ so that it is small enough to fit inside $f^{-2}(D_{p_2})$
(b) move $f(D_{p_1})$ inside $f^{-2}(D_{p_2})$.

We will complete **(a)** and **(b)** over an entire subsequence D_{p_n} of the discs D_n. Then sub-discs of this subsequence D_{p_n} will be part of our wandering domains - a disc is moved near the origin, then is iterated towards another disc farther from the origin, which is again returned near the origin, etc...

(a) is accomplished by adjusting the parameters (d_n) - the powers that 'crush' the size of the disc D_n, and **(b)** is accomplished by adjusting the parameters (w_n) - the centers of $f(D_n)$. This takes some work as one needs parameters that accomplish **(a)** and **(b)** simultaneously over this entire subsequence D_{p_n}.

Now we begin constructing a transcendental function with uncountable singular set and a wandering domain. Fix some choice of the parameters $(\lambda, (d_n)_{n\geq 1}, (w_n)_{n\geq 1})$ as in Theorem 2.1. We notice that the real line is preserved under the map f. We will need the following estimates to establish the existence of the discs $f^{-n}(D_{p_n})$ we discussed informally above. We note that the proof of the following Lemma is quite close to that of Lemma 3.2 in [FGJ15]. It is included because the author feels the notation and constructions introduced therein are essential to the understanding of this section.

Lemma 2.2. *Let $f, \phi, (\lambda, (d_n)_{n\geq 1}, (w_n)_{n\geq 1})$ be as in Theorem 2.1, and $t \in [1/2, 5/8]$. Suppose the following estimate holds:*

$$\forall x \geq 0, \quad \frac{d\phi}{dx}(x) \geq \frac{10}{\lambda} \tag{2.2}$$

Then the orbit of t under iteration of f escapes to infinity. Moreover there exists a sequence of Euclidean discs $(U_n^t)_{n\geq 1}$, together with a subsequence of positive integers $(p_n^t)_{n\geq 1}$, so that for every $n \geq 1$:

(a) U_n^t has radius $.009(\frac{d}{dx}f^n(t))^{-1}$ with $(\frac{d}{dx}f^n(t))^{-1} \leq 50^{-n}$

(b) U_n^t is contained in the disc centered at t and of radius $20(\frac{d}{dx}f^n(t))^{-1}$

(c) $f^k(U_n^t) \subseteq S^+ \cap \mathbb{H}_+$ for every $0 \leq k \leq n-1$, and

(d) $f^n(U_n^t) \subseteq \frac{1}{4}\tilde{D}_n^t$ where $\frac{1}{4}\tilde{D}_n^t := \{z \in \mathbb{C} \,|\, |z - z_{p_n^t}| \leq 1/4\}$

Proof. Let $(x_k^t = f^k(t))_{k\geq 1}$ denote the iteration of t under f. We have the following computation:

$$\frac{d}{dx}f(x) = \frac{d}{dx}\cosh(\lambda \sinh(\phi(x))) =$$

$$\sinh(\lambda \sinh(\phi(x)))\lambda \cosh(\phi(x))\frac{d}{dx}\phi(x) \geq$$

$$\lambda\phi(x)\lambda\frac{d}{dx}\phi(x)$$

where we have used the fact that $\sinh(r) \geq r$ and $\cosh(r) \geq 1$ for $r \geq 0$. Integrating our assumption (2.2) we have that $\phi(x) - \phi(0) = \phi(x) \geq \frac{10}{\lambda}x$ so that:

$$\frac{d}{dx}f(x) \geq 100x \text{ and } f(x) \geq 50x^2 - 1 \tag{2.3}$$

In particular one may verify the orbit $(x_k^t)_{k\geq 0}$ escapes to infinity. Moreover from (2.3) one may compute that:

$$\forall k \geq 0, \;\; x_{k+1}^t - x_k^t \geq 11 \text{ and } \frac{d}{dx}f(x_k^t) \geq 100 \cdot \frac{1}{2} = 50$$

Now it follows inductively that $\frac{d}{dx}f^n(t) \geq 50^n$. We define the sequence (p_n^t) so that for every n, $|x_n^t - a_{p_n^t}|$ is minimal. (Recall a_n is the real part of a vertex along ∂S^+). We define $\tilde{D_n^t} := D_{p_n^t}$. Now we state the geometric facts that $\frac{1}{4}\tilde{D_n^t} \subseteq D(x_n^t, 5)$ and that $D(x_n^t, 10)$ does not intersect $\mathbb{D}$. So then $D(x_n^t, 10)$ doesn't contain any singular values of f. This means that f^{-1} has an injective inverse branch on $D(x_n^t, 10)$ mapping $D(x_n^t, 5)$ onto a neighborhood of x_{n-1}^t.

We would like to estimate a radius in which this neighborhood of x_{n-1}^t is contained. Recall some of Koebe's distortion estimates that for a univalent function F in the unit disc with $F(0) = 0, F'(0) = 1$, that for any $z \in \mathbb{D}$:

$$\textbf{(a)} \qquad |F(z)| \leq \frac{|z|}{(1-|z|)^2}$$

$$\textbf{(b)} \qquad \frac{1-|z|}{(1+|z|)^3} \leq |F'(z)|$$

Using (a) with

$$F(z) = \frac{f^{-1}(10z + x_n^t) - f^{-1}(x_n^t)}{10 \cdot \frac{d}{dx} f^{-1}(x_n^t)}$$

gives us that f^{-1} maps $D(x_n^t, 5)$ onto a neighborhood of x_{n-1}^t contained in a disk of radius

$$10\frac{1/2}{(1-1/2)^2}\frac{d}{dx}f^{-1}(x_n^t) \leq 20/50 \leq \pi/2$$

Hence, $\frac{1}{4}\tilde{D_n}$ has a preimage under f in $S^+ \cap D(x_{n-1}, 5)$. One may iterate this process n times to obtain a preimage of $\frac{1}{4}\tilde{D_n}$ under f^n close to $x_0^t = t$. Again one uses (a) from Koebe's theorem with the function

$$F(z) = \frac{(f^n)^{-1}(10z + x_n^t) - (f^n)^{-1}(x_n^t)}{10\frac{d}{dx}(f^n)^{-1}(x_n^t)}$$

to estimate that this n^{th} preimage is contained in a disc centered at t of radius:

$$20(\frac{d}{dx}f^n(t))^{-1} \leq \frac{20}{50^n}$$

Now one may use (b) of Koebe's theorem to prove that the n^{th} preimage of $\frac{1}{4}\tilde{D_n}$ contains a disc of radius:

$$\frac{1}{4}\cdot\frac{1}{4}\cdot\left(\frac{d}{dz}(f^n)^{-1}(z_{p_n^t})\right) \geq \frac{1}{16}\frac{1-5/10}{(1+5/10)^3}\left(\frac{d}{dx}(f^n)^{-1}(x_n^t)\right) \geq .009\left(\frac{d}{dx}f^n\,(t)\right)^{-1}$$

$\square$

Next we lift a lemma from [FGJ15] whose proof we will omit (a very similar statement Lemma 4.6 is proven in Section 4). This lemma states that there are universal parameters that satisfy the hypotheses in our previous lemma about the derivative of ϕ.

Lemma 2.3. *There exist a positive real number $\lambda^0 \in \pi\mathbb{N}^*$ and an exponentially increasing sequence $(d_n^0)_{n\geq 1}$ in $2\mathbb{N}^*$ so that for every choice of parameters $(\lambda, (d_n)_{n\geq 1}, (w_n)_{n\geq 1})$ with $\lambda \geq \lambda^0$ and $d_n \geq d_n^0$ over all n, condition (2.2) of Lemma 2.2 holds, so that the discs $(U_n^t)_{n\geq 1}$ specified in lemma 2.2 exist.*

So by taking parameters $(\lambda^0, (d_n^0)_{n\geq 1}, (w_n^0 := 1/2)_{n\geq 1})$ as in the above lemma, and *any* $t \in [1/2, 5/8]$, we have this sequence of discs U_n^t so that $f^k(U_n^t)$ stays in S^+ for $k < n$ and $f^n(U_n^t) \subseteq \frac{1}{4}\tilde{D}_n^t$. Moreover since ϕ is close to the identity, $\phi(f^n(U_n^t)) \subseteq \frac{1}{2}\tilde{D}_n^t$. So from our definitions of f in the disc components, we know $f^{n+1}(U_n^t) \subseteq D(w_{p_n^t}^0 = 1/2, (1/2)^{d_{p_n^t}})$.

What we do not have control over yet are the further iterates $f^k(U_n^t)$ for $k > n+1$. In [Bis15] and [FGJ15] the procedure taken is to further adjust f so that $f^{n+1}(U_n^t) \subseteq U_{n+1}^t$. One achieves this by carefully adjusting the critical values (w_n). In this way it is understood how all iterates of the discs U_n^t behave and indeed it is not difficult to show then that they belong to a wandering domain. In this case the singular set would be ± 1, t and a sequence converging to t. We will achieve an uncountable $\mathcal{S}(f)$ by instead repeatedly adjusting f so that $f^{n+1}(U_n^t) \subseteq U_{n+1}^{t'}$ for some $t' \in [1/2, 5/8], t' \neq t$. Namely we fix now some (any) dense sequence (t_n) in $[1/2, 5/8]$, and we will work to ensure that $f^{n+1}(U_n^{t_n}) \subseteq U_{n+1}^{t_{n+1}}$. We will need two more lemmas to do this. The first lemma enables us to choose $(d_n)_{n\geq 1}$ large enough so that $f^{n+1}(U_n^{t_n})$ is crushed sufficiently small to be able to fit into $U_{n+1}^{t_{n+1}}$ (without compromising the rest of the construction):

Lemma 2.4. *Fix $(\lambda^0, (d_n^0)_{n\geq 1}, (w_n^0 := 1/2)_{n\geq 1})$. Then there exists a sequence of positive real numbers $(r_n)_{n\geq 1}$ not depending on t so that for every new choice of parameters $(d_n)_{n\geq 1}$ with $d_n \geq d_n^0$ for every $n \geq 1$, the corresponding maps f and ϕ satisfy the condition of Lemma 2.2 as well, and*

$$\forall n \geq 1, \quad .009 \left(\frac{d}{dx} f^n(t)\right)^{-1} \geq r_n$$

In particular, we may assume that for all such parameters and all $t \in [1/2, 5/8]$, every Euclidean disc U_n^t in Lemma 2.2 has radius larger or equal than r_n, and consequently we may choose (d_n) so that for all t:

$$\left(\frac{1}{2}\right)^{d_{p_n^t}} < r_{n+1}$$

We omit the proof since it is nearly identical to that of Lemma 3.4 in [FGJ15] after observing that it suffices to establish the estimate for $t = 5/8$. We will now need to let $p_n := p_n^{t_n}$ to simplify notation (In words, p_n is the index of the disc D_{p_n} which is iterated into by f in n steps from $U_n^{t_n}$.)

We have established that we may choose the sequence (d_n) sufficiently large so that $f^{n+1}(U_n^{t_n})$ is crushed small enough to be able to fit inside $U_{n+1}^{t_{n+1}}$. Namely $f^n(U_n^{t_n})$ is contained in a disc of radius $1/4$, $\phi(U_n^{t_n})$ is contained in a disc of radius $1/2$, which is then sent inside a disc of radius $(1/2)^{d_{p_n}}$ by $\sigma \circ \eta(z) = \rho_{p_n}((z - z_{p_n})^{d_{p_n}})$, and $(1/2)^{d_{p_n}}$ is less than the radius of the next disc $U_{n+1}^{t_{n+1}}$. The last thing we have to do is adjust the critical values $(w_n)_{n\geq 1}$ so that $f^{n+1}(U_n^{t_n})$ actually does land inside $U_{n+1}^{t_{n+1}}$. Namely if we denote w_n' to be the center of $U_n^{t_n}$, we need to make sure:

$$f^{n+1}(U_n^{t_n}) \subseteq D(w_{n+1}', (1/2)^{d_{p_n}})$$

From which it will be clear by our previous work that:

$$f^{n+1}(U_n^{t_n}) \subseteq D(w_{n+1}', (1/2)^{d_{p_n}}) \subseteq D(w_{n+1}', r_{n+1}) \subseteq U_{n+1}^{t_{n+1}}$$

For this we need the following lemma:

Lemma 2.5. *There is a choice of the parameters $(\lambda, (d_n)_{n\geq 1}, (w_n)_{n\geq 1})$ satisfying the hypotheses of Lemma 2.2, so that for large enough n we have:*

$$f^{n+1}(U_n^{t_n}) \subset U_{n+1}^{t_{n+1}}$$

Proof. Let $(\lambda^0, (d_n^0)_{n\geq 1}, (w_n^0 := 1/2)_{n\geq 1})$ as in Lemma 2.4. We have that $f(U_1^{t_1}) \subseteq \frac{1}{4}\tilde{D}_1^{t_1}$. We will adjust f so that $\rho_{p_1}(0) = w_2'$ (recall that w_2' is defined to be the center of $U_2^{t_2}$) rather than $\rho_{p_1}(0) = 1/2$. We call this new function f_1. We note that there is a corresponding correction in ϕ to ensure that f_1 is still holomorphic. However, even though now we have that $f_1(\tilde{D}_1^{t_1}) \subseteq U_2^{t_2}$ it may no longer be the case that $f_1(U_1^{t_1}) \subseteq \frac{1}{4}\tilde{D}_1^{t_1}$ because of the correction in ϕ. But indeed we can fix d_{p_1} as large as we like by Lemma 2.4 so that the dilatation of ϕ is concentrated on an annulus with as small of an area as we wish. For example, choose this area so small so that:

$$\sup\{|f_1(z) - f(z)| : z \in D(t_1, 1)\} < s_1 << 1$$

Here we are using a fact we formulate more precisely at the beginning of Section 4 - namely that a quasiconformal map whose dilatation is supported on a small set is close to the identity. In any case, now we know that $f_1(U_1^{t_1}) \subseteq \frac{1}{4}\tilde{D}_1^{t_1}$ and $f_1^2(U_1^{t_1}) \subseteq U_2^{t_2}$.

Indeed we may proceed iteratively in this way. At step n we adjust ρ_{p_n} so that $\rho_{p_n}(0) = w_{n+1}'$. By choosing large enough (d_{p_n}), one may ensure that the correction of f in discs centered at $\{x_0^t, x_1^t, ..., x_{n+1}^t\}$ of radius 1 is less than $s_n << 1$, over all $t \in \{t_1, ..., t_{n+1}\}$. Doing this we can see that for $k \leq n$,

$$f^{k+1}(U_k^{t_k}) \subset U_{k+1}^{t_{k+1}}$$

We choose the sequence (s_n) so that it will sum to be less than some $\epsilon > 0$. This ensures that the limit function under this iterative procedure satisfies:

$$f^{n+1}(U_n^{t_n}) \subset U_{n+1}^{t_{n+1}}$$

over all n, as needed.

□

We conclude this section by arguing that the domains $U_n := U_n^{t_n}$ are contained in the Fatou set, and that they are contained in different Fatou components. This will establish that the function $f \in \mathcal{B}$ we have constructed above has a wandering domain. It is not so hard to see that the domains U_n are contained in the Fatou set - for any subsequence (f^{n_k}) defined on some U_n there will be a subsequence converging to infinity, or if (f^{n_k}) does not contain such a subsequence, there will be a further subsequence converging to a constant function.

Now suppose by way of contradiction that two of the domains U_{n_1}, U_{n_2} were contained in the same Fatou component, with $n_1 < n_2$. But then $f^{n_1}(U_{n_1})$ would be above the horizontal line $y = \pi$, whereas $f^{n_1}(U_{n_2})$ would be contained in S^+ (below the line $y = \pi$). Thus the Fatou component containing U_{n_1}, U_{n_2} would have to cross $y = \pi$ (which belongs to the Julia set of f since $f(y = \pi) \subset \mathbb{R}$), and this is a contradiction.

So we have constructed $f \in \mathcal{B}$ with wandering domain U and uncountable $\mathcal{S}(f)$. It is also clear that $\Lambda(U, f)$ is uncountable since $\Lambda(U, f)$ contains the accumulation set of any critical point of f. Our next section will prove that the Fatou components in this construction (or those constructions in [Bis15] or [FGJ15]) must all be bounded.

3. Boundedness of Fatou Components in Bishop's Construction

Lemma 3.1. *The Julia set of f contains the real line: $\mathbb{R} \subseteq \mathcal{J}(f)$.*

Proof. : In the Eremenko-Lyubich class $\mathcal{B}$ it is always true that the Julia set is the closure of the escaping set ([EL92]). Since $f \in \mathcal{B}$, it suffices to show that for $x > 0$, $f^n(x) \to \infty$. We show this by first assuming $x > 1$ is fixed and establishing that $\cosh(\lambda \sinh(\phi(x))) - x$ is always bigger than some fixed constant so that x iterates to infinity.

$$\cosh(\lambda \sinh(\phi(x))) - x \overset{\lambda>1}{>} \cosh(\sinh(\phi(x))) - x \overset{\sinh(x)>x}{>}$$

$$\cosh(\phi(x)) - x \overset{\phi'(x)>1}{>} \cosh(x) - x$$

Now we estimate the derivative of $\cosh(x) - x$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cosh(x) - x) = \sinh(x) - 1 > x - 1 > 0$$

This establishes that for fixed $x > 1$, $\cosh(\lambda \sinh(\phi(x))) - x$ is always bigger than some fixed constant, so $f^n(x)$ iterates to infinity, as needed. If $x > 0$, $\cosh(\lambda \sinh(\phi(x))) > 1$ which then iterates to infinity.

□

$\mathcal{J}(f)$ is forwards and backwards invariant. This will help us establish which parts of the graph belong to $\mathcal{J}(f)$. One must be careful however, since the previously pictured graph (with straight euclidean lines) is not sent to $\mathbb{R}$. Rather pieces of a quasiconformally distorted graph are sent to $\mathbb{R}$ by f.

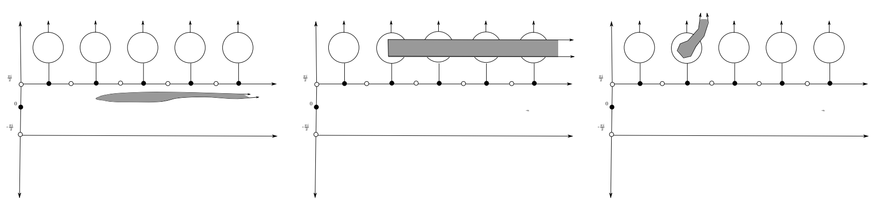
But one should keep in mind the quasiconformal distortion is uniformly close to the identity. Notice that in particular the parts of the graph which correspond to straight lines are sent to $\mathbb{R}$ by f. This means these straight lines (quasiconformally distorted) are part of the Julia set. We begin to establish that the wandering Fatou components of f must be bounded (note that this is not a proposition about the *orbits* of these Fatou components which are certainly unbounded):

Proposition 3.2. *Any wandering domain of f must be bounded (as a set).*

Proof. : Recall we have already somewhat described the wandering domains of f in Section 2 - the wandering domains contain the discs U_n and their iterates under f. [FGJ15] establishes that this wandering domain, and its reflections, are the only wandering domains for f. However *a priori* it is possible that the wandering domains are not bounded discs but rather unbounded domains containing these discs. But our previous lemma about $\mathcal{J}(f)$ tells us any such unbounded Fatou component can not cross the 'straight-line' portions of our graph (although it may cross the 'disc' portions of the graph which are not mapped into $\mathbb{R}$ by f). This leaves only a few possibilities (Illustrated in Figure 10) and we will show one-by-one that each possibility can't exist.

First of all, suppose we had an unbounded Fatou component contained inside S^+ (as in Figure 10a). We will argue that this unbounded Fatou component would eventually map to an unbounded component containing a small disk around some $a_n + i\pi$. Suppose not, namely assume we have an unbounded Fatou component $U \subset S^+$ so that $f(U)$ is bounded. Recall that in S^+, we know our map is $f(z) = \cosh(\lambda(\sinh(\phi(z))))$. There are two ways that $f(U)$ could be bounded, either (1) $\lambda(\sinh(\phi(U)))$ is vertically unbounded (i.e. has nonempty intersection with $|y| > n$ for all $n > 0$) or (2) $\lambda(\sinh(\phi(U)))$ is unbounded to the left (i.e. has nonempty intersection with $x < -n$ for all $n > 0$). But in case (1) it must then be the case that $\cosh(\lambda(\sinh(\phi(U))))$ has nonempty intersection with the real and imaginary axes (a contradiction since $\mathbb{R} \subset \mathcal{J}(f)$) and $i\mathbb{R} \subset \mathcal{J}(f)$). As for case (2), if $\lambda(\sinh(\phi(U)))$ is unbounded to the left, we know that for all $n \geq 1$, $f^n(U)$ must have nonempty intersection with any open neighborhood of 0. On the other hand, $f^n(U)$ leaves S^+ for some n, so that some iterate of such a Fatou component U would have to cross the boundary of S^+, a contradiction, since the boundary of S^+ belongs to $\mathcal{J}(f)$.

So we only need to consider the case when our unbounded Fatou component contains a small disk around $a_n + i\pi$. Here there are two possibilities. (Remember a Fatou component can't cross the 'straight-line' segments of the graph). Either the Fatou component crosses infinitely many discs D_n (Figure 10b), or the Fatou component is vertically unbounded (Figure 10c).



(A) unbounded inside S^+ (B) horizontally unbounded (C) vertically unbounded

FIGURE 10

Consider one of the vertically unbounded components V of the plane with the graph removed (illustrated in Figure 11). The construction of f relies on the Riemann map from V to the right half plane. Before arguing that cases (B) and (C) can not occur, we will need to understand how this Riemann map behaves. Notice that if we replace the two half-discs of the boundary of V with straight line segments, we can write down explicitly the Riemann map. After a rotation and translation, the Riemann map is $z \to \lambda \sinh(z)$. It is sensible therefore that this slight geometric perturbation of the boundary of V does not affect the Riemann map near infinity. We give a more rigorous argument for this.

Lemma 3.3. *Consider the Riemann map g from V to the half-strip obtained by replacing the half-discs in the boundary of V with straight line segments. Normalize g so that the corners of the half strip are fixed and infinity is fixed. Then g is uniformly close to the identity near infinity. (See Figure 12).*

Proof. : We use an argument based on the theory of harmonic measure. Assume the width of the strip is 1 and the two corners are $0, 1$. Define $w(x, y) = x$ on $\partial V \cap \mathbb{R}$, $w(x, y) = 0$ on the left portion of ∂V and $w(x, y) = 1$ on the right portion of ∂V. Let $g = u + iv$. Let w also denote the harmonic extension inside V. We first

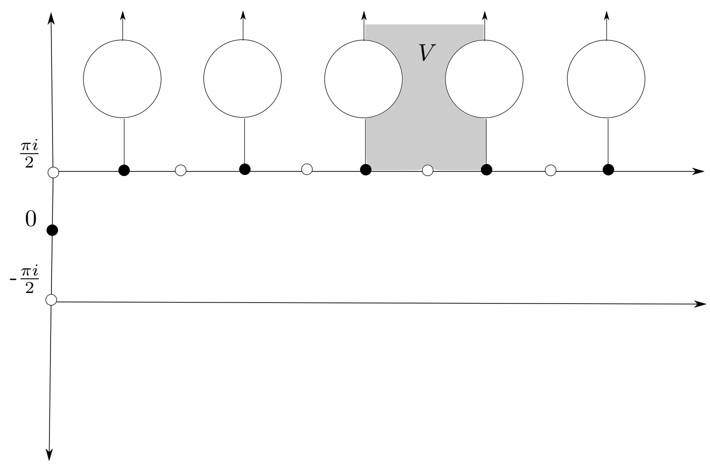


FIGURE 11. The vertically unbounded component V.

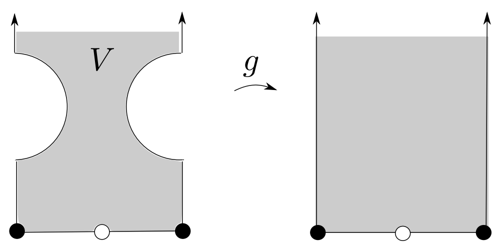


FIGURE 12. The map g from V to a half-strip.

estimate $(w-u)(z)$ for $z \in V$. We denote **bottom** as the part of the boundary of V lying on the line $y = \pi/2$, and **top** as the rest of the boundary of V. $\mathrm{d}\omega_z$ denotes harmonic measure on ∂V at the point $z \in V$.

$$|(w-u)(z)| \leq \int_{\mathbf{bottom}} |w-u|\,(\zeta)\mathrm{d}\omega_z(\zeta) + \int_{\mathbf{top}} |w-u|\,(\zeta)\mathrm{d}\omega_z(\zeta)$$

as $z \to \infty$, $\mathrm{d}\omega_z(\mathbf{bottom}) \to 0$ so that the left summand above vanishes. On the other hand the summand on the right also vanishes because w, u agree on **top**. This establishes that $u(z) \to \mathrm{real}(z)$ as $z \to \infty$. It remains to be shown that $v(z) \to \mathrm{imag}(z)$ as $z \to \infty$.

To establish this we recall the following estimate for harmonic functions h:

$$|\nabla h(z)| \leq \frac{C}{r} \int_{\mathbb{D}(z,r)} |h|$$

Now letting $h = u - w$ we have that

$$|\nabla(u-w)(z)| \leq \frac{C}{r}\int_{\mathbb{D}(z,r)} |u-w|$$

and we just finished proving that the integrand tends to zero as $z \to \infty$. Then the Cauchy-Riemann equations allow us to deduce that $v(z) \to \text{imag}(z)$ as $z \to \infty$, as needed.

□

Now we understand how our map f behaves in each region of the plane. In particular in the region V, the Riemann map discussed above is applied, followed by a rotation/dilation, followed by $z \to \cosh(\lambda \sinh(z))$ (up to a quasiconformal perturbation).

We return to our argument that the Fatou components of f must be bounded, picking up with case (B). Consider a disc D_n, and let V_n be the vertical R-component neighboring D_n and D_{n+1}. Notice that in V_n there is a preimage curve of $(+1, \infty)$ tending to infinity vertically. Moreover, there are d_n preimage segments of $[-1, 1]$ in the disc D_n, and one of these preimage segments has as one of its endpoints $a_n + i(\pi - 1)$ (the bottom of the disc D_n). Thus, by continuity, there is a preimage of $[-1, \infty)$ connecting $a_n + i(\pi - 1)$ to ∞ contained in $D_n \cup V_n$. Since any Fatou component can not cross this curve, case (B) can not happen.

What we are left to show is that a vertically unbounded Fatou component U (Figure 10c) can not exist. Again, we proceed by way of contradiction, using a hyperbolic geometry argument. We know that all forward iterates $f^n(U)$ remain in the upper half plane. This allows us to estimate the hyperbolic distance in $f^n(U)$ in terms of the hyperbolic distance in the upper half plane:

$$d_{f^n(U)}(f^n(z), f^n(w)) > d_{\mathbb{H}}(f^n(z), f^n(w)) \tag{3.1}$$

Schwarz's lemma indicates that the left hand side is bounded above by the hyperbolic distance in the Fatou component U, and so:

$$d_U(z, w) > d_{\mathbb{H}}(f^n(z), f^n(w)) \text{ over all } n \tag{3.2}$$

Our contradiction will consist of showing in fact the right hand side is not bounded, namely by showing the sequence of euclidean distances $|f^n(z) - f^n(w)|$ is unbounded. Indeed in S^+ our function f acts as $\exp \circ \exp$ which increases $|z - w|$ under iteration, and we have just finished showing that in the half strip V our function f essentially rotates/dilates V into S^+ and exponentiates.

To be more precise, we can consider $x, y \in \mathbb{R}$ with $0 < x < y$ and consider how $|x - y|$ behaves under iteration of the exponential. Let $g(x) := e^x$. Then:

$$|g(x) - g(y)| = \int_x^y e^t \mathrm{d}t > \int_x^y e^x \mathrm{d}t > e^x\,|x - y| \tag{3.3}$$

So indeed

$$|g^n(x) - g^n(y)| > e^{nx}\,|x - y| \to \infty \text{ as } n \to \infty \tag{3.4}$$

Similar estimates hold if we replace $0 < x < y$ with $z, w \in S^+$ and replace $g(z) := e^z$ with $g(z) := \cosh(\lambda \sinh(z))$ - namely the euclidean distance $|f^n(z) - f^n(w)|$ is increasing by a definite factor with n provided $z, w \in S^+$. We claim that the image $f(U) \subset S^+$ must be unbounded to the right (i.e. has nonempty intersection with $|x| > n$ over all $n > 0$). Indeed, the argument is similar to the one given in case (B) that f can not map an unbounded domain in S^+ to a bounded one. Since the map f differs in definitions on V and S^+ only by a dilation, rotation and the Riemann map g, the same arguments apply.

So we may choose $z \in \tilde{D_n} \cap U$ and $w \in V \cap U$ so that real$(f(w)) >>$ real$(f(z))$. It is clear that for the first n iterations $\left|f^k(z) - f^k(w)\right|$ increases by a definite factor. But $f^n(z), f^n(w)$ lie in a vertical half-strip. However we know that $f^{n+1}(z)$ returns near $1/2$, whereas lemma 3.3 indicates $f^{n+1}(w) \approx f^{n+1}(|w|)$, so again $\left|f^{n+1}(z) - f^{n+1}(w)\right|$ is increased by a definite factor.

Together these estimates indicate that as $k \to \infty$, $|f^k(z) - f^k(w)| \to \infty$, and this is our needed contradiction. This concludes our proof that the wandering Fatou components of f must be bounded.

$\square$

4. A Transcendental Function with Unbounded Wandering Fatou Components

Our goal in this section will be to produce a function $f \in \mathcal{B}$ with *unbounded* wandering Fatou components. This function will have finitely many critical values but infinitely many asymptotic values. But before we begin we need two results - the first is a variant of Bishop's theorem [Bis15] that we discussed in the introduction.

Suppose as before that we have some infinite tree T with alternate vertices labeled ± 1, and we denote the components of $\mathbb{C} \setminus T$ by Ω_j. Instead of mapping each Ω_j conformally to the right half-plane $\mathbb{H}_r$ by a map τ_j followed by cosh, let us map certain components Ω_j conformally to the *left*-half plane $\mathbb{H}_l$ followed by the exponential map onto $\mathbb{D}$, and then a quasiconformal self-map of $\mathbb{D}$ shifting the asymptotic value 0. This procedure is illustrated in Figure 13.

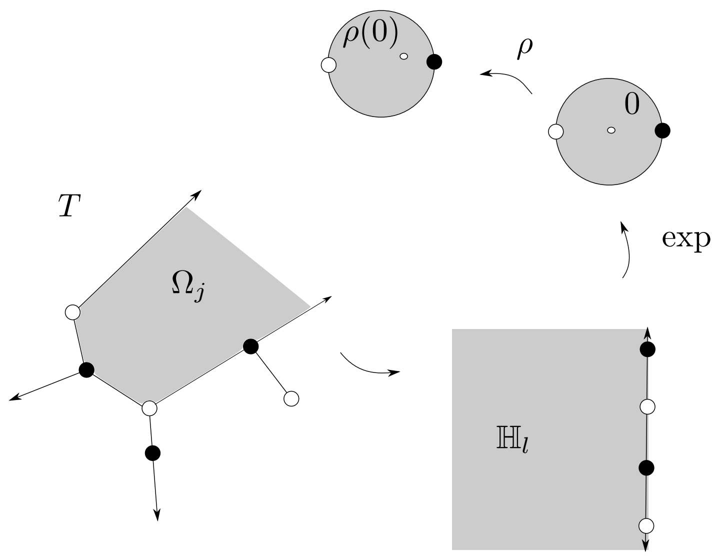


Figure 13. The map f on an L-component.

These components that are mapped to $\mathbb{H}_l$ will be called *L-components* and those that are mapped to $\mathbb{H}_r$ as before are called R-components. The L-components play the role of the D-components in the previously stated version of the theorem. With this we can state the version of theorem 7.1 from [Bis15] we need:

Theorem 4.1. *Let T be an unbounded connected graph and let τ be a conformal map defined on each complementary domain $\mathbb{C} \setminus T$ as above. Assume that:*

(i) No two L-components of $\mathbb{C} \setminus T$ share a common edge.
(ii) T is bipartite with uniformly bounded geometry.
(iii) The map τ on a L-component maps edges to intervals of length 2π on $\mathbb{H}_l$ with vertices in $2\pi i\mathbb{Z}$

(iv) On R-components the τ-sizes of all edges is uniformly bounded from below.

Then there is an $r_0 > 0$, a transcendental f, and a K-quasiconformal map ϕ of the plane, with K depending only on the uniformly bounded geometry constants, so that $f = \sigma \circ \tau \circ \phi$ off $T(r_0)$. Moreover the only singular values of f are the critical values ± 1 - corresponding to the vertices of T, and those asymptotic values assigned by the L-components.

The second result we need before beginning our construction is a precise formulation of the fact (we have already used) that if the dilatation of a quasiconformal map ϕ is supported on a 'small set', then ϕ should be close to the identity. First we formulate precisely what we mean by a 'small set'.

Definition 4.2. A measurable set $E \subseteq \mathbb{R}^2$ is said to be (ϵ, h) thin if

$$\text{area}(E \cap D(z,1)) \leq \epsilon \cdot h(|z|)$$

over all $z \in \mathbb{C}$, where $h : [0,\infty) \to [0,\pi^2]$ is a decreasing function so that:

$$\int_0^\infty h(r) r^n \mathrm{d}r < \infty$$

over all n.

For our purposes we will be able to take $\epsilon = 1, h = \exp$. This definition is used in the following result:

Lemma 4.3. (Bishop, personal communication) *Suppose $\phi : \mathbb{C} \to \mathbb{C}$ is K-quasiconformal and ϕ fixes $0, 1$, and $\phi(\mathbb{R}) \subseteq \mathbb{R}$. Furthermore suppose ϕ is conformal in the strip $\{x + iy : |y| < \pi/2\}$. Let $E = \{z : \mu(z) \neq 0\}$ where μ is the dilatation of ϕ and suppose E is (ϵ, h)-thin. If ϵ is sufficiently small (depending on K, h), then:*

$$\frac{1}{C} \leq |f'(x)| \leq C$$

for all $x \in \mathbb{R}$ where C depends on K, h and ϵ is otherwise independent of f. If we fix K, h and let $\epsilon \to 0$, then $C \to 1$.

We return to the task of producing a function $f \in \mathcal{B}$ with unbounded wandering Fatou components. Our strategy will be to apply Bishop's theorem to a tree that we now construct. We start with the region S^+ as before:

$$S^+ := \{x + iy \in \mathbb{C} : x > 0, |y| < \pi/2\}$$

that is mapped conformally to $\mathbb{H}_r$ by $z \to \lambda \sinh(z)$, and then holomorphically to $\mathbb{C} \setminus [-1, 1]$ by cosh as illustrated in Figure 6. The vertices on this strip are defined in Section 1. In particular we still have the vertices $(a_n \pm i\pi/2)$ where:

$$n\pi - 10^{-1} < a_n \leq n\pi$$

But we will replace the regions D_n by half-strips H_n:

$$H_n := \{x + iy \in \mathbb{C} : a_n - 1 < |x| < a_n + 1 \text{ and } |y| > \pi - 1\}$$

The domains H_n are just rotations/translations/scalings of the domain S^+. So in particular we already have vertices defined on H_n. Moreover H_n is mapped conformally to $\mathbb{H}_r$ by $z \to \lambda_n \sinh(\text{similarity}(z))$ where similarity(z) is a rotation/translation/scale, and $\lambda_n > 0$. But in fact we will want the H_n domains to play the role of L-components in applying Bishop's theorem. So we use instead the fact that H_n is mapped conformally to the *left* half-plane $\mathbb{H}_l$ by $z \to -\lambda_n \sinh(\text{similarity}(z))$. Then by post-composing with the exponential, H_n is mapped inside the unit disc $\mathbb{D}$. This procedure is illustrated in Figure 14.

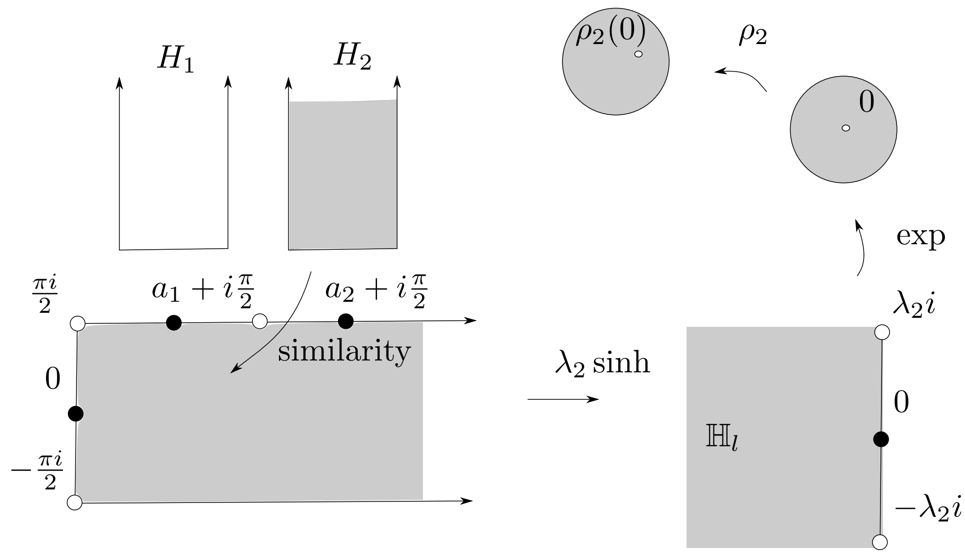


FIGURE 14. The map f on an L-component in our construction.

Exactly as in Section 1 we will further postcompose with a quasiconformal map ρ_n of the unit disc. We fix these quasiconformal maps ρ_n so that they fix the boundary of $\mathbb{D}$ and $\rho_n(0) = w_n$ where w_n is a parameter to be fixed later in a neighborhood $\mathcal{N}_{1/2}$ of $1/2$. Furthermore we ensure ρ_n is conformal in $\frac{3}{4}\mathbb{D}$ and ρ_n is K_ρ-quasiconformal where K_ρ does not depend on n.

We let $\sigma \circ \tau|_{H_n}$ denote the composition $z \to \rho_n(\exp(-\lambda_n \sinh(\text{similarity}(z))))$ mapping H_n to $\mathbb{D}$. Notice that this map has the asymptotic value $\rho_n(0)$ coming from applying $\sigma \circ \tau$ to the curve $\gamma(t) = a_n + i\pi t$ approaching ∞ in H_n. It is also important to note that the dilatation of the map $\sigma \circ \tau$ is supported on smaller sets with increasing λ_n.

Lastly we construct vertical segments connecting H_n to S^+ and a vertical segment connecting $i\pi/2$ to ∞. These are the same segments constructed in Section

1 and have the same vertices. This whole construction is reflected in the real and imaginary axes to produce the tree pictured in Figure 15.

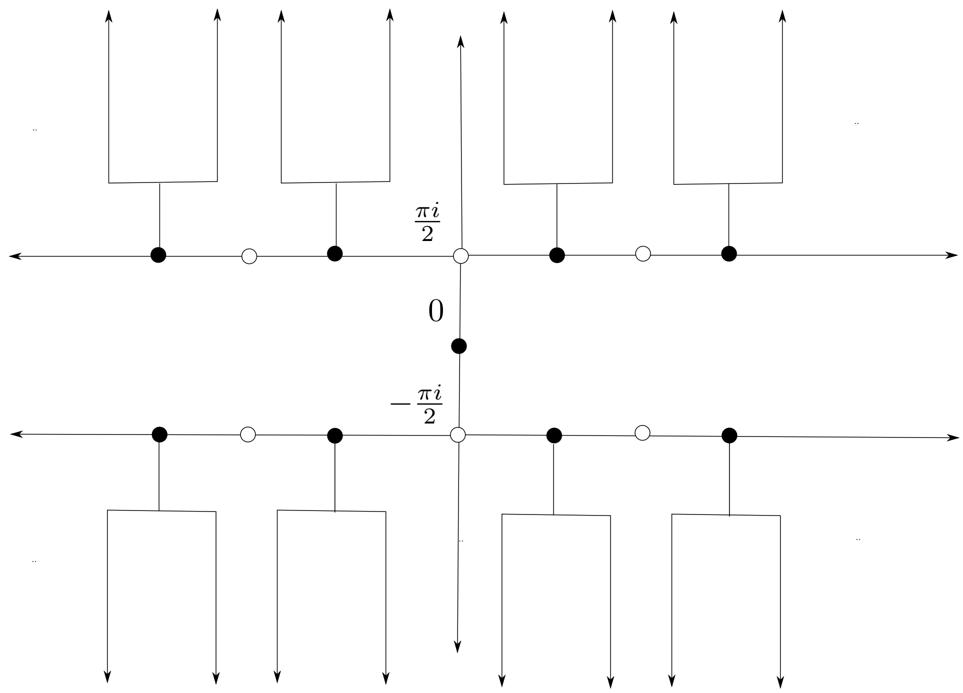


FIGURE 15. The graph to which we apply Bishop's folding theorem.

We invoke Bishop's theorem to produce an entire function that extends the above definitions on S^+ and H_n up to a quasiconformal perturbation:

Theorem 4.4. *For every choice of the parameters* $(\lambda, (\lambda_n)_{n\geq 1}, (w_n)_{n\geq 1})$ *so that* $\lambda, \lambda_n \in \pi N^*, w_n \in \mathcal{N}_{1/2}$ *over all* n*, there exists a transcendental entire function* f *and a quasiconformal map* $\phi : \mathbb{C} \to \mathbb{C}$ *so that:*

(a) for every $z \in \mathbb{C}$, $f(\overline{z}) = \overline{f(z)}$ *and* $f(-z) = f(z)$*;*

(b) $f \circ \phi^{-1}$ *extends the maps* $(\sigma \circ \tau)|_{S^+}$ *and* $(\sigma \circ \tau)|_{H_n}$ *for every* $n \geq 1$*:*

$$f(z) = \begin{cases} \cosh(\lambda \sinh(\phi(z))) & \text{if } \phi(z) \in S^+ \\ \rho_n(\exp(\sinh(\text{similarity}(z))) & \text{if } \phi(z) \in H_n \end{cases} \tag{4.1}$$

(c) f *has two critical values* ± 1*; and its set of asymptotic values is* $\{w_n : n \geq 1\}$ *(hence* f *is in class* $\mathcal{B}$*).*

(d) $\phi(0) = 0, \phi(\mathbb{R}) = \mathbb{R}$, ϕ *is conformal in* S^+ *and its dilatation is uniformly bounded above by a universal constant* $K > 1$ *which does not depend on the parameters.*

The proof of this theorem would consist of verifying that this tree satisfies the uniformly bounded geometry conditions. But indeed, this work has already been done - this tree is obtained by copying-and-pasting various parts of the graph in Section 2.

Our strategy will be to choose parameters $(\lambda, (\lambda_n)_{n\geq 1}, (w_n)_{n\geq 1})$ so that we can guarantee there will be a wandering domain, and we will mostly follow the logic given in Section 1. Let's talk informally about our strategy before we begin any proof. Our wandering domains will contain thin strips in a subsequence of the domains H_n.

Each of these thin strips is mapped to a bounded domain by $\sigma \circ \tau$ near 1/2 inside $\mathbb{D}$. Moreover, there is the disc D_n in each thin strip that has an n^{th} preimage near 1/2 under $\sigma \circ \tau$. This is illustrated in Figure 16.

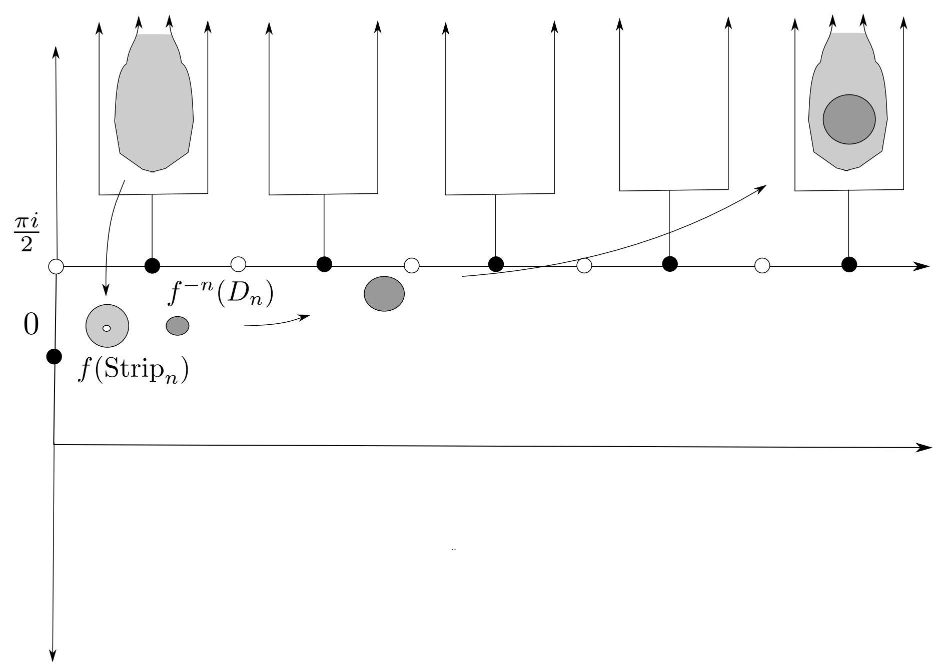


FIGURE 16. The map f before carefully choosing the parameters.

As in Section 2, we need to be able to do two things:

(a) shrink the image $f(\text{Strip}_n)$ so that it is small enough to fit inside $f^{-n}(D_n)$
(b) perturb $f(\text{Strip}_n)$ so that it actually lies inside $f^{-n}(D_n)$.

(a) is accomplished by choosing the parameters (λ_n) sufficiently large - notice that as $\lambda_n \to \infty$, the image of the strip shrinks to the point $\rho_n(0)$. So in fact (λ_n) plays the role of the exponential powers (d_n) in Section 2. On the other hand **(b)** is accomplished by adjusting the values $\rho_n(0)$ as in Section 2. And as in Section 2, the difficulty is that these choices are interdependent.

We proceed more rigorously with the following adaptation of Lemma 2.2 whose proof remains unchanged in our current setting by taking $t = 1/2$:

Lemma 4.5. *Let $f, \phi, (\lambda, (\lambda_n)_{n\geq 1}, (w_n)_{n\geq 1})$ as in Theorem 3.1. If the following estimates hold:*

$$\forall x \geq 0, \quad \frac{d\phi}{dx}(x) \geq \frac{10}{\lambda} \tag{4.2}$$

Then the orbit of $1/2$ under iteration of f escapes to infinity, and there exists a sequence of Euclidean discs $(U_n)_{n\geq 1}$, together with a subsequence of positive integers $(p_n)_{n\geq 1}$ so that for every $n \geq 1$:

(a) U_n has radius $.009(\frac{d}{dx}f^n(1/2))^{-1}$ with $(\frac{d}{dx}f^n(1/2))^{-1} \leq 50^{-n}$

(b) U_n is contained in the disc centered at $1/2$ and of radius $20(\frac{d}{dx}f^n(t))^{-1}$

(c) $f^k(U_n) \subseteq S^+ \cap \mathbb{H}_+$ for every $0 \leq k \leq n-1$, and

(d) $f^n(U_n) \subseteq \frac{1}{4}\tilde{D}_n \subseteq D_n \subseteq H_n$ where $\frac{1}{4}\tilde{D}_n := \{z \in \mathbb{C} | |z - z_{p_n}| \leq 1/4\}$

Next we prove the analogue of Lemma 2.3 establishing that there are universal parameters satisfying the hypotheses of the previous lemma:

Lemma 4.6. *There exists a positive real number $\lambda^0 \in \pi\mathbb{N}^*$ and an increasing sequence (λ_n^0) in $\pi\mathbb{N}^*$ so that for every choice of parameters $(\lambda, (\lambda_n)_{n\geq 1}, (w_n)_{n\geq 1})$ with $\lambda \geq \lambda^0, \lambda_n \geq \lambda_n^0$ condition (4.2) of Lemma 4.2 holds, and hence the euclidean disks U_n exist.*

Proof. This is a consequence of Lemma 4.3. Notice that as $\lambda, (\lambda_n)$ increase, the vertices in the tree move closer together, so that the neighborhood $T(r)$ where the quasiconformal folding takes place converges to the tree T in the Hausdorff metric. This means that we may take $\lambda^0, (\lambda_n^0)$ large enough so that $T(r)$ is $(1, h(x) = \exp(-x))$ thin. Lemma 4.3 then applies to tell us that there is a constant C so that $\phi'(x) > 1/C$. By taking λ^0 larger than $\max\{10C, \lambda^0\}$, we are assured that condition (4.2) of Lemma 4.5 holds. □

We now define the strips $\text{Strip}_n \subseteq H_n$ that we discussed informally at the beginning of this section. The domains Strip_n will turn out to be contained in our unbounded wandering domains. We choose any such unbounded domain Strip_n that contains $\frac{1}{2}\mathbb{D}_n$ and so that

$$-\lambda_n(\sinh(\text{similarity}(\text{Strip}_n))) \cap \{x + iy \in \mathbb{C} : x > -n\}$$

is bounded for all $n \in \mathbb{N}$. This means that $-\lambda_n(\sinh(\text{similarity}(\text{Strip}_n)))$ is horizontally unbounded but not vertically. Recall that similarity(z) here is a similarity from H_n to S^+. We are ensured then that

$$\text{as } \lambda_n \to \infty, \ \operatorname{diam}(\exp(-\lambda_n(\sinh(\text{similarity}(\text{Strip}_n))))) \to 0.$$

Let us consider a choice of parameters $(\lambda^0, (\lambda_n^0)_{n\geq 1}, (w_n := 1/2)_{n\geq 1})$ coming from Lemma 4.6. Lemma 4.5 applies then to tell us there exist these discs U_n near $1/2$ so that $f^k(U_n) \subseteq S^+$ for $1 \leq k \leq n-1$, and $f^n(U_n) \subseteq \frac{1}{4}\tilde{D}_n \subseteq H_{p_n}$. In fact since ϕ is close to the identity (we can ensure this by perhaps taking λ_n^0 larger) we know $\phi(f^n(U_n)) \subseteq \frac{1}{2}D_n$, and so $f^{n+1}(U_n) \subseteq f(\text{Strip}_{p_n}) \subseteq D(w_{p_n}^0 = 1/2, r)$ where the radius r tends to zero as λ_n^0 tends to ∞. We need to ensure we can take λ_n^0 large enough so that this radius r is less than the radius of the next disc U_{n+1} without compromising the rest of the construction. This is the purpose of the next lemma which establishes universal lower bounds on the radii of U_{n+1}.

Lemma 4.7. *Fix the parameters $(\lambda^0, (\lambda_n^0)_{n\geq 1}, (w_n := 1/2)_{n\geq 1})$. Then there exists a sequence of positive real numbers (r_n) so that for every new choice of parameters $(\lambda_n) \geq (\lambda_n^0)$ the corresponding maps f, ϕ satisfy equation (4.2) and:*

$$\forall n \geq 1, \ .009\left(\frac{d}{dx}f^n\left(\frac{1}{2}\right)\right)^{-1} \geq r_n$$

In particular, we may assume that for all such parameters, every Euclidean disk U_n in Lemma 4.2 has radius larger than or equal to r_n, and consequently we may choose λ_n so that the image $f(\text{Strip}_{p_n})$ has diameter less than r_{n+1}.

Proof. For every $n \geq 1$, let $(\lambda_n^j)_{j\geq 1}$ denote any sequence of increasing sequences of positive integers with $(\lambda_n^j) \geq (\lambda_n^0)$ for all $j \geq 1$. For each fixed j, the sequence $(\lambda_n^j)_{n\geq 1}$ yields a quasiconformal ϕ_j according to Theorem 4.4. But notice that the dilatation constant of ϕ_j does not depend on j, also by Theorem 4.4. This means by compactness that there is a subsequence ϕ_{j_l} converging to some ϕ in compact subsets of $\mathbb{C}$. And since $f_j(z) = (\cosh(\lambda \sinh(\phi_j(z))))$ in S^+, we know that (f_{j_l}) converges in compact subsets of S^+ to $f(z) = (\cosh(\lambda \sinh(\phi(z))))$. This means that the radii $r_n = \left(\frac{d}{dx}f^n\left(\frac{1}{2}\right)\right)^{-1}$ of the discs U_n must have a positive lower bound for each fixed $n \geq 1$.

□

Now we have to adjust the quasiconformal maps ρ_n so that the image $f(\text{Strip}_{p_n})$ actually lands inside the next disc U_{n+1}. We let w_n' denote the center of the disc U_n. So namely we need to adjust f so that:

$$f(\text{Strip}_{p_n}) \subset D(w_{n+1}', r_{n+1})$$

From which we will deduce by the above lemma that

$$f^{n+1}(U_n) \subset f(\text{Strip}_{p_n}) \subset D(w_{n+1}', r_{n+1}) \subset U_{n+1}$$

We do so in the following lemma:

Lemma 4.8. *There is a choice of the parameters $(\lambda, (\lambda_n)_{n\geq 1}, (w_n)_{n\geq 1})$ satisfying the hypotheses of Lemma 4.2, so that for large enough n we have:*

$$f^{n+1}(U_n) \subset f(\text{Strip}_{p_n}) \subset U_{n+1}$$

Proof. Let $(\lambda^0, (\lambda_n^0)_{n\geq 1}, (w_n^0 := 1/2)_{n\geq 1})$ as in Lemma 2.4. We have that $f(U_1) \subseteq \frac{1}{4}\tilde{D}_1$. We will adjust f so that $\rho_{p_1}(0) = w_2'$ (recall that w_2' is defined to be the center of U_2) rather than $\rho_{p_1}(0) = 1/2$. We call this new function f_1. We note that there is a corresponding correction in ϕ to ensure that f_1 is still holomorphic. However even though now we have that $f_1(\mathrm{Strip}_{p_1}) \subseteq U_2$ it may no longer be the case that $f_1(U_1) \subseteq \frac{1}{4}\tilde{D}_1$ because of the correction in ϕ. But indeed we can fix λ_{p_1} as large as we like by Lemma 2.4 so that the dilatation of ϕ is concentrated on a region with as small of an area as we wish (small in the sense of definition 4.2). For example choose this area so small that:

$$\sup\{|f_1(z) - f(z)| : z \in D(1/2, 1)\} < s_1 << 1$$

Now we know that $f_1(U_1) \subseteq \frac{1}{4}\tilde{D}_1$ and $f_1^2(U_1) \subseteq f(\mathrm{Strip}_{p_n}) \subseteq U_2$.

Indeed we may proceed iteratively in this way. At step n we adjust ρ_{p_n} so that $\rho_{p_n}(0) = w_{n+1}'$. By choosing large enough (λ_n), one may ensure that the correction of f in discs centered at $\{1/2, f(1/2), ..., f^n(1/2)\}$ of radius 1 is less than $s_n << 1$. Doing this we can see that for $k \leq n$,

$$f^{k+1}(U_k) \subset f(\mathrm{Strip}_{p_k}) \subseteq U_{k+1}$$

We choose the sequence (s_n) so that it will sum to be less than some $\epsilon > 0$. This ensures that the limit function under this iterative procedure satisfies:

$$f^{n+1}(U_n) \subset f(\mathrm{Strip}_n) \subset U_{n+1}$$

over all n, as needed.

□

We remark that the domains Strip_{p_n} are contained in the Fatou set, and that they are contained in different Fatou components. Indeed the reasoning is nearly identical to the analogous proof given in Section 2 so we omit it. This establishes that the function $f \in \mathcal{B}$ we have constructed above has unbounded wandering domains.

We also would like to remark that according to [Her98] the image of a Fatou component can miss at most one point in the image Fatou component. This means that the strips Strip_{p_n} are contained in a Fatou component but do not comprise the entire Fatou component. Let U_{p_n} be the Fatou component containing Strip_{p_n}. Indeed Strip_{p_n} maps to only a bounded subset of $\mathrm{Strip}_{p_{n+1}}$, so that there must be an unbounded preimage of $\mathrm{Strip}_{p_{n+1}}$ inside U_{p_n}. In fact U_{p_n} must contain an unbounded preimage of Strip_{p_k} for each $k > n$.

Lastly, a word about why this answers question 3 of [OS16]. Namely we claim that for this function f there is an unbounded wandering domain in $BU(f)$, all of whose iterates are unbounded. Indeed it is clear that the Fatou components U_{p_n} are unbounded since they contain the unbounded sets Strip_{p_n}. Moreover since $\mathrm{Strip}_{p_n} \subset BU(f)$, by Theorem 1.1 of [OS16] it is true that $U_{p_n} \subset BU(f)$. Also by the aforementioned result of [Her98] we know that $f(U_{p_n})$ can miss at most one point of the image Fatou component. It follows then that the function f we have constructed answers question 3 of [OS16].

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