# 1 Complexity of reals in $\omega$-models of set theory

In [MS70] D. Martin and R. Solovay show that if $\aleph_1^{L[r]} = \aleph_1$ and $\mathrm{MA}_{\aleph_1}$ holds, then every uncountable set of reals is co-analytic. Thus by a ccc forcing over L, a model of set theory is obtained which satisfies

$$\mathbb{R} \cap \mathrm{L} \text{ is uncountable, co-analytic, and } \mathbb{R}^{\mathrm{L}} \neq \mathbb{R}$$

H. Friedman asked in [Fri75, problem 86] whether it is consistent to have

$$\mathbb{R} \cap \mathrm{L} \text{ is uncountable, analytic, and } \mathbb{R}^{L} \neq \mathbb{R}$$

In [VW98] B. Velickovič and H. Woodin investigate this question of of Friedman. THEOREM 4 of [VW98] answers Friedman's question negatively by showing that if $M$ is an inner model of set theory with $\mathbb{R}^M$ analytic, then

$$(*) \qquad \text{either } \aleph_1^M \text{ is countable or } \mathbb{R}^M = \mathbb{R}$$

Increasing the absoluteness between $V$ and $M$ admits a corresponding increase the complexity of $\mathbb{R}^M$. For example, $(*)$ holds with "$\mathbb{R}^M$ *is analytic*" replaced by "$\mathbb{R}^M$ *is* $\underset{\sim}{\Sigma}^1_2$" assuming "$\forall x \in \mathbb{R}\, x^\sharp$ *exists*" (see THEOREM 5 [VW98]). Assuming that *there are $n$ Woodin cardinals with a measurable cardinal above*, $(*)$ holds when "$\mathbb{R}^M$ *is* $\underset{\sim}{\Sigma}^1_{n+2}$" (see THEOREM 6 [VW98]). These results can be viewed as instances of a single result once a definition has been made.

DEFINITION 1.0.1. A set of reals $A$ $*$*-absolute* if for each $\mathbb{P}$ there is a $\mathbb{P}$-name $\dot{A}_{\mathbb{P}}$ such that whenever $\mathbb{P}_1$ and $\mathbb{P}_2$ are posets with $G_1 \subseteq \mathbb{P}_1$ generic over V and $G_2 \subseteq \mathbb{P}_2$ generic over $\mathrm{V}[G_1]$, then

$$\left\langle \mathrm{HC}, \in, A \right\rangle^{\mathrm{V}[G_1]} \prec_{\Sigma_1} \left\langle \mathrm{HC}, \in, A \right\rangle^{\mathrm{V}[G_1][G_2]}$$

where $A^{\mathrm{V}[G]} \overset{\text{df}}{=} \dot{A}_{\mathbb{P}}[G]$ whenever $G \subseteq \mathbb{P}$ is generic over V. ⊣

NOTATION. For $\kappa$ a regular cardinal there is a variant of $*$-absoluteness which I will denote $*_\kappa$*-absoluteness* which restrict to $\mathbb{P}$ of size at most $\kappa$. Similarly define $*_{<\kappa}$*-absoluteness*. ⊣

THEOREM 1.0.2. Let $M$ be an inner model of set theory with $A = \mathbb{R}^M$ and suppose that $A$ is $*_{\aleph_1}$-absolute, then either $\aleph_1^M$ is countable or $\mathbb{R}^M = \mathbb{R}$. ⊣

*Proof.* The proof of this is a good warm up for the proof of ❑

The following is useful in aplications of this theorem.

DEFINITION 1.0.3. $A \subseteq \mathbb{R}$ is *universally Baire* (UB) if for every poset $\mathbb{P}$ there are trees $T$ and $T^*$ such that

$$A = p[T] = \mathbb{R} \setminus p[T^*]$$

$$V^{\mathbb{P}} \models p[T] = \mathbb{R} \setminus p[T^*]$$ ⊣

REMARK. As with $*$-absoluteness there are the restricted notions of $\kappa$*-universally Baire* ($\mathrm{UB}_\kappa$) and $<\kappa$-*universally Baire* ($\mathrm{UB}_{<\kappa}$) for $\kappa$ a regular cardinal. ⊣

LEMMA 1.0.4. If $A \subseteq \mathbb{R}$ is UB (resp. $\mathrm{UB}_\kappa$ or $\mathrm{UB}_{<\kappa}$), then $A$ is $*$-absolute (resp. $*_\kappa$-absolute or $*_{<\kappa}$-absolute). $\dashv$

*Proof.* For each regular cardinal $\kappa$ choose $T_\kappa$ and $T^*_\kappa$ trees such that

$$A = p[T_\kappa] = \mathbb{R} \setminus p[T^*_\kappa]$$

and

$$V^{\mathrm{Col}(\omega,\kappa)} \models p[T_\kappa] = \mathbb{R} \setminus p[T^*_\kappa]$$

For $\mathbb{P} \in \mathrm{H}(\kappa^+)$ set

$$\dot{A}_\mathbb{P} \stackrel{\text{df}}{=} \{\langle p, \sigma\rangle : \sigma \subseteq \mathbb{P} \times \omega^{<\omega} \;\&\; p \Vdash \sigma \in p[T_\kappa]\}$$

To see that this is well defined it suffices to show that if ❑

THEOREM 4 of [VW98] follows from THEOREM 1.0.2 and the following classical theorem of Schoenfield.

THEOREM. (Schoenfield) $\underset{\sim}{\Sigma}^1_1$ sets are universally Baire. $\dashv$

THEOREM 5 of [VW98] follows from THEOREM 1.0.2 and the following theorem of D. Martin.

THEOREM 1.0.5. Suppose for all $x$ in $\mathrm{H}(\kappa^+)$, $x^\sharp$ exists. Then $\underset{\sim}{\Sigma}^1_2$ is $\kappa$-universally Baire. $\dashv$

Finally THEOREM 6 of [VW98] follows from THEOREM 1.0.2 and the following theorem due to D. Martin, J. Steel, and H. Woodin.

THEOREM 1.0.6. Suppose there are $n$ Woodin cardinals $\delta_1 < \cdots < \delta_n$ and a measurable cardinal above. Then $\underset{\sim}{\Sigma}^2_{n+2}$ is $<\!\delta_1$-universally Baire.

DEFINITION 1.0.7. Let $\prec$ be a linear order on a set, also denoted by $\prec$, with cofinality a limit ordinal. $T \subseteq \prec^{<\omega}$ is $\prec$*-superperfect* if for all $s \in T$, there is $i \geq \mathrm{lh}(s)$ such that

$$\big|\{t(i) : t \in T \;\&\; t \supseteq s\}\big| \text{ is cofinal in } \prec \quad \dashv$$

If $T$ is $\prec$-superperfect and $s \in T$, let $s^+$ be the shortest $t \in T$ suth that $t \supseteq s$ and $t$ is a **splitting node** of $T$, i.e., $\big|\{t(i) : t \in T \;\&\; t \supseteq s\}\big|$ is cofinal in $\prec$.

THEOREM 1.0.8. Suppose that $T$ is $\prec$-superperfect and $\mathcal{D} = \{D_i : i \in \omega\}$ is a set of dense subsets of $T$. Then for any $a \subseteq \omega$, there are $\mathcal{D}$ generic branches $b_0$, $b_1$, and $b_2$ such that $a$ is "continuously" coded by $\langle b_0, b_1, b_2, \prec\rangle$.

REMARK. The use of three generic branches is optimal in a sense to be described below. $\dashv$

*Proof.* The decoding is simple enough. Inductively define $n^j_i$ for $j < \omega$ and $i < 3$ as follows:

$$\begin{aligned} n^0_i &= \mathrm{lh}(\langle\rangle^+) \text{ for } i = 0, 1, 2 \\ n^{j+1}_i &= \inf\{n : b_2(n^j_2) \prec b_i(n)\} \text{ for } i = 0, 1 \\ n^{j+1}_2 &= \inf\{n : b_2(n) \prec b_1(n^{j+1}_0)\} \end{aligned}$$

and

$$a = \{j : b_0(n_0^j) \prec b_1(n_1^j)\}$$

The construction of an appropriate triple $\langle b_0, b_1, b_2\rangle$ is a simple matter. Prior to the $j^{\text{th}}$ round of the construction there will be

$$\begin{aligned} b_i^j &\in D_l \text{ for all } l < j \\ n_i^j &= \mathrm{lh}(b_i^{j+}) \\ S_i^j &= \{t(m) : t \in T \ \&\ t \supseteq b_i^j \ \&\ m < n_i^j\} \end{aligned}$$

where initially

$$\begin{aligned} b_i^0 &= \langle\rangle \\ n_i^0 &= \mathrm{lh}(\langle\rangle^+) \\ S_i^0 &= \{t(m) : t \in T \ \&\ m < n_i^0\} \end{aligned}$$

For definiteness suppose $j \in a$. (If $j \notin a$ just reverse the construction of $b_0^{j+1}$ and $b_1^{j+1}$ in what follows.)

- Pick $b_0^{j+1} \in D_j$ with $b_0^{j+1} \supsetneqq b_0^{j+}$. Let $n_0^{j+1} = \mathrm{lh}(b_0^{j+})$ and let $S_0^{j+1} = \{t(m) : t \in T \ \&\ t \supseteq b_0^j \ \&\ m < n_0^{j+1}\}$. By definition $S_0^{j+1}$ is bounded on $\prec$.
- Pick $b_1^1 \in D_0$ with $\mathrm{lh}(b_1^1) \supsetneqq \{\}^+$ and $b_1^1(n_1^0) \succ b_0^1(n_0^0)$. This can be done as $\{t(n_1^0) : t \in T\}$ is cofinal in $\prec$. As above, set $n_1^1 = \mathrm{lh}(b_1^{1+})$ and set $S_1^1 = \{t(m) : t \in T \ \&\ t \supseteq b_1^1 \ \&\ m < n_1^1\}$. Again, $S_1^1$ is bounded on $\prec$.
- Pick $b_2^1 \in D_0$ with $b_2^1(n_2^0)$ bounding $S_0^1 \cup S_1^1$. Set $n_2^1 = \mathrm{lh}(b_2^{1+})$ and set $S_2^1 = \{t(m) : t \in T \ \&\ t \supseteq b_2^1 \ \&\ m < n_2^1\}$.

❑

THEOREM 1.0.9. Let $\kappa$ be a measurable cardinal, $G \subseteq \mathrm{Col}(\omega, {<}\kappa)$, and suppose $R \subseteq \mathbb{R}_G$ is $\mathrm{OD}^{\mathrm{V}[G]}(\mathbb{R}_G \cup V)$ and satisfies $(*)$ "*some mild closure conditions*". Then either

- $\mathrm{card}(\omega_1^R) = \aleph_0^{V[G]}$, or
- $R = \mathbb{R}^{V[G]}$

*Proof.* Fix $r_0 \in \mathbb{R}_G$, $a_0 \in V$, and a $\Sigma_1$ formula $\theta(v_0, v_1, v_2)$ such that

$$\begin{aligned} R(x) &\leftrightarrow \mathrm{V}[G]|\alpha_0 \models \theta(x, r_0, a_0) \\ &\leftrightarrow \mathrm{V}[r_0][x] \models \text{“} \Vdash_{\mathrm{Col}(\omega,<\kappa)} \mathrm{V}[\dot{G}]|\check{\alpha}_0 \models \theta(\check{x}, \check{r}_0, \check{a}_0)\text{”} \end{aligned}$$

Let $\alpha_1 \gg \kappa, \alpha_0$ be large enough that $V_{\alpha_1} \models \mathrm{ZF}^-$ and

$$R(x) \leftrightarrow \mathrm{V}[r_0][x]|\alpha_1 \models \text{“} \Vdash_{\mathrm{Col}(\omega,<\kappa)} \mathrm{V}[\dot{G}] \models \theta(\check{x}, \check{r}_0, \check{a}_0)\text{”}$$

❑

Let $S$ be a set of of ordinals coding $V_{\alpha_1}$ and $\theta'$ express "$\Vdash_{\mathrm{Col}(\omega,<\kappa)} \mathrm{V}[\dot{G}] \models \theta(\check{x}, \check{r}_0, \check{a}_0)$", then

$$R(x) \leftrightarrow L[S, r_0, x] \models \theta^*(x, r_0, a_0)$$

# References


[Fri75] Harvey Friedman, *One hundred and two problems in mathematical logic*, J. Symbolic Logic **40** (1975), 113–129. MR 51 #5254

[MS70] D. A. Martin and R. M. Solovay, *Internal Cohen extensions*, Ann. Math. Logic **2** (1970), no. 2, 143–178. MR 42 #5787

[VW98] Boban Velickovic and W. Hugh Woodin, *Complexity of reals in inner models of set theory*, Ann. Pure Appl. Logic **92** (1998), no. 3, 283–295. MR 99f:03067