Extreme values of some continuous nowhere differentiable functions

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Abstract

We consider the functions $T_n(x)$ defined as the n-th partial derivative of Lebesgue's singular function $L_a(x)$ with respect to a at $a = \frac{1}{2}$. This sequence includes a multiple of the Takagi function as the case $n = 1$. We show that T_n is continuous but nowhere differentiable for each n, and determine the Hölder order of T_n. From this, we derive that the Hausdorff dimension of the graph of T_n is one. Using a formula of Lomnicki and Ulam, we obtain an arithmetic expression for $T_n(x)$ using the binary expansion of x, and use this to find the sets of points where T_2 and T_3 take on their absolute maximum and minimum values. We show that these sets are topological Cantor sets. In addition, we characterize the sets of local maximum and minimum points of T_2 and T_3.

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1. *Introduction*

Let $L_a(x)$ be Lebesgue's singular function with a real parameter a $(0 < a < 1, a \neq \frac{1}{2})$. As is well known, $L_a(x)$ is strictly increasing and has a derivative equal to zero almost everywhere. In 1991, Sekiguchi and Shiota [**14**] proved that $L_a(x)$ is an analytic function with respect to a for each fixed x in $[0, 1]$, and studied the functions

$$T_{r,n}(x) := \frac{1}{n!} \left.\frac{\partial^n L_a(x)}{\partial a^n}\right|_{a=r}, \qquad n = 1, 2, 3, \ldots, \quad 0 < r < 1.$$

In this paper, we consider the case $r = \frac{1}{2}$ and define

$$T_n(x) := \frac{1}{n!} \left.\frac{\partial^n L_a(x)}{\partial a^n}\right|_{a=\frac{1}{2}}, \qquad n = 1, 2, 3, \ldots. \tag{1·1}$$

Figure 1 shows the graph of T_n, for $n = 1, 2, 3$ and 4. We will show that the T_n are continuous and nowhere differentiable functions whose graphs have Hausdorff dimension

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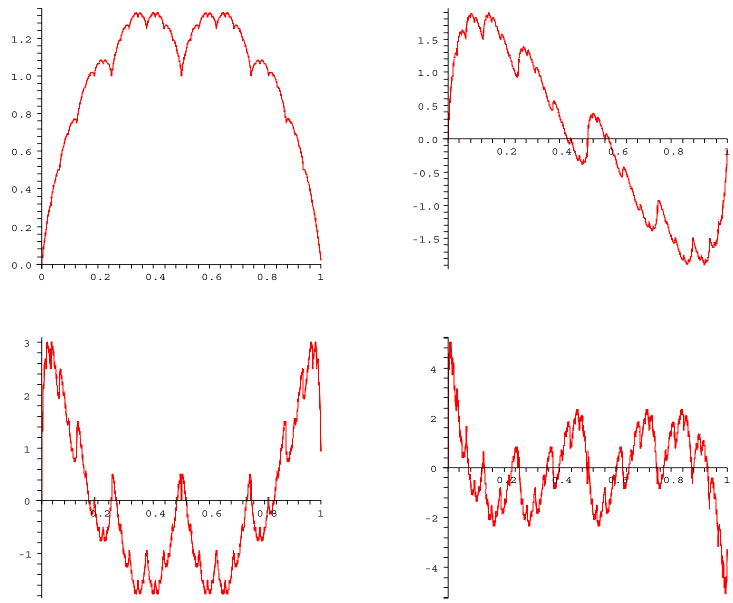

Fig. 1. Graphs of T_1 (top left), T_2 (top right), T_3 (bottom left) and T_4 (bottom right).

one. Our main purpose, however, is to study the sets of points at which T_n takes on its maximum and minimum values.

The functions T_n are more than a mathematical curiosity. For example, it was shown by Hata and Yamaguti [**2**] that T_1 is two times the Takagi function, which is known to have several applications in physics. (See, for instance, Tasaki, Antoniou and Suchanecki [**16**].) Moreover, Okada, Sekiguchi and Shiota [**11**, **12**] showed that the relationship between $T_{r,n}$ and L_a has interesting applications to the binary digital sum, the power sum, and the exponential sum problems in number theory. Lastly, Kawamura [**5**] showed a close relationship between the coordinate functions of Lévy dragon curve and T_n.

The Takagi function, introduced first by Takagi in 1903 [**15**], is one of the simplest examples of a continuous, nowhere differentiable function. It is given by

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{2^k}\psi(2^k x), \qquad 0 \le x \le 1, \tag{1·2}$$

where $\psi(x) = |x - \lfloor x + \frac{1}{2} \rfloor|$. The Takagi function was rediscovered independently by other mathematicians, e.g. Knopp in 1918, Hobson in 1926, Van der Waerden in 1930, Hildebrandt in 1933, and De Rham in 1957. It is known alternatively as the Van der Waerden function, or the Takagi-Van der Waerden function.

Several authors have studied properties of the Takagi function. For example, in 1959, Kahane [**4**] showed that the set of points at which the Takagi function attains its maximum value is a Cantor set, and the maximum value is $\frac{2}{3}$. In 1986, Mauldin and Williams [**10**] introduced a new geometric property of a function: convex Lipschitz of order θ, and used this to show, among other things, that the graph of the Takagi function has Hausdorff dimension one. Kairies [**3**] gave several functional equations which the Takagi function satisfies, and discussed their relationship. A more general class of functions, called the *Takagi class*, was introduced by Hata and Yamaguti [**2**]. A function f belongs to the Takagi class if it satisfies a system of infinitely many difference equations of the form

$$f\left(\frac{2j+1}{2^{k+1}}\right) - \frac{1}{2}\left\{f\left(\frac{j}{2^k}\right) + f\left(\frac{j+1}{2^k}\right)\right\} = C_k, \tag{1·3}$$

for $0 \le j \le 2^k - 1$, $k = 0, 1, 2, \cdots$, with the boundary conditions $f(0) = f(1) = 0$. Here, $\{C_k\}$ is any given numerical sequence. When $C_k = 2^{-(k+1)}$, f is the Takagi function. Hata and Yamaguti proved that (1·3) has a unique continuous solution if and only if $\sum_{k=0}^{\infty} |C_k| < \infty$. Kono [**8**] investigated the regularity and the differentiability of functions of the Takagi class, concluding that f has no finite derivative at any point if $\limsup_{k\to\infty} 2^k |C_k| > 0$.

The organization of this article is as follows. Section 2 gives the functional equations and a system of infinitely many difference equations having T_n as a solution. These equations show that the T_n are not self-affine in the sense of Kono [**7**], and do not belong to the Takagi class when $n \ge 2$. Section 3 gives an arithmetic expression for $T_n(x)$ using the binary expansion of x. This representation is the key to many of the later results of the paper. In section 4, we show that

$$T_n(x+y) - T_n(x) = O(|y|(\log(1/|y|))^n) \qquad \text{as} \quad y \to 0,$$

from which it follows that the Hausdorff dimension of the graph of T_n is one. In section 5, we prove that the T_n are nowhere differentiable.

Section 6 treats the problem of finding maximum and minimum points of T_n. It begins by reviewing Kahane's result on the maximum points of the Takagi function, which is one half times T_1. Not surprisingly, the problem becomes more difficult as n increases. However, using the arithmetic expression from section 3, we can - at least in principle - find the extremal points of T_n by studying the roots and local extrema of a particular sequence of functions, made up of an exponential factor and a polynomial factor of degree n. Unfortunately, this analysis is feasible only for $n = 2$ or 3, as no simple expressions are available for the roots of the polynomial when $n \ge 4$. We find that both the sets of maximum points and the sets of minimum points of T_2 and T_3 are topological Cantor sets, and are therefore uncountably large. (By contrast, we conjecture that the sets of maximum and minimum points of T_n are finite when $n \ge 4$.) Specifically, the maximum points of T_2 are exactly those points of the form

$$x = .00 \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} 010 \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} 01010 \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} 0101010 \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} 010101010 \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} \cdots,$$

while the minimum points are obtained by interchanging zeros and ones in the above pattern. Similarly, the maximum points of T_3 lying in the interval $[0, \frac{1}{2}]$ are exactly those points of the form

$$x = .0000 \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} 0 \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} \underbrace{010}\,\underbrace{010} \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} \underbrace{010} \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} \underbrace{01010}\,\underbrace{01010} \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} \underbrace{01010} \begin{cases} 01 \\ \text{or} \\ 10 \end{cases}$$

$$\underbrace{0101010}\,\underbrace{0101010} \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} \underbrace{0101010} \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} \underbrace{010101010}\,\underbrace{010101010} \begin{cases} 01 \\ \text{or} \\ 10 \end{cases} \cdots .$$

Interestingly, the set of minimum points of T_3 is found to coincide with the set of maximum points of T_1. The graphs in Figure 1 illustrate this coincidence.

In subsection 6·4, we introduce a general procedure for constructing a point x with a "large" value of $T_n(x)$. This algorithm, which places every next "1" in the binary expansion of x so as to achieve the greatest immediate increase in the value of $T_n(x)$, will be called the *max-greedy algorithm*. A dual version, the *min-greedy algorithm*, produces a "small" value of $T_n(x)$. We show that these greedy algorithms actually attain the maximum and minimum values of T_2 and T_3. Numerical evidence suggests that the greedy algorithms continue to be optimal for larger values of n, but we have not been able to prove this.

In subsection 6·5, we consider the local extrema of $T_n(x)$. We show how a dense set of local extreme points can be obtained from any global extreme point, and give complete characterizations of the local maximum and minimum points of T_2 and T_3. The paper ends with a discussion of some unsolved problems and conjectures.

2. *Functional equations*

First, we derive functional equations for the T_n. In 1983, Yamaguti and Hata [**18**] proved the following general theorem.

THEOREM 2·1 (Yamaguti-Hata, 1983). *Let $(t,x) \in (-1,1) \times [0,1]$, $\psi : [0,1] \to [0,1]$ and $g : [0,1] \to \mathbf{R}$. The functional equation $F(t,x) = tF(t,\psi(x)) + g(x)$ has a unique bounded solution $F(t,x)$, which is given by $F(t,x) = \sum_{n=0}^{\infty} t^n g(\psi^n(x))$.*

As an example of this theorem, Yamaguti and Hata showed that $T_1(x)$ is the unique bounded solution of the following functional equation:

$$T_1(x) = \begin{cases} \frac{1}{2}T_1(2x) + 2x, & 0 \le x \le \frac{1}{2}, \\ \frac{1}{2}T_1(2x-1) + 2(1-x), & \frac{1}{2} \le x \le 1. \end{cases} \tag{2·1}$$

It was shown by De Rham [**13**] that $L_a(x)$ is the unique continuous solution of the functional equation

$$L_a(x) = \begin{cases} aL_a(2x), & 0 \le x \le \frac{1}{2}, \\ (1-a)L_a(2x-1) + a, & \frac{1}{2} \le x \le 1. \end{cases} \tag{2·2}$$

Differentiating this n times with respect to a gives

$$\frac{\partial^n L_a(x)}{\partial a^n} = \begin{cases} a\frac{\partial^n L_a(2x)}{\partial a^n} + n\frac{\partial^{n-1} L_a(2x)}{\partial a^{n-1}}, & 0 \le x < \frac{1}{2}, \\ \\ (1-a)\frac{\partial^n L_a(2x-1)}{\partial a^n} - n\frac{\partial^{n-1} L_a(2x-1)}{\partial a^{n-1}}, & \frac{1}{2} \le x \le 1, \end{cases}$$

for $n = 2, 3, \cdots$. Thus, by Definition 1·1 we have the following result.

PROPOSITION 2·2. *For each $n \geq 2$, $T_n(x)$ is the unique bounded solution, given the function T_{n-1}, of the following functional equation.*

$$T_n(x) = \begin{cases} \frac{1}{2}T_n(2x) + T_{n-1}(2x), & 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{2}T_n(2x-1) - T_{n-1}(2x-1), & \frac{1}{2} \leq x \leq 1. \end{cases} \tag{2·3}$$

From the above equation, we can see that the graphs of the T_n have the following symmetry properties.

COROLLARY 2·3. *For each $n \in \mathbf{N}$ and $x \in [0,1]$, $T_n(x) = (-1)^{n-1}T_n(1-x)$.*

COROLLARY 2·4. *For each $n \geq 2$, $T_n(x)$ satisfies the following functional equations.*

$$T_n\left(\frac{x}{2}\right) + T_n\left(\frac{x+1}{2}\right) = T_n(x), \qquad 0 \leq x \leq 1, \tag{2·4}$$

$$T_n\left(\frac{x}{2}\right) - T_n\left(\frac{x+1}{2}\right) = 2T_{n-1}(x), \qquad 0 \leq x \leq 1. \tag{2·5}$$

Conversely, (2·4) *and* (2·5) *uniquely determine T_n, given T_{n-1}.*

The functional equation (2·1) yields the following system of infinitely many difference equations having $T_1(x)$ as a unique bounded solution:

$$T_1\left(\frac{2j+1}{2^{k+1}}\right) - \frac{1}{2}\left\{T_1\left(\frac{j}{2^k}\right) + T_1\left(\frac{j+1}{2^k}\right)\right\} = 2^{-k}, \tag{2·6}$$

for $0 \leq j \leq 2^k - 1$, $k = 0, 1, 2, \cdots$, with the boundary conditions $T_1(0) = T_1(1) = 0$. By the remark following (1·3), this implies that T_1 is twice the Takagi function.

Similarly, from the functional equation (2·3), we have

COROLLARY 2·5. *If $n \geq 2$, then $T_n(x)$ satisfies the following system of infinitely many difference equations.*

$$T_n\left(\frac{2j+1}{2^{k+1}}\right) - \frac{1}{2}\left\{T_n\left(\frac{j}{2^k}\right) + T_n\left(\frac{j+1}{2^k}\right)\right\} = T_{n-1}\left(\frac{j+1}{2^k}\right) - T_{n-1}\left(\frac{j}{2^k}\right), \tag{2·7}$$

for $0 \leq j \leq 2^k - 1$, $k = 0, 1, 2, \cdots$, with the boundary conditions $T_n(0) = T_n(1) = 0$.

Proof. From (2·1) and (2·3), we derive that $T_n(0) = T_n(1) = 0$ for all $n \in \mathbf{N}$. From (2·3), it follows that

$$T_n(\tfrac{1}{2}) - \tfrac{1}{2}\{T_n(0) + T_n(1)\} = \tfrac{1}{2}T_n(1) + T_{n-1}(1) = 0 = T_{n-1}(1) - T_{n-1}(0).$$

It is now straightforward to show (2·7) by induction on k. (The induction step involves using (2·3) in the forward direction for each term in the left side of (2·7), applying the induction hypothesis to those terms having a numerator $2j+1$, rearranging terms, and finally using (2·3) in the reverse direction with $n-1$ in place of n.)

A different proof of equation (2·7) was given by Sekiguchi and Shiota [**14**]. Note that the right hand side of (2·7) depends on j as well as on k. Therefore, T_n does not belong to the Takagi class when $n \geq 2$.

Finally, we note that the functions $T_n, n \in \mathbf{N}$ are not self-affine in the sense of Kono. Recall that a function $f : [0,1] \to \mathbf{R}$ is called *self-affine in the sense of Kono* if there exist

$a \in \mathbf{N}$ and $b > 0$ such that $f((x+i)/a) - f(i/a) = f(x)/b$ or $-f(x)/b$ for all $0 \le x < 1$ and $i = 0, 1, \cdots, a-1$. Kono [**7**] studied the differentiability and the Hölder order of self-affine functions. In 1990, Urbanski [**17**] showed how to calculate the Hausdorff dimension of the graph of any continuous self-affine function.

3. *An arithmetic expression for $T_n(x)$*

Below we derive an expression for $T_n(x)$ that lies on the basis of many of the later results of this paper.

Let the binary expansion of $x \in [0,1]$ be denoted by

$$x = \sum_{k=1}^{\infty} \omega_k 2^{-k}, \qquad \omega_k = \omega_k(x) \in \{0, 1\}.$$

For those $x \in [0,1]$ having two binary expansions, we choose the expansion which is eventually all zeroes. As an exception, fix $\omega_k = 1$ for every k if $x = 1$. Let $q_k = q_k(x) = \sum_{j=1}^{k} \omega_j$; in other words, q_k is the number of 1's occurring in the first k binary digits of x. By convention, $q_0 = 0$.

The arithmetic expression follows by diffentiating the following formula of Lomnicki and Ulam [**9**]:

$$L_a(x) = \frac{a}{1-a} \sum_{k=1}^{\infty} \omega_k a^{k-q_k} (1-a)^{q_k}, \qquad 0 \le x \le 1, \tag{3·1}$$

where $0 < a < 1$.

Theorem 3·1. *For each $n \in \mathbf{N}$, $T_n(x)$ has the following representation:*

$$T_n(x) = \sum_{k=1}^{\infty} \omega_k \left(\frac{1}{2}\right)^{k-n} \sum_{i=0}^{n} (-1)^i \binom{q_k - 1}{i} \binom{k - q_k + 1}{n - i}, \qquad 0 \le x \le 1.$$

Proof. If $f(a) = (1-a)^m$ and $g(a) = a^l$, Leibniz's rule implies

$$\begin{aligned}
\frac{d^n}{da^n}\{f(a)g(a)\} &= \sum_{i=0}^{n} \binom{n}{i} f^{(i)}(a) g^{(n-i)}(a) \\
&= \sum_{i=0}^{n} \binom{n}{i} \binom{m}{i} \binom{l}{n-i} i!(n-i)!(1-a)^{m-i} a^{l-n+i} (-1)^i \\
&= n! \sum_{i=0}^{n} \binom{m}{i} \binom{l}{n-i} (1-a)^{m-i} a^{l-n+i} (-1)^i.
\end{aligned}$$

Using this with $m = q_k - 1$ and $l = k - q_k + 1$ gives, from (1·1) and (3·1):

$$\begin{aligned}
T_n(x) &= \frac{1}{n!} \sum_{k=1}^{\infty} \omega_k \, \frac{\partial^n}{\partial a^n} \left\{ (1-a)^{q_k - 1} a^{k - q_k + 1} \right\} \Big|_{a=\frac{1}{2}} \\
&= \sum_{k=1}^{\infty} \omega_k \left(\frac{1}{2}\right)^{k-n} \sum_{i=0}^{n} (-1)^i \binom{q_k - 1}{i} \binom{k - q_k + 1}{n - i}.
\end{aligned}$$

Theorem 3·1 shows that, for example,

$$T_1(x) = \frac{\partial L_a(x)}{\partial a}\Big|_{a=\frac{1}{2}} = \sum_{k=1}^{\infty} \omega_k \left(\frac{1}{2}\right)^{k-1} \{k + 2 - 2q_k\},$$

a formula that has already appeared in [**5**].

Definition 3·2. *For each $n \in \mathbf{N}$, define the function*

$$f_n(t,q) := \left(\frac{1}{2}\right)^{t-n} \sum_{i=0}^{n} (-1)^i \binom{q-1}{i} \binom{t-q+1}{n-i}, \qquad t \in \mathbf{R}, \quad q \in \mathbf{N},$$

where as usual,

$$\binom{t}{i} := \frac{t(t-1)(t-2)\cdots(t-i+1)}{i!}, \qquad t \in \mathbf{R}, \quad i \in \mathbf{N} \cup \{0\}.$$

Lemma 3·3. *Let f and f' be arbitrary functions from $\{(k,q) \in \mathbf{N} \times \mathbf{N} : k \geq q\}$ to $\mathbf{R}$, and let $S(x) = \sum_{k=1}^{\infty} \omega_k(x) f(k, q_k(x))$, and $S'(x) = \sum_{k=1}^{\infty} \omega_k(x) f'(k, q_k(x))$. Then $S(x) = S'(x)$ for all x if and only if $f(k,q) = f'(k,q)$ for all $k \in \mathbf{N}$ and $1 \leq q \leq k$.*

Proof. Let $W(x) = S(x) - S'(x)$ and $h(k,q) = f(k,q) - f'(k,q)$. Suppose $W(x) = 0$ for all x. Especially, $0 = W(2^{-k}) = h(k,1)$ for all k. Similarly, $0 = W(2^{-1} + 2^{-k}) = h(1,1) + h(k,2) = h(k,2)$ for all $k \geq 2$, etc. Continuing this argument yields that $h(k,q) = 0$ for $q \in \mathbf{N}$ and $k \geq q$. The lemma follows.

Corollary 3·4. *For all $k, q \in \mathbf{N}$ and $n \geq 2$,*

$$f_n(k,q) - f_n(k+1,q) = f_n(k+1,q+1), \tag{3·2}$$

Proof. Since (2·4) shows that

$$\sum_{k=1}^{\infty} \omega_k f_n(k+1, q_k) + \left\{ \sum_{k=1}^{\infty} \omega_k f_n(k+1, q_k+1) + f_n(1,1) \right\} = \sum_{k=1}^{\infty} \omega_k f_n(k, q_k),$$

and $f_n(1,1) = 0$ if $n \geq 2$, (3·2) follows from Lemma 3·3.

Of course, Corollary 3·4 can also be proved using identities for binomial coefficients.

4. *Hölder order and Hausdorff dimension of T_n*

In this section, we establish the Hölder order of T_n, and show that the graph of T_n has Hausdorff dimension one.

Theorem 4·1. *For each $n \in \mathbf{N}$, there exist positive constants C_n and δ_n such that if $0 \leq x < x + y \leq 1$ and $y < \delta_n$, then*

$$|T_n(x+y) - T_n(x)| \leq C_n y (\log_2(1/y))^n.$$

Moreover, the function $y(\log_2(1/y))^n$ can not be replaced by a function of smaller order.

Proof. We will show that if $0 \leq x < x + y \leq 1$ and $y < 2^{-n}$, then

$$|T_n(x+y) - T_n(x)| \leq \frac{2^{2n+3}}{n!} y (\log_2(1/y))^n.$$

Note first that for all k and q,

$$|f_n(k,q)| \leq 2^{-(k-n)} \sum_{i=0}^{n} \binom{q-1}{i} \binom{k-q+1}{n-i} = 2^{-(k-n)} \binom{k}{n}.$$

Thus, if $\xi \in [0,1]$, $j/2^N \le \xi < (j+1)/2^N$ and $N \ge n$, then by Theorem 3·1,

$$
\begin{aligned}
|T_n(\xi) - T_n(j/2^N)| &= \left| \sum_{k=N+1}^{\infty} \omega_k(\xi) f_n(k, q_k(\xi)) \right| \\
&\le \sum_{k=N+1}^{\infty} |f_n(k, q_k(\xi))| \le \sum_{k=N}^{\infty} 2^{-(k-n)} \binom{k}{n} \\
&= 2^{-(N-n)} \binom{N}{n} \sum_{k=N}^{\infty} 2^{-(k-N)} \frac{k(k-1)\cdots(k-n+1)}{N(N-1)\cdots(N-n+1)} \\
&= 2^{-(N-n)} \binom{N}{n} \sum_{l=0}^{\infty} 2^{-l} \left(1 + \frac{l}{N}\right)\left(1 + \frac{l}{N-1}\right) \cdots \left(1 + \frac{l}{N-n+1}\right) \qquad (4\cdot1) \\
&\le 2^{-(N-n)} \binom{N}{n} \sum_{l=0}^{\infty} 2^{-l} \left(1 + \frac{l}{n}\right)\left(1 + \frac{l}{n-1}\right) \cdots (1+l) \\
&= 2^{-(N-n)} \binom{N}{n} \sum_{l=0}^{\infty} 2^{-l} \binom{n+l}{n} = 2^{-(N-n)} \binom{N}{n} 2^{n+1} \\
&\le \frac{2^{2n+2}}{n!} N^n 2^{-(N+1)}.
\end{aligned}
$$

Now fix x and y satisfying $0 \le x < x+y \le 1$ and $y < 2^{-n}$, and let N be the integer such that $2^{-(N+1)} \le y < 2^{-N}$. Note that $N \ge n$. There are two cases to consider.

Case 1. There exists an integer j such that

$$\frac{j}{2^N} \le x < x+y < \frac{j+1}{2^N}.$$

Then the first N digits of x, $x+y$ and $j/2^N$ are the same, so by (4·1),

$$
\begin{aligned}
|T_n(x+y) - T_n(x)| &\le |T_n(x) - T_n(j/2^N)| + |T_n(x+y) - T_n(j/2^N)| \\
&\le 2 \cdot \frac{2^{2n+2}}{n!} N^n 2^{-(N+1)} \\
&\le \frac{2^{2n+3}}{n!} y(\log_2(1/y))^n.
\end{aligned}
$$

Case 2. There exists an integer j such that

$$\frac{j-1}{2^N} \le x < \frac{j}{2^N} \le x+y < \frac{j+1}{2^N}.$$

In this case, use Corollary 2·3 and (4·1) to obtain

$$
\begin{aligned}
|T_n(x+y) - T_n(x)| &\le |T_n(x+y) - T_n(j/2^N)| + |T_n(j/2^N) - T_n(x)| \\
&= |T_n(x+y) - T_n(j/2^N)| + |T_n(1-x) - T_n(1-j/2^N)| \\
&\le \frac{2^{2n+3}}{n!} y(\log_2(1/y))^n.
\end{aligned}
$$

Finally, to see that the bound is best possible, take $x = 0$ and $y = 2^{-N}$, and observe that

$$T_n(2^{-N}) = f_n(N, 1) = 2^{-(N-n)} \binom{N}{n}.$$

Hence, there exists a constant A_n such that, for all sufficiently large N,

$$T_n(2^{-N}) \geq A_n 2^{-N} N^n = A_n y(\log_2(1/y))^n.$$

This completes the proof.

A consequence of Theorem 4·1 is, that for every $\varepsilon > 0$ there exists a constant $C_{n,\varepsilon}$ such that if $0 \leq x < x + y \leq 1$, then $|T_n(x+y) - T_n(x)| \leq C_{n,\varepsilon} y^{1-\varepsilon}$. A standard argument (e.g. Falconer [**1**, Theorem 8.1]) thus implies the following:

COROLLARY 4·2. *For each $n \in \mathbf{N}$, the graph of T_n has Hausdorff dimension one.*

5. *Nowhere-differentiability of T_n*

Since T_1 is two times the Takagi function, it is nowhere differentiable. In this section, we show that the same is true for T_n when $n \geq 2$.

THEOREM 5·1. *For each $n \in \mathbf{N}$, T_n is continuous but nowhere differentiable.*

Proof. The continuity of T_n follows from Theorem 4·1. To establish the nowhere-differentiability, let x_0 be any point in $[0,1]$, and consider two cases.

Case 1. If x_0 is not a dyadic rational, then there exist integers $j_1, j_2, \ldots$ such that

$$\frac{j_k}{2^k} < x_0 < \frac{j_k + 1}{2^k}, \qquad k \in \mathbf{N}.$$

Define

$$P_n(k,j) := \frac{T_n\left((j+1)\cdot 2^{-k}\right) - T_n\left(j \cdot 2^{-k}\right)}{2^{-k}}, \qquad k \in \mathbf{N}, \quad j = 0, 1, \ldots, 2^k - 1.$$

We will show by induction on n that $\lim_{k\to\infty} P_n(k, j_k)$ does not exist as a real number. For $n = 1$, this was proved by Takagi in 1903. So let $n \in \mathbf{N}$, and suppose that $\lim_{k\to\infty} P_n(k, j_k)$ does not exist. From (2·7), we derive that

$$\{P_{n+1}(k+1, 2j_k) - P_{n+1}(k+1, 2j_k+1)\} = 4P_n(k, j_k), \tag{5·1}$$

for $k \in \mathbf{N}$. Let A be the set of those indices k such that $j_{k+1} = 2j_k$, and note that $\mathbf{N} \backslash A$ consists of those indices k such that $j_{k+1} = 2j_k + 1$. Since x_0 is not a dyadic rational, both A and $\mathbf{N} \backslash A$ are infinite.

Aiming for a contradiction, suppose there exists a finite number m such that

$$\lim_{k\to\infty} P_{n+1}(k, j_k) = m. \tag{5·2}$$

Reindexing the sequence $\{(k, j_k)\}$ and separating it into two subsequences gives

$$\lim_{k\to\infty,\, k\in A} P_{n+1}(k+1, 2j_k) = m, \tag{5·3}$$

$$\lim_{k\to\infty,\, k\notin A} P_{n+1}(k+1, 2j_k+1) = m. \tag{5·4}$$

Since

$$P_{n+1}(k+1, 2j_k) + P_{n+1}(k+1, 2j_k+1) = 2P_{n+1}(k, j_k),$$

(5·2), (5·3) and (5·4) imply that, in fact,

$$\lim_{k\to\infty} P_{n+1}(k+1, 2j_k) = \lim_{k\to\infty} P_{n+1}(k+1, 2j_k+1) = m.$$

But by (5·1), this means that $\lim_{k\to\infty} P_n(k, j_k) = 0$, contradicting the induction hypothesis.

Case 2. Assume next that x_0 is a dyadic rational, say $x_0 = j/2^N$. Since for $k > N$,

$$T_n(x_0 + 2^{-k}) - T_n(x_0) = f_n(k, q_N(x_0) + 1),$$

and, for fixed q, $f_n(k, q)$ equals 2^{n-k} times a polynomial in k with a positive leading coefficient, we have

$$\lim_{k\to\infty} \frac{T_n(x_0 + 2^{-k}) - T_n(x_0)}{2^{-k}} = +\infty. \tag{5·5}$$

Similarly, using (5·5) with $1 - x_0$ in place of x_0 gives, by Corollary 2·3,

$$\begin{aligned}\lim_{k\to\infty} & \frac{T_n(x_0 - 2^{-k}) - T_n(x_0)}{-2^{-k}} \\ &= \lim_{k\to\infty} \frac{(-1)^{n-1}\left\{T_n(1 - x_0 + 2^{-k}) - T_n(1 - x_0)\right\}}{-2^{-k}} = (-1)^n \cdot \infty.\end{aligned}$$

Therefore, we conclude that T_n does not have a finite derivative anywhere.

6. *Maximum and minimum values of T_n*

We call a point $x \in [0, 1]$ a *maximum point* (resp. *minimum point*) of T_n if T_n takes on its maximum (resp. minimum) value at x. We denote the set of maximum points of T_n by S_n^+, and the set of minimum points by S_n^-. The set S_1^+ was described by Kahane in 1959:

THEOREM 6·1 (Kahane, 1959 [**4**]). *The maximum value of $T_1(x)$ is $4/3$, and a point $x \in [0, 1]$ with binary expansion $x = .\omega_1\omega_2\cdots$ is a maximum point of T_1 if and only if $\omega_{2j-1} + \omega_{2j} = 1$ for $j = 1, 2, \cdots$.*

In other words, the binary expansion of any maximum point of T_1 is of the form

$$x = .\begin{cases}01\\ \text{or}\\ 10\end{cases}\begin{cases}01\\ \text{or}\\ 10\end{cases}\begin{cases}01\\ \text{or}\\ 10\end{cases}\begin{cases}01\\ \text{or}\\ 10\end{cases}\cdots.$$

Thus, the set S_1^+ is a self-similar Cantor set with Hausdorff dimension $\frac{1}{2}$, and it is not difficult to see that the $\frac{1}{2}$-dimensional Hausdorff measure of S_1^+ is positive and finite. (Of course, the minimum value of $T_1(x)$ is zero, and S_1^- is the finite set $\{0, 1\}$.)

The above result can easily be understood visually. Figure 2 shows how the graph of T_1 can be constructed step by step from (1·2). (Recall that $T_1(x) = 2T(x)$.) We can see immediately that the binary expansion of any maximum point should start with $x = .01$ or $x = .10$, and continue following the same rule.

However, finding maximum (or minimum) points of T_n is more difficult for $n \geq 2$. The problem is, that T_n for $n \geq 2$ does not belong to the Takagi class, and hence, it is not clear how to find extrema of T_n directly from its functional equation.

We begin with a general result concerning the nature of maximum and minimum points.

PROPOSITION 6·2. *Let $n \geq 2$. Any maximum or minimum point of T_n is not a dyadic rational.*

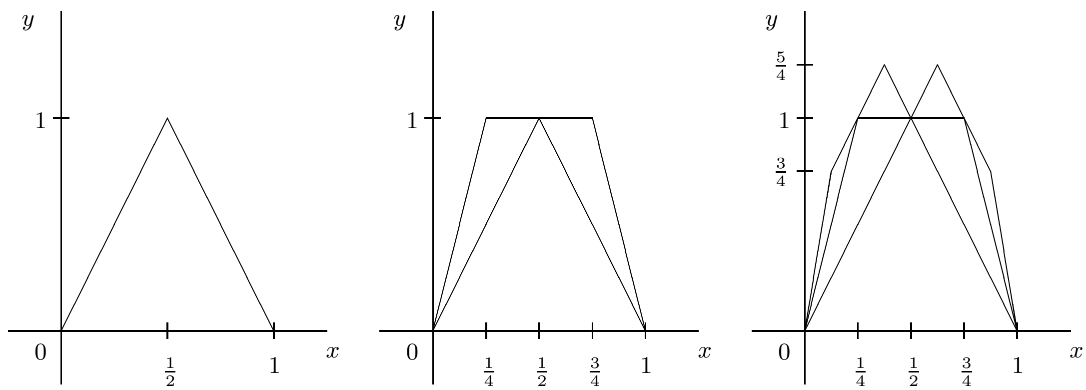

Fig. 2. The first three steps of the construction of $T_1(x)$

Proof. Let x be a dyadic rational in $[0,1]$. Consider first the case $x = 0$. It is easy to check that $T_n(2^{-n}) = 1$, $T_n(0) = 0$, and $T_n(2^{-1} + 2^{-n}) = -1$. It follows that $x = 0$ is not a maximum or minimum point of T_n. By symmetry, neither is $x = 1$. Suppose therefore that $x \in (0,1)$. Let q be the number of ones in the binary expansion of x, and let k_0 be the position of the last "1". We will construct points x' and x'' such that $T_n(x') > T_n(x) > T_n(x'')$. First, recall from section 3 that $T_n(x) = \sum_{k=1}^{\infty} \omega_k f_n(k, q_k)$, and note that $f_n(k, q+1)$ is strictly positive for sufficiently large k. Choose $l > k_0$ such that $f_n(l, q+1) > 0$, and let $x' = x + 2^{-l}$. Then $T_n(x') = T_n(x) + f_n(l, q+1) > T_n(x)$.

To find x'', consider two cases. If $f_n(l, q+1) < 0$ for some $l > k_0$, then $T_n(x)$ can be made smaller by adding a "1" in position l, so take $x'' = x + 2^{-l}$. Otherwise, $f_n(k, q+1) \geq 0$ for all $k > k_0$, and therefore, by (3·2),

$$f_n(k+1, q) = f_n(k, q) - f_n(k+1, q+1) \leq f_n(k, q)$$

for all $k \geq k_0$. Since $f_n(k, q)$ is eventually positive, it follows that $f_n(k_0, q) > 0$. This means $T_n(x)$ will be made smaller by removing the "1" in position k_0, so take $x'' = x - 2^{-k_0}$. In both cases, $T_n(x'') < T_n(x)$.

Proposition 6·2 says that we need only consider points $x \in [0,1]$ having a non-terminating binary expansion. Observe that for each such x, there is a unique, strictly increasing sequence $\{k_q\}_{q \in \mathbf{N}}$, such that $x = \sum_{q=1}^{\infty} 2^{-k_q}$. Thus, we can write

$$T_n(x) = \sum_{q=1}^{\infty} f_n(k_q, q). \tag{6·1}$$

Which sequence(s) $\{k_q\}$ will maximize the above sum? Which will minimize it? To answer these questions, an analysis of the functions $f_n(\cdot, q)$, for all $q \in \mathbf{N}$, is necessary. This analysis turns out to be feasible only for the cases $n = 2$ and $n = 3$.

6·1. *Local extrema of $f_2(\cdot, q)$ and $f_3(\cdot, q)$*

We begin by examining the zeros and local extrema of the functions $f_2(\cdot, q)$ and $f_3(\cdot, q)$. Figures 3 and 4 show the general shapes of the graphs of these functions for fixed q.

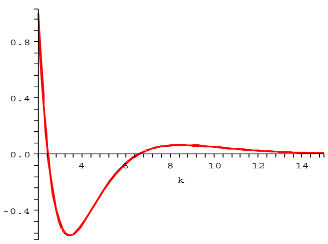

Fig. 3. Graph of $f_2(k,q)$, shown here for $q=3$

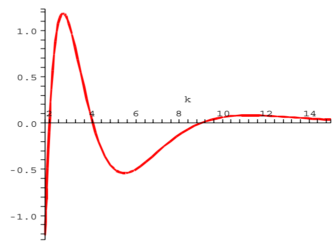

Fig. 4. Graph of $f_3(k,q)$, shown here for $q=3$

DEFINITION 6·3. *For each $q \in \mathbf{N}$, define the numbers*

$$k_2^+(q) := \left\lceil \frac{4q-1+\sqrt{8q+1}}{2} \right\rceil, \qquad k_2^-(q) := \left\lfloor \frac{4q-1-\sqrt{8q+1}}{2} \right\rfloor + 1,$$

and

$$k_3^+(q) := \left\lceil \frac{4q+1+\sqrt{24q+1}}{2} \right\rceil, \qquad k_3^-(q) := \left\lfloor \frac{4q+1-\sqrt{24q+1}}{2} \right\rfloor + 1,$$

where as usual, $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the largest integer less than or equal to x and the smallest integer greater than or equal to x, respectively.

LEMMA 6·4. *The following statements hold for each fixed q.*

(i) $f_2(t,q)$ has two real roots given by $t = (4q-3 \pm \sqrt{8q-7})/2$.

(ii) $\lim_{t\to-\infty} f_2(t,q) = \infty$, and $f_2(t,q)$ eventually decreases to zero as $t \to \infty$.

(iii) Over integer values of k, $f_2(k,q)$ has a local maximum at $k_2^+(q)$, and a global minimum at $k_2^-(q)$. Moreover, if (and only if) $8q+1$ is square, the local maximum is also attained at $k_2^+(q)+1$, and the minimum is also attained at $k_2^-(q)-1$.

Proof. (i) The zeros of $f_2(t,q)$ follow easily from the expansion

$$f_2(t,q) = 2^{-(t-1)}[t^2 - (4q-3)t + 4(q-1)^2]. \tag{6·2}$$

(ii) From (6·2), it is clear that $\lim_{t\to-\infty} f_2(t,q) = \infty$ and $\lim_{t\to\infty} f_2(t,q) = 0$. Since $f_2(t,q)$ has two roots, it is eventually positive. Hence, it eventually decreases to zero.

(iii) Over integer values of k, $f_2(k,q)$ will have a local maximum at any point k with

$$f_2(k-1,q) < f_2(k,q) \quad \text{and} \quad f_2(k,q) \geq f_2(k+1,q).$$

By (3·2), this condition is equivalent to

$$f_2(k,q+1) < 0 \leq f_2(k+1,q+1).$$

It follows that $k = \lceil t \rceil - 1$, where t is the "+" root of $f_2(t,q+1)$. By part (i), this means $k = k_2^+(q)$. Similarly, $f_2(k,q)$ has a local minimum at any point k with $f_2(k,q) < f_2(k+1,q)$ and $f_2(k-1,q) \geq f_2(k,q)$, or in other words, with $f_2(k+1,q+1) < 0 \leq f_2(k,q+1)$. Thus k is the integer part of the "$-$" root of $f_2(t,q+1)$, which by part (i) equals $k_2^-(q)$.

Finally, if $8q+1$ is square, then $f_2(t,q+1)$ has integer roots and, hence, $f_2(k,q+1) = f_2(k+1,q+1)$ both for $k = k_2^+(q)$ and $k = k_2^-(q)-1$.

LEMMA 6·5. *The following statements hold for each fixed q.*

(i) $f_3(t,q)$ has three real roots: $t=2(q-1)$ and $t=(4q-1\pm\sqrt{24q-23})/2$. The root $2(q-1)$ is the middle root for all $q\geq 2$.

(ii) $\lim_{t\to-\infty} f_3(t,q)=-\infty$, and $f_3(t,q)$ eventually decreases to zero as $t\to\infty$.

(iii) Over integer values of k, $f_3(k,q)$ has a tie for local minimum at $2q-1$ and $2q$, and local maxima at $k_3^-(q)$ and $k_3^+(q)$. Moreover, if (and only if) $24q+1$ is square, the local maxima are also attained at $k_3^-(q)-1$ and $k_3^+(q)+1$, respectively.

Proof. (i) The roots of $f_3(t,q)$ follow easily from the expansion

$$f_3(t,q)=6^{-1}2^{-(t-3)}\{t-2(q-1)\}[t^2-(4q-1)t+4q^2-8q+6]. \tag{6·3}$$

(ii) Equation (6·3) shows that $\lim_{t\to-\infty} f_3(t,q)=-\infty$ and $\lim_{t\to\infty} f_3(t,q)=0$. Since $f_3(t,q)$ has three roots, it is eventually positive. Hence, it eventually decreases to zero.

(iii) This is similar to the proof of Lemma 6·4 (iii).

6·2. *Global extrema of T_2*

THEOREM 6·6. *(i) The minimum value of T_2 is given by*

$$\min_{0\leq x\leq 1} T_2(x)=\sum_{q=1}^{\infty} f_2(k_2^-(q),q)=-1.8923882908550019\cdots.$$

(ii) The minimum points of T_2 are exactly those points x that can be written as

$$x=\frac{3}{4}+\sum_{r=2}^{\infty}\left[\left(\frac{1}{2}\right)^{r^2-\varepsilon_r}+\sum_{j=1}^{r}\left(\frac{1}{2}\right)^{r^2+2j-1}\right],$$

where $\{\varepsilon_r\}_{r\geq 2}$ is any sequence in $\{0,1\}^{\mathbf{N}}$. In particular, S_2^- is a topological Cantor set.

Proof. Recall from Lemma 6·4 that the function $k\to f_2(k,q)$ has an absolute minimum value at $k_2^-(q)$. Since $k_2^-(q)$ is strictly increasing in q, the point $x=\sum_{q=1}^{\infty}2^{-k_2^-(q)}$ minimizes $T_2(x)$ in view of (6·1). Part (i) of the theorem follows.

To see part (ii), note that the sequence $\{k_2^-(q)\}$ satisfies

$$k_2^-(q+1)-k_2^-(q)=\begin{cases}1, & \text{if } 8q+1 \text{ is a square,}\\ 2, & \text{otherwise.}\end{cases} \tag{6·4}$$

Furthermore, by Lemma 6·4, $f_2(k_2^-(q),q)=f_2(k_2^-(q)-1,q)$ if and only if $8q+1$ is a square. This happens exactly when $q=r(r+1)/2$ for some $r\in\mathbf{N}$, in which case $k_2^-(q)=r^2$, and we can choose either $k_q=r^2$ or $k_q=r^2-1$. The exception is the case $q=1$, where $r=1$ and we must choose $k_1=1$.

Since $T_2(x)=-T_2(1-x)$ for all x, we immediately obtain

COROLLARY 6·7. *(i) The maximum value of T_2 is given by*

$$\max_{0\leq x\leq 1} T_2(x)=1.8923882908550019\cdots.$$

(ii) The maximum points of T_2 are exactly those points x that can be written as

$$x=\sum_{r=2}^{\infty}\left[\left(\frac{1}{2}\right)^{r^2-\varepsilon_r}+\sum_{j=1}^{r-1}\left(\frac{1}{2}\right)^{r^2+2j}\right],$$

where $\{\varepsilon_r\}_{r\geq 2}$ is any sequence in $\{0,1\}^{\mathbf{N}}$. In particular, S_2^+ is a topological Cantor set.

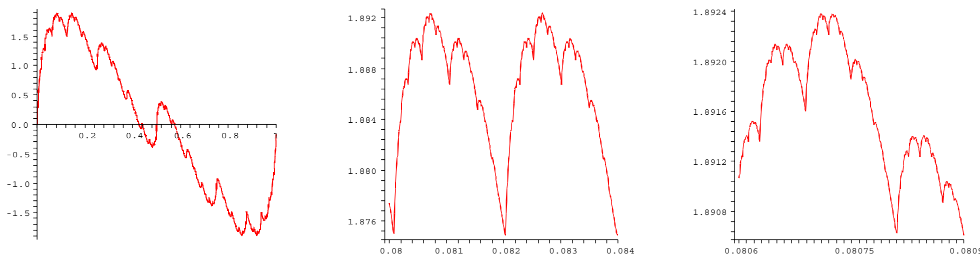

Fig. 5. Zooming in on the maximum of $T_2(x)$

From the above results we see that the binary expansion of any maximum point of T_2 is of the form

$$x = .00 \left\{\begin{matrix} 01 \\ \text{or} \\ 10 \end{matrix}\right. 010 \left\{\begin{matrix} 01 \\ \text{or} \\ 10 \end{matrix}\right. 01010 \left\{\begin{matrix} 01 \\ \text{or} \\ 10 \end{matrix}\right. 0101010 \left\{\begin{matrix} 01 \\ \text{or} \\ 10 \end{matrix}\right. 010101010 \left\{\begin{matrix} 01 \\ \text{or} \\ 10 \end{matrix}\right. \cdots .$$

Similarly, the binary expansion of any minimum point is obtained by interchanging zeros and ones in the above pattern.

Figure 5 illustrates the repeated binary splitting of the set of maximum points. The first panel appears to show two maximum points on the far left. The second panel zooms in on the leftmost maximum, and shows that it consists in fact of two separate maxima. The last panel zooms in on the leftmost of these, again revealing two new maximum points, etc. As the second panel shows, entire portions of the graph are sometimes repeated exactly. This behavior should not be surprising and can, in fact, be found anywhere in the graph by just zooming in sufficiently. The third panel illustrates that local maxima too, like their global counterparts, tend to come in "pairs" (really pairs of infinite clusters of local maxima at the same level). We will say more on local extrema later.

REMARK 6·8. Corollary 6·7 implies that, contrary to S_1^+, the set S_2^+ has Hausdorff dimension zero. For, at each stage r the set S_2^+ is covered by 2^{r-1} intervals of length 2^{-r^2}. Hence, for each $s > 0$, $\mathcal{H}^s(S_2^+) \le 2^{r-1-sr^2} \to 0$ as $r \to \infty$, where $\mathcal{H}^s$ denotes Hausdorff measure. Likewise, the set S_2^- has Hausdorff dimension zero.

6·3. *Global extrema of* T_3

While finding the extrema of T_2 was relatively straightforward, things are less clear for T_3. First, because the graph of T_3 is line-symmetric rather than point-symmetric, there is no direct relationship between the maximum and minimum values of T_3. Second, the function $f_3(\cdot, q)$ has two distinct local maxima, while its local minimum is not global. It is therefore not immediately clear how the sequence $\{k_q\}$ should be chosen in order to maximize or minimize $T_3(x)$.

We state our main results in the next two theorems. The proofs will follow after a few technical lemmas have been established.

THEOREM 6·9. *(i) The maximum value of T_3 is given by*

$$\max_{0 \le x \le 1} T_3(x) = \sum_{q=1}^{\infty} f_3(k_3^-(q), q) = 3.0388253219604675 \cdots .$$

(ii) The binary expansion of any maximum point in $[\frac{1}{2}, 1]$ has the form

$$x = .1111 \begin{cases} 01 \\ or \\ 10 \end{cases} 1 \begin{cases} 01 \\ or \\ 10 \end{cases} \underbrace{101}\underbrace{101} \begin{cases} 01 \\ or \\ 10 \end{cases} \underbrace{101} \begin{cases} 01 \\ or \\ 10 \end{cases} \underbrace{10101}\underbrace{10101} \begin{cases} 01 \\ or \\ 10 \end{cases} \underbrace{10101} \begin{cases} 01 \\ or \\ 10 \end{cases}$$
$$\underbrace{1010101}\underbrace{1010101} \begin{cases} 01 \\ or \\ 10 \end{cases} \underbrace{1010101} \begin{cases} 01 \\ or \\ 10 \end{cases} \underbrace{101010101}\underbrace{101010101} \begin{cases} 01 \\ or \\ 10 \end{cases} \cdots,$$

while the maximum points in $[0, \frac{1}{2}]$ are obtained by interchanging zeros and ones in the above pattern.

Remark 6·10. By an argument similar to that in Remark 6·8, it follows that the set S_3^+ is a topological Cantor set with Hausdorff dimension zero.

Theorem 6·11. *(i) The minimum value of T_3 is given by*

$$\min_{0 \le x \le 1} T_3(x) = \sum_{q=1}^{\infty} f_3(2q, q) = -\sum_{q=1}^{\infty} 4^{-(q-2)}(q-1) = -\frac{16}{9}.$$

(ii) The minimum points of T_3 are exactly those points $x = .\omega_1\omega_2\cdots$ with $w_{2j-1} + w_{2j} = 1$ for all $j \in \mathbf{N}$.

Observe that the set S_3^- is the same as the set S_1^+. Thus, S_3^- has Hausdorff dimension $\frac{1}{2}$, and has positive and finite measure with respect to $\mathcal{H}^{1/2}$.

We will first prove Theorem 6·9. This requires a few preliminary lemmas, as well as some additional notation. Define

$$t^+(q) := \frac{4q + 1 + \sqrt{24q+1}}{2}, \qquad t^-(q) := \frac{4q + 1 - \sqrt{24q+1}}{2},$$

so that $k_3^+(q) = \lceil t^+(q) \rceil$, and $k_3^-(q) = \lfloor t^-(q) \rfloor + 1$.

Lemma 6·12. *For all $q \in \mathbf{N}$,*

$$f_3(k_3^+(q), q) \le 2f_3(t^+(q), q).$$

Proof. Fix q, and let $g(t) := g_q(t) := 2^{t-3} f_3(t, q)$. Note that $g(t)$ is a cubic polynomial with a positive leading coefficient, having three real roots. Since $t^+(q)$ lies to the right of the rightmost root, it follows that $g(t)$ increases on $t \ge t^+(q)$. Since $t^+(q) \le k_3^+(q) \le t^+(q) + 1$, we obtain

$$\begin{aligned} f_3(k_3^+(q), q) &= 2^{-(k_3^+(q)-3)} g(k_3^+(q)) \le 2^{-(k_3^+(q)-3)} g(t^+(q) + 1) \\ &\le 2^{-(t^+(q)-3)} g(t^+(q) + 1) = 2f_3(t^+(q) + 1, q) \\ &= 2f_3(t^+(q), q). \end{aligned}$$

Lemma 6·13. *Let $g_q(t) := 2^{t-3} f_3(t, q)$. Then*

$$\begin{aligned} g_q(t^+(q)) &= 2q + 3 + \sqrt{24q+1}, \\ g_q(t^-(q)) &= 2q + 3 - \sqrt{24q+1}. \end{aligned}$$

Proof. Routine calculation.

LEMMA 6·14. *For all* $q \geq 3$,

$$k_3^-(q) \leq k < 2(q-1) \quad \Longrightarrow \quad f_3(k,q) > f_3(k_3^+(q),q). \tag{6·5}$$

Proof. Recall from Lemma 6·5 that $2(q-1)$ is the middle zero of $f_3(\cdot,q)$. Thus, $f_3(\cdot,q)$ is decreasing on $[k_3^-(q), 2(q-1)]$, and the hypothesis $k_3^-(q) \leq k < 2(q-1)$ implies

$$f_3(k,q) \geq f_3(2q-3,q) = 64 \cdot 2^{-2q}(q-2).$$

From Lemmas 6·12 and 6·13,

$$f_3(k_3^+(q),q) \leq 2f_3(t^+(q),q) = 16 \cdot 2^{-t^+(q)}(2q+3+\sqrt{24q+1}),$$

and, since $t^+(q) = 2q + \frac{1}{2} + \frac{1}{2}\sqrt{24q+1}$, the conclusion will follow provided that

$$4(q-2) \geq \left(\frac{1}{2}\right)^{\frac{1}{2}+\frac{1}{2}\sqrt{24q+1}}(2q+3+\sqrt{24q+1}), \qquad q \geq 3. \tag{6·6}$$

The right hand side of this inequality is decreasing on $q \geq 3$, as can be seen from its derivative. When $q = 3$, the right hand side evaluates to .6421, and the left hand side to 4. This proves (6·6), and the lemma.

Proof of Theorem 6·9 Let $x = \sum_{q=1}^{\infty} 2^{-k_q}$ be any maximum point of T_3. Since $T_3(1-x) = T_3(x)$, we may assume without loss of generality that $x \geq \frac{1}{2}$, so $k_1 = 1 = k_3^-(1)$. Suppose that $k_2 > 2$, and consider two possibilities. If $k_2 = 3$, then since $f_3(3,2) = -1 < 0 = f_3(2,2)$, it is strictly better to shift the second "1" to position 2, leaving the rest of the binary expansion of x fixed. On the other hand, if $k_2 \geq 4$, then since $f_3(3,1) = 1 > 0 = f_3(1,1)$, it is strictly better to move the first "1" into position 3 (again leaving the other binary places unchanged). Since we assumed that x was a maximum point, we conclude that $k_2 = 2 = k_3^-(2)$.

Note that for all $q \geq 3$, Lemma 6·14 implies that $f_3(k,q)$ attains its maximum over integer values of k at $k_3^-(q)$. Therefore,

$$T_3(x) = \sum_{q=1}^{\infty} f_3(k_q,q) \leq \sum_{q=1}^{\infty} f_3(k_3^-(q),q),$$

and since $k_3^-(q)$ is strictly increasing in q, the first statement of the theorem follows.

To prove the pattern of zeros and ones in the binary expansions of maximum points, we first show that

$$k_3^-(q+1) - k_3^-(q) \in \{1,2\}, \qquad q \in \mathbf{N}. \tag{6·7}$$

This follows from the expression

$$t^-(q+1) - t^-(q) = 2 - \frac{12}{\sqrt{24q+25}+\sqrt{24q+1}},$$

which, for $q \geq 1$, implies that $1 \leq t^-(q+1) - t^-(q) < 2$. Hence, $k_3^-(q+1) - k_3^-(q) = \lfloor t^-(q+1) \rfloor - \lfloor t^-(q) \rfloor \in \{1,2\}$.

From the definition of $k_3^-(q)$, we see that $k_3^-(q+1) - k_3^-(q) = 1$ if and only if

$$\left\lfloor \tfrac{1}{2} - \tfrac{1}{2}\sqrt{24q+1} \right\rfloor - \left\lfloor \tfrac{1}{2} - \tfrac{1}{2}\sqrt{24(q+1)+1} \right\rfloor = 1,$$

which happens if and only if there exists some integer m such that

$$\tfrac{1}{2}\sqrt{24q+1}-\tfrac{1}{2}\le m<\tfrac{1}{2}\sqrt{24(q+1)+1}-\tfrac{1}{2}. \tag{6·8}$$

Partially solving this double inequality for q gives

$$6q\le m(m+1)<6(q+1).$$

There are now six cases, based on the remainder of m modulo 6:

$$\begin{array}{llll}
\text{(i)} & m=6r & \Rightarrow\quad m(m+1)=6(6r^2+r) & \Rightarrow\quad q=r(6r+1);\\
\text{(ii)} & m=6r+1 & \Rightarrow\quad m(m+1)=6(6r^2+3r)+2 & \Rightarrow\quad q=3r(2r+1);\\
\text{(iii)} & m=6r+2 & \Rightarrow\quad m(m+1)=6(6r^2+5r+1) & \Rightarrow\quad q=(3r+1)(2r+1);\\
\text{(iv)} & m=6r+3 & \Rightarrow\quad m(m+1)=6(6r^2+7r+2) & \Rightarrow\quad q=(3r+2)(2r+1);\\
\text{(v)} & m=6r+4 & \Rightarrow\quad m(m+1)=6(6r^2+9r+3)+2 & \Rightarrow\quad q=(3r+3)(2r+1);\\
\text{(vi)} & m=6r+5 & \Rightarrow\quad m(m+1)=6(r+1)(6r+5) & \Rightarrow\quad q=(r+1)(6r+5).
\end{array}$$

In cases (i), (iii), (iv) and (vi), equality holds in the left part of (6·8), which implies that $24q+1$ is a square. Hence, by Lemma 6·5 (iii), there is an arbitrary choice between $k_3^-(q)$ and $k_3^-(q)-1$. In cases (ii) and (v) there is no choice, and the binary expansion of x must have another "1" immediately following the q-th "1".

We now turn to the minimum value of $T_3(x)$. Recall from Lemma 6·5 that for fixed q, $f_3(k,q)$ has a shared local minimum at $k=2q-1$ and $k=2q$. However, this local minimum is, in general, not the global minimum of $f_3(k,q)$, since the graph falls to $-\infty$ to the left. The following lemma will help show that the minimum value of $T_3(x)$ is nonetheless obtained by taking $k_q=2q-1$ or $k_q=2q$ for every q.

LEMMA 6·15. *For each k and each q,*

$$f_3(k,q)<f_3(2q,q)\quad\Longrightarrow\quad f_3(k+1,q+1)<f_3(2(q+1),q+1). \tag{6·9}$$

Proof. Note first that $f_3(2q,q)=-(1/4)^{q-2}(q-1)$, so

$$\frac{f_3(2q+2,q+1)}{f_3(2q,q)}=\frac{q}{4(q-1)}\le\frac{1}{2},\qquad\text{for } q\ge 2.$$

Next, the hypothesis $f_3(k,q)<f_3(2q,q)$ implies that $k\le 2(q-1)$ and $f_3(k,q)<0$. Hence, from (6·3),

$$4q^2-4qk-8q+k^2+k+6>0. \tag{6·10}$$

By routine calculation,

$$f_3(k,q)-2f_3(k+1,q+1)=4\cdot 2^{-k}(4q^2-4qk-8q+k^2+3k+4).$$

Thus, (6·10) implies that $f_3(k,q)\ge 2f_3(k+1,q+1)$, and since $f_3(k,q)<0$, we have

$$\frac{f_3(k+1,q+1)}{f_3(k,q)}\ge\frac{1}{2}\ge\frac{f_3(2q+2,q+1)}{f_3(2q,q)},\qquad q\ge 2.$$

This implies (6·9) for $q\ge 2$, since all four quantities in (6·9) are negative when $f_3(k,q)<f_3(2q,q)$. For $q=1$, the implication is trivial: the hypothesis is false for each $k\in\mathbf{N}$, since $f_3(k,1)\ge 0=f_3(2,1)$.

Table 1. *The maximum and minimum values of T_1, T_2 and T_3, with the cardinalities and Hausdorff dimensions of the sets of maximum and minimum points.*

n	$\max T_n(x)$	$\min T_n(x)$	$\lvert S_n^+\rvert$	$\lvert S_n^-\rvert$	$\dim_H S_n^+$	$\dim_H S_n^-$
1	$4/3$	0	$2^{\aleph_0}$	2	$1/2$	0
2	$1.892388\cdots$	$-1.892388\cdots$	$2^{\aleph_0}$	$2^{\aleph_0}$	0	0
3	$3.038825\cdots$	$-16/9$	$2^{\aleph_0}$	$2^{\aleph_0}$	0	$1/2$

Proof of Theorem 6·11 Let $x = \sum_{q=1}^{\infty} 2^{-k_q}$ be a minimum point of T_3. Suppose, by way of contradiction, that $f_3(k_{q_0}, q_0) < f_3(2q_0, q_0)$ for some q_0. Then by Lemma 6·15,

$$f_3(k_{q_0}+j, q_0+j) < f_3(2(q_0+j), q_0+j), \qquad \text{all } j \in \mathbf{N}.$$

Since the local minimum of $f_3(\cdot, q_0+j)$ is at $k = 2(q_0+j)$, this implies that

$$f_3(k_{q_0}+j, q_0+j) \le f_3(k, q_0+j), \qquad \text{all } k \ge k_{q_0}+j. \tag{6·11}$$

(See Figure 4.) Now define a point x' by $\omega_k(x') = \omega_k(x)$ for $k \le k_{q_0}$, and $\omega_k(x') = 1$ for all $k > k_{q_0}$. For each j, it must be the case that $k_{q_0+j} \ge k_{q_0}+j$. Thus, by (6·11),

$$T_3(x') = \sum_{q=1}^{q_0} f_3(k_q, q) + \sum_{j=1}^{\infty} f_3(k_{q_0}+j, q_0+j) \le \sum_{q=1}^{\infty} f_3(k_q, q) = T_3(x).$$

But, since the binary expansion of x' is eventually all ones, x' is dyadic rational. Hence, by Proposition 6·2, $T_3(x') > T_3(x)$. This contradiction proves part (i).

Part (ii) follows from the fact that $f_3(2q-1, q) = f_3(2q, q)$ for all $q \in \mathbf{N}$.

The results of this section are summarized in Table 1.

6·4. *Greedy algorithms*

The above results regarding maximum and minimum points of T_2 and T_3 each required a different approach. It would be desirable to have one general method to compute extreme values of $T_n(x)$ for arbitrary n, and to arbitrary depth. Consider first the problem of maximizing $T_n(x)$. A glance at the representation (6·1) suggests the following algorithm. For $q = 1, 2, 3, \dots$, put the q-th "1" in a location k which maximizes the value of $f_n(k, q)$ over all available locations k. In other words, choose

$$k_1 = \min\{k \in \mathbf{N} : f_n(k, 1) \ge f_n(j, 1)\, \forall j \in \mathbf{N}\},$$

and inductively, for $q = 2, 3, \dots$, take

$$k_q = \min\{k > k_{q-1} : f_n(k, q) \ge f_n(j, q)\, \forall j > k_{q-1}\}.$$

This algorithm, which always achieves the greatest immediate increase in the value of $T_n(x)$, will be called the *max-greedy algorithm*. Similarly, we define a *min-greedy algorithm* to be the algorithm that selects

$$k_1 = \max\{k \in \mathbf{N} : f_n(k, 1) \le f_n(j, 1)\, \forall j \in \mathbf{N}\},$$

and inductively, for $q = 2, 3, \dots$,

$$k_q = \max\{k > k_{q-1} : f_n(k, q) \le f_n(j, q)\, \forall j > k_{q-1}\}.$$

Heuristically, the min-greedy algorithm produces a "small" value of $T_n(x)$. Note that while the max-greedy algorithm chooses the smallest index in case of a tie, the min-greedy algorithm chooses the largest index. This distinction will simplify the presentation of our results.

We say the max-greedy algorithm is *optimal* for T_n if the resulting point $x = \sum_{q=1}^{\infty} 2^{-k_q}$ is a maximum point of T_n. Optimality of the min-greedy algorithm is defined analogously. Our goal is to show that the two greedy algorithms are optimal for both T_2 and T_3. (That this is not clear *a priori* follows from the fact that the greedy algorithms can usually not select the *overall* extrema of the functions $f_n(k, q)$.) Note that the max-greedy algorithm is clearly optimal for T_1.

LEMMA 6·16. *(a) When $n = 2$, the max-greedy algorithm selects $k_q = k_2^+(q)$, and the min-greedy algorithm selects $k_q = k_2^-(q)$.*

(b) When $n = 3$, the max-greedy algorithm selects $k_q = k_3^+(q)$, and the min-greedy algorithm selects $k_q = 2q$.

Proof. First, let $n = 2$. That the min-greedy algorithm selects $k_q = k_2^-(q)$ is obvious, so we need only consider the max-greedy algorithm. Note that $f_2(k, 1) = \left(\frac{1}{2}\right)^{k-2} \binom{k}{2}$, which for $k \geq 1$ is maximized at $k = k_2^+(1)$. Now let $q \in \mathbf{N}$, and suppose the max-greedy algorithm has selected $k_q = k_2^+(q)$. Clearly, the choice $k_{q+1} = k_2^+(q+1)$ is available. Note also that it is not possible to choose k_{q+1} to the left of the leftmost root of $f_2(\cdot, q+1)$. Thus (see Figure 3), the max-greedy algorithm selects $k_{q+1} = k_2^+(q+1)$.

Similar arguments apply to the proof of part (b).

PROPOSITION 6·17. *The sequences $\{k_n^+(q)\}_{q\in\mathbf{N}}$ and $\{k_n^-(q)\}_{q\in\mathbf{N}}$ form a partition of $\mathbf{N}$, both for $n = 2$ and for $n = 3$.*

Proof. Note that from the proofs of Lemmas 6·4 and 6·5, we have the representations

$$\begin{aligned} k_2^+(q) &= \max\{k : f_2(k, q+1) < 0\}, \\ k_2^-(q) &= \min\{k : f_2(k+1, q+1) < 0\}, \\ k_3^+(q) &= \max\{k : f_3(k, q+1) < 0\}, &\quad (6·12) \\ k_3^-(q) &= \min\{k : f_3(k+1, q+1) > 0\}. &\quad (6·13) \end{aligned}$$

We prove the proposition for $n = 3$. The argument for $n = 2$ is similar.

a) *Disjointness.* Let $k \in \mathbf{N}$, and suppose, by way of contradiction, that there exist q_1 and q_2 such that $k = k_3^+(q_1) = k_3^-(q_2)$. Note that this implies

$$2q_1 < k < 2q_2. \quad (6·14)$$

From (6·12) and (6·13) it follows that $f_3(k, q_1 + 1) < 0$, and $f_3(k, q_2 + 1) \leq 0$. By (6·14) and the expansion

$$f_3(k, q+1) = 6^{-1}2^{-(k-3)}(k - 2q)[k^2 - (4q+3)k + 4q^2 + 2],$$

it follows that

$$\begin{aligned} k^2 - (4q_1 + 3)k + 4q_1^2 + 2 &< 0, \\ k^2 - (4q_2 + 3)k + 4q_2^2 + 2 &\geq 0. \end{aligned}$$

Subtracting these inequalities gives $4(q_2 - q_1)\{k - (q_1 + q_2)\} < 0$, and so

$$k - (q_1 + q_2) < 0. \quad (6·15)$$

On the other hand, using (6·12) and (6·13) with the roles of q_1 and q_2 interchanged yields that $f_3(k+1, q_1+1) \geq 0$ and $f_3(k+1, q_2+1) > 0$. In view of (6·14), it follows that

$$k^2 - (4q_1+1)k + 4q_1^2 - 4q_1 \geq 0,$$
$$k^2 - (4q_2+1)k + 4q_2^2 - 4q_2 < 0.$$

Subtracting these inequalities gives $4(q_2 - q_1)\{k - (q_1+q_2) + 1\} > 0$, and so

$$k - (q_1 + q_2) + 1 > 0. \tag{6·16}$$

However, since $k - (q_1 + q_2)$ is integer, (6·15) and (6·16) contradict each other. Hence, the sequences $\{k_3^+(q)\}$ and $\{k_3^-(q)\}$ are disjoint.

b) *Covering.* Recall from (6·7) that

$$k_3^-(q+1) - k_3^-(q) \in \{1, 2\}, \qquad q \in \mathbf{N}. \tag{6·17}$$

Similarly, by rewriting the difference $t^+(q+1) - t^+(q)$, it can be seen that

$$k_3^+(q+1) - k_3^+(q) \in \{2, 3\}, \qquad q \in \mathbf{N}. \tag{6·18}$$

Observe that $k_3^-(q) = q$ for $q \in \{1, 2, 3, 4\}$, and $k_3^+(1) = 5$. It is now easy to see, using (6·17), (6·18) and the already established disjointness, that $\{k_3^+(q)\}_{q\in\mathbf{N}}$ and $\{k_3^-(q)\}_{q\in\mathbf{N}}$ together cover $\mathbf{N}$. The details are left to the interested reader.

THEOREM 6·18. *The max-greedy algorithm attains the maxima of T_2 and T_3. The min-greedy algorithm attains the minima of T_2 and T_3.*

Proof. By Proposition 6·17, we have

$$x = \sum_{q=1}^{\infty} 2^{-k_n^+(q)} \quad \Longrightarrow \quad 1 - x = \sum_{q=1}^{\infty} 2^{-k_n^-(q)} \qquad (n = 2, 3).$$

The theorem now follows from Lemma 6·16 and Theorems 6·6, 6·9 and 6·11, using the appropriate type of symmetry in each case.

6·5. *Local extrema.*

So far, we have only considered absolute maxima and minima of T_n. What can be said about *local* extrema? How large are the sets of local maximum and minimum points of T_n? This subsection provides some answers. First, we show how a dense set of local extreme points of T_n can be obtained using the binary expansion of any global extreme point. Then we give a complete characterization of the sets of local maximum and minimum points of T_2 and T_3.

DEFINITION 6·19. *Let x and x' be points in $(0, 1)$. We say that x is a* finite binary shuffle *of x' if there exists a positive integer K such that $q_k(x) = q_k(x')$ for all $k \geq K$.*

Thus, x is a finite binary shuffle of x' if and only if x can be obtained from x' by rearranging the first K binary digits of x'. Note that this condition is stronger than the condition "the binary expansions of x and x' agree beyond some index K".

Kahane [**4**] proved that the local minimum points of T_1 are exactly the dyadic rationals, and the local maximum points are exactly those points $x = .\omega_1\omega_2\cdots$ satisfying $\omega_1 + \omega_2 + \cdots + \omega_{2k} = k$, for all large enough k. In our terminology, the last condition is equivalent to x being a finite binary shuffle of a global maximum point of T_1. Generalizing this idea, we obtain a sufficient condition for a point x to be a local maximum point of T_n.

THEOREM 6·20. *Let $x \in (0,1)$. If x is a finite binary shuffle of some global maximum (minimum) point x' of T_n, then x is a local maximum (minimum) point of T_n.*

Proof. Let x be a finite binary shuffle of a global maximum point x' of T_n, and let K be the number from Definition 6·19. Suppose that x is not a local maximum point of T_n. Since the binary expansion of x eventually matches that of x', it is non-terminating by Proposition 6·2. Thus, there exists a point y whose first K binary places match those of x, such that $T_n(y) > T_n(x)$. This implies that

$$\sum_{k=K+1}^{\infty} \omega_k(y) f_n(k, q_k(y)) > \sum_{k=K+1}^{\infty} \omega_k(x) f_n(k, q_k(x)). \tag{6·19}$$

Define a point y' by

$$\omega_k(y') = \begin{cases} \omega_k(x'), & k \le K, \\ \omega_k(y), & k > K. \end{cases}$$

It is not difficult to see that $q_K(y') = q_K(x') = q_K(x) = q_K(y)$, and therefore, $q_k(y') = q_k(y)$ for all $k > K$. Thus, by (6·19),

$$\begin{aligned} T_n(y') &= \sum_{k=1}^{K} \omega_k(y') f_n(k, q_k(y')) + \sum_{k=K+1}^{\infty} \omega_k(y') f_n(k, q_k(y')) \\ &= \sum_{k=1}^{K} \omega_k(x') f_n(k, q_k(x')) + \sum_{k=K+1}^{\infty} \omega_k(y) f_n(k, q_k(y)) \\ &> \sum_{k=1}^{K} \omega_k(x') f_n(k, q_k(x')) + \sum_{k=K+1}^{\infty} \omega_k(x) f_n(k, q_k(x)) \\ &= T_n(x'), \end{aligned}$$

contradicting the assumption that x' is a global maximum point of T_n. The argument for local minima is similar.

As a consequence, the sets of local maximum (or minimum) points of T_n are dense in $[0,1]$ for every n. When n equals 2 or 3, a stronger statement holds: every interval contains uncountably many local maximum and minimum points (at least one for each global maximum or minimum point).

The condition in Theorem 6·20 is not always necessary, as the next result shows.

THEOREM 6·21. *T_2 has a local minimum at $x = 0$, and a local maximum at $x = 1$.*

Proof. For $k \in \mathbf{N}$, let $m(k) = \min\{T_2(x) : 2^{-k} \le x \le 2^{-(k-1)}\}$. We will show that $m(k) \ge 0$ for all sufficiently large k. Fix k, and let $x \in [2^{-k}, 2^{-(k-1)}]$ be a point such that $T_2(x) = m(k)$. If $x = 2^{-(k-1)}$, then $T_2(x) = f_2(k-1, 1)$, which is positive for all sufficiently large k. Assume therefore that $x < 2^{-(k-1)}$, so that in the representation $x = \sum_{q=1}^{\infty} 2^{-k_q}$, we have $k_1 = k$.

For $q \in \mathbf{N}$, let $\hat{k}(q) = 2q - \frac{3}{2} + \frac{1}{2}\sqrt{8q-7}$, the rightmost root of $f_2(\cdot, q)$. Define $\hat{q} = \min\{q : k + q - 1 < \hat{k}(q)\}$. Then $\hat{q} < \infty$, and $f_2(k_q, q) \ge 0$ for all $q < \hat{q}$. When k is large enough, we have

$$\frac{3k}{4} \le \hat{q} \le k. \tag{6·20}$$

We will now construct a lower bound for the sum $\sum_{q=\hat{q}}^{\infty} f_2(k_q, q)$, which contains all of the negative terms in the series $\sum_{q=1}^{\infty} f_2(k_q, q)$, if any are present.

Using (6·2), and by considering the minimum of the polynomial $t^2-(4q-3)t+4(q-1)^2$, we find that

$$f_2(k,q) \geq 2^{-(k-1)}(-2q+7/4), \qquad q \in \mathbf{N},\, k \geq q. \tag{6·21}$$

By choosing k large enough, we can ensure that $\hat{q}$ is sufficiently large so that $k_2^-(q) \geq 3q/2+1$ for all $q \geq \hat{q}$. By (6·21), it follows that for $q \geq \hat{q}$,

$$f_2(k_q,q) \geq f_2(k_2^-(q),q) \geq 2^{-(k_2^-(q)-1)}(-2q+7/4) \geq 2^{-3q/2}(-2q+7/4),$$

where the last inequality follows since $-2q+7/4<0$. Hence,

$$\sum_{q=\hat{q}}^{\infty} f_2(k_q,q) \geq \sum_{q=\hat{q}}^{\infty} 2^{-3q/2}(-2q+7/4) = -2^{-3\hat{q}/2}(A\hat{q}+B),$$

where A and B are calculable constants, with $A>0$. Note that $3\hat{q}/2 \geq 9k/8$ by (6·20), so that $f_2(k,1) = 2^{-(k-1)}(k^2-k) > 2^{-3\hat{q}/2}(A\hat{q}+B)$ for all large enough k. Therefore,

$$m(k) = \sum_{q=1}^{\infty} f_2(k_q,q) \geq f_2(k,1) + \sum_{q=\hat{q}}^{\infty} f_2(k_q,q) > 0,$$

for all sufficiently large k. This implies that T_2 has a local minimum at $x=0$. The local maximum at $x=1$ follows by symmetry.

THEOREM 6·22. *A point $x \in (0,1)$ is a local maximum (minimum) point of T_2 if and only if x is a finite binary shuffle of some global maximum (minimum) point of T_2.*

Proof. We need only prove the "forward" direction of the theorem, and do so first for minima. Let $x \in (0,1)$ be a local minimum point of T_2. For $q \in \mathbf{N}$, let $\mathcal{M}(q)$ be the set of all indices k that minimize $f_2(k,q)$. That is, $\mathcal{M}(q) = \{k_2^-(q)-1, k_2^-(q)\}$ if $8q+1$ is square, and $\mathcal{M}(q) = \{k_2^-(q)\}$ otherwise.

Note first that x cannot be dyadic rational, for if it were, then the proof of Theorem 5·1 would imply that $T_2(x-2^{-m}) < T_2(x) < T_2(x+2^{-m})$ for all large enough m. Hence, there exists a dyadic interval $I=[r/2^l,(r+1)/2^l]$, such that T_2 attains its absolute minimum over I at x, and x is interior to I.

Since $T_2(x) \leq T_2(r/2^l)$, there is an index $k_0>l$ with $\omega_{k_0}(x)=1$ and $f_2(k_0,q_{k_0}(x)) \leq 0$. Let $q_0 := q_{k_0}(x)$, so that $k_{q_0}=k_0$. Define the number

$$j^* := \min\{j : k_0+j \leq k_2^-(q_0+j)\}.$$

Note that j^* is finite since $k_2^-(q)$ increases in steps of 2 for infinitely many q. Now for $1 \leq j < j^*$, k_0+j is to the right of $k_2^-(q_0+j)$, and on or to the left of the rightmost root of $f_2(\cdot,q_0+j)$. (Since this is true for $j=0$ by the assumption $f_2(k_0,q_0) \leq 0$ (see Figure 3), and the rightmost root of $f_2(\cdot,q)$ increases by at least 1 for each q.) It follows that

$$f_2(k_0+j,q_0+j) \leq f_2(k,q_0+j), \qquad 1 \leq j < j^*,\, k \geq k_0+j.$$

Since $k_{q_0+j} \geq k_0+j$ for all j, we obtain

$$\sum_{q>q_0} f_2(k_q,q) \geq \sum_{j=1}^{j^*-1} f_2(k_0+j,q_0+j) + \sum_{j=j^*}^{\infty} f_2(k_2^-(q_0+j),q_0+j),$$

with equality if and only if $k_{q_0+j} = k_0 + j$ for all $1 \leq j < j^*$, and $k_{q_0+j} \in \mathcal{M}(q_0 + j)$ for all $j \geq j^*$. But then, by Theorem 6·6, x is a finite binary shuffle of some global minimum point.

Next, let x be a local *maximum* point of T_2. Then $1 - x$ is a local minimum point, whose binary expansion is obtained from that of x by complementation. From the above proof, it follows that $1-x$ is a finite binary shuffle of some global minimum point $1-x'$. But then x is also a finite binary shuffle of x', and since x' is a global maximum point of T_2, the proof of the theorem is complete.

A similar result holds for the local maxima of T_3, though for this case, the argument is slightly more subtle.

THEOREM 6·23. *A point x is a local maximum point of T_3 if and only if x is a finite binary shuffle of some global maximum point of T_3.*

The proof uses the following sets, defined for $q \in \mathbf{N}$. If $24q+1$ is a square, let $\mathcal{M}^+(q) = \{k_3^+(q), k_3^+(q)+1\}$ and $\mathcal{M}^-(q) = \{k_3^-(q)-1, k_3^-(q)\}$. Otherwise, let $\mathcal{M}^+(q) = \{k_3^+(q)\}$ and $\mathcal{M}^-(q) = \{k_3^-(q)\}$.

Proof of Theorem 6·23 Let x be a local maximum point of T_3. Then there exists a dyadic interval $I = [r/2^l, (r+1)/2^l]$, such that T_3 attains its absolute maximum over I at x. As in the proof of Theorem 6·22, x is dyadic irrational. We can therefore choose the interval I small enough so that $q_0 := q_l(x) \geq 3$.

Now consider two cases. First, suppose that $k_q := k_q(x) \geq 2(q-1)$ for all $q > q_0$. Then it follows immediately that

$$\sum_{q>q_0} f_3(k_q, q) \leq \sum_{q>q_0} f_3(k_3^+(q), q), \tag{6·22}$$

with equality if and only if $k_q \in \mathcal{M}^+(q)$ for all $q > q_0$. By the results of the previous subsection, x must be a finite binary shuffle of some global maximum point of T_3.

Next, suppose that $k_q < 2(q-1)$ for at least one $q > q_0$. Fix this value of q. Now for each $j \in \mathbf{N}$, we must have $k_{q+j} \geq k_q + j$. And, by a repeated application of Lemma 6·14,

$$f_3(k_q + j, q+j) > f_3(k_3^+(q+j), q+j), \tag{6·23}$$

as long as $k_q + j \geq k_3^-(q+j)$. Thus, the only way for x to be an absolute maximum point over I given the condition $k_q < 2(q-1)$, is that $k_{q+j} = k_q + j$ for all j with $k_q + j \geq k_3^-(q+j)$, and $k_{q+j} \in \mathcal{M}^-(q+j)$ for all other j. But in view of Theorem 6·9, this implies x is a finite binary shuffle of some global maximum point of T_3.

THEOREM 6·24. *A point $x \in [0,1]$ is a local minimum point of T_3 if and only if x is either a dyadic rational, or a finite binary shuffle of some global minimum point of T_3.*

Proof. If x is a local minimum point of T_3, then it must be an absolute minimum point over some dyadic interval. By essentially the same argument as in the proof of Theorem 6·11, it follows that either (a) the binary expansion of x is eventually all ones, in which case x is dyadic rational; or (b) the binary expansion of x satisfies $k_q \in \{2q-1, 2q\}$ for all q beyond some index, in which case x is a finite binary shuffle of some global minimum point of T_3.

Conversely, let x_0 be any dyadic rational, and let q_0 denote the number of ones in the binary expansion of x_0. We will show that

$$T_3(x_0) = \min\{T_3(x) : x_0 \le x \le x_0 + \delta\}, \qquad \text{for some } \delta > 0. \tag{6·24}$$

Applying the same argument to the point $1 - x_0$, and using the line symmetry of the graph of T_3, then yields that T_3 has a local minimum at x_0.

For $k \in \mathbf{N}$, let $I_k = [x_0 + 2^{-k}, x_0 + 2^{-(k-1)}]$, and let $m(k) = \min\{T_3(x) : x \in I_k\}$. We may assume that k is large enough so that if $x_0 = j/2^l$ in lowest terms, then $k > l + 1$. If the minimum value of T_3 over I_k is taken on at one of the endpoints, then $m(k) - T_3(x_0)$ is either $f_3(k-1, q_0+1)$ or $f_3(k, q_0+1)$, and so $m(k) \ge T_3(x_0)$ for sufficiently large k.

Otherwise, the argument from the second part of the proof of Proposition 6·2 implies that the minimum over I_k is attained at some dyadic irrational point $x = \sum_{q=1}^{\infty} 2^{-k_q}$. For $q \in \mathbf{N}$, let $\hat{k}(q) = 2q - \frac{1}{2} + \frac{1}{2}\sqrt{24q - 23}$, the rightmost root of $f_3(\cdot, q)$. Assume $k \ge \hat{k}(q_0 + 1)$, and let $\hat{q} = \min\{q : k + q - q_0 - 1 < \hat{k}(q)\}$. Then $q_0 + 1 < \hat{q} < \infty$, and $f_3(k_q, q) \ge 0$ for all $q_0 < q < \hat{q}$. By choosing k large enough, we can further ensure that $\hat{q} \ge \frac{2}{3}(k - q_0)$.

Since x is dyadic irrational, the proof of Theorem 6·11 shows that $f_3(k_q, q) \ge f_3(2q, q)$ for all $q \ge \hat{q}$. Thus,

$$\sum_{q=\hat{q}}^{\infty} f_3(k_q, q) \ge \sum_{q=\hat{q}}^{\infty} f_3(2q, q) = -\sum_{q=\hat{q}}^{\infty} 4^{-(q-2)}(q-1) = -(64/9)2^{-2\hat{q}}(3\hat{q} - 2).$$

Since $\hat{q} \ge \frac{2}{3}(k - q_0)$, the magnitude of the last expression goes to zero faster than $f_3(k, q_0 + 1)$ as $k \to \infty$. Hence, for all sufficiently large k,

$$T_3(x) - T_3(x_0) = \sum_{q > q_0} f_3(k_q, q) \ge f_3(k, q_0 + 1) + \sum_{q=\hat{q}}^{\infty} f_3(k_q, q) > 0,$$

and so $m(k) > T_3(x_0)$. This implies (6·24).

6·6. *Results and conjectures for the extrema of T_n*

PROPOSITION 6·25. *For every $n \ge 2$,*

$$\frac{2^n}{\sqrt{\pi n}} \sim \left(\frac{1}{2}\right)^n \binom{2n}{n} < \max_{0 \le x \le 1} T_n(x) < 2^{n-1}.$$

Proof. The lower bound follows since

$$\max_{0 \le x \le 1} T_n(x) > T_n(2^{-2n}) = f_n(2n, 1) = \left(\frac{1}{2}\right)^n \binom{2n}{n}.$$

By Stirling's formula, $\left(\frac{1}{2}\right)^n \binom{2n}{n} \sim 2^n/\sqrt{\pi n}$ as $n \to \infty$.

To see the upper bound, let $M_n := \max_{0 \le x \le 1} |T_n(x)|$, and note that by (2·3),

$$|T_n(x)| \le \tfrac{1}{2} M_n + M_{n-1}, \qquad 0 \le x \le 1, \quad n \ge 2.$$

It follows that $M_n \le 2M_{n-1}$. By Theorem 6·6 and Corollary 6·7, we have $M_2 < 2$ and, hence, $M_n < 2^{n-1}$ for all $n \ge 2$.

CONJECTURE A.

(i) The max-greedy algorithm is optimal for T_n, for all $n \in \mathbf{N}$.

(ii) The min-greedy algorithm is optimal for T_{2n-1}, for all $n \in \mathbf{N}$.

Table 2. *Comparisons of the value obtained by the max-greedy algorithm (G_n), and the lower bound $L_n = \left(\frac{1}{2}\right)^n \binom{2n}{n}$ of Proposition* 6·25.

n	L_n	G_n	G_n/L_n
4	4.375	5.2212	1.1934
5	7.875	9.2478	1.1743
6	14.438	16.736	1.1592
10	180.43	203.19	1.1261
20	1.3146×10^5	1.4349×10^5	1.0915
50	8.9610×10^{13}	9.4896×10^{13}	1.0590
100	7.1430×10^{28}	7.4434×10^{28}	1.0421
200	6.4067×10^{58}	6.5984×10^{58}	1.0299

Conjecture A is based not only on the fact that the greedy algorithms are optimal for T_1, T_2 and T_3, but also on extensive numerical experimentation. (The min-greedy algorithm does not appear to be optimal for T_n when n is even and $n > 2$. Of course, in this case the minimum is directly related to the maximum, via the point-symmetry of the graph of T_n.)

Table 2 shows the values obtained by the max-greedy algorithm (G_n), as well as the lower bound from Proposition 6·25 (L_n). The last column gives the ratio of these values, which seems to tend to 1 as $n \to \infty$. This suggests the following guess.

CONJECTURE B. As $n \to \infty$,

$$\max_{0 \le x \le 1} T_n(x) \sim \frac{2^n}{\sqrt{\pi n}}. \tag{6·25}$$

Lastly, we address the number of maximum and minimum points of T_n.

PROPOSITION 6·26. *Let $n \in \mathbf{N}$. Provided the max-greedy algorithm is optimal for T_n, T_n has at least two distinct maximum points in the interval $[0, \frac{1}{2}]$.*

Proof. Note that $f_n(k, 1) = \left(\frac{1}{2}\right)^{k-n} \binom{k}{n}$, which is maximized simultaneously at $k = 2n - 1$ and $k = 2n$. If the max-greedy algorithm is optimal, either value of k can be chosen as k_1, and each choice results in a different final point x.

CONJECTURE C. For each $n \ge 4$, T_n has only finitely many maximum points, and only finitely many minimum points.

To illustrate this conjecture, we consider the maximum points of T_4. Assuming the max-greedy algorithm is optimal, each k_q must be chosen at a local maximum of the corresponding function $f_4(\cdot, q)$ in order to produce a maximum point. If there were to be infinitely many maximum points, there would have to be a tie for a local maximum of $f_4(\cdot, q)$ for infinitely many q. By (3·2), this means the equation

$$f_4(k + 1, q + 1) = 0 \tag{6·26}$$

would have to have infinitely many integer solutions. Setting $p := k - q + 1$, (6·26) can be written as $\sum_{i=0}^{4} (-1)^i \binom{p}{i}\binom{q}{4-i} = 0$, and the further substitution $x = p - q$, $y = p + q$

transforms this into

$$x^4 - 6x^2y + 8x^2 + 3y^2 - 6y = 0, \tag{6·27}$$

or equivalently,

$$\begin{gathered} z^2 - 6zy + 3y^2 + 8z - 6y = 0, \\ z = x^2. \end{gathered} \tag{6·28}$$

Equation (6·28) does have infinitely many integer solutions (z, y), and they can be found by a standard method [**6**, Chapter 11]. For $m \in \mathbf{Z}$, let

$$\begin{aligned} u_m &= \pm\frac{1}{2}\left\{(4+\sqrt{6})(5+2\sqrt{6})^m + (4-\sqrt{6})(5-2\sqrt{6})^m\right\}, \\ v_m &= \pm\frac{1}{2\sqrt{6}}\left\{(4+\sqrt{6})(5+2\sqrt{6})^m - (4-\sqrt{6})(5-2\sqrt{6})^m\right\}, \end{aligned} \tag{6·29}$$

where the signs of u_m and v_m are independent. Then

$$(z, y) = \left(\frac{u_m + 3v_m + 1}{2}, \frac{v_m + 3}{2}\right) \tag{6·30}$$

is an integer solution to (6·28), and conversely, each solution to (6·28) must be of the form (6·30). The question is, for how many of these solutions z is a square. An exhaustive computer search showed that the only solution pairs (6·30) with $|m| \le 1000$ (and any choice of signs in (6·29)) for which z is a square in fact have $|m| \le 5$. They are $(0,0)$, $(0,2)$, $(1,1)$, $(1,3)$, $(4,2)$, $(4,8)$, $(9,3)$, $(9,17)$, $(36,8)$, $(36,66)$, $(8281,1521)$ and $(8281,15043)$. We guess therefore that (6·27) has only finitely many integer solutions. Numerical computations also indicate that only two of the above solution pairs (z, y) correspond to a pair (k, q) such that the max-greedy algorithm selects $k_q = k$. These are $(36, 8)$ (corresponding to $k = 7, q = 1$) and $(8281, 1521)$ (corresponding to $k = 1520, q = 715$). Thus, it seems as though T_4 has precisely 4 distinct maximum points.

Finally, we point out that, while the minimum of $T_3(x)$ was attained by choosing $k_q = 2q - 1$ or $k_q = 2q$ for each q, this sequence does not work for higher-indexed T_n's, even though the function $f_n(\cdot, q)$ has a local extremum at $2q$ for every odd n. For instance, when $n = 5$, $f_5(\cdot, q)$ has in fact a local *maximum* at $2q$. For $n = 7$, $f_7(\cdot, q)$ does have a local minimum at $2q$, but an easy calculation shows that $\sum_{q=1}^{\infty} f_7(2q, q) = -256/81 > -4$, whereas $f_7(10, 2) = -6$.

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