On the classification of self-similar sets determined by two contractions on the plane,

- Dedicated to the late Professor Masaya Yamaguti.-

Kiko Kawamura

Department of Mathematics,
University of North Texas, Denton, TX 76203, U.S.A.

Abstract

We classify *binary self-similar sets*, which are compact sets determined by two contractions on the plane, into four classes from the viewpoint of functional equations. In this classification, we can not only show close relationships among functions with self-similarity but also give solutions to a few open problems in other field.

Key words:
functional equation, self-similar set, nowhere differentiable function

Introduction

In the history of mathematics, we have seen some discoveries of strange functions, which gave us a strong impact; for examples, *the Takagi function* [19], constructed as a simple example of a nowhere differentiable but continuous function, *the Von Koch curve* [10], a continuous Jordan curve, which admits no tangent line anywhere, and *the Lévy curve* [12], which is a continuous curve but with positive area, and so on. Each of these curves was discovered independently and initially, no relationships between them were known for a long time.

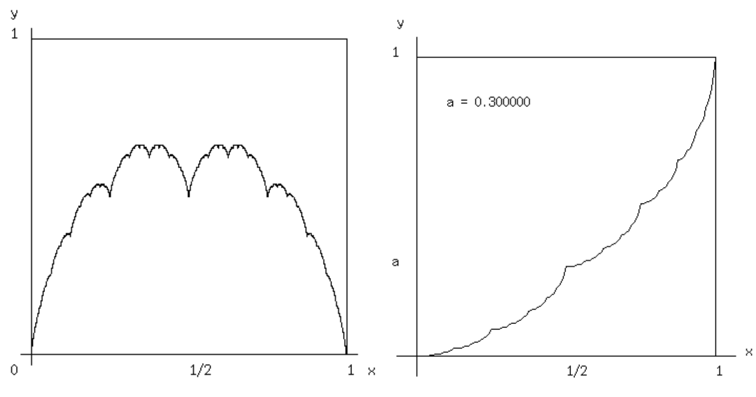

Figure 1: The Takagi function and Lebesgue's singular function

However in 1984, Hata and Yamaguti showed the following beautiful relationship between the Takagi function $T^1(x)$ and Lebesgue's singular function $M_a^1(x)$ (Figure 1), which is a monotone increasing continuous function whose derivative is zero almost everywhere [22].

$$2T^1(x) = \left.\frac{\partial M_a^1(x)}{\partial a}\right|_{a=1/2}, \tag{1}$$

where a is a real parameter with $0 < a < 1$.

A generalization of this relation was considered by Sekiguchi-Shiota in 1991. They computed the k-th partial derivative of $M_a^1(x)$ with respect to the real parameter a [16], and showed that it has a nice application to an open problem about digital sums [18].

Also, Tasaki, Antoniou and Suchanecki pointed out Hata-Yamaguti's result has some valuable applications in physics [20].

The purpose of this paper is to extend Hata-Yamaguti's results by finding close relationships among other strange functions from the viewpoint of the theory of fractal geometry.

Recall a definition of self-similar sets. We say that a set X is self-similar, if it is a unique empty compact solution of the set equation $X = \psi_0(X) \cup \psi_1(X) \cup \cdots \cup \psi_{m-1}(X)$ for some finite similarity contractions $\psi_0, \psi_1, \ldots, \psi_{m-1}$ on $\mathbb{R}^n$. In this paper, we define *binary self-similar sets* as self-similar sets defined by two similarity contractions on the plane and classify them into four classes determined by the form of their functional equation. This leads to a classification of self-similar sets that has not been studied yet. Although many studies on the classification of self-similar sets have been investigated from the viewpoint of connectedness (Ex.[5], [1]), it is still very difficult to determine if a given self-similar set is connected.

Binary self-similar sets are the simplest case of self-similar sets; however, they include many interesting special cases; for instance, *the Lévy curve*, *the Von Koch curve* (Figure 2) and *Pólya's space filling curve*, which is a simiplified version of the Peano curve. Also, we show that our main theorems have nice applications to several open problems in other fields.

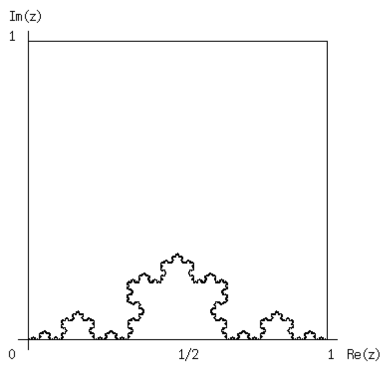

Figure 2: The Von Koch curve

Section 1 gives a discussion about the classfication of binary self-similar sets.

First, we introduce the following four functional equations.

$$G^1_{\alpha,\gamma}(x) = \begin{cases} \alpha G^1_{\alpha,\gamma}(2x), & 0 \le x < 1/2, \\ \gamma G^1_{\alpha,\gamma}(2x-1) + (1-\gamma), & 1/2 \le x \le 1, \end{cases} \tag{2}$$

$$G^2_{\alpha,\gamma}(x) = \begin{cases} \alpha \overline{G^2_{\alpha,\gamma}(2x)} & 0 \le x < 1/2, \\ \gamma \overline{G^2_{\alpha,\gamma}(2x-1)} + (1-\gamma), & 1/2 \le x \le 1, \end{cases} \tag{3}$$

$$G^3_{\alpha,\gamma}(x) = \begin{cases} \alpha G^3_{\alpha,\gamma}(2x), & 0 \le x < 1/2, \\ \gamma \overline{G^3_{\alpha,\gamma}(2x-1)} + (1-\gamma), & 1/2 \le x \le 1, \end{cases} \tag{4}$$

$$G^4_{\alpha,\gamma}(x) = \begin{cases} \alpha\overline{G^4_{\alpha,\gamma}(2x)}, & 0 \le x < 1/2, \\ \gamma G^4_{\alpha,\gamma}(2x-1) + (1-\gamma), & 1/2 \le x \le 1, \end{cases} \tag{5}$$

where α and γ are complex parameters satisfying $|\alpha| < 1$, $|\gamma| < 1$.

It can be proved that there exists a unique bounded solution $G^i_{\alpha,\gamma}(x)$ $(i = 1, 2, 3, 4)$ for each functional equation. These solutions are complex-valued real functions. The closure of their images are binary self-similar sets, because each functional equation represents a pair of two similar contractions on the plane. In addition, we see that any binary self-similar set can be determined by contractions represented by these four functional equations. In other words, any binary self-similar set can be expressed as the closure of the image of a solution of one of these functional equations; therefore, any binary self-similar set can be classified into one of four classes.

We also obtain explicit formulas for a unique bounded solution $G^i_{\alpha,\gamma}(x)$ $(i = 1, 2, 3, 4)$ by a number-theoretical expression. These are our main theorems (See Theorem 1.2 - Theorem 1.6 below). Although $G^i_{\alpha,\gamma}(x)$ $(i = 1, 2, 3, 4)$ are discontinuous functions except when $\gamma = 1 - \alpha$, their differentiability with respect to the complex parameters α and γ can be proved from main theorems.

In section 2, as an extension of Hata-Yamaguti's results, we find a relationship between $G^1_{\alpha,1-\alpha}(x)$ and *the Takagi function*, $T^1(x)$, as follows.

$$\operatorname{Im} \left.\frac{\partial G^1_{\alpha,1-\alpha}(x)}{\partial \alpha_I}\right|_{\alpha=1/2} = 2T^1(x), \tag{6}$$

where α_I is the imaginary part of α satisfing $|\alpha| < 1$.

Furthermore, we prove that the real part of $G^1_{\alpha,1-\alpha}(x)$ can be expressed in terms of the derivatives of even order of Lebesgue's singular function, and the imaginary part is expressed in terms of the derivatives of odd order of Lebesgue's singular function.

From (1) and (6), we know that $G^1_{\alpha,1-\alpha}(x)$ has a close relationship with a nowhere differentiable function, $T^1(x)$, and a singular function, $M^1_a(x)$. Then a question arises: how about $G^2_{\alpha,1-\alpha}(x)$, $G^3_{\alpha,1-\alpha}(x)$ and $G^4_{\alpha,1-\alpha}(x)$?

To provide an answer for this question, we introduce real-valued functions $T^i(x)$ and $M^i_a(x)$ $(i = 2, 3, 4)$ satisfying equations analogous to (1) and (6). We analyze the behavior of these functions. It is interesting to observe how the behavior of $T^i(x)$ and $M^i_a(x)$ $(i = 1, 2, 3, 4)$ depends on the number of complex conjugate terms of (2) – (5). In other words, these real functions clarify the essential difference among four classes from the view point of real analysis.

In section 3, we mention a relationship between the Cantor function and binary self-similar sets. We define a *generalized Cantor function*, $F_{a,b}(x)$, having two real parameters a and b satisfying $|a| < 1$ and $|1 - a - b| < 1$. In case $a = 1/3, b = 1/3$, $F_{a,b}(x)$ is the Cantor function. Next, we define $E_{a,b}(x) = \sup\{y \in [0, 1]; F_{a,b}(y) = x\}$. This function is the generalized inverse function of $F_{a,b}(x)$, and it is known that $E_{a,0}(x)$ is Lebesgue's singular function [8], [9]. From the viewpoint of the functional equations, we have $E_{a,b}(x) = G^i_{a,1-a-b}(x)$, $(i = 1, 2, 3, 4)$.

Section 4 gives some applications of our main theorems.

First, we give a solution to the open problem of exponential sums in number theory. Explicit formulas of exponential sums have been investigated by many authors for a long time. In 1998, Muramoto et al. gave a part of this solution using the representation of Lebesgue's singular function [15]. We show how their results can be generalized using a main theorem of section 1.

Next, we solve an open problem in ergodic theory posed by Mizutani and Ito in 1987 [14]. They investigated a set of four *Dragons* by using the algebraic methods of Dekking and point out an interesting open problem about *the Lévy curve.* We show that our main theorem is powerful for analyzing *the Lévy curve*, and a relationship between *the Lévy curve* and *Dragon* (Figure 3) is clarified.

Lastly, we study *Dragon* from the viewpoint of the functional equation again and compare with the *Lévy curve.* From section 3, we recognized *the Lévy curve* as the image of a complex-valued continuous function $G^1_{1/2+i/2,1/2-i/2}(x)$ and characterized it by *the Takagi function* and *Lebesgue's*

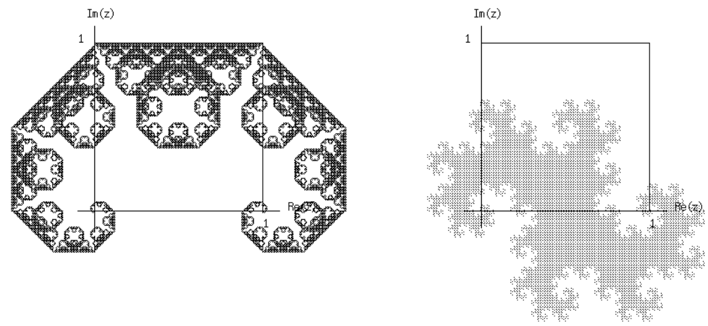

Figure 3: The Levy curve and Dragon

singular function. Similarly, *Dragon* can be viewed as the image of a discontinuous complex-valued function $G^1_{1/2+i/2,1/2+i/2}(x)$, and we can characterize it in terms of Rademacher series and *self-affine dust.*

Throughout this paper "singular function" means a monotone increasing continuous function whose derivative is zero almost everywhere.

1 Classification of binary self-similar sets

The history of systematic mathematical research on self-similar sets dates back to 1981, when Hutchinson considered the non-empty compact set $X \subset \mathbb{R}^n$ satisfying the following set equation.

$$X = \psi_0(X) \cup \psi_1(X) \cup \cdots \cup \psi_{m-1}(X), \tag{7}$$

where $\psi_0, \psi_1, \ldots, \psi_{m-1}$ are similarity contractions on $\mathbb{R}^n$.

(Recall that a map $\psi : \mathbb{R}^n \to \mathbb{R}^n$ is a *similarity contraction* iff there exists a constant number $L(\psi) \in (0,1)$ so that the equality $\|\psi(x) - \psi(y)\| = L(\psi)\|x - y\|$ holds for any $x, y \in \mathbb{R}^n$).

Besides, Hutchinson proved essentially the following important theorem [7].

Theorem 1.1. *For any finite family of similarity contractions, there exists a unique compact solution of* (7)*.*

If $\psi_0, \psi_1, \ldots, \psi_{m-1}$ satisfy with the open set condition ; (there exists a non-empty open set U such that $\psi_i(U) \subset U$ and $\psi_i(U) \cup \psi_j(U), i \neq j$), then many more properties of the set X can be determined. [3], [7].

In this paper, we say that a set X is self-similar, if X is a unique empty compact solution of (7) and define *binary self-similar sets* as follows.

Definition 1.1. A set X is binary self-similar, iff X is a non-empty compact set $X \subset \mathbb{C}$ satisfying

$$X = \psi_1(X) \cup \psi_2(X),$$

where ψ_1, ψ_2 are similarity contractions on $\mathbb{C}$.

It is well known that any similarity contraction can be expressed as a composition of scaling maps, rotations and reflections. Two similarity contractions $\psi_1, \psi_2 : \mathbb{C} \to \mathbb{C}$ can be normalized so that $z = 0$ is the fixed point of ψ_1, and $z = 1$ is the fixed point of ψ_2 without loss of generality. This leads to four different cases:

$$\begin{cases} \psi_1(z) &= \alpha z, \\ \psi_2(z) &= \gamma z + (1-\gamma), \end{cases} \tag{8}$$

$$\begin{cases} \psi_1(z) &= \alpha \overline{z}, \\ \psi_2(z) &= \gamma \overline{z} + (1-\gamma), \end{cases} \tag{9}$$

$$\begin{cases} \psi_1(z) &= \alpha z, \\ \psi_2(z) &= \gamma \overline{z} + (1-\gamma), \end{cases} \tag{10}$$

$$\begin{cases} \psi_1(z) &= \alpha \overline{z}, \\ \psi_2(z) &= \gamma z + (1-\gamma), \end{cases} \tag{11}$$

where α and γ are complex parameters satisfying $|\alpha| < 1$, $|\gamma| < 1$.

In other words, any binary self-similar set is determined by one of these four pairs of contractions. For instance, *the Lévy curve* [12], a continuous curve but with positive area, is obtained by (8), if $\alpha = 1/2 + i/2$ and $\gamma = 1/2 - i/2$. Also, *the von Koch curve* [10], a continuous Jordan curve admitting no tangent line anywhere, and *Pólya's space filling curve* are obtained by (9); more exactly, if $\alpha = 1/2 + (\sqrt{3}/6)i$ and $\gamma = 1/2 - (\sqrt{3}/6)i$, *the von Koch curve* is given, and if $\alpha = 1/2 + i/2$ and $\gamma = 1/2 - i/2$, *Pólya's space filling curve* is given [23].

Now, we introduce the following four functional equations:

$$G^1_{\alpha,\gamma}(x) = \begin{cases} \alpha G^1_{\alpha,\gamma}(2x), & 0 \le x < 1/2, \\ \gamma G^1_{\alpha,\gamma}(2x-1) + (1-\gamma), & 1/2 \le x \le 1, \end{cases} \tag{12}$$

$$G^2_{\alpha,\gamma}(x) = \begin{cases} \alpha \overline{G^2_{\alpha,\gamma}(2x)}, & 0 \le x < 1/2, \\ \gamma \overline{G^2_{\alpha,\gamma}(2x-1)} + (1-\gamma), & 1/2 \le x \le 1, \end{cases} \tag{13}$$

$$G^3_{\alpha,\gamma}(x) = \begin{cases} \alpha G^3_{\alpha,\gamma}(2x), & 0 \le x < 1/2, \\ \gamma \overline{G^3_{\alpha,\gamma}(2x-1)} + (1-\gamma), & 1/2 \le x \le 1, \end{cases} \tag{14}$$

$$G^4_{\alpha,\gamma}(x) = \begin{cases} \alpha \overline{G^4_{\alpha,\gamma}(2x)}, & 0 \le x < 1/2, \\ \gamma G^4_{\alpha,\gamma}(2x-1) + (1-\gamma), & 1/2 \le x \le 1, \end{cases} \tag{15}$$

where α and γ are complex parameters satisfying $|\alpha| < 1$, $|\gamma| < 1$.

Note that these functional equations resemble the following general functional equations G. de Rham studied in 1957 [4].

$$f(x) = \begin{cases} \psi_1(f(2x)), & 0 \le x \le 1/2, \\ \psi_2(f(2x-1)), & 1/2 \le x \le 1, \end{cases} \tag{16}$$

where ψ_1, ψ_2 are contractions on $\mathbb{R}^2$.

He showed that (16) has a unique continuous solution $f(x)$ if and only if $\psi_2(p_1) = \psi_1(p_2)$, where p_1, p_2 are the unique fixed points of ψ_1 and ψ_2, respectively. This result was generalized to the case of finitely many contractions by M. Hata in 1985 [5].

If $\gamma = 1-\alpha$, we can see that each functional equation of (12) - (15) has a unique continuous solution from de Rham's theorem. It is also clear that the image of this solution is a compact set and coincides with the corresponding binary self-similar set. However, what is the situation in the other cases? More exactly, if $\gamma \neq 1-\alpha$, does there exist a solution of (12) - (15) ?

Before stating our result about this, some notations need to be introduced. We let α and γ denote complex parameters satisfying $|\alpha| < 1$, $|\gamma| < 1$.

The real part of α is denoted by α_R, and the imaginary part of α is denoted by α_I. Similarly, γ_R is the real part of γ, and γ_I is the imaginary part of γ. Furthermore, the binary expansion of $x \in [0,1]$ and its quaternary expansion are defined as follows.

Definition 1.2. The binary expansion of $x \in [0,1]$ is denoted by

$$x = \sum_{n=1}^{\infty} \omega_n 2^{-n}, \qquad \omega_n = \omega_n(x) \in \{0,1\}.$$

For these $x \in [0,1]$ which have two binary expansions we choose the expansion which is even tually all zeroes. However, fix $\omega_n = 1$ for every n if $x = 1$.

Let $q(x,n) = \sum_{k=1}^{n} \omega_k$; in other words, $q(x,n)$ is the number of 1's occurring in the first n binary digits of x. By convention, $q(x,0) = 0$.

Definition 1.3. The quaternary expansion of $x \in [0,1]$ is denoted by

$$x = \sum_{n=1}^{\infty} \xi_n 4^{-n}, \qquad \xi_n = \xi_n(x) \in \{0,1,2,3\}.$$

For these $x \in [0,1]$ which have two quaternary expansions we choose the expansion which is even tually all zeroes. However, fix $\xi_n = 3$ for every n if $x = 1$.

Let p_k (k=0,1,2,3) be the number of k's occurring in the first n quaternary digits of x.

First, we consider an explicit formula for a solution of (12).

Theorem 1.2. *There exists a unique bounded solution of* (12)*, and it has the following expression.*

$$G^1_{\alpha,\gamma}(x) = (1-\gamma) \sum_{n=1}^{\infty} \omega_n \alpha^{n-1-q(x,n-1)} \gamma^{q(x,n-1)}, \qquad 0 \le x \le 1, \tag{17}$$

Proof. First, we show that (17) is a solution of (12). Note that $\omega_1 = 0$ and $q(2x, n-1) = q(x,n)$ if $0 \le x < 1/2$, and $\omega_1 = 1$ and $q(2x-1, n-1) = q(x,n) - 1$ if $1/2 \le x \le 1$. It is now an easy exercise to prove that (17) satisfies (12).

Suppose f is bounded solution of (12). Then by induction of k, we have the following equation uniquely.

$$f\Big(\sum_{n=1}^{\infty} \omega_n 2^{-n}\Big) = \alpha^{k-1-q(x,k-1)} \gamma^{q(x,k-1)} f\Big(\sum_{n=k}^{\infty} \omega_n 2^{-n}\Big) + (1-\gamma) \sum_{n=1}^{k-1} \omega_n \alpha^{n-1-q(x,n-1)} \gamma^{q(x,n-1)} \tag{18}$$

Since $|\alpha| < 1$ and $|\gamma| < 1$, we can take the limit on both sides of (18) to obtain $f = G^1_{\alpha,\gamma}$. □

Example 1.1. In 1934, Z. Lomnicki and S. Ulam showed that Lebesgue's singular function $M^1_a(x)$ has the following representation [13].

$$M^1_a(x) = \frac{a}{1-a} \sum_{n=1}^{\infty} \omega_n a^{n-q(x,n)} (1-a)^{q(x,n)}, \tag{19}$$

where $0 < a < 1$ and $a \neq 1/2$.

Since

$$\sum_{n=1}^{\infty}\omega_n a^{n-q(x,n-1)}(1-a)^{q(x,n-1)} = \sum_{n=1}^{\infty}\omega_n a^{n+1-q(x,n)}(1-a)^{q(x,n)-1},$$

we have

$$M_a^1(x) = G_{a,1-a}^1(x).$$

Corollary 1.2.1. *If $\gamma \neq 1-\alpha$, $G^1_{\alpha,\gamma}(x)$ is discontiunuous at $x = l/2^k$, for every $1 \leq k$ and $1 \leq l \leq 2^k - 1$.*

Proof. If $\gamma = 1-\alpha$, it is clear that $G^1_{\alpha,\gamma}(x)$ is continuous from de Rham's results.

From Theorem 1.2, we have

$$G^1_{\alpha,\gamma}(1-0) = (1-\gamma)\sum_{n=1}^{\infty}\gamma^{n-1} = 1.$$

From (12),

$$\begin{aligned} G^1_{\alpha,\gamma}(1/2-0) &= \alpha G^1_{\alpha,\gamma}(1-0) = \alpha, \\ G^1_{\alpha,\gamma}(1/2) &= 1-\gamma. \end{aligned}$$

Recursively, from (12),
$G^1_{\alpha,\gamma}(x)$ is discontinuous at $x \in \{l/2^k\}_{k=1,2,\ldots,1\leq l\leq 2^k-1}$. □

It induces an interesting problem to suppose a solution of (12) is unbounded.

Theorem 1.3. *There are $2^{\mathfrak{c}}$ unbounded solutions of* (12)*.*

Proof. Let

$$T(x) = \begin{cases} 2x, & 0 \leq x < 1/2, \\ 2x-1, & 1/2 \leq x \leq 1. \end{cases}$$

A completely invarient set $S \subset [0,1]$ has the property that if $x \in S, T(x) \in S$ and $T^{-1}(x) \subset S$. For any $x \in [0,1]$, there is a smallest completely invarient set containing x.

Let P be the family of smallest completely invarient sets. Each set in P is countable. Note if $S_1, S_2 \in P$ and $S_1 \cap S_2 \neq$, then $S_1 = S_2$. Therefore, P is a partition $[0,1]$.

Since T has only countably many periodic points, only the countably many elements of P contains a periodic point.

Let $\mathfrak{c}$ be the cardinal of $2^{\aleph_0}$. For each $\alpha \prec \mathfrak{c}$, let $X_\alpha \in S_\alpha$. For each α, choose a value for $f(X_\alpha)$ if f satisfies the equation (12). This determines the value of f on S_α. Note that since S_α does not contain a periodic point we choose any value for $f(X_\alpha)$ that we wish. Also the value of f on S_α has no influence on the value of f anywhere else.

Thus, there are as many solutions of (12) as there are functions $\mathfrak{c}$ into $\mathbb{R}$. (i.e. there are $2^{\mathfrak{c}}$ solutions of (12).)

□

Next, we give an explicit formula for a solution of (20).

Theorem 1.4. *There exists a unique bounded solution of* (13)*, and it has the following expression.*

$$G^2_{\alpha,\gamma}(x) = c(\xi_1) + \sum_{n=1}^{\infty} c(\xi_{n+1})\alpha^{p_0+p_1}(\overline{\alpha})^{p_0+p_2}\gamma^{p_2+p_3}(\overline{\gamma})^{p_1+p_3}, \qquad 0 \leq x \leq 1, \tag{20}$$

where

$$c(\xi_n) = \begin{cases} 0, & \xi_n = 0, \\ \alpha\overline{(1-\gamma)}, & \xi_n = 1, \\ (1-\gamma), & \xi_n = 2, \\ \gamma\overline{(1-\gamma)} + (1-\gamma), & \xi_n = 3. \end{cases}$$

Proof. Note that (13) is equivalent to the following.

$$G^2_{\alpha,\gamma}(x) = \begin{cases} |\alpha|^2 G^2_{\alpha,\gamma}(4x), & 0 \le x < 1/4, \\ \alpha\overline{\gamma} G^2_{\alpha,\gamma}(4x-1) + \alpha\overline{(1-\gamma)}, & 1/4 \le x < 1/2, \\ \overline{\alpha}\gamma G^2_{\alpha,\gamma}(4x-2) + (1-\gamma), & 1/2 \le x < 3/4, \\ |\gamma|^2 G^2_{\alpha,\gamma}(4x-3) + \gamma\overline{(1-\gamma)} + (1-\gamma), & 3/4 \le x \le 1. \end{cases} \tag{21}$$

It is straighforward that (20) satisfies (21). Therefore, (20) is a solution of (13).

It can be proved that (20) is a unique bounded soluion in the same way as the proof of Theorem 1.2. □

Third, we derive an explicit formula for a solution of (14): $G^3_{\alpha,\gamma}(x)$. To do this, it is helpful to consider the real and imaginary part of $G^3_{\alpha,\gamma}(x)$. Define the vector-valued function

$$\mathbb{Y}(x) = \begin{pmatrix} \operatorname{Re} G^3_{\alpha,\gamma}(x) \\ \operatorname{Im} G^3_{\alpha,\gamma}(x) \end{pmatrix}.$$

Then $\mathbb{Y}(x)$ is the unique solution of the following functional equation.

$$\mathbb{Y}(x) = \begin{cases} \begin{pmatrix} \alpha_R & -\alpha_I \\ \alpha_I & \alpha_R \end{pmatrix} \mathbb{Y}(2x) + \begin{pmatrix} 0 \\ 0 \end{pmatrix}, & 0 \le x < 1/2, \\ \begin{pmatrix} \gamma_R & \gamma_I \\ \gamma_I & -\gamma_R \end{pmatrix} \mathbb{Y}(2x-1) + \begin{pmatrix} 1-\gamma_R \\ -\gamma_I \end{pmatrix}, & 1/2 \le x \le 1. \end{cases}$$

Hence, we can obtain the following theorem.

Theorem 1.5. *There exists a unique bounded solution of* (14) *and it has the following expression.*

$$\begin{pmatrix} \operatorname{Re} G^3_{\alpha,\gamma}(x) \\ \operatorname{Im} G^3_{\alpha,\gamma}(x) \end{pmatrix} = \left(\sum_{n=1}^{\infty} \omega_n \begin{pmatrix} \alpha_R & -\alpha_I \\ \alpha_I & \alpha_R \end{pmatrix}^{n-1-q(x,n-1)} \begin{pmatrix} \gamma_R & \gamma_I \\ \gamma_I & -\gamma_R \end{pmatrix}^{q(x,n-1)} \right) \begin{pmatrix} 1-\gamma_R \\ -\gamma_I \end{pmatrix}.$$

where $x \in [0,1]$.

In the same way, we can also obtain an explicit formula for $G^4_{\alpha,\gamma}(x)$.

Theorem 1.6. *There exists a unique bounded solution of* (15)*, and it has the following expression.*

$$\begin{pmatrix} \operatorname{Re} G^4_{\alpha,\gamma}(x) \\ \operatorname{Im} G^4_{\alpha,\gamma}(x) \end{pmatrix} = \left(\sum_{n=1}^{\infty} \omega_n \begin{pmatrix} \alpha_R & \alpha_I \\ \alpha_I & -\alpha_R \end{pmatrix}^{n-1-q(x,n-1)} \begin{pmatrix} \gamma_R & -\gamma_I \\ \gamma_I & \gamma_R \end{pmatrix}^{q(x,n-1)} \right) \begin{pmatrix} 1-\gamma_R \\ -\gamma_I \end{pmatrix}.$$

For comparision, we give expressions for the real and imaginary part of $G^1_{\alpha,\gamma}(x)$ and $G^2_{\alpha,\gamma}(x)$.

Remark 1.6.1. $G^1_{\alpha,\gamma}(x)$ and $G^2_{\alpha,\gamma}(x)$ also have the following expressions.

$$\begin{pmatrix} \operatorname{Re} G^1_{\alpha,\gamma}(x) \\ \operatorname{Im} G^1_{\alpha,\gamma}(x) \end{pmatrix} = \left(\sum_{n=1}^{\infty} \omega_n \begin{pmatrix} \alpha_R & -\alpha_I \\ \alpha_I & \alpha_R \end{pmatrix}^{n-1-q(x,n-1)} \begin{pmatrix} \gamma_R & -\gamma_I \\ \gamma_I & \gamma_R \end{pmatrix}^{q(x,n-1)} \right) \begin{pmatrix} 1-\gamma_R \\ -\gamma_I \end{pmatrix}.$$

$$\begin{pmatrix} \operatorname{Re} G^2_{\alpha,\gamma}(x) \\ \operatorname{Im} G^2_{\alpha,\gamma}(x) \end{pmatrix} = \left(\sum_{n=1}^{\infty} \omega_n \begin{pmatrix} \alpha_R & \alpha_I \\ \alpha_I & -\alpha_R \end{pmatrix}^{n-1-q(x,n-1)} \begin{pmatrix} \gamma_R & \gamma_I \\ \gamma_I & -\gamma_R \end{pmatrix}^{q(x,n-1)} \right) \begin{pmatrix} 1-\gamma_R \\ -\gamma_I \end{pmatrix}.$$

Recall that $G^i_{\alpha,\gamma}(x)$ $(i = 1, 2, 3, 4)$ are discontinuous functions except when $\gamma = 1 - \alpha$; therefore, it is clear they are not differentiable if $\gamma \neq 1 - \alpha$.

However, using Theorem 1.2 - Theorem 1.6, we obtain the following result concering the differentiability with respect to α and γ.

Corollary 1.6.1. *For each fixed $x \in [0, 1]$, $G^1_{\alpha,\gamma}(x)$ is an analytic function with respect to α, γ in a complex domain $E = \{(\alpha, \gamma) \in \mathbb{C}^2; |\alpha| < 1, |\gamma| < 1\}$, but the other $G^i_{\alpha,\gamma}(x), (i = 2, 3, 4)$ are not analytic.*

Consider the closure of the image of a unique bounded solution $G^i_{\alpha,\gamma}(x), (i = 1, 2, 3, 4)$. It is clearly a binary self-similar set. Furthermore, we can see that any binary self-similar set can be determined by contractions represented by four functional equations (12) - (15) and be classified into one of four classes completely.

2 An extension of Hata and Yamaguti's result

In 1903, T. Takagi discovered an example of a nowhere differentiable continuous function that was much simpler than a well-known example discovered by K. Weierstrass. It is called the Takagi function.

The Takagi function $T^1(x)$ is defined by

$$T^1(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \varphi^n(x), \qquad 0 \leq x \leq 1,$$

where

$$\varphi(x) = \begin{cases} 2x, & 0 \leq x \leq 1/2, \\ 2(1-x), & 1/2 \leq x \leq 1, \end{cases} \tag{22}$$

and φ^n is the n-fold iteration of φ. It is known that $\varphi(x)$ is a typical chaotic dynamical system.

On the other hand, G. de Rham studied the following functional equations:

$$M^1_a(x) = \begin{cases} aM^1_a(2x), & 0 \leq x \leq 1/2, \\ (1-a)M^1_a(2x-1) + a, & 1/2 \leq x \leq 1, \end{cases} \tag{23}$$

where $a \neq 1/2$ and $0 < a < 1$. He showed that the unique continuous solution $M^1_a(x)$ of (23) is Lebesgue's singular function. It is well-known that $M^1_a(x)$ is a strictly increasing function whose derivative is zero almost everywhere.

In 1984, Hata-Yamaguti studied $T^1(x)$ and $M^1_a(x)$ and discovered the following interesting connection [22].

$$2T^1(x) = \left. \frac{\partial M^1_a(x)}{\partial a} \right|_{a=\frac{1}{2}}.$$

The functional equation having $T^1(x)$ as a unique continuous solution is as follows. [22]

$$T^1(x) = \begin{cases} (1/2)T^1(2x) + x, & 0 \leq x \leq 1/2, \\ (1/2)T^1(2x-1) + (1-x), & 1/2 \leq x \leq 1. \end{cases} \tag{24}$$

This discovery came as a big surprise, because each function had been discovered independently and initially and no relationships between them were known for a long time. Furthermore, T. Sekiguchi and Y. Shiota considered a generalization of this result in 1991. They computed the k-th partial derivative of $M_a(x)$ with respect to a. More precisely, they defined $T^1_{a,k}(x)$ by

$$T^1_{a,k}(x) = \frac{1}{k!} \frac{\partial^k M^1_a(x)}{\partial a^k}, \qquad (k = 1, 2, \dots), \tag{25}$$

where $0 < a < 1$ and $a \neq 1/2$ and proved that $T^1_{a,k}(x)$ is a nowhere differentiable but continuous function [16]. Besides, they found that (25) has a nice application to an open problem concerning digital sums in number theory [18].

Now, we study the relationship between $T^1_{a,k}(x)$ and $G^1_{\alpha,1-\alpha}(x)$.

Lemma 2.1. *The function $G^1_{\alpha,1-\alpha}(x)$ is related to $T^1_{\alpha_R,k}(x)$ by*

$$\left.\frac{\partial^k G^1_{\alpha,1-\alpha}(x)}{\partial \alpha_I^k}\right|_{\alpha=\alpha_R} = i^k k! T^1_{\alpha_R,k}(x), \qquad (k = 0, 1 \ldots).$$

Proof. From Corollary 1.6.1,

$$\frac{\partial}{\partial \alpha_I} G^1_{\alpha,1-\alpha}(x) = i \frac{\partial}{\partial \alpha_R} G^1_{\alpha,1-\alpha}(x) = i \frac{d}{d\alpha} G^1_{\alpha,1-\alpha}(x).$$

Since $\frac{dG^1_{\alpha,1-\alpha}(x)}{d\alpha}$ is also an analytic function with respect to α, we have

$$\frac{\partial}{\partial \alpha_I}\left(\frac{\partial}{\partial \alpha_I} G^1_{\alpha,1-\alpha}(x)\right) = i \frac{\partial}{\partial \alpha_R}\left(i \frac{\partial}{\partial \alpha_R} G^1_{\alpha,1-\alpha}(x)\right) = i \frac{d}{d\alpha}\left(i \frac{d}{d\alpha} G^1_{\alpha,1-\alpha}(x)\right).$$

Iterating gives

$$\frac{\partial^k G^1_{\alpha,1-\alpha}(x)}{\partial \alpha_I^k} = i^k \frac{\partial^k G^1_{\alpha,1-\alpha}(x)}{\partial \alpha_R^k}.$$

If $\alpha_I = 0$, then $G^1_{\alpha,1-\alpha}(x) = M^1_{\alpha_R}(x)$.

Hence, for $\alpha_R \in (0, 1)$,

$$\left.\frac{\partial^k G^1_{\alpha,1-\alpha}(x)}{\partial \alpha_I^k}\right|_{\alpha=\alpha_R} = i^k \frac{\partial^k G^1_{\alpha,1-\alpha}(x)}{\partial \alpha_R^k} = i^k k! T^1_{\alpha_R,k}(x).$$

□

Example 2.1. If $k = 1$,

$$\left.\frac{\partial G^1_{\alpha,1-\alpha}(x)}{\partial \alpha_I}\right|_{\alpha=1/2} = 2iT^1(x).$$

Since $T^1(x)$ is a real valued function, we obtain the following relationship between the Takagi function and $G^1_{\alpha,1-\alpha}(x)$.

$$\operatorname{Im} \left.\frac{\partial G^1_{\alpha,1-\alpha}(x)}{\partial \alpha_I}\right|_{\alpha=1/2} = 2T^1(x).$$

Furthermore, we can see the difference between the real part and imaginary part of $G^1_{\alpha,1-\alpha}(x)$ from Lemma 2.1.

Proposition 2.2. *We have*

$$\operatorname{Re} G^1_{\alpha,1-\alpha}(x) = M^1_{\alpha_R}(x) + \sum_{n=1}^{\infty} \alpha_I^{2n} (-1)^n T^1_{\alpha_R,2n}(x),$$

$$\operatorname{Im} G^1_{\alpha,1-\alpha}(x) = \sum_{n=0}^{\infty} \alpha_I^{2n+1} (-1)^n T^1_{\alpha_R,2n+1}(x), \qquad 0 \leq x \leq 1.$$

Proof. From Corollary 1.6.1, $G^1_{\alpha,1-\alpha}(x)$ is analytic with respect to $\alpha \in W = \{z \in \mathbb{C}; |z| < 1, |1-z| < 1\}$. Therefore, it has the Taylor expansion with $\alpha = \alpha_R \in (0,1)$.

$$\begin{aligned} G^1_{\alpha,1-\alpha}(x) = G^1_{\alpha_R,1-\alpha_R}(x) + \alpha_I \left.\frac{\partial G^1_{\alpha,1-\alpha}(x)}{\partial \alpha_I}\right|_{\alpha=\alpha_R} + \frac{\alpha_I^2}{2!} \left.\frac{\partial^2 G^1_{\alpha,1-\alpha}(x)}{\partial \alpha_I^2}\right|_{\alpha=\alpha_R} + \cdots \\ + \frac{\alpha_I^{n-1}}{(n-1)!} \left.\frac{\partial^{n-1} G^1_{\alpha,1-\alpha}(x)}{\partial \alpha_I^{n-1}}\right|_{\alpha=\alpha_R} + R_n(x). \end{aligned}$$

From Lemma 2.1,

$$\begin{aligned} G^1_{\alpha,1-\alpha}(x) &= M^1_{\alpha_R}(x) + i\alpha_I T^1_{\alpha_R,1}(x) + (-1)\alpha_I^2 T^1_{\alpha_R,2}(x) + (-i)\alpha_I^3 T^1_{\alpha_R,3}(x) + \cdots \\ &= M^1_{\alpha_R}(x) + \sum_{n=1}^{\infty} \alpha_I^{2n}(-1)^n T^1_{\alpha_R,2n}(x) + i\sum_{n=0}^{\infty} \alpha_I^{2n+1}(-1)^n T^1_{\alpha_R,2n+1}(x). \end{aligned}$$

□

Proposition 2.2 shows that the real part of $G^1_{\alpha,1-\alpha}(x)$ can be expressed in terms of the derivatives of even order of Lebesgue's singular function, and the imaginary part is expressed in terms of the derivatives of odd order of Lebesgue's singular function.

From Example 2.1, we know that the first class of complex-valued functions $\{G^1_{\alpha,1-\alpha}(x)\}$ has a connection with the Takagi function $T^1(x)$.

Next, we investigate the other classes $\{G^i_{\alpha,1-\alpha}(x)\}$ $(i = 2,3,4)$ in a similar way. Based on the relationship between $T^1(x)$ and $G^1_{\alpha,1-\alpha}(x)$, we can define each function $T^i(x)$ $(i = 2,3,4)$ as follows :

$$T^i(x) = \frac{1}{2}\,\mathrm{Im} \left.\frac{\partial G^i_{\alpha,1-\alpha}(x)}{\partial \alpha_I}\right|_{\alpha=1/2}.$$

since each $G^i_{\alpha,1-\alpha}(x)$ is differentiable with respect to α_I.

First, we analyze $T^2(x)$. $T^2(x)$ is the unique solution of the following functional equation.

$$T^2(x) = \begin{cases} (-1/2)T^2(2x) + x, & 0 \le x \le 1/2, \\ (-1/2)T^2(2x-1) + (1-x), & 1/2 \le x \le 1. \end{cases} \tag{26}$$

Because from (13),

$$\left.\frac{\partial G^2_{\alpha,1-\alpha}(x)}{\partial \alpha_I}\right|_{\alpha=\frac{1}{2}} = \begin{cases} i\overline{G^2_{\alpha,1-\alpha}(2x)} + \dfrac{1}{2} \left.\overline{\dfrac{\partial G^2_{\alpha,1-\alpha}(2x)}{\partial \alpha_I}}\right|_{\alpha=\frac{1}{2}}, & 0 \le x < 1/2, \\ -i\overline{G^2_{\alpha,1-\alpha}(2x-1)} + \dfrac{1}{2} \left.\overline{\dfrac{\partial G^2_{\alpha,1-\alpha}(2x-1)}{\partial \alpha_I}}\right|_{\alpha=\frac{1}{2}} + i, & 1/2 \le x \le 1. \end{cases}$$

Since $G^2_{1/2,0}(x) = x$,

$$\left.\frac{\partial G^2_{\alpha,1-\alpha}(x)}{\partial \alpha_I}\right|_{\alpha=\frac{1}{2}} = \begin{cases} \dfrac{1}{2} \left.\overline{\dfrac{\partial G^2_{\alpha,1-\alpha}(2x)}{\partial \alpha_I}}\right|_{\alpha=\frac{1}{2}} + 2ix, & 0 \le x < 1/2, \\ \dfrac{1}{2} \left.\overline{\dfrac{\partial G^2_{\alpha,1-\alpha}(2x-1)}{\partial \alpha_I}}\right|_{\alpha=\frac{1}{2}} + 2i(1-x), & 1/2 \le x \le 1. \end{cases}$$

Therefore, (26) can be obtained.

Recall the following three theorems.

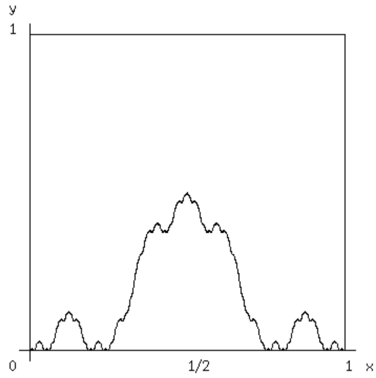

Figure 4: $T^2(x)$

Theorem 2.1 (Yamaguti-Hata [21]). *Let $(t,x) \in (-1,1) \times [0,1]$, $\psi : [0,1] \to [0,1]$ and $g : [0,1] \to \mathbb{R}$. The functional equation $F(t,x) = tF(t,\psi(x)) + g(x)$ has an unique solution $F(t,x)$, which is given by $F(t,x) = \sum_{n=0}^{\infty} t^n g(\psi^n(x))$.*

Theorem 2.2 (Hata-Yamaguti [22]). *The series $f(x) = \sum_{n=0}^{\infty} c_n \varphi^n(x)$ with $f(0) = f(1) = 0$, is defines a continuous function, iff $\sum_{n=0}^{\infty} |c_n| < \infty$.*

Theorem 2.3 (Kono [11]). *The series $f(x) = \sum_{n=0}^{\infty} c_n \varphi^n(x)$ with $f(0) = f(1) = 0$, has no finite derivative at any point, iff $\lim_{n\to\infty} \sup 2^n |c_n| > 0$.*

Applying the above theorems gives the following proposition.

Proposition 2.3. *$T^2(x)$ is a nowhere differentiable but continuous function having the following expression.*

$$T^2(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\varphi^n(x)}{2^n}, \qquad x \in [0,1]. \tag{27}$$

Proof. In Theorem 2.1, let

$$\psi(x) = \varphi(x) = \begin{cases} 2x, & 0 \leq x \leq 1/2, \\ 2(1-x), & 1/2 \leq x \leq 1, \end{cases}$$

$g(x) = \varphi(x)/2$, and $t = -1/2$.

We have the following equation which corresponds to (26).

$$F(-\frac{1}{2}, x) = -\frac{1}{2} F(-\frac{1}{2}, \varphi(x)) + \frac{\varphi(x)}{2}.$$

Therefore, we have

$$T^2(x) = F(-\frac{1}{2}, x) = \sum_{n=0}^{\infty} (-\frac{1}{2})^n \frac{\varphi^{n+1}(x)}{2}.$$

From Theorem 2.2 and Theorem 2.3, it is clear that $T^2(x)$ is a nowhere differentiable but continuous function. ∎

Figures 1 and 4 show the graphs of $T^1(x)$ and $T^2(x)$. Although both $T^1(x)$ and $T^2(x)$ are symmetric, continuous and nowhere differentiable, their graphs are quite different. Observe that the graph of $T^2(x)$

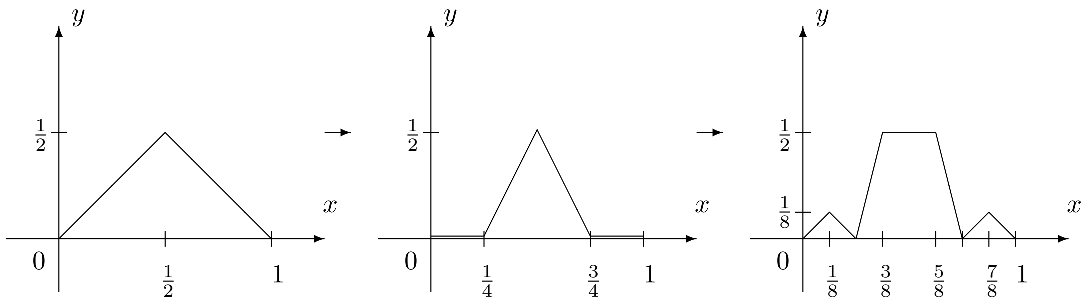

Figure 5: Construction of $T^2(x)$

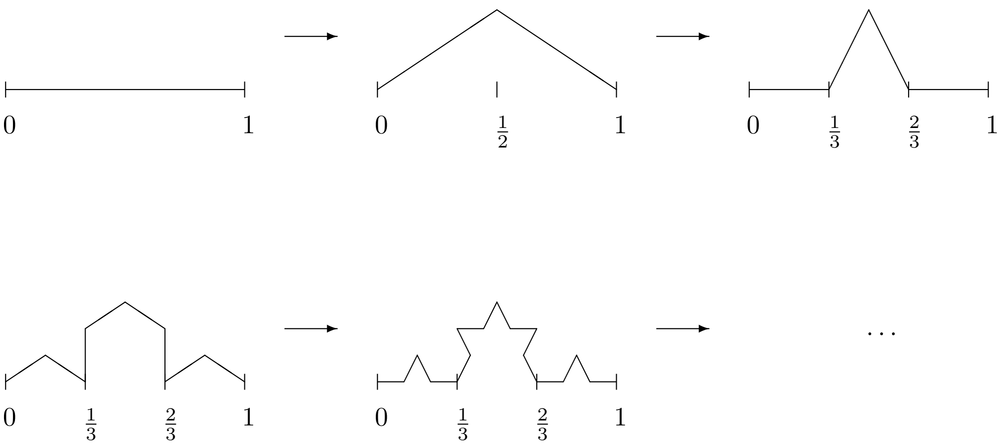

Figure 6: Construction of the Von Koch curve

appears to be somewhat similar to the Von Koch curve, a binary self-similar set, given as the image of $G^2_{\alpha,1-\alpha}(x)$, $(\alpha = 1/2 + (\sqrt{3}/6)i)$.

This is not surprising. Figure 5 shows how the graph of $T^2(x)$ can be constructed step by step from (27). Figure 6 shows the first five steps of the construction of the Von Koch curve. The analogy between the two constructions is evident.

Thus, $T^2(x)$ is a real valued function whose graph has essentially the same geometric structure as the binary self-similar sets, given as the image of $G^2_{\alpha,1-\alpha}(x)$.

Notice, however, that the graph of $T^2(x)$ is not itself a binary self-similar set.

Similarly, we study $T^3(x)$.

$T^3(x)$ is the unique solution of the following functional equation.

$$T^3(x) = \begin{cases} (1/2)T^3(2x) + x, & 0 \le x \le 1/2, \\ (-1/2)T^3(2x-1) + (1-x), & 1/2 \le x \le 1. \end{cases} \tag{28}$$

Since (28) was not studied in Theorem 2.1, we must consider finding an expression of $T^3(x)$ directly from (28).

Before stating this result, some notations need to be introduced. The binary expansion of $j \in \mathbb{N}$ is denoted by $j = \sum_{n=0}^{\infty} j_n 2^n$ with $j_n \in \{0, 1\}$, and $[x]$ denotes the greatest integer less than or equal to x.

Define
$$s_j = (-1)^{s(j)}, \tag{29}$$
where $s(j) = \sum_{n=0}^{\infty} j_n$.

Lemma 2.4. *$T^3(x)$ has the following exact expression.*
$$T^3(x) = \sum_{n=1}^{\infty} s_{[2^{n-1}x]} \frac{\varphi^n(x)}{2^n}, \qquad x \in [0, 1]. \tag{30}$$

Proof. Note that $s_{[2^n x]} = -s_{[2^n x - 2^{n-1}]}$ if $1/2 \leq x \leq 1$. It is straightforward to prove that (30) satisfies (28). □

Applying Theorem 2.2 and Theorem 2.3 to (30) gives the following theorem.

Theorem 2.4. *$T^3(x)$ is a nowhere differentiable but continuous function.*

Figure 7 shows the graph of $T^3(x)$. The graph of $T^3(x)$ is not symmetric, because the coefficient of $T^3(x)$ depends on not only n but also on x.

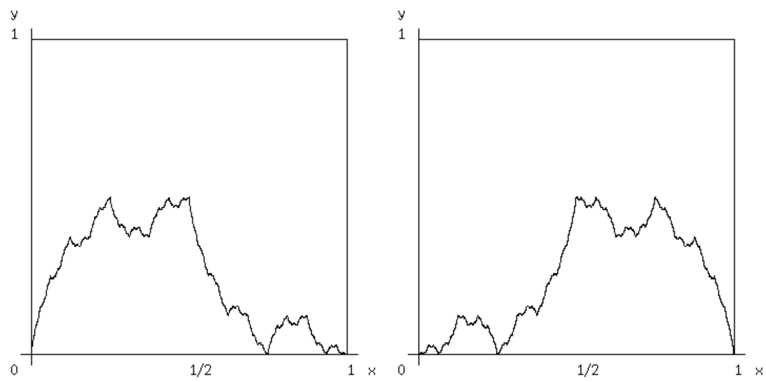

Figure 7: $T^3(x)$ and $T^4(x)$

Finally, we should mention $T^4(x)$. Since the following relationship between $T^3(x)$ and $T^4(x)$ holds
$$T^4(x) = T^3(1 - x),$$
it follows easily from (30) that
$$T^4(x) = \sum_{n=1}^{\infty} (-1)^{n-1} s_{[2^{n-1}x]} \frac{\varphi^n(x)}{2^n}, \qquad x \in [0, 1].$$

Obviously, $T^4(x)$ is also a nowhere differentiable but continuous function, and it is a unique continuous solution of the following functional equation.
$$T^4(x) = \begin{cases} (-1/2)T^4(2x) + x, & 0 \leq x \leq 1/2, \\ (1/2)T^4(2x - 1) + (1 - x), & 1/2 \leq x \leq 1. \end{cases} \tag{31}$$

The graph of $T^i(x)$ $(i = 1, 2, 3, 4)$ can be expressed as a compact set Y satisfying the set equation $Y = \psi_1(Y) \cup \psi_2(Y)$, where

$$\begin{cases} \psi_1(z) &= \lambda_1 z + \lambda_2 \overline{z}, \\ \psi_2(z) &= \overline{\lambda_1} z + \overline{\lambda_2}\overline{z} + (1 - \overline{\lambda_1} - \overline{\lambda_2}), \end{cases} \quad \text{if} \quad i = 1,$$

$$\begin{cases} \psi_1(z) &= \lambda_2 z + \lambda_1 \overline{z}, \\ \psi_2(z) &= \overline{\lambda_2} z + \overline{\lambda_1}\overline{z} + (1 - \overline{\lambda_1} - \overline{\lambda_2}), \end{cases} \quad \text{if} \quad i = 2,$$

$$\begin{cases} \psi_1(z) &= \lambda_1 z + \lambda_2 \overline{z}, \\ \psi_2(z) &= \overline{\lambda_2} z + \overline{\lambda_1}\overline{z} + (1 - \overline{\lambda_1} - \overline{\lambda_2}), \end{cases} \quad \text{if} \quad i = 3,$$

$$\begin{cases} \psi_1(z) &= \lambda_2 z + \lambda_1 \overline{z}, \\ \psi_2(z) &= \overline{\lambda_1} z + \overline{\lambda_2}\overline{z} + (1 - \overline{\lambda_1} - \overline{\lambda_2}), \end{cases} \quad \text{if} \quad i = 4,$$

with $\lambda_1 = 1/2 + i/4$, $\lambda_2 = i/4$.

Note that ψ_1 and ψ_2 are contractions but not similar maps.

Next, based on the relationship between $T^1(x)$ and $M_a^1(x)$, given by Hata-Yamaguti, we find singular functions $M_a^i(x)$ such that

$$2T^i(x) = \left.\frac{\partial M_a^i(x)}{\partial a}\right|_{a=1/2}, \qquad (i = 2, 3, 4).$$

We let a denote a real parameter satisying $0 < a < 1$ and $a \neq 1/2$.

First, we define a real function $M_a^2(x)$ as follows.

Definition 2.1. $M_a^2(x)$ is the unique continuous solution of the following infinitely many difference equations.

$$M_a^2\left(\frac{2j+1}{2^{k+1}}\right) = \left\{\frac{1}{2} - (-1)^k\left(a - \frac{1}{2}\right)\right\} M_a^2\left(\frac{j}{2^k}\right) + \left\{\frac{1}{2} + (-1)^k\left(a - \frac{1}{2}\right)\right\} M_a^2\left(\frac{j+1}{2^k}\right),$$

where $0 \leq j \leq 2^k - 1$, $(k = 0, 1, 2, \cdots)$.

The boundary conditions are $M_a^2(0) = 0$ and $M_a^2(1) = 1$.

Note that Lebesgue's singular function $M_a^1(x)$ also can be expressed as the unique continuous solution of the following difference equations.

$$M_a^1\left(\frac{2j+1}{2^{k+1}}\right) = (1-a) M_a^1\left(\frac{j}{2^k}\right) + a M_a^1\left(\frac{j+1}{2^k}\right),$$

where $0 \leq j \leq 2^k - 1$, $(k = 0, 1, 2, \cdots)$. The boundary conditions are $M_a^1(0) = 0$ and $M_a^1(1) = 1$.

It may be interesting to compare the difference between $M_a^1(x)$ and $M_a^2(x)$.

From this definition, an exact expression of $M_a^2(x)$ is given by the same idea as in Theorem 1.4.

Lemma 2.5. *$M_a^2(x)$ can be expressed in terms of the quaternary expansion of x as follows.*

$$M_a^2(x) = c(\omega_1) + \sum_{n=1}^{\infty} c(\omega_{n+1}) a^{p_0+2p_1+p_3}(1-a)^{p_0+2p_2+p_3}, \qquad 0 \leq x \leq 1,$$

where

$$c(\omega_n) = \begin{cases} 0, & \omega_n = 0, \\ a(1-a), & \omega_n = 1, \\ a, & \omega_n = 2, \\ 1 - a(1-a), & \omega_n = 3. \end{cases}$$

Recall Definition 1.3. From this lemma, it can be proved that $M_a^2(x)$ is a singular function. Moreover, for fixed x, $M_a^2(x)$ is differentiable with respect to a.

On the other hand, for $T^2(x)$, the corresponding difference equations are obtained from (26).

Lemma 2.6. *$T^2(x)$ is the unique solution of the following infinitely many difference equations.*

$$T^2\left(\frac{2j+1}{2^{k+1}}\right) = \frac{1}{2}\left\{T^2\left(\frac{j}{2^k}\right) + T^2\left(\frac{j+1}{2^k}\right)\right\} + \frac{(-1)^k}{2^{k+1}}, \tag{32}$$

where $0 \le j \le 2^k - 1$, $(k = 0, 1, 2, \dots)$. The boundary condition is $T^2(0) = T^2(1) = 0$.

Proof. We can prove that $T^2(x)$ satisfies (32) by (26) and the mathematical induction. □

Theorem 2.5. *We have*

$$2T^2(x) = \left.\frac{\partial M_a^2(x)}{\partial a}\right|_{a=1/2}.$$

Proof. From Definition 2.1, we see $M_{1/2}^2(x) = x$. Therefore,

$$\left.\frac{\partial M_a^2}{\partial a}\left(\frac{2j+1}{2^{k+1}}\right)\right|_{a=1/2} = \frac{1}{2}\left.\frac{\partial M_a^2}{\partial a}\left(\frac{j}{2^k}\right)\right|_{a=1/2} + \frac{1}{2}\left.\frac{\partial M_a^2}{\partial a}\left(\frac{j+1}{2^k}\right)\right|_{a=1/2} + \frac{(-1)^k}{2^k},$$

where

$$\left.\frac{\partial M_a^2(0)}{\partial a}\right|_{a=1/2} = \left.\frac{\partial M_a^2(1)}{\partial a}\right|_{a=1/2} = 0.$$

Applying Lemma 2.6 completes the proof. □

In the same way, we study $M_a^3(x)$.

Definition 2.2. $M_a^3(x)$ is the unique solution of the following infinitely many difference equations.

$$\begin{aligned} M_a^3\left(\frac{4j+1}{2^{k+2}}\right) &= \left\{\frac{1}{2} - s_j\left(a - \frac{1}{2}\right)\right\} M_a^3\left(\frac{j}{2^k}\right) \\ &\quad + \left\{\frac{1}{2} + s_j\left(a - \frac{1}{2}\right)\right\} M_a^3\left(\frac{2j+1}{2^{k+1}}\right). \\ M_a^3\left(\frac{4j+3}{2^{k+2}}\right) &= \left\{\frac{1}{2} + s_j\left(a - \frac{1}{2}\right)\right\} M_a^3\left(\frac{2j+1}{2^{k+1}}\right) \\ &\quad + \left\{\frac{1}{2} - s_j\left(a - \frac{1}{2}\right)\right\} M_a^3\left(\frac{j+1}{2^k}\right), \end{aligned}$$

where $0 \le j \le 2^k - 1$, $(k = 0, 1, 2, \cdots)$ and s_j is given by (29). The boundary conditions are $M_a^3(0) = 0, M_a^3(1) = 1$ and $M_a^3(1/2) = a$.

Lemma 2.7. *$M_a^3(x)$ can be expressed in terms of the binary expansion of x as follows.*

$$M_a^3(x) = \sum_{n=1}^{\infty} \omega_n a^{\alpha(x,n)}(1-a)^{\beta(x,n)}, \qquad 0 \le x \le 1.$$

Here, $\alpha(x,n)$ and $\beta(x,n)$ are defined by

1. $\alpha(x,1) = 1, \beta(x,1) = 0.$

2. For $n \geq 2$,

$$\alpha(x,n) = p^{'}(x,1) + \frac{(n-1) + \sum_{k=1}^{n-1} s_{[2^k x]}}{2}, \qquad \beta(x,n) = q^{'}(x,1) + \frac{(n-1) - \sum_{k=1}^{n-1} s_{[2^k x]}}{2},$$

where

$$p^{'}(x,1) = \begin{cases} 1, & 0 \leq x < 1/2, \\ 0, & 1/2 \leq x \leq 1. \end{cases} \qquad q^{'}(x,1) = \begin{cases} 0, & 0 \leq x < 1/2, \\ 1, & 1/2 \leq x \leq 1. \end{cases}$$

From this Lemma, we see that $M_a^3(x)$ is also singular. Also, for fixed x, $M_a^3(x)$ is differentiable with respect to a.

From (28), the following difference equations can be derived.

Lemma 2.8. *$T^3(x)$ is the unique continuous solution of the following infinitely many difference equations.*

$$T^3\left(\frac{4j+1}{2^{k+2}}\right) = \frac{1}{2}\left\{T^3\left(\frac{j}{2^k}\right) + T^3\left(\frac{2j+1}{2^{k+1}}\right)\right\} + \frac{s_j}{2^{k+2}},$$
$$T^3\left(\frac{4j+3}{2^{k+2}}\right) = \frac{1}{2}\left\{T^3\left(\frac{2j+1}{2^{k+1}}\right) + T^3\left(\frac{j+1}{2^k}\right)\right\} - \frac{s_j}{2^{k+2}}.$$

where $0 \leq j \leq 2^k - 1$, ($k = 0,1,2,\ldots$).

The boundary conditions are $T^3(0) = T^3(1) = 0$ and $T^3(1/2) = 1/2$.

Proof. Note $s_j = -s_{j-2^n}$ if $2^n \leq j \leq 2^{n+1} - 1$. Use (28) and induction. □

From Lemma 2.8 and Definition 2.2, then we obtain the following theorem.

Theorem 2.6. *We have*

$$2T^3(x) = \left.\frac{\partial M_a^3(x)}{\partial a}\right|_{a=1/2}.$$

Proof. Analogous to the proof of Theorem 2.5. □

Lastly, we define $M_a^4(x)$ as the unique continuous solution of the following difference equations.

$$M_a^4\left(\frac{4j+1}{2^{k+2}}\right) = \left\{\frac{1}{2} + (-1)^k s_j\left(a - \frac{1}{2}\right)\right\} M_a^4\left(\frac{j}{2^k}\right) + \left\{\frac{1}{2} - (-1)^k s_j\left(a - \frac{1}{2}\right)\right\} M_a^4\left(\frac{2j+1}{2^{k+1}}\right).$$
$$M_a^4\left(\frac{4j+3}{2^{k+2}}\right) = \left\{\frac{1}{2} - (-1)^k s_j\left(a - \frac{1}{2}\right)\right\} M_a^4\left(\frac{2j+1}{2^{k+1}}\right) + \left\{\frac{1}{2} + (-1)^k s_j\left(a - \frac{1}{2}\right)\right\} M_a^4\left(\frac{j+1}{2^k}\right).$$

where $0 \leq j \leq 2^k - 1$, $(k = 0,1,\ldots)$, and the boundary condition are $M_a^4(0) = 0$, $M_a^4(1) = 1$ and $M_a^4(1/2) = a$.

Since $T^4(x) = T^3(1-x)$, it is clear from Theorem 2.6 that $M_a^4(x)$ has the following relationship with $T^4(x)$.

$$2T^4(x) = \left.\frac{\partial M_a^4(x)}{\partial a}\right|_{a=1/2}.$$

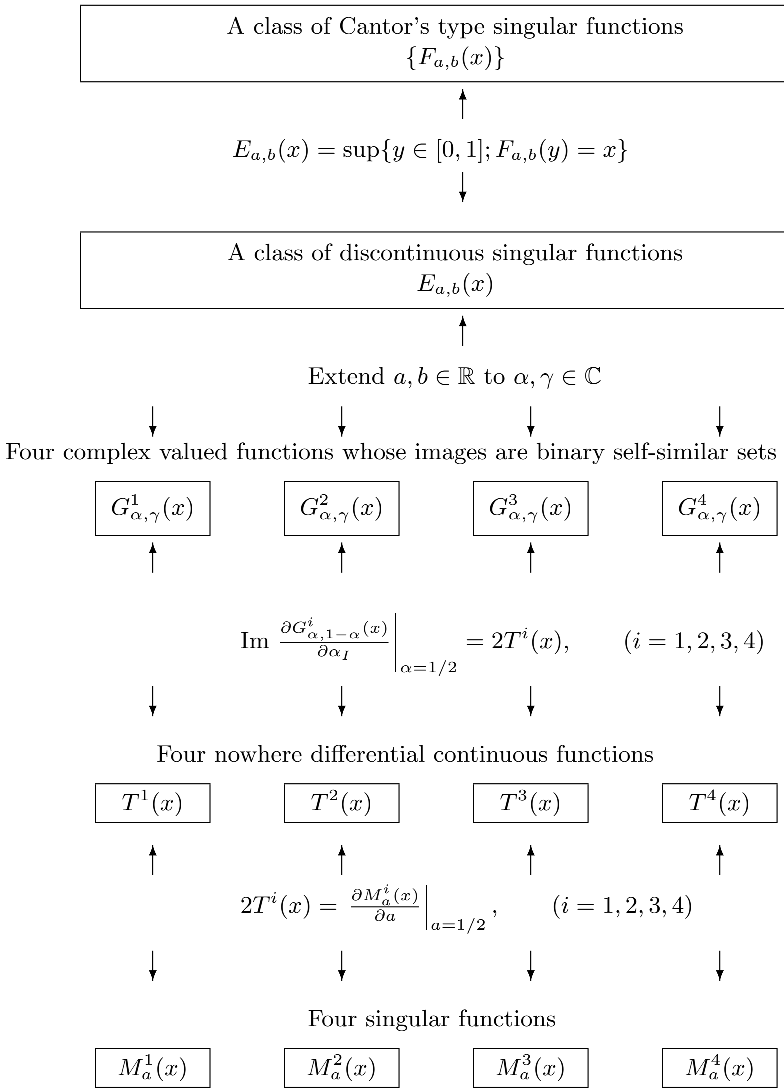

Figure 8: Relationships between functions in this paper

3 Relationship between Cantor's function and binary self-similar sets

In this section, we mention a close relationship between the Cantor function $C(x)$ and binary self-similar sets.

First, we introduce a function $F_{a,b}(x)$ with two real parameters a and b as the unique solution of the following functional equation. (See Figure 9)

$$F_{a,b}(x) = \begin{cases} \frac{1}{2}F_{a,b}(\frac{x}{a}), & 0 \le x \le a, \\ \frac{1}{2}, & a \le x \le a+b, \\ \frac{1}{2}F_{a,b}(\frac{x-a-b}{1-a-b}) + \frac{1}{2}, & a+b \le x \le 1, \end{cases}$$

where $0 < a < 1$, $0 \le b < 1$ and $0 \le a+b < 1$.

This function was studied in [8] and [9]. In these papers, it was proved that $F_{a,b}(x)$ is singular, and the following relationships were obtained:

$$F_{a,0}^{-1}(x) = M_a^1(x), \qquad 0 \le x \le 1,$$

where $a \neq 1/2$, and

$$F_{1/3,1/3}(x) = C(x), \qquad 0 \le x \le 1.$$

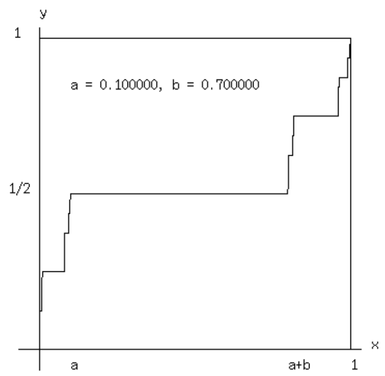

Figure 9: $F_{a,b}(x)$

Next, define $E_{a,b}(x)$ as the generalized inverse function of $F_{a,b}(x)$.

$$E_{a,b}(x) := \sup\{y \in [0,1]; F_{a,b}(y) = x\}.$$

Since $E_{a,b}(x)$ satisfies the functional equation:

$$E_{a,b}(x) = \begin{cases} aE_{a,b}(2x), & 0 \le x < 1/2, \\ (1-a-b)E_{a,b}(2x-1) + (a+b), & 1/2 \le x \le 1, \end{cases} \tag{33}$$

it follows that $E_{a,b}(x) = G^i_{a,1-a-b}(x)$, $(i = 1,2,3,4)$.

Theorem 3.1. *$E_{a,b}(x)$ is singular function. If $b > 0$, $E_{a,b}(x)$ is discontinuous at $x = l/2^k$, for every $1 \le k$ and $1 \le l \le 2^k - 1$.*

Proof. From Corollary 1.2.1, the discontinuity of $E_{a,b}(x)$ is clear.

We prove the singularity of $E_{a,b}(x)$. If $b = 0$, $E_{a,b}(x)$ is Lebesgue's singular function. Since $F_{a,b}(x)$ is a monotone increasing function, it follows that $E_{a,b}(x)$ is a strictly increasing function. Therefore, $E_{a,b}(x)$ is of bounded variation. In short, $E_{a,b}(x)$ is differentiable at almost everywhere. We show that its derivative is zero.

Suppose that $E_{a,b}(x)$, $(b \neq 0)$, is differentiable at a point x_0 in interval $[0,1]$.

Then, for each k, there exists l_k satisfying

$$\frac{l_k}{2^k} < x_0 < \frac{l_k+1}{2^k}$$

and

$$\frac{E_{a,b}(\frac{l_k+1}{2^k}) - E_{a,b}(\frac{l_k}{2^k})}{2^{-k}} < \infty.$$

Since $E_{a,b}(x) = G^i_{a,1-a-b}(x)$, $(i = 1, 2, 3, 4)$, from Theorem 1.2, we have

$$\frac{E_{a,b}(\frac{l_k+1}{2^k}) - E_{a,b}(\frac{l_k}{2^k})}{2^{-k}} \leq \frac{E_{a,0}(\frac{l_k+1}{2^k}) - E_{a,0}(\frac{l_k}{2^k})}{2^{-k}}.$$

Since $E_{a,0}(x)$ is Lebesgue's singular function,

$$\lim_{k\to\infty} \frac{E_{a,0}(\frac{l_k+1}{2^k}) - E_{a,0}(\frac{l_k}{2^k})}{2^{-k}} = 0.$$

From the above inequality, we see that its derivative is zero almost everywhere. This completes the proof. □

4 Applications

In this section, we give three applications of our main theorems.

The first application is to solve an open problem of an exponential sum in number theory. An exponential sum is defined by

$$F(\xi, N) = \sum_{j=0}^{N-1} e^{\xi s(j)}, \qquad N \in \mathbb{N}, \qquad \xi \in \mathbb{C},$$

where $j = \sum_{n=0}^{\infty} j_n 2^n$, $j_n \in \{0, 1\}$, and $s(j) = \sum_{n=0}^{\infty} j_n$.

The explicit formula of exponential sum gives several applications to other fields. For instance,

$$F(\log 2, N) = \sum_{j=0}^{N-1} 2^{s(j)},$$

represents the number of odd numbers appearing in the first N lines of Pascal's triangle. It has been applied in computer science.

Although many authors tried to find simpler representations of $F(\xi, N)$, this problem remained open for about 60 years. Finally, in 1998, Muramoto et al. found the following direct relationship with Lebesgue's singular function $M^1_a(x)$ (See Example 1.1).

Theorem 4.1 (Muramoto-Okada-Sekiguchi-Shiota [15])**.** *Let $t = \log_2 N$. Let $[t]$ and $\{t\}$ denote the integer and decimal part of t, respectively. Evidently $2^{[t]} < N < 2^{[t]+1}$ and $1/2 \leq N/2^{[t]+1} = 1/2^{1-\{t\}} < 1$. For every real number $\xi \neq 0$,*

$$F(\xi, N) = \frac{1}{a^{[t]+1}} M^1_a\left(\frac{1}{2^{1-\{t\}}}\right), \qquad N \in \mathbb{N},$$

where $a = 1/(1 + e^{\xi})$.

Note $M^1_a(x) = G^1_{a,1-a}(x)$. It follows that Theorem 4.1 can be generalized by using Theorem 1.2.

Theorem 4.2. *Let ξ be a complex number, and let $\alpha = 1/(1 + e^{\xi})$. If $|\alpha| < 1$ and $|1 - \alpha| < 1$, then*

$$F(\xi, N) = \frac{1}{\alpha^{[t]+1}} G^1_{\alpha,1-\alpha}\left(\frac{1}{2^{1-\{t\}}}\right), \qquad N \in \mathbb{N}.$$

Proof. From Theorem 1.2, we have

$$\begin{aligned} G^1_{\alpha,1-\alpha}\left(\frac{1}{2^{1-\{t\}}}\right) &= G^1_{\alpha,1-\alpha}\left(\frac{N}{2^{[t]+1}}\right) \\ &= \sum_{n=0}^{N-1}\left\{G^1_{\alpha,1-\alpha}\left(\frac{n+1}{2^{[t]+1}}\right) - G^1_{\alpha,1-\alpha}\left(\frac{n}{2^{[t]+1}}\right)\right\} \\ &= \sum_{n=0}^{N-1}\alpha^{[t]+1-s(n)}(1-\alpha)^{s(n)} \\ &= \alpha^{[t]+1}\sum_{n=0}^{N-1}e^{\xi s(n)} \\ &= \alpha^{[t]+1}F(\xi,N). \end{aligned}$$

□

Remark 4.2.1. If $\xi = \pi i$,

$$F(\pi i, N) = \sum_{j=0}^{N-1}(-1)^{s(j)}. \tag{34}$$

We observe that (34) seems to be similar to the explicit expresion (30) for $T^3(x)$.

Next, as the second application of our main theorem, we give a solution to an open problem raised by Mizutani and Ito in 1987.

Before stating this problem, their results need to be introduced.

Mizutani-Ito showed the following [14]. They defined

$$W = \{(\delta_1, \delta_2, \delta_3, \dots) \in \{0, 1, -1, i, -i\}^{\mathbb{N}}\},$$

as the set of sequences satisfying the revolving condition: For all k, $\delta_{k+1} = 0$, or $\delta_{k+1} = (-i)\delta_{j_0}$, where $j_0 = \max\{j \in N; \delta_j \neq 0, j \leq k\}$.

A set X was defined as follows.

$$X = \{\sum_{k=1}^{\infty}\delta_k(1+i)^{-k}; (\delta_1, \delta_2, \delta_3, \dots) \in W\}.$$

Recall that Dragon is a binary self-similar set, constructed by the pair of similar contractions (8) if $\alpha = 1/2 + i/2$ and $\gamma = 1/2 + i/2$. The Hausdorff dimension is 2, and its parallel translations fill the plane. Dragon was regarded as completely unrelated things with *the Lévy curve.*

In their paper, Mizutani and Ito showed that the set X is a union of four Dragon X_i $(i = 0, 1, 2, 3)$ by using the algebraic method of Dekking [2]. In addition, they pointed out the following interesting problem.

Define a set X^* as follows.

$$X^* = \{\sum_{k=1}^{\infty}\overline{\delta_k}(1+i)^{-k}; (\delta_1, \delta_2, \delta_3, \dots) \in W\}.$$

Obviously, the set $\{(\overline{\delta_1}, \overline{\delta_2}, \overline{\delta_3}, \dots); (\delta_1, \delta_2, \delta_3, \dots) \in W\}$ satisfies the inverse revolving condition. Figure 10 shows a computer simulation of X^*. The shape of this figure led Mizutani and Ito to conjecture that X^* is a collection of the Lévy curves.

We give a proof of their conjecture. It can be obtained from our main theorem.

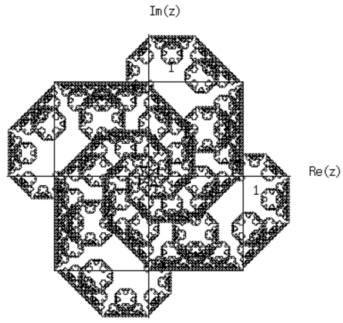

Figure 10: X^*

Let $\overline{X^*}$ be the complex conjugate of the set X^*. Then we have

$$\overline{X^*} = \{\sum_{k=1}^{\infty} \delta_k (1+i)^{-k} i^k; (\delta_1, \delta_2, \delta_3, \dots) \in W\}.$$

Also, we define

$$L_k = \{i^k G^1_{1/2+i/2,1/2-i/2}(x); x \in [0,1]\}, \qquad (k = 0,1,2,3).$$

Recall that the image of $G^1_{1/2+i/2,1/2-i/2}(x)$ is the Lévy curve.

Theorem 4.3.

$$\overline{X^*} = \cup_{k=0,1,2,3} L_k, \tag{35}$$

where each $L_k (k = 0,1,2,3)$ denotes the Lévy curve.

Proof. By Theorem 1.2, we have

$$\begin{aligned} G^1_{1/2+i/2,1/2-i/2}(x) &= i \sum_{n=1}^{\infty} \omega_n (\frac{1}{2})^n (1+i)^{n-q(x,n)} (1-i)^{q(x,n)} \\ &= i \sum_{n=1}^{\infty} \omega_n (\frac{1}{2})^n (1+i)^{n-q(x,n)} (-i(1+i))^{q(x,n)} \\ &= i \sum_{n=1}^{\infty} \omega_n (\frac{1}{2})^n (1+i)^{2n} (1+i)^{-n} (-i)^{q(x,n)} \\ &= i \sum_{n=1}^{\infty} \omega_n (-i)^{q(x,n)} (1+i)^{-n} i^n. \end{aligned}$$

Here, we put $\xi_n = i\omega_n(-i)^{q(x,n)}$. Then we see that $\xi_n \in \{0,1,-1,i,-i\}$ satisfies the revolving condition.

Hence, this completes the proof. □

Lastly, as the third application, we study a discontinuous complex-valued function $G^1_{\alpha,\alpha}(x)$ and compare with $G^1_{\alpha,1-\alpha}(x)$. Note that *Dragon* can be given as the image of $G^1_{\alpha,\alpha}(x)$, $(\alpha = 1/2 + i/2)$, and *the Leúy curve* is given as the image of $G^1_{\alpha,1-\alpha}(x)$, $(\alpha = 1/2 + i/2)$.

First, recall the property of $G^1_{\alpha,1-\alpha}(x)$. It is a continuous complex-valued function having the following relationship with the the Takagi function $T^1(x)$ and Lebesgue's singular function $M^1_a(x)$.

$$\mathrm{Im}\left.\frac{\partial G^1_{\alpha,1-\alpha}(x)}{\partial \alpha_I}\right|_{\alpha=1/2} = 2T^1(x) = \left.\frac{\partial M^1_a(x)}{\partial a}\right|_{a=1/2}. \tag{36}$$

Then a question arises: how about $G^1_{\alpha,\alpha}(x)$?

Although $G^1_{\alpha,\alpha}(x)$ is a discontinuous complex-valued function, it is analytic with respect to $\alpha \in D = \{z \in \mathbb{C}; |z| < 1\}$.

Define a real-valued function $K^1(x)$ by

$$K^1(x) := \frac{1}{2}\,\mathrm{Im}\left.\frac{\partial G^1_{\alpha,\alpha}(x)}{\partial \alpha_I}\right|_{\alpha=1/2} = \frac{1}{2}\left.\frac{\partial G^1_{\alpha_R,\alpha_R}(x)}{\partial \alpha_R}\right|_{\alpha_R=1/2}. \tag{37}$$

It is surprising that *self-affine dust*, which is a well-known self-similar set, appears as the graph of $K^1(x)$ (See Figure 11), and also the Rademacher series appear as the graph of $G^1_{\alpha_R,\alpha_R}(x)$. More exactly,

$$\begin{aligned} G^1_{\alpha_R,\alpha_R}(x) &= \frac{1-\alpha_R}{\alpha_R}\sum_{n=1}^{\infty}\omega_n\alpha_R^n \\ &= \frac{1}{2} - \frac{1-\alpha_R}{2\alpha_R}\sum_{n=1}^{\infty}\alpha_R^n\phi_n(x), \qquad \alpha_R \in (-1,1), \end{aligned}$$

where $\phi_n(x)$ is a Rademacher function and $\phi_n(x) = (-1)^{\omega_n} = 1 - 2\omega_n$.

Next, compare $T^1(x)$ and $K^1(x)$. Although $T^1(x)$ is continuous and $K^1(x)$ is not, both functions have the following similar representations.

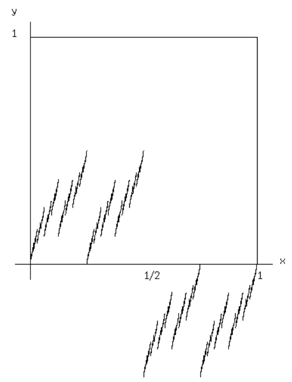

Figure 11: $K^1(x)$

Proposition 4.1. *$T^1(x)$ and $K^1(x)$ can be expressed by*

$$T^1(x) = \sum_{n=1}^{\infty}\omega_n\frac{p(x,n) - \{q(x,n) - 2\}}{2^n}, \qquad 0 \le x \le 1,$$

$$K^1(x) = \sum_{n=1}^{\infty}\omega_n\frac{p(x,n) + \{q(x,n) - 2\}}{2^n}, \qquad 0 \le x \le 1,$$

where $p(x,n) = n - q(x,n)$.

Proof. From Example 1.1 and Theorem 1.2, we have

$$M_a^1(x) = \frac{a}{1-a}\sum_{n=1}^{\infty}\omega_n a^{n-q(x,n)}(1-a)^{q(x,n)},$$

$$G^1_{\alpha_R,\alpha_R}(x) = \sum_{n=1}^{\infty}\omega_n(1-\alpha_R)\alpha_R^{n-1}.$$

From (36) and (37), it follows

$$\begin{aligned}
\left.\frac{\partial M_a^1(x)}{\partial a}\right|_{a=1/2} &= \sum_{n=1}^{\infty}\omega_n\{(p(x,n)+1)(\frac{1}{2})^{n-1} + (1-q(x,n))(\frac{1}{2})^{n-1}\} \\
&= \sum_{n=1}^{\infty}\omega_n\frac{p(x,n)-q(x,n)+2}{2^{n-1}}. \\
\left.\frac{\partial G^1_{\alpha_R,\alpha_R}(x)}{\partial \alpha_R}\right|_{\alpha_R=1/2} &= \sum_{n=1}^{\infty}\omega_n\{-(\frac{1}{2})^{n-1} + (n-1)(\frac{1}{2})^{n-1}\} \\
&= \sum_{n=1}^{\infty}\omega_n\frac{p(x,n)+q(x,n)-2}{2^{n-1}}.
\end{aligned}$$

This completes the proof. □

Note that $K^1(x)$ is the unique solution of the following functional equation.

$$K^1(x) = \begin{cases}(1/2)K^1(2x) + x, & 0 \le x \le 1/2, \\ (1/2)K^1(2x-1) - (1-x), & 1/2 \le x \le 1.\end{cases}$$

We observe that this functional equation is similar to (24), (26), (28) and (31).

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