Background and terminology

Throughout assume that $\mathcal{M}_1^\sharp$ exists and let $\mathcal{M}_1$ be the result of iterating the last extender of $\mathcal{M}_1^\sharp$ OR many times.

External System

Fix a real $t_0 \in \mathbb{R}$ such that $\mathcal{M}_1^\sharp \in \mathrm{L}[t_0]$. Let κ_0 be the least inaccessible of $\mathrm{L}[t_0]$ and take G_0 generic over $\mathrm{L}[t_0]$ for the Levy collapse of everything below κ_0 to ω and finally set $\mathbb{R}_0 = \mathbb{R} \cap \mathrm{L}[t_0, G_0]$.

REMARK. "least inaccesible" could be replace by "2nd inaccessible", "ω^{th} inaccesible", "least Mahlo", etc.; essentially any definable cardinal would suffice. $\dashv$

LEMMA. $\mathrm{HOD}^{\mathrm{L}[t_0,G_0]} = \mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)}$

Proof. ❑

The goal is to calculate $\mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)}$ as a "fine structural model", albeit, in a slightly different hierarchy than the usual hierarchy of Mitchell-Steel mice [MS94b]. From the point of view of V it is not hard to say what the final calculation is. Call a premouse M **good** provided

- There is an ordinal $\delta^M < \kappa_0 = \omega_1^{\mathrm{L}(\mathbb{R}_0)}$ such that $M \models$ "δ^M is Woodin" and $M = \mathrm{L}[M|\delta^M]$.
- For all $\lambda < \delta^M$, $\mathrm{L}[M|\lambda] \models$"$\lambda$ is not Woodin".

A **good mouse** is just a countably iterable good premouse. (In general the only difference between mouse and premouse is that a mouse is *sufficiently iterable* iterable and in our setting countable iterability suffices.) The model $\mathrm{L}(\mathbb{R}_0)$ can recognize it's good premice, but it can not recognize it's good mice, in particular, in $\mathrm{L}(\mathbb{R}_0)$ no good premouse is countably iterable. This is stronger than saying $\mathrm{L}(\mathbb{R}_0)$ can not determine which good premouse is $\mathcal{M}_1$, it says precisely that in $\mathrm{L}(\mathbb{R}_0)$ there are no good mice.

LEMMA 0.0.1. In $\mathrm{L}(\mathbb{R}_0)$ there are no good countably iterable good premice.

Proof. Suppose M is a countably iterable good premouse in $\mathrm{L}(\mathbb{R}_0)$, then $\mathrm{L}(\mathbb{R}_0) \models \forall r \in \mathbb{R}\,(r^\sharp$ exists). This is nonsense as $t_0^\sharp$ could not possibly belong to $\mathrm{L}(\mathbb{R}_0)$. ❑

Good mice naturally form a directed system. If M is good and N is a non-dropping countable iterate of M, then N is good and moreover by the Dodd-Jensen Lemma the map $\pi_{M,N}$ depends only on M and N and not on the particular iteration leading from M to N. This gives rise to a natural directed system $\mathcal{D}^e = \langle D^e, <^e \rangle$ with index set $D^e = \{M : M$ is a good mouse $\}$ and $M <^e N$ in case N is a non-dropping iterate of M. For $M <^e N$ let $\pi_{M,N}$ be the corresponding iteration map. Using Dodd-Jensen again get that

$$M <^e N <^e R \Rightarrow \pi_{M,R} = \pi_{N,R} \circ \pi_{M,N}$$

in other words the iteration maps commute so the direct limit $\mathcal{M}^* = \operatorname{dir}\lim \mathcal{D}^e$ exists. Moreover, if $\mathcal{M}_1 = M_0 <^e M_1 <^e M_3 \cdots$ is chosen cofinal in $\mathcal{D}^e$, then $\mathcal{M}^*$ is just the limit of the M_i's which is wellfounded since $\mathcal{M}_1$ is iterable.

It turns out that $\mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)} = M^*[B^*]$ where B^* is a collection of branches through certain iteration trees on M^*. Thus $\mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)}$ is a fine structural mouse together with a fragment of an iteration strategy on that mouse. This is the starting point for a new hierarchy of mice.

The point of the following sequence of definitions and lemata is simply to define B^*. Call a limit length iteration tree $\mathcal{T}$ on a good mouse **maximal** if $\mathrm{L}(M(\mathcal{T})) \models$ "$\delta(\mathcal{T})$ is Woodin". A tree $\mathcal{T}$ is **short** if for all limit $\lambda < \mathrm{lh}(\mathcal{T})$, $\mathcal{T} \restriction \lambda$ is non-maximal. For non-maximal limit length $\mathcal{T}$ define $\mathcal{Q}(\mathcal{T})$ to be the least initial segment of $\mathrm{L}(M(\mathcal{T}))$ so that definably over $\mathcal{Q}(\mathcal{T})$ there is a witness to the fact that $\delta(\mathcal{T})$ is not Woodin in $\mathrm{L}(M(\mathcal{T}))$. For the following definition I will used a modified notion of initial segment of a premouse. For M and N premice set $M \trianglelefteq^w N$ if either $M = N|o(M)$ or $M = N\|o(M)$. With this notion $M|\beta \triangleleft^w M\|\beta$ whenever $M\|\beta$ is active.

LEMMA 0.0.2. Let $\mathcal{T}$ be non-maximal, then there is at most one cofinal wellfounded branch b through $\mathcal{T}$ such that $\mathcal{Q}(\mathcal{T}) \trianglelefteq^w M_b^{\mathcal{T}}$. Moreover, if b is such a branch, then $b \in \mathrm{L}(M, \mathcal{T})$.

Proof. Suppose that b and c are two distinct cofinal wellfounded branches such that $\mathcal{Q}(\mathcal{T}) \trianglelefteq^w M_b^{\mathcal{T}} \cap M_c^{\mathcal{T}}$. Clearly, $\mathcal{Q}(\mathcal{T}) \notin M_b^{\mathcal{T}} \cap M_c^{\mathcal{T}}$ so without loss of generality assume $\mathcal{Q}(\mathcal{T}) \notin M_b^{\mathcal{T}}$. Their are two possibilities, either $\mathcal{Q}(\mathcal{T}) = M_b^{\mathcal{T}}$ or $M_b^{\mathcal{T}}$ is active and $\mathcal{Q}(\mathcal{T}) = M_b^{\mathcal{T}-}$, where N^- is just N less its top extender whenever N is an active premouse. In either case b drops (since otherwise $\mathcal{Q}(\mathcal{T}) \in M_b^{\mathcal{T}}$).

Suppose first that $\mathcal{Q}(\mathcal{T}) = M_b^{\mathcal{T}}$. In this case $M_b^{\mathcal{T}} \trianglelefteq^w M_c^{\mathcal{T}}$ by assumption and the standard argument shows that $M_b^{\mathcal{T}} \neq M_c^{\mathcal{T}}$ so $M_b^{\mathcal{T}} \triangleleft^w M_c^{\mathcal{T}}$. $M_b^{\mathcal{T}} \notin M_c^{\mathcal{T}}$ since $M_c^{\mathcal{T}} \models$ "$\delta(\mathcal{T})$ is a cardinal". It must be that $M_c^{\mathcal{T}} = M_b^{\mathcal{T}-}$, but this too yields a contradiction since $\mathrm{ult}(M_b^{\mathcal{T}}, E_{\text{top}}^{M_b^{\mathcal{T}}})$"$\delta(\mathcal{T})$ is a cardinal" yet $M_b^{\mathcal{T}} \triangleleft \mathrm{ult}(M_b^{\mathcal{T}}, E_{\text{top}}^{M_b^{\mathcal{T}}})$ and $M_b^{\mathcal{T}}$ trivially codes a collapse of $\delta(\mathcal{T})$ since b drops.

The other case is that $\mathcal{Q}(\mathcal{T}) = M_b^{\mathcal{T}-}$. As above

, then $M_c^{\mathcal{T}}$ would see that $\mathrm{card}(\delta(\mathcal{T})) < \delta(\mathcal{T})$ which contradicts, the fact that $M_c^{\mathcal{T}} \models$ "$\delta(\tau)$ is a cardinal". InSo $M_c^{\mathcal{T}} = M_b^{\mathcal{T}}$ and b and c both drop, but this yields the standard contradiction.

Suppose b is a/the cofinal wellfounded branch through $\mathcal{T}$ with $\mathcal{Q}(\mathcal{T}) \trianglelefteq M_b^{\mathcal{T}}$. If $M_b^{\mathcal{T}}$ is set sized, i.e. b drops, then by absoluteness such a branch b_g can be found in $\mathrm{L}(M, \mathcal{T})[g]$ for g collapsing $\mathrm{card}(M_b^{\mathcal{T}})$ to $\aleph_0$. The above argument shows that $b_g = b$ for all g hence $b_g \in \mathrm{L}(M, \mathcal{T})$ as desired. So assume that b does not drop, then for all γ there are generic branches c such that $\gamma \in \mathrm{wfp}(M_c^{\mathcal{T}})$. In particular take $\gamma = o(\mathcal{Q}(\mathcal{T}))$. Suppose c and c' are distinct generic cofinal γ-wellfounded branches with $\mathcal{Q}(\mathcal{T}) \in \mathrm{wfp}(M_c^{\mathcal{T}}) \cap \mathrm{wfp}(M_{c'}^{\mathcal{T}})$. As above we get a contradiction since $\delta(\mathcal{T})$ is Woodin with respect to all $A \subseteq \delta(\mathcal{T})$ common to both $M_c^{\mathcal{T}}$ and $M_{c'}^{\mathcal{T}}$, but $\mathcal{Q}(\mathcal{T})$ provides a counter example. So we have that for all generic γ-wellfounded cofinal non-dropping branches are the same hence they are all actually wellfounded and hence they are just b and b is in $\mathrm{L}(M, \mathcal{T})$ as desired. ❑

For $\mathcal{T}$ a non-maximal tree on good premouse M if there is a cofinal wellfounded branch b through $\mathcal{T}$ such that $\mathcal{Q}(\mathcal{T}) \trianglelefteq M_b^{\mathcal{T}}$, then call this branch b the **true branch** of $\mathcal{T}$ and denote it by $b_{\mathcal{T}}$. Call a short iteration tree $\mathcal{T}$ on a good premouse M **good** if $\mathcal{T}$ is according to the **true (partial) iteration strategy**

Γ_M on M, i.e., for limit $\lambda < \mathrm{lh}(\mathcal{T})$, $[0,\lambda]^{\mathcal{T}} = b_{\mathcal{T}\restriction\lambda}$. Call a limit length tree $\mathcal{T}$ **suitable** if $\mathcal{T}$ is short and $\{\alpha : [0,\alpha]^{\mathcal{T}}$ does not drop$\}$ is cofinal $\mathrm{lh}(\mathcal{T})$.

LEMMA 0.0.3. Let $\mathcal{T}$ be a suitable tree on good M, then $\mathcal{T}$ is according to the true strategy on M.

Proof. We need only show $\mathcal{Q}(\mathcal{T}) \trianglelefteq M^{\mathcal{T}}_{\lambda}$ for all limit $\lambda < \mathrm{lh}(\mathcal{T})$. Let $\nu = \mathrm{lh}(E^{\mathcal{T}}_{\lambda})$ and $\lambda' > \lambda$ so that $[0,\lambda']^{\mathcal{T}}$ does not drop. Then $\mathcal{Q}(\mathcal{T}\restriction\lambda) \in \mathrm{H}(\nu)^{M^{\mathcal{T}}_{\lambda'}} \subseteq \mathrm{H}(\nu)^{M^{\mathcal{T}}_{\lambda}}$. ❑

If $\mathcal{T}$ arrises in a successful comparison of good M with a second good N and $\mathrm{lh}(\mathcal{T}) = \theta + 1$ where θ is a limit ordinal, then $\mathcal{T}\restriction\theta$ is suitable and hence according to Γ_M. If $\mathcal{T}\restriction\theta$ is non-maximal, then clearly $\mathcal{Q}(\mathcal{T}\restriction\theta) \triangleleft M^{\mathcal{T}}_{\theta}$ so actually $\mathcal{T}$ is according to Γ_M. If $\mathcal{T}\restriction\theta$ is maximal, then of course $\mathcal{T}$ is no longer according to Γ_M as $\mathcal{Q}(\mathcal{T}\restriction\theta)$ does not exist.

The notion of true strategy exends to maximal trees via the following lemma. The point here is that for non-maximal suitable trees $\mathcal{T}$, if $b_{\mathcal{T}}$ exists, then $b_{\mathcal{T}} \in \mathrm{L}(\mathbb{R}_0)$, whereas for maximal trees, this is definately not the case.

LEMMA 0.0.4. Suppose that $\mathcal{T}$ is a maximal tree on good M, then there is at most one cofinal wellfounded branch $b_{\mathcal{T}}$ (in V or any extension of V).

Proof. Suppose that b and c are two cofinal wellfounded branches.

FINISH ❑

Let κ^* be the least inaccessible of M^* above $\delta^* = \delta^{M^*}$ and let

$$U = \{\mathcal{T} \in M^*|\kappa^* : \mathcal{T} \text{ is a good tree on } M^*\}$$
$$B = \{b_{\mathcal{T}} : \mathcal{T} \in U\}$$

THEOREM 0.0.5. (Woodin) $\mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)} = \mathrm{HOD}^{\mathrm{L}[t_0,G_0]} = M^*[B]$

The proof that $M^* \subseteq \mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)}$ requires internalizing the directed system and will be dealt with in the next section. We shall now work on showing $\mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)} \subseteq M^*[B]$. To begin with we reduce the calculation to showing $\mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)} \cap \mathcal{P}(\Theta_0) \subseteq M^*[B]$.

LEMMA 0.0.6. $\mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)} = \mathrm{L}[S_0]$ for some $S_0 \subseteq \Theta_0 = \omega_2^{\mathrm{L}[t_0,G_0]}$

Proof. This is essentially Vopenka. Let $\mathbb{B} = \{\langle A_0, A_1, \ldots, A_{n-1}\rangle : A_i \subseteq \mathbb{R}_0$ is $\mathrm{OD}^{\mathrm{L}(\mathbb{R}_0)}\}$. Set $\vec{A} \leq \vec{B}$ if $\mathrm{lh}(\vec{A}) \geq \mathrm{lh}(\vec{B})$ and for all $i < \mathrm{lh}(\vec{B})$, $A_i \subseteq B_i$. View $\mathbb{B}$ as a poset in $\mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)}$ (I'll finish this later - the point is $\mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)} = \mathrm{L}[\mathbb{B},\tau]$ where τ is the name for a real in $\mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)}[\dot{G}]$.) ❑

Because of this lemma all we must do to show $\mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)} \subseteq M^*[B]$ is calculate S_0 in $M^*[B]$. The first step is to prove “part” of the desired result, this corresponds to Steel's work on $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}$ under $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ in [Ste95].

LEMMA 0.0.7. With everything as above

1. $\mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)}|\delta^* = M^*|\delta^*$

2. $\delta^* = \Theta_0$

Proof. For (1.) we must show that for each $a \subseteq \gamma < \delta^*$ such that $a \in \mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)}$, then $a \in M^*$. Suppose this fails, then in $\mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)}$ we may look for a minimal hence definable without parameters $a \subseteq \gamma < \delta^*$. This is because M^* is definable without parameters in $\mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)}$. (This will also be showed in the next section.) Fix ϕ so that for $\alpha < \delta^*$

$$\xi \in a \Leftrightarrow \mathrm{L}(\mathbb{R}_0) \models \phi(\xi)$$

Fix $M \in \mathcal{D}^e$ such that t_0 is generic for $\mathbb{B}^M$ (Woodin's free extender algebra as defined in M) and such that $\xi \in \mathrm{rng}(\pi_{M,M^*})$.

FINISH

So we have

$$\xi \in a \Leftrightarrow M \models \text{“}\,\mathrm{Col}(\omega, {<}\kappa_M) \Vdash \mathrm{L}(\dot{R}) \models \phi\big(\pi_{M,M^*}(\bar{\xi})\big)\text{”}$$

hence as $\pi_{M^*,M^{**}}(\xi) = \pi_{M^*,M^{**}} \circ \pi_{M,M^*}(\bar{\xi})$

$$\xi \in a \Leftrightarrow M^* \models \text{“}\,\mathrm{Col}(\omega, {<}\kappa^*) \Vdash \mathrm{L}(\dot{R}) \models \phi\big(\dot{\pi}_{M^*,\dot{M}^{**}}(\xi)\big)\text{”}$$

So the issue is to show that $\pi_{M^*,M^{**}} \restriction \delta^* \in M^*$. For this it suffices to show that $\pi_{M^*,M^{**}} \restriction \delta^*$ is in $\mathrm{L}(\mathbb{R}_G)$ where $G \subseteq \mathrm{Col}(\omega, {<}\kappa^*)$ generic over M^* and moreover, that $\pi_{M^*,M^{**}} \restriction \delta^*$ is independent any particular G.

For (2.) first notice that if $\Theta_0 < \delta^*$, then $S_0 \in M$ so $\mathrm{HOD}^{\mathrm{L}(\mathbb{R}_0)} \subseteq M^*$ which we know is false. Suppose, then that $\delta^* < \Theta_0$ ❏

Internal Directed System

For each good M and $s \in \mathrm{OR}^{<\omega}$ define

$$\gamma_{M,s} = \sup\{\alpha < \delta^M : \alpha \text{ is definable in } M|\sup s + \omega \text{ from the parameter } s\}$$
$$X_{M,s} = \Sigma_1\text{- Hull}^{M|\sup s+\omega}(\gamma_s \cup s)$$

CLAIM. $X_{M,s} \cap \delta^M = \gamma_{M,s}$

Proof. Suppose α is Σ_1 definable in $M|\sup s + \omega$ from $\vec{\beta} \in \gamma_{M,s}^{<\omega}$ and s. Let $\xi < \gamma_{M,s}$ be Σ_1 definable in $M|\sup s + \omega$ from s with $\vec{\beta} \in \xi^{<\omega}$. Fix $n < \omega$ such that ξ is Σ_1 definable from s in $M|\sup s + n$ and α is Σ_1 definable over $M|\sup s + n$ from $\vec{\beta}$ and s. Define

$$\bar{\xi} = \sup\{\eta : \eta \text{ is definable from parameters } < \xi \text{ in } M|\sup s + n\}$$

Since δ^M is regular in M, $\bar{\xi} < \delta^M$ moreover $\bar{\xi}$ is Σ_1 definable from s in $M|\sup s + \omega$ so $\bar{\xi} < \gamma_{M,s}$. Clearly $\alpha < \bar{\xi}$ by choice of ξ and n, hence $\alpha < \gamma_{M,s}$ as desired. ❏

Suppose that M is a good mouse and that $\mathcal{T}$ is a maximal suitable tree on M with a cofinal branch in V that fixes a finite sequence of ordinals s. An easy absoluteness argument shows that if g is generic over $\mathrm{L}(M,\mathcal{T})$ for collapsing all ordinals in s, δ^M, and $\mathrm{lh}(\mathcal{T})$, then in $\mathrm{L}(M,\mathcal{T})$ there is a cofinal non-dropping branch b such that b is ***s*-wellfounded** ($s \subseteq \mathrm{wfp}(M_b^{\mathcal{T}})$) and b fixes s.

DEFINITION 0.0.8. A s-wellfounded non-dropping cofinal branch through a maximal suitable tree $\mathcal{T}$ on good M will simply be called an ***s*-good** branch of $\mathcal{T}$.

LEMMA 0.0.9. Let $\mathcal{T}$ be a maximal suitable tree on good M and suppose b and c are s-good branches of $\mathcal{T}$. Then the embeddings $i_b^{\mathcal{T}}$ and $i_c^{\mathcal{T}}$ agree on $X_{M,s}$.

Proof. Let b and c be as in the statement of the theorem and let $M^{\mathcal{T}} = \mathrm{L}(M(\mathcal{T}))$ as usual. Fix a generic g so that b and c are in $\mathrm{L}(M, \mathcal{T})[g]$. Let ξ be the largest point in common with both branches and let $\kappa = \min\{\mathrm{crit}(i_{\xi,b}^{\mathcal{T}}), \mathrm{crit}(i_{\xi,c}^{\mathcal{T}})\}$. The standard "zipper" argument shows that $\delta(\mathcal{T}) \cap \mathrm{rng}(i_{\xi,b}^{\mathcal{T}}) \cap \mathrm{rng}(i_{\xi,c}^{\mathcal{T}}) = \kappa$. Since $X_{M_b^{\mathcal{T}},s} \subseteq \mathrm{rng}(i_{\xi,b}^{\mathcal{T}})$ and $X_{M_c^{\mathcal{T}},s} \subseteq \mathrm{rng}(i_{\xi,c}^{\mathcal{T}})$

$$\gamma_{M^{\mathcal{T}},s} \subseteq \delta(\mathcal{T}) \cap \mathrm{rng}(i_{\xi,b}^{\mathcal{T}}) \cap \mathrm{rng}(i_{\xi,c}^{\mathcal{T}}) \subseteq \kappa$$

Thus

$$i_b^{\mathcal{T}} \restriction \gamma_{M_\xi^{\mathcal{T}},s} = i_c^{\mathcal{T}} \restriction \gamma_{M_\xi^{\mathcal{T}},s} = \mathrm{id} \restriction \gamma_{M_\xi^{\mathcal{T}},s}$$

which implies

$$i_{\xi,b}^{\mathcal{T}} \restriction X_{M_\xi^{\mathcal{T}},s} = i_{\xi,c}^{\mathcal{T}} \restriction X_{M_\xi^{\mathcal{T}},s}$$

and so $i_b^{\mathcal{T}} \restriction X_{M,s} = i_c^{\mathcal{T}} \restriction X_{M,s}$ as claimed. ❑

As an imediate corollary notice that if $M \in \mathrm{L}(\mathbb{R}_0)$ is good and iterable (in V) via its unique iteration strategy Γ_M and $\mathcal{T} \in \mathrm{L}(\mathbb{R}_0)$ is a tree on M according to Γ_M, then $\mathrm{L}(\mathbb{R}_0)$ sees that $\mathcal{T}$ is according to $\Gamma_{M,0}$. Moreover, $\mathrm{L}(\mathbb{R}_0)$ knows that $M^{\mathcal{T}} = \mathrm{L}(M(\mathcal{T}))$ is the final model of $\mathcal{T}$, i.e., $M^{\mathcal{T}} = M_{b_{\mathcal{T}}}^{\mathcal{T}}$ where $b_{\mathcal{T}} = \Gamma_M(\mathcal{T})$, but the branch $b_{\mathcal{T}}$ need not be in $\mathrm{L}(\mathbb{R}_0)$ (and in general can not be). The preceeding lemma says even though $\mathrm{L}(\mathbb{R}_0)$ fails to see $b_{\mathcal{T}}$ it succeeds in seeing $i_{b_{\mathcal{T}}}^{\mathcal{T}} \restriction X_{M,s}$ for all s fixed by $i_{b_{\mathcal{T}}}^{\mathcal{T}}$. The next lemma states that $b_{\mathcal{T}}$ can be reconstructed from the fixed points of $i_{b_{\mathcal{T}}}^{\mathcal{T}}$. The construction is local and gives

$$\mathrm{L}(M, \mathcal{T})[b_{\mathcal{T}}] = \mathrm{L}(M, \mathcal{T})[\Gamma]$$

where Γ is any sufficiently rich collection of fixed points of $i_{b_{\mathcal{T}}}^{\mathcal{T}}$.

Let $\mathcal{T}$ be a maximal suitable tree on a suitable premouse M. Suppose that Γ is a class of ordinals such that for all $s \in \Gamma^{<\omega}$ there is an s-good branch through $\mathcal{T}$. Define

$$b_s^{\mathcal{T}} = \bigcap \{ c : c \text{ is an } s\text{-good branch through } \mathcal{T} \}$$

and

$$\xi_s^{\mathcal{T}} = \sup b_s^{\mathcal{T}}, \text{ i.e., } b_s^{\mathcal{T}} = [0, \xi_s^{\mathcal{T}}]^{\mathcal{T}} \text{ since } b_s \text{ is closed}$$

also set

$$b^{\mathcal{T}} = \bigcup_{s \in \Gamma^{<\omega}} b_s^{\mathcal{T}} \quad \text{and} \quad \xi^{\mathcal{T}} = \sup_{s \in \Gamma^{<\omega}} \xi_s^{\mathcal{T}}$$

Notice that for $s, s' \in \Gamma^{<\omega}$, $s \subseteq s'$ implies $b_s^{\mathcal{T}} \subseteq b_{s'}^{\mathcal{T}}$ and $\xi_s^{\mathcal{T}} \leq \xi_{s'}^{\mathcal{T}} \leq \xi^{\mathcal{T}}$.

LEMMA 0.0.10. $b^{\mathcal{T}}$ is a cofinal wellfounded branch through $\mathcal{T}$ and hence $b^{\mathcal{T}} = b_{\mathcal{T}}$ is the true branch of $\mathcal{T}$ by Lemma **??**.

Proof. This will follow by showing that $M(\mathcal{T}\restriction\xi^{\mathcal{T}})$ is a Σ^1_2-Woodin, i.e., $\mathrm{L}(M(\mathcal{T}\restriction\xi^{\mathcal{T}}))\models$"$\delta(\mathcal{T})$ is Woodin". As usual set $M^{\mathcal{T}}=\mathrm{L}(M(\mathcal{T}))$ to be the *true* final model of $\mathcal{T}$. Notice that whenever b is s-good, then $\gamma_s^{M_b^{\mathcal{T}}}=\gamma_s^{M^{\mathcal{T}}}$ and $X_s^{M_b^{\mathcal{T}}}=X_s^{M^{\mathcal{T}}}$. There are now a couple of cases to consider.

CASE 1. There is some $s\in\Gamma^{<\omega}$ such that all s-good generic branches agree (and hence agree with $b_{\mathcal{T}}$ as well).

In this case there is nothing to do as $b_{\mathcal{T}}\in\mathrm{L}(M,\mathcal{T})$ by a simple forcing argument.

CASE 2. For all $s\in\Gamma^{<\omega}$ there are distinct s-good branches.

For each $s\in\Gamma^{<\omega}$ set
$$\mu_s^{\mathcal{T}}=\inf\{\mathrm{crit}(i_b^{\mathcal{T}}):b\text{ is a }s\text{-good}\}$$
and notice that $s\supseteq t$ implies $\mu_s^{\mathcal{T}}\geq\mu_t^{\mathcal{T}}$. Let b and c be two s-good branches. We may assume that $\xi_s^{\mathcal{T}}=\sup(b\cap c)$. As in LEMMA 0.0.4 $\gamma_{M,\xi_s^{\mathcal{T}}}=\gamma_{M^{\mathcal{T}},s}\leq\mu_s^{\mathcal{T}}$ and $i_b^{\mathcal{T}}\restriction X_{M^{\mathcal{T}},\xi_s^{\mathcal{T}}}:X_{M^{\mathcal{T}},\xi_s^{\mathcal{T}}}\to X_{M^{\mathcal{T}},s}$ and $i_c^{\mathcal{T}}\restriction X_{M^{\mathcal{T}},\xi_s^{\mathcal{T}}}:X_{M^{\mathcal{T}},\xi_s^{\mathcal{T}}}\to X_{M^{\mathcal{T}},s}$ agree. wa

❑

Appendix A

The purpose of this appendix is to give proofs of some of the *standard* results mentioned. First I will establish some notation. For this discussion a premouse may be fine structural, coarse, or various sort of Doddages satisfying some coherence condition. Just to review a bit, coarse premice are defined in [MS94a] as follows:

DEFINITION. $M=\langle M,\in,\delta^M\rangle$ is a **coarse premouse** if M is a model of "*ZC* $+\ \Sigma_2$-*Collection* $+\ \Sigma_\omega$-*Collection for relation* $R\subseteq a\times M$ *with* $a\in M|\delta^M$" where $M|\alpha=V_\alpha^M$ and ZC is "ZFC $-$ Collection".

Recall that a **coarse iteration tree** $\mathcal{T}$ of length θ on coarse M is a structure
$$\mathcal{T}=\langle T,\langle E_\alpha,\nu_\alpha:\alpha+1<\theta\rangle\rangle$$
satisfying

1. $T\subseteq\theta\times\theta$ is a **tree order** on θ, i.e., for every $\alpha+1<\theta$, there is an immediate T-predicessor denoted $T(\alpha+1)$ and for every limit $\lambda<\theta$, $[0,\lambda]_T=\{\beta:\beta T\lambda\}$ is cofinal in λ.
2. E_α is an extender from the M_α-sequence, where $M_0=M$, $M_{\alpha+1}=\mathrm{ult}(M_{T(\alpha+1)},E_\alpha)$, and for λ a limit, $M_\lambda=\mathrm{dir\,lim}_{\eta\in[0,\lambda]_T}M_\eta$.
3. For all $\alpha+1<\beta+1<\theta$, $\mathrm{lh}(E_\alpha)>\mathrm{lh}(E_\beta)$ and $\nu_\alpha\leq\nu_\beta$ and $\nu_\alpha+1<\mathrm{lh}(E_\alpha)$. (Normality)
4. For $\alpha+1<\theta$, $T(\alpha+1)$ is the minimal $\gamma\leq\alpha$ such that $\mathrm{crit}(E_\alpha)\leq\nu_\alpha$. (Non-overlapping)

All iteration trees $\mathcal{T}$ will be assumed to be normal and non-overlapping.

A coarse premouse M is **good** if

1. $M = \mathrm{L}(M|\delta^M) \models$ "δ^M is Woodin"

2. For all $\alpha < \delta^M$, $\mathrm{L}(M|\alpha) \models$ "α is not Woodin"

A **good** coarse iteration tree on a coarse mouse M is just a normal non-overlapping short iteration tree on M.

LEMMA 0.0.11. Let M be a good premouse (fine or coarse) containing all of the ordinals. Let $\mathcal{T}$ be a good iteration tree on M. If $\mathrm{lh}(\mathcal{T}) = \theta + 1$ is a successor ordinal, then $\mathcal{T}$ can be freely extended to a good tree of length $\theta + 2$. Otherwise $\mathrm{lh}(\mathcal{T}) = \theta$ is a limit ordinal and either

1. $\mathcal{T}$ has a unique cofinal branch $b_{\mathcal{T}}$ such that $\mathcal{T} \oplus b_{\mathcal{T}}$ is good, moreover, $b_{\mathcal{T}} \in \mathrm{L}(M, \mathcal{T})$

or else

2. $\mathcal{T}$ is maximal, i.e. $\mathrm{L}(M(\mathcal{T})) \models$ "$\delta(\mathcal{T})$ is Woodin", in which case $\mathcal{T}$ may fail to have a cofinal wellfounded branch b such that $\mathcal{T} \oplus b$ is good, however, there are *arbitrarily good* cofinal branches generic over $\mathrm{L}(M, \mathcal{T})$, i.e. for any α there is a $\mathrm{L}(M, \mathcal{T})$-generic cofinal branch b with $\alpha \in \mathrm{wfp}(M_b^{\mathcal{T}})$. Moreover, if there is a cofinal wellfounded branch (anywhere), then it is unique, so again denote it by $b_{\mathcal{T}}$. If $b_{\mathcal{T}}$ exists and in addition $\langle M, \mathcal{T}\rangle^\sharp$ exists, then $b_{\mathcal{T}} \in \mathrm{L}(\langle M, \mathcal{T}\rangle^\sharp)$. $\dashv$

This lemma essentially follows from the following three lemmas which in turn are consequences of Theorem 4.4 and Theorem 2.2 [MS94a] together with an absoluteness argument.

LEMMA 0.0.12. Let $M = \mathrm{L}(M|\delta^M)$ be a premouse, $\mathcal{T}$ an iteration tree of limit length on M, and $\alpha \in \mathrm{OR}$. Then for all sufficiently large θ, $M^{\mathrm{Col}(\omega,\theta)}$ contains a maximal branch through $\mathcal{T}$ with $\alpha \in \mathrm{wfp}(M_b^{\mathcal{T}})$.

Proof. Fix $\mathcal{T}$ and M and work in $\mathrm{L}(M, \mathcal{T})$. Let θ be a regular cardinal greater than $|M|\delta^M|$, α, and $\mathrm{lh}(\mathcal{T})$ such that $M|\theta$ is a premouse. Let G be $\mathrm{Col}(\omega, \theta)$ generic. Set $N = M|\theta$ and suppose that with $\mathcal{T}$ viewed as a tree on N there is no α wellfounded branch. In $\mathrm{L}(M, \mathcal{T})[G]$ the statement "*there is a countable tree on N with no α wellfounded maximal branch*" is a $\Sigma^1_2(r)$ statement fo any real r coding N and α, thus this statement is absolute to $M[G]$. So fix $p \in \mathrm{Col}(\omega, \theta)$ and a name $\dot{\mathcal{T}}$ so that

$$M \models p \Vdash \text{"}\dot{\mathcal{T}} \text{ is a countable tree on } \check{N} \text{ with no } \check{\alpha} \text{ wellfounded maximal branch"}$$

Now work entirely in M. Fix λ a regular cardinal of M bigger than θ with $M|\lambda \prec_{\Sigma_n} M$ for some n large enouph so that

$$M|\lambda \models p \Vdash \text{"}\dot{\mathcal{T}} \text{ is a countable tree on } \check{N} \text{ with no } \check{\alpha} \text{ wellfounded maximal branch"}$$

Let $\{N, \dot{t}, p, \theta\} \subseteq X \prec M|\lambda$ with $|X|^M = \omega$ and let $\pi : \bar{M} \simeq X \prec M|\lambda$ be the uncollapse embedding and let $\bar{N} = \pi^{-1}[N]$. Let $\bar{G} \in M$ be $\bar{M}$-generic for $\mathrm{Col}(\omega, \bar{\theta})$ with $\bar{p} \in \bar{G}$. Then $\bar{\mathcal{T}}$ be the countable tree in M on $\bar{N}$ determined by $\bar{G}$ and $\pi^{-1}[\dot{\mathcal{T}}]$. Since $\pi \restriction \bar{N} : \bar{N} \to N$, Theorem 4.4 [MS94a] yields a maximal wellfounded branch b through $\bar{\mathcal{T}}$. The branch b exists in M so by absoluteness there is an $\bar{\alpha}$ wellfounded branch in $\bar{M}[\bar{G}]$. This contradicts the fact that

$$\bar{M} \models \bar{p} \Vdash \text{"}\bar{\dot{\mathcal{T}}} \text{ is a countable tree on } \bar{\check{N}} \text{ with no } \bar{\check{\alpha}} \text{ wellfounded maximal branch"}$$

We have that the assumption "*there is no α wellfounded maximal branch through $\mathcal{T}$ as viewed as a tree on N*" is false in $\mathrm{L}(M, \mathcal{T})[G]$. So let b be an α wellfounded maximal branch through $\mathcal{T}$ when viewed as a tree on N. All that is left is to show that b induces a maximal α wellfounded maximal branch when $\mathcal{T}$ is viewed as a tree on M, Let $M = \mathrm{L}(M|\delta^M)$ be a premouse, $\mathcal{T}$ an iteration tree of limit length on M, and $\alpha \in \mathrm{OR}$. Then for all sufficiently large θ, $M^{\mathrm{Col}(\omega,\theta)}$ contains a maximal branch through $\mathcal{T}$ with $\alpha \in \mathrm{wfp}(M_b^{\mathcal{T}})$.but this is trivial as $N_b^{\mathcal{T}} \lhd M_b^{\mathcal{T}}$. ❑

A similar argument also using Theorem 4.4 [MS94a] shows:

LEMMA 0.0.13. Let $M = \mathrm{L}(M|\delta^M)$ be a premouse and $\mathcal{T}$ be an iteration tree of successor length $\gamma + 1$ on M. Then for E an extender from $M_\gamma^{\mathcal{T}}$ such that E is appropriate for extending $\mathcal{T}$ to a $\gamma + 2$ length tree on M, one of the following occur:

1. $\mathrm{ult}(M_{\gamma+1}^{\mathcal{T}*}, E)$ is wellfounded, where $M_{\gamma+1}^{\mathcal{T}*}$ is the unique model to which E should be applied.
2. For all α, for all sufficiently large θ, $M^{\mathrm{Col}(\omega,\theta)}$ contains a maximal branch through $\mathcal{T}$ with $\alpha \in \mathrm{wfp}(M_b^{\mathcal{T}})$. $\dashv$

The "Moreover" part of (2.) in Lemma 0.0.11 requires the following preliminary lemma.

LEMMA 0.0.14. Suppose that $\mathcal{T}$ is a maximal good tree on a good premouse M and that b is a wellfounded cofinal branch, then b is non-dropping (if $\mathcal{T}$ and M are fine structural) and b is the unique.

Proof. First suppose that M and $\mathcal{T}$ are fine structural and b is a cofinal branch such that whenever $M_b^{\mathcal{T}}\|\xi$ is acive, then $M_b^{\mathcal{T}}|\xi$ is Σ_2^1-full, then trivially $M_b^{\mathcal{T}} \trianglelefteq \mathrm{L}(M(\mathcal{T}))$ and so b can't have dropped.

Suppose that there are two cofinal branches b and c, then let $\Gamma = \{\alpha : i_b^{\mathcal{T}}(\alpha) = i_c^{\mathcal{T}}(\alpha) = \alpha\}$. ❑

References

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