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3{ 9@Lw[†] 2OM85*;R[‡]
wd@ics.nara-wu.ac.jp kawamura@e.ics.nara-wu.ac.jp

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Koch 6J@~$O Euclid J?LL>e$NE57?E*$J<+8JAj;w=89g$H7FCNil$F$$$k!% Koch Eg$O Koch 6J@~$N;0DN%3%T!<$G0O$^lkJD=89gG"$k!#$3lirr7W;;2DG=@-$N4QE@$+$iD4$Y$k!%K\O@J8$G$O!$8EE5E*7W;;2DG=@-2r@O$N1~MQ$H7F!$ Euclid 6u4V>e$N6J@~$N7W;;2DG=@-$HJD=89g$N7W;;2DG=@-$rDj5A$7!$ Koch 6J@~$O7W;;2DG=$J6J@~G"$j!$ Koch 6J@~$b Koch Eg$b7W;;2DG=$JJD=89g$G$"$k3H$r<($9!%

Computability of Koch Curve and Koch Island

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Abstract

Koch curve is known as a typical self-similar set on Euclidean plane. Koch island is a closed set surrounded by three copies of Koch curve. We investigate them from the viewpoint of computability. In this paper, we define computability of a curve and that of a closed set as an application of classical computable analisys to Euclidean spaces and show that Koch curve is a computable curve and both Koch curve and Koch island are computable closed sets.

1 Introduction

Our aim in this paper is finding fundamental mathematical tools to investigate self-similar sets from the viewpoint of computability.

First of all, we should determine what "computability" means. We have already obtained a mathematical theory to investigate computability of real functions[6][7], which is often referred as "classical computable analysis". To investigate curves and closed sets in Euclidean spaces from the viewpoint of computability, we should apply classical computable analysis to curves and closed sets. In other words, we should define computability of a curve and computability of a closed set by using the terms on classical computable analysis.

Defining computability of a curve is a straightforward task. A curve is (or can be identified with) a continuous function from an interval. We define a curve to be computable if it is a computable function from the interval $[0, 1]$.

For example, any segment with computable endpoints is constructed from a computable curve since it is constructed from a computable function $f : [0, 1] \to \mathbb{R}^q$ such that $f(t) = (1 - t)a + tb$ where a and b are the endpoints.

There is no loss of generality in the restriction of the domain to $[0,1]$. If $[a,b]$ is an interval with computable endpoints, then the computability of $f : [a,b] \to \mathbb{R}^q$ is equivalent to the computability of $\tilde{f} : [0,1] \to \mathbb{R}^q$ defined by $\tilde{f}(t) = f(a+(b-a)t)$ for $t \in [0,1]$.

On the other hand, it is not so straightforward in case of closed sets. Let X be a closed set on $\mathbb{R}^q$. Define $f_X : \mathbb{R}^q \to \mathbb{R}$ by

$$f_X(x) = \begin{cases} 0 & \text{if } x \in X, \\ 1 & \text{if } x \notin X. \end{cases} \tag{1}$$

Since any computable function is continuous, f_X is not computable unless X is either $\emptyset$ or $\mathbb{R}^q$. This function f_X is useless for our aim.

Rather than (1), we may find, for a closed set X on $\mathbb{R}^q$, a computable function $g : \mathbb{R}^q \to \mathbb{R}$ satisfying

$$\begin{aligned} g(x) &= 0 \qquad \text{if } x \in X, \\ g(x) &\neq 0 \qquad \text{if } x \notin X. \end{aligned} \tag{2}$$

For example, for $\{(x_1,x_2) \in \mathbb{R}^2 : {x_1}^2 + {x_2}^2 = 1\}$, the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x_1,x_2) = {x_1}^2 + {x_2}^2 - 1$ is such a function.

It may thus be useful for our aim to define a closed set X on $\mathbb{R}^q$ to be computable if there exists a computable function satisfying the condition (2).

Be careful that these definitions are position-sensitive. In other words, all of the curves (closed sets) congruent to a computable curve (closed set) are not computable. For example, let a be an uncomputable real. Then, $\{(x_1,x_2) \in \mathbb{R}^2 : (x_1 - a)^2 + {x_2}^2 = 1\}$ is not a computable closed set although it is congruent to the computable closed set $\{(x_1,x_2) \in \mathbb{R}^2 : {x_1}^2 + {x_2}^2 = 1\}$.

Finding a function satisfying the condition (2) has the following application. Let X be a closed set on $\mathbb{R}^q$ such that there exists a function $g : \mathbb{R}^q \to \mathbb{R}$ satisfying (2) and x a computable point on $\mathbb{R}^q$. If $x \notin X$, then there exists an effective procedure that shows this. More precisely, there exists a semi-algorithm that terminates with returning `false` for an input $(n_1,\ldots,n_q) \in \mathbb{N}^q$ if $(x_1,\ldots,x_q) \notin X$ where $x_1,\ldots,x_q$ are the computable reals indexed with $n_1,\ldots,n_q$ respectively.

There is however no semi-algorithm that terminates with returning `true` for an input $(n_1,\ldots,n_q) \in \mathbb{N}^q$ if $(x_1,\ldots,x_q) \in X$ where $x_1,\ldots,x_q$ are the same as above since there is no semi-algorithm that terminates with returning `true` for an input $n \in \mathbb{N}$ if n indexes 0. There is either no semi-algorithm that terminates with returning `false` for an input $(n_1,\ldots,n_q) \in \mathbb{N}^q$ *if and only if* $(x_1,\ldots,x_q) \notin X$ where $x_1,\ldots,x_q$ are the same as above since "there exists a computable real that is indexed with n" is not a recursively enumerable predicate on n.

In case of $X = \{(x_1,x_2) \in \mathbb{R}^2 : {x_1}^2 + {x_2}^2 = 1\}$, and $g : \mathbb{R}^2 \to \mathbb{R}$ defined by $g(x_1,x_2) = {x_1}^2 + {x_2}^2 - 1$, the closed set X is a Jordan closed curve and the computable function g satisfies a stronger condition:

$$\begin{aligned} g(x) &= 0 \qquad \text{if } x \text{ is on } X, \\ g(x) &< 0 \qquad \text{if } x \text{ is inside } X, \\ g(x) &> 0 \qquad \text{if } x \text{ is outside } X. \end{aligned} \tag{3}$$

Similarly to the case of condition (2), finding a function satisfying the stronger condition (3) has the following application. Let X be a Jordan closed curve such that there exists a function $g : \mathbb{R}^q \to \mathbb{R}$ satisfying (3) and x a computable point on $\mathbb{R}^q$. If x is not on X, then there exists an effective procedure that determines whether x is inside X or outside X. More precisely, there exists a semi-algorithm that terminates with returning `inside` or `outside` for an input $(n_1,\ldots,n_q) \in \mathbb{N}^q$ if $(x_1,\ldots,x_q)$ is inside X or outside X respectively where $x_1,\ldots,x_q$ are the computable reals indexed with $n_1,\ldots,n_q$ respectively.

The next thing we should do is recalling what "self-similarity" means. A set is *self-similar* if it is constructed from some miniatures of the whole[3][4][5]. More precisely, for any finitely many contractions $F_1,\ldots,F_m$ on $\mathbb{R}^q$, the set equation

$$X = F_1(X) \cup \cdots \cup F_m(X)$$

has a unique nonempty compact solution. Any set that is a solution of a set equation of this form is called a self-similar set[4][5].

We will introduce three closed sets named Koch curve, Koch coastline, and Koch island. Koch curve is a self-similar set. Koch coastline and Koch island are closed sets constructed from Koch curve. The first closed set we introduce is *Koch curve*. In Figure 1, let $\triangle abc$ be an isosceles triangle with lengths $\|a-b\| = \|a-c\| = 1/\sqrt{3}$ and $\|b-c\| = 1$. Let a_1 and a_2 be the points trisecting the edge bc. Let T_0 and T_1 be the similarity transformations that map $\triangle abc$ onto $\triangle a_1ba$ and $\triangle a_2ac$ respectively. *Koch curve* is the unique nonempty compact solution of the set equation

$$X = T_0(X) \cup T_1(X).$$

Figure 2 illustrates Koch curve.

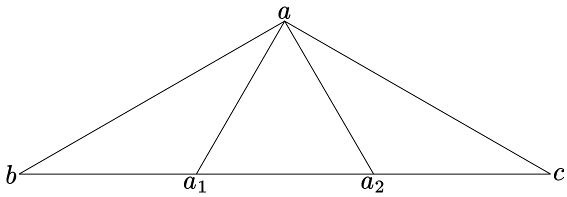

Figure 1: Similarity transformations for constructing Koch curve

We now briefly recall some of the properties of Koch curve. Koch curve is a self-similar nonempty compact set. Although Koch curve has a null area, it has an infinite length. Koch curve is a Jordan arc. There is no differentiable curve that constructs Koch curve. These properties can be easily checked from the definition.

The second closed set we introduce is *Koch coastline*. Koch coastline is a closed set constructed from three copies of Koch curve as Figure 3. Koch coastline is a Jordan closed curve.

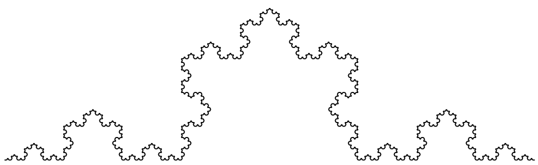

Figure 2: Koch curve

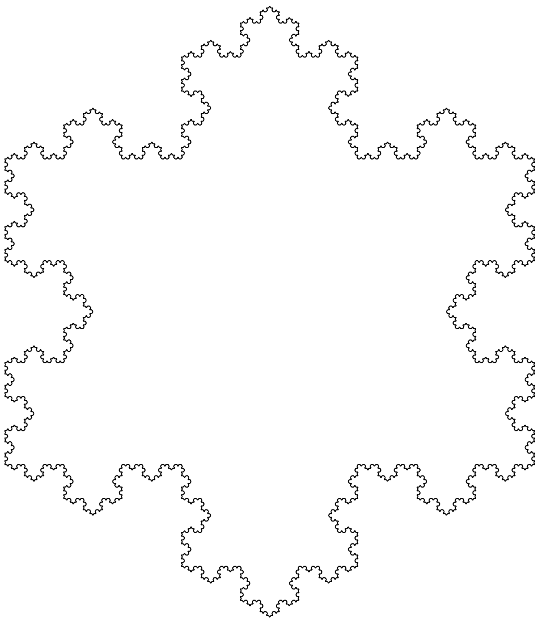

Figure 3: Koch coastline

The last closed set we introduce is *Koch island*. Denote Koch coastline by γ and the inner domain of γ by D. Koch island is defined to be $\gamma \cup D$.

We can easily show that a segment with computable endpoints is constructed from a computable curve. Then a question arises; how about Koch curve? Both a segment and Koch curve are Jordan arcs. One of the major differences between a segment and Koch curve is that Koch curve is a totally non-differentiable curve while a segment is a differentiable curve. Another is that Koch curve has an infinite length while a segment has a finite length. Do these differences affect to the computability of Koch curve? The same question arises on Koch coastline comparing with a circle. A similar question on computability of closed sets also arises on Koch island comparing with a closed disc.

We have obtained the answers to these questions. Koch curve is constructed from a computable curve. So is Koch coastline. Koch island is a computable closed set. One of the most important facts in showing this is that each of these closed sets is a limit of unions of computable segments or computable triangles.

2 Preliminary

We use the terminology on classical computable analysis in [6]. We will identify a point (x, y) and a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ throughout this paper. We write:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \qquad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

First, we will define Koch curve.

Definition 2.1. With $T_0, T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T_0(x) = \frac{1}{\sqrt{3}} R_{\pi/6} J x,$$

$$T_1(x) = \frac{1}{\sqrt{3}} R_{-\pi/6} J (x - e_1) + e_1,$$

Koch curve is the unique nonempty compact solution of the equation

$$X = T_0(X) \cup T_1(X).$$

There are many ways to construct Koch curve. We will introduce one which starts with a segment. With the transformations T_0 and T_1 in Definition 2.1, we define Γ_n recursively by

$$\Gamma_0 = \{(t,0) : t \in [0,1]\},$$
$$\Gamma_{n+1} = T_0(\Gamma_n) \cup T_1(\Gamma_n).$$

Then Koch curve coincides with $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} \Gamma_n}$.

Calculation of some of the beginning terms yields the following.

- Γ_0 is a segment connecting $(0,0)$ and $(1,0)$.
- Γ_1 is a polygonal line connecting $(0,0)$, $(1/2, 1/2\sqrt{3})$, and $(1,0)$ in this order.
- Γ_2 is a polygonal line connecting $(0,0)$, $(1/3,0)$, $(1/2,1/2\sqrt{3})$, $(2/3,0)$, and $(1,0)$ in this order.
- Γ_3 is a polygonal line connecting $(0,0)$, $(1/6,1/6\sqrt{3})$, $(1/3,0)$, $(1/3,1/3\sqrt{3})$, $(1/2,1/2\sqrt{3})$, $(2/3,1/3\sqrt{3})$, $(2/3,0)$, $(5/6,1/6\sqrt{3})$, and $(1,0)$ in this order.
- Γ_4 is a polygonal line connecting $(0,0)$, $(1/9,0)$, $(1/6,1/6\sqrt{3})$, $(2/9,0)$, $(1/3,0)$, $(7/18,1/6\sqrt{3})$, $(1/3,1/3\sqrt{3})$, $(4/9,1/9\sqrt{3})$, $(1/2,1/2\sqrt{3})$, $(5/9,1/9\sqrt{3})$, $(2/3,1/3\sqrt{3})$, $(11/18,1/6\sqrt{3})$, $(2/3,0)$, $(7/9,0)$, $(5/6,1/6\sqrt{3})$, $(8/9,0)$, and $(1,0)$ in this order.

Figure 4 illustrate these steps.

By using Koch curve, we will establish the following definition.

Definition 2.2. With $T, T' : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(x) = R_{-2\pi/3}x + e_1,$$
$$T'(x) = R_{2\pi/3}(x - e_1),$$

Koch coastline is $\gamma \cup T(\gamma) \cup T'(\gamma)$ where γ denotes Koch curve.

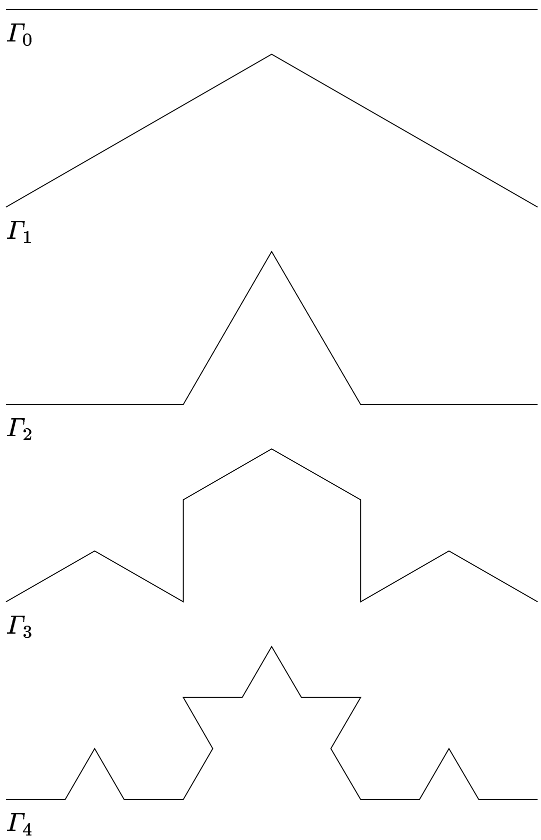

Figure 4: Some beginning steps of construction of Koch curve

Koch coastline is a Jordan closed curve. Thus the following is well-defined.

Definition 2.3. Denote the inner domain of Koch coastline by D. Then, *Koch island* is $\overline{D}$.

3 Computability of curves and closed sets

3.1 general results

As explained in the introduction, our first task is defining the computability of curves and closed sets on an Euclidean space. We proceed this task by using the traditional definitions on the computability of real functions.

In this paper, we consider a curve on $\mathbb{R}^q$ to be a continuous function from an interval to $\mathbb{R}^q$. We say a curve f constructs a set γ if $f(I) = \gamma$ where I is the domain of f.

In addition, we define computable curves and computable closed sets as follows.

Definition 3.1. A *computable curve* on $\mathbb{R}^q$ is a computable function $f : [0, 1] \to \mathbb{R}^q$.

Definition 3.2. A *computable closed set* on $\mathbb{R}^q$ is a subset of $\mathbb{R}^q$ such that there exists a computable function $g : \mathbb{R}^q \to \mathbb{R}$ satisfying

$$\begin{aligned} g(x) &= 0 && \text{if } x \in X, \\ g(x) &\neq 0 && \text{if } x \notin X. \end{aligned}$$

A computable closed set is a closed set since it is an inverse image of a closed set by a continuous function.

For a nonempty subset S of $\mathbb{R}^q$, we define $\underline{d}_S : \mathbb{R}^q \to \mathbb{R}$ by

$$\underline{d}_S(x) = \inf\{\|x - y\| : y \in S\}.$$

The function $\underline{d}_S$ is well-defined since the set $\{\|x-y\| : y \in S\}$ is nonempty and bounded below.

Lemma 3.1. *Let $\{S_n\}$ be an arbitrary sequence of nonempty subsets of $\mathbb{R}^q$. Then, the sequence of functions $\{\underline{d}_{S_n}\}$ is effectively uniformly continuous.*

Proof. Let x and y be arbitrary points on $\mathbb{R}^q$.

For any $n \in \mathbb{N}$ and any $\varepsilon > 0$, there exists $u \in S_n$ such that $\|y - u\| < \underline{d}_{S_n}(y) + \varepsilon$. Then we have

$$\begin{aligned} \underline{d}_{S_n}(x) - \underline{d}_{S_n}(y) &< \|x - u\| - \|y - u\| + \varepsilon \\ &\leq \|x - y\| + \varepsilon. \end{aligned}$$

From the arbitrariness of ε, we obtain

$$\underline{d}_{S_n}(x) - \underline{d}_{S_n}(y) \leq \|x - y\|.$$

By exchanging x and y in this argument, we obtain

$$\underline{d}_{S_n}(y) - \underline{d}_{S_n}(x) \leq \|x - y\|.$$

Hence we have

$$|\underline{d}_{S_n}(x) - \underline{d}_{S_n}(y)| \leq \|x - y\|.$$

This implies that $\{\underline{d}_{S_n}\}$ is effectively uniformly continuous. □

Theorem 3.1. *Any computable curve on $\mathbb{R}^q$ constructs a computable closed set.*

Proof. Let $f : [0, 1] \to \mathbb{R}^q$ be a computable function. We will show that $\underline{d}_{f([0,1])}$ is a computable function satisfying the condition in Definition 3.2.

It is immediate from $f([0, 1])$ being a closed set that $\underline{d}_{f([0,1])}(x) = 0$ iff $x \in f([0, 1])$. We easily obtain that $\underline{d}_{f([0,1])}$ is effectively uniformly continuous as a special case of Lemma 3.1. The remaining is sequential computability of $\underline{d}_{f([0,1])}$.

Let $\{x_n\}$ be an arbitrary computable sequence of points on $\mathbb{R}^q$. We will check that $\{\underline{d}_{f([0,1])}(x_n)\}_{n\in\mathbb{N}}$ is a computable sequence of reals in order to show that $\underline{d}_{f([0,1])}$ is sequentially computable. Define $g_n : [0, 1] \to \mathbb{R}$ for $n \in \mathbb{N}$ by

$$g_n(t) = \|x_n - f(t)\|.$$

It is obvious that $\{g_n\}$ is a computable sequence of functions. By using the effective version of Max-Min Theorem, we obtain that $\{\min_{t\in[0,1]} g_n(t)\}_{n\in\mathbb{N}}$ is a computable sequence of reals. Namely, $\{\underline{d}_{f([0,1])}(x_n)\}$ is a computable sequence of reals. Hence $\underline{d}_{f([0,1])}$ is sequentially computable.

Now we have established that $\underline{d}_{f([0,1])}$ is a computable function satisfying the condition in Definition 3.2. Thus $f([0, 1])$ is a computable closed set. □

3.2 Koch curve and Koch coastline

We are now ready to investigate computability of Koch curve, etc.

Lemma 3.2. *In the notation of Definition 2.1, define $f_n : [0, 1] \to \mathbb{R}^2$ recursively by*

$$f_0(t) = \begin{pmatrix} t \\ 0 \end{pmatrix},$$

$$f_{n+1}(t) = \begin{cases} T_0(f_n(2t)) & \textit{if } t \in [0, 1/2], \\ T_1(f_n(2t - 1)) & \textit{if } t \in [1/2, 1]. \end{cases}$$

Then, $\{f_n\}$ is a computable sequence of functions.

Proof. By induction on n, it is straightforward to show that each f_n is well-defined and satisfies

$$f_n(t) = (T_{b_{n-1}} \circ \cdots \circ T_{b_0})(f_0(2^n t - K)) \quad \text{if } t \in [K/2^n, (K + 1)/2^n]$$

where $b_0, \ldots, b_{n-1} \in \{0, 1\}$ and $2^0 b_0 + \cdots + 2^{n-1} b_{n-1} = K$.

Clearly, $\{f_n\}$ is effectively uniformly continuous. We however find a difficulty here in showing that $\{f_n\}$

is sequentially computable. Calculation of an integer K and a bit string $b_0, \dots, b_{n-1}$ from a real t is not effective.

Take any computable sequence $\{t_k\} \subset [0,1]$. There exists a double sequence of rationals $\{r_{jk}\}$ such that $r_{jk} \to t_k$ effectively in n, j, and k as $j \to \infty$. To overcome the difficulty, we shall investigate the triple sequence of points $\{f_n(r_{jk})\}$.

We have

$$f_n(r_{jk}) = (T_{b_{n-1}} \circ \cdots \circ T_{b_0})(f_0(2^n r_{jk} - K)) \quad \text{if } r_{jk} \in [K/2^n, (K+1)/2^n]$$

where $b_0, \dots, b_{n-1} \in \{0,1\}$ and $2^0 b_0 + \cdots + 2^{n-1} b_{n-1} = K$. In this case, computation of K and $b_0, \dots, b_{n-1}$ from n, j and k *is* effective since the relation $\le$ in $\mathbb{Q}$ is effective. More precisely, we can construct a recursive function that computes $\langle K, b_0, \dots, b_{n-1}\rangle$ from n, j, and k by using a recursive function that corresponds to the relation $\le$ in $\mathbb{Q}$. Hence $\{f_n(r_{jk})\}$ is a computable triple sequence of points.

We are ready to show sequential continuity of $\{f_n(t_k)\}$. Since $r_{jk} \to t_k$ effectively in j and k as $j \to \infty$ and $\{f_n\}$ is effectively uniformly continuous, we obtain that $f_n(r_{jk}) \to f_n(t_k)$ effectively in n, j, and k as $j \to \infty$. We conclude that $\{f_n(t_k)\}$ is a computable double sequence of points since it is a limit of a computable and effectively convergent sequence of points. □

Lemma 3.3. *In the notation of Lemma 3.2, the sequence of functions $\{f_n\}$ is effectively uniformly convergent as $n \to \infty$.*

Proof. As a preparation, we show, by induction on n, that for any $n \in \mathbb{N}$ and any $t \in [0,1]$,

$$\|f_n(t) - f_{n+1}(t)\| \le \frac{1}{2\sqrt{3}} \left(\frac{1}{\sqrt{3}}\right)^n.$$

For induction base, evaluate $\|f_0(t) - f_1(t)\|$. We have

$$\|f_0(t) - f_1(t)\| = \frac{t}{\sqrt{3}} \qquad \text{if } t \in [0, 1/2],$$
$$\|f_0(t) - f_1(t)\| = \frac{1-t}{\sqrt{3}} \qquad \text{if } t \in [1/2, 1].$$

Thus, for all $t \in [0,1]$,

$$\|f_0(t) - f_1(t)\| \le \frac{1}{2\sqrt{3}}.$$

For induction step, suppose for any $t \in [0,1]$,

$$\|f_n(t) - f_{n+1}(t)\| \le \frac{1}{2\sqrt{3}} \left(\frac{1}{\sqrt{3}}\right)^n.$$

and evaluate $\|f_{n+1}(t) - f_{n+2}(t)\|$. From the induction hypothesis, we have

$$\begin{aligned}\|f_{n+1}(t) - f_{n+2}(t)\| &= \frac{1}{\sqrt{3}} \|f_n(2t) - f_{n+1}(2t)\| \\ &\qquad \text{if } t \in [0, 1/2],\end{aligned}$$

$$\begin{aligned}\|f_{n+1}(t) - f_{n+2}(t)\| &= \frac{1}{\sqrt{3}} \|f_n(2t-1) - f_{n+1}(2t-1)\| \\ &\qquad \text{if } t \in [1/2, 1].\end{aligned}$$

Thus for any $t \in [0,1]$

$$\|f_{n+1}(t) - f_{n+2}(t)\| \le \frac{1}{2\sqrt{3}} \left(\frac{1}{\sqrt{3}}\right)^{n+1}.$$

We have finished the preparation.

The result above implies that $\{f_n\}$ converges uniformly to a continuous function as $n \to \infty$. Using f for the limit, we have

$$\begin{aligned}\|f_n(t) - f(t)\| &\le \sum_{k=n}^{\infty} \|f_k(t) - f_{k+1}(t)\| \\ &\le \sum_{k=n}^{\infty} \left(\frac{1}{2\sqrt{3}} \left(\frac{1}{\sqrt{3}}\right)^k\right) \\ &= \frac{\sqrt{3}+1}{4} \left(\frac{1}{\sqrt{3}}\right)^n.\end{aligned}$$

We hence conclude that $\{f_n\}$ is effectively uniformly convergent as $n \to \infty$. □

Theorem 3.2. *Koch curve is constructed from a computable curve.*

Proof. In the notation of Lemma 3.2, define $f : [0,1] \to \mathbb{R}^2$ by

$$f(t) = \lim_{n \to \infty} f_n(t).$$

From Lemmas 3.2 and 3.3, we obtain that f is well-defined and computable. Furthermore, $f([0,1])$ constructs Koch curve since

$$\begin{aligned} f_0([0,1]) &= \{(t,0) : t \in [0,1]\}, \\ f_{n+1}([0,1]) &= T_0(f_n([0,1])) \cup T_1(f_n([0,1])), \end{aligned}$$

and

$$f([0,1]) = \bigcap_{k=1}^{\infty} \overline{\bigcup_{n=k}^{\infty} f_n([0,1])}.$$

□

We obtain the following two corollaries of Theorem 3.2.

Corollary 3.2.1. *Koch coastline is constructed from a computable curve.*

Proof. By using Patching Theorem. □

Corollary 3.2.2. *Both Koch curve and Koch coastline are computable closed sets.*

Proof. By using Theorem 3.1. □

3.3 Koch island and the Jordan domains of Koch coastline

For arbitrary points $a, b, c \in \mathbb{R}^2$, we denote by $\triangle abc$ the interior of the triangle whose vertexes are a, b, and c.

Lemma 3.4. *If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are computable sequence of points on $\mathbb{R}^2$, then $\{\underline{d}_{\triangle a_n b_n c_n}\}_{n\in\mathbb{N}}$ is a computable sequence of functions.*

Proof. It is immediate from Lemma 3.1 that $\{\underline{d}_{\triangle a_n b_n c_n}\}$ is effectively uniformly continuous. It remains to show that $\{\underline{d}_{\triangle a_n b_n c_n}\}$ is sequentially computable.

Since

$$\overline{\triangle a_n b_n c_n} = \{a_n + t(b_n - a_n) + u(c_n - a_n) : t, u \in [0,1]\},$$

we have

$$\underline{d}_{\triangle a_n b_n c_n}(x) = \min \{\|a_n + t(b_n - a_n) + u(c_n - a_n) - x\| : (t,u) \in [0,1] \times [0,1]\}.$$

From an argument similar to that in the proof of Theorem 3.1, we obtain that if $\{x_k\}$ is a computable sequence of points on $\mathbb{R}^2$, then, $\{\underline{d}_{\triangle a_n b_n c_n}(x_k)\}$ is a computable double sequence of reals. Thus we have established that $\{\underline{d}_{\triangle a_n b_n c_n}\}$ is a computable sequence of functions. □

In the notation of Definition 2.2 and Lemma 3.2, define γ_n by

$$\gamma_n = f_{2n}([0,1]) \cup T(f_{2n}([0,1])) \cup T'(f_{2n}([0,1])).$$

It is straightforward from the definition that each γ_n is a Jordan closed curve. We denote the inner domain of γ_n by D_n.

Lemma 3.5. *$\{\underline{d}_{D_n}\}$ is a computable sequence of functions.*

Proof. Some tedious manipulation yields that there exist three sequences of points $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ and a recursive function $e : \mathbb{N} \to \mathbb{N}$ such that

$$\overline{D_n} = \bigcup_{k \le e(n)} \overline{\triangle a_n b_n c_n}.$$

Thus

$$\underline{d}_{D_n}(x) = \min_{k \le e(n)} \underline{d}_{\triangle a_n b_n c_n}(x).$$

From Lemma 3.4 and some manipulation on "$\min_{k\le e(n)}$", we obtain that $\{\underline{d}_{D_n}\}$ is a computable sequence of functions. □

We denote Koch coastline by γ and the inner domain of γ by D.

Lemma 3.6. *The sequence of functions $\{\underline{d}_{D_n}\}$ converges to $\underline{d}_D$ uniformly and effectively in n as $n \to \infty$.*

Proof. Let n be an arbitrary nonnegative integer and x an arbitrary point.

Some tedious calculation yields that $D = \bigcup_{n=0}^{\infty} D_n$. Thus

$$\underline{d}_D(x) \le \underline{d}_{D_n}(x).$$

Some tedious calculation again yields that for any $y \in D_{n+1}$, there exists $z \in D_n$ such that

$$\|y - z\| \le \frac{1}{2\sqrt{3}} \left(\frac{1}{\sqrt{3}}\right)^n.$$

For any $x \in \mathbb{R}^2$ and any $\varepsilon > 0$, there exists $y \in D$ such that

$$\|x - y\| < \underline{d}_D(x) + \varepsilon.$$

For this y, since $y \in \bigcup_{n=0}^{\infty} D_n$, there exists $n_0 \in \mathbb{N}$ such that $y \in D_{n_0}$.

- In case of $n_0 \le n$, we have $y \in D_n$. Thus

$$\underline{d}_{D_n}(x) \le \|x - y\| < \underline{d}_D(x) + \varepsilon.$$

- In case of $n_0 > n$, we construct y_k for $n \leq k \leq n_k$ such as $y_k \in D_k$ as follows.
  - Define $y_{n_0} = y$.
  - If we have already defined $y_{k+1} \in D_{k+1}$, then there is $z \in D_k$ such that

$$\|y_{k+1} - z\| < \frac{1}{2\sqrt{3}} \left(\frac{1}{\sqrt{3}} \right)^k .$$

    Choose one of such z to be y_k.

  Then, we have

$$\begin{aligned} \underline{d}_{D_n}(x) &\leq \|x - y_n\| \\ &\leq \|x - y\| + \sum_{k=n}^{n_0-1} \|y_{k+1} - y_k\| \\ &< \underline{d}_D(x) + \varepsilon + \sum_{k=n}^{n_0-1} \frac{1}{2\sqrt{3}} \left(\frac{1}{\sqrt{3}} \right)^k . \end{aligned}$$

In both cases, we have

$$\underline{d}_{D_n}(x) < \underline{d}_D(x) + \varepsilon + \frac{\sqrt{3}+1}{4} \left(\frac{1}{\sqrt{3}} \right)^n .$$

Then, the arbitrariness of ε implies

$$\underline{d}_{D_n}(x) \leq \underline{d}_D(x) + \frac{\sqrt{3}+1}{4} \left(\frac{1}{\sqrt{3}} \right)^n .$$

Now we have established that

$$|\underline{d}_{D_n}(x) - \underline{d}_D(x)| \leq \frac{\sqrt{3}+1}{4} \left(\frac{1}{\sqrt{3}} \right)^n .$$

This concludes that $\{\underline{d}_{D_n}\}$ converges to $\underline{d}_D$ uniformly and effectively in n as $n \to \infty$. □

As an immediate consequence of Lemma 3.5 and Lemma 3.6, we obtain the following theorem.

Theorem 3.3. *$\underline{d}_D$ is a computable function.*

The following corollaries hold.

Corollary 3.3.1. *Koch island $\overline{D}$ is a computable closed set.*

Proof. Immediate from $\underline{d}_D(x) = 0$ iff $x \in \overline{D}$. □

Corollary 3.3.2. *Koch coastline γ is a computable closed set with a function g satisfying the condition:*

$$\begin{aligned} g(x) &= 0 && \text{if } x \text{ is on } \gamma, \\ g(x) &< 0 && \text{if } x \text{ is inside } \gamma, \\ g(x) &> 0 && \text{if } x \text{ is outside } \gamma. \end{aligned}$$

Proof. Set $g(x) = 2\underline{d}_D(x) - \underline{d}_\gamma(x)$. □

4 Conclusion

Both Koch curve and koch Coastline are constructed from computable curves. All of Koch curve, koch Coastline and Koch island are computable closed sets. Furthermore, there is a computable function which separates inner and outer domains of Koch coastline.

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