Project Description

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1. Introduction

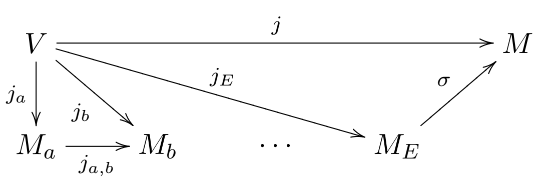
This proposal concerns the interaction between inner model theory, descriptive set theory, and models of determinacy. There are three projects described. The first project is primarily pedantic and simply aims to write up and publish results due to the author. The first result concerns obtaining strong models of determinacy and hence inner models for large cardinals from the existence of certain ideals on the first uncountable cardinal and the second result concerns the complexity of the reduction relation among simple Boolean equivalence relations. The second project aims to improve the earlier on strength from ideals. In particular the goal is to make a certain part of the argument more widely applicable. If successful this work would move forward several other projects which rely on the similar techniques and are currently stalled at a particular point. The third project, to be done in conjunction with Steve Jackson at UNT, is an attempt to isolate the cardinals of $\mathrm{L}(\mathbb{R})$ via their large cardinal properties in $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}$. This project might help in Jackson's inductive analysis of the regular cardinals $\mathrm{L}(\mathbb{R})$. Each of the 2^{nd} and 3^{rd} project will employ techniques related to Woodin's analysis of HOD as a fine structural inner model. Success on any of these projects would yield interesting as well as fruitful results for those working in this field.

Descriptive Set Theory. Broadly stated descriptive set theory is the study of *definable* subsets of *Polish spaces* (complete, separable, metric spaces). Every Polish space is Borel isomorphic to Euclidean space $\mathbb{R}$ or $\mathbb{N}$ with the discrete topology. The symbol $\mathbb{R}$ will be used somewhat ambiguously for the ambient Polish space and the term "real" will refer to any element $\mathbb{R}$. Typical Polish spaces used in descriptive set theory are $\mathbb{N}$ (which will be denoted by ω), ω^ω = *infinite sequences of non-negative integers* (*Baire space*), 2^ω = *infinite binary sequences* (*Cantor space*), or countable products of these. At the very bottom of the descriptive set theoretic hierarchy are the Borel sets, next come projections of Borel subsets of the plane, these are the **analytic** sets (denoted $\utilde{\Sigma}^1_1$), complements of $\utilde{\Sigma}^1_1$ sets are called **co-analytic** (denoted $\utilde{\Pi}^1_1$). Moving upward, the **projective sets** are defined via the operations of projection and complementation. Specifically, the $\utilde{\Sigma}^1_{n+1}$ sets are the projection of $\utilde{\Pi}^1_n$ sets, and the $\utilde{\Pi}^1_{n+1}$ sets are complements of $\utilde{\Sigma}^1_{n+1}$ sets. Equivalently, $A \subseteq \mathbb{R}$ is projective if A is *first order definable* over the structure $\langle V_{\omega+1}, \in\rangle$, i.e. there is a formula φ and real z such that "$x \in A \iff \langle V_{\omega+1}, \in\rangle \models \varphi(x, z)$". The projective hierarchy can be extended to transfinite levels $\utilde{\Pi}^1_\alpha$ for $\alpha \in \mathrm{OR}$. Every set in $\mathrm{L}(\mathbb{R})$ falls in this generalized projective hierarchy, moreover, under the axiom of determinacy every set of reals falls somewhere in this hierarchy. The pointclass $\utilde{\Sigma}^2_1$ consists of all those sets $A \subseteq \mathbb{R}$ defined by existential second order quantification over $\langle V_{\omega+1}, \in\rangle$, in other words $A \in \utilde{\Sigma}^2_1$ if there is a formula φ such that "$x \in A \iff \exists B \subseteq \mathbb{R}\, \langle V_{\omega+1}, \in\rangle \models \varphi(A, x)$". Generally questions about Borel sets are answered in ZFC. Already at the level of analytic sets there are questions independent of ZFC, for example is every function whose graph is $\utilde{\Pi}^1_1$ Lebesgue measurable. Moreover, as described below questions concerning the *determinacy* of analytic sets already involve *large cardinals*.

Determinacy. Each set A of reals naturally determines an infinite perfect information game G_A on ω. In G_A two players I and II take turns playing integers so that after ω many moves together they have built a real $x = \langle n_0, n_1, \cdots\rangle$. The real x is a winning run for I if $x \in A$, else it is a win for II. A strategy is a function $\sigma : \omega^{<\omega} \to \omega$. A strategy σ is winning for I if whenever player I follows σ, then the result is a win for I. In other words σ is a I winning strategy if whenever player II plays $y = n_0, n_1, \cdots$ and I always responds with $\sigma(\langle n_0, \ldots, n_{i-1}\rangle)$, then $\sigma * y = \langle \sigma(\emptyset), n_0, \sigma(\langle n_0\rangle), n_1, \sigma(\langle n_0, n_1\rangle), \cdots\rangle \in A$. Similarly define the notion of winning strategy for player II. A set $A \subseteq \mathbb{R}$ is determined if one of the players has a winning strategy in

G_A. For example $A = \mathbb{R}$ is determined since player I has a winning strategy in $G_{\mathbb{R}}$, namely, the strategy "*always play* 0" is winning for I. Determinacy of open and closed games, i.e. A is open or closed, was proved by Gale and Stewart [GS53]. The determinacy of all Borel games is already a significant theorem due to Martin [Mar85]. The proof of Borel determinacy requires ω_1 iterations of the powerset operation [Fri71]. Determinacy for projections of Borel sets, i.e. analytic sets, already exhibits large cardinal strength [Har78] as discussed below. The *Axiom of Determinacy* (AD) is the assertion that all games are determined. The axiom of determinacy implies all sets are Lebesgue measurable and have the property of Baire as well as "*there is no uncountable sequence of distinct reals*" so clearly this axiom contradicts choice and hence is *false* in V (the universe of all sets).

LARGE CARDINALS. A *large cardinal axiom* asserts the existence of a non-trivial elementary embedding $j : V \to M$ and further stipulates that M should resemble V to some degree. The least ordinal moved by j is called the critical point of j denoted $\mathrm{crit}(j)$. The amount of agreement between M and V determines the strength of the axiom. For instance if M is not even required to be wellfounded, then there is no strength at all as any non-principle ultrafilter on any infinite can be used in conjunction with an ultrapower construction to produce a non-trivial embedding $j : V \to M$. On the other hand $V = M$ is impossible as shown by Kunen [Kun71]. If M is wellfounded, then identify M with its transitive collapse. If $j : V \to M$ is a non-trivial elementary embedding with M transitive, then $\kappa = \mathrm{crit}(j)$ is called a **measurable cardinal**. For j as above $\mu_j = \{a \subseteq \kappa : \kappa \in j(a)\}$ is a κ-complete ultrafilter (two valued measure) on $\mathcal{P}(\kappa)$. Moreover, any κ-complete ultrafilter μ on $\mathcal{P}(\kappa)$ induces a transitive model $M_\mu = \mathrm{ult}(V, \mu)$ and elementary embedding $j_\mu : V \to M_\mu$ with $\kappa = \mathrm{crit}(j)$. Moving up the large cardinal hierarchy a cardinal κ is called **$\boldsymbol{\lambda}$-strong** if there is $j : V \to M$ such that $V_{\lambda+1} = V^M_{\lambda+1}$. A cardinal κ is measurable just in case κ is κ-strong while saying κ is $\kappa + 1$-strong is already much stronger than simply saying κ is measurable. The notion of λ-strong can't be captured by a single ultrapower construction, but it can be captured by a directed system of ultrapowers. If κ is λ-strong as witnessed by $j : V \to M$, then $E_a = \{x : x \subseteq [\kappa]^{\mathrm{card}(a)} \ \&\ a \in j(x)\}$ is a non-principle κ-complete ultrafilter on $\mathcal{P}([\kappa]^{\mathrm{card}(a)})$ for each $a \in [\lambda]^{<\omega}$. The measure E_a induces a map $j_a : V \to M_a = \mathrm{ult}(V, E_a)$ such that for $a \subseteq b \in [\lambda]^{<\omega}$ there is a natural $j_{a,b} : M_a \to M_b$ so that the following commutes:



The resulting directed system $\langle M_a, j_{a,b}\rangle_{a \subseteq b \in [\lambda]^{<\omega}}$ has a direct limit model M_E and there is are embeddings $V \xrightarrow{j_E} M_E \xrightarrow{\sigma} M$ such that $\sigma \restriction \lambda + 1 = \mathrm{id} \restriction \lambda + 1$ and $\sigma_E \circ j_E = j$. Hence $j_E : V \to M_E$ and $V_{\lambda+1} = V^{M_E}_{\lambda+1}$ so j_E witnesses that κ is λ-strong. Clearly $E = \{(a, x) : a \in [\lambda]^{<\omega} \ \&\ a \in j(x)\}$ *codes* enough of j to witness that κ is λ-strong. The object E is called **the $(\boldsymbol{\kappa}, \boldsymbol{\lambda})$-extender derived from $\boldsymbol{j}$**, the model M_E is called the **ultrapower by $\boldsymbol{E}$**, and j_E is called the ultrapower embedding.

With the exception of the wellfoundedness of M_E the extender E can be simply characterized by its first order properties over $V_{\kappa+1}$ for example "*for each* $a \in [\lambda]^{<\omega}$, E_a *is a measure on* $[\kappa]^{|a|}$" and "*for* $a \subseteq b \in [\lambda]^{<\omega}$, E_b *projects to* E_a", i.e. for $a \subseteq b \in [\lambda]^{<\omega}$ and $A \subseteq [\kappa]^{|a|}$, $A \in E_a \iff A^{a,b} \in E_b$ where for $u \in [\kappa]^{|b|}$, $u^{b,a}$ sits in u the same way a sits in b (so $u \mapsto u^{b,a}$ is the obvious projection map $\pi_{b,a} : [\kappa]^{|b|} \to [\kappa]^{|a|}$), for $A \subseteq [\kappa]^{|a|}$, $A^{a,b} = \pi^{-1}_{b,a}[A] = \{u \in [\kappa]^{|b|} : u^{b,a} \in A\}$. An object satisfying all of the first order properties of being an extender over a transitive model N is called a **pre-extender** over N. Suppose E is a (κ, λ)-extender over a transitive model M and N is a transitive model with $V^N_{\kappa+1} = V^M_{\kappa+1}$, then E is a (κ, λ)-*pre-extender* on N and we can form $j^N_E : N \to N_E$ as described above, although there is no guarantee that N_E will be wellfounded.

WOODIN CARDINALS. Woodin cardinals are a slightly different breed of large cardinals in that they are not the critical points of embeddings. Let $\kappa < \lambda < \delta$ be cardinals and let $A \subseteq \delta$, κ is **λ-A-strong** if there is $j : V \to M$ such that $\kappa = \text{crit}(j)$, $j(\kappa) > \lambda$, $V_{\lambda+1} = V^M_{\lambda+1}$, and $j(A) \cap \lambda = A \cap \lambda$. Call κ **<δ-A-strong** provided κ is λ-A-strong for all $\lambda < \delta$. A cardinal δ is a **Woodin cardinal** if for all $A \subseteq \delta$, $\{\kappa : \kappa \text{ is } {<}\delta\text{-}A\text{-strong}\}$ is unbounded in δ.

MICE. (*This description of mice and iterability is incomplete and even a bit inaccurate in the details. The goal is to give the reader a vague idea of what a mouse is. In particular the issue of* dropping *will be completely ignored.* The description here is based on Mitchell-Steel [MS94, Ste99a].) By relativism Godel's constructible hierarchy to a (κ, λ)-extender E (over V) the inner model $\mathrm{L}[E]$ is obtained in which κ remains λ-strong. Generalizing this further $\vec{E} = \langle E_\alpha : \alpha < \xi \rangle$ is a **coherent sequence of extenders** provided that for $\alpha < \xi$ such that $E_\alpha \neq \emptyset$, E_α is a (κ_α, α)-pre-extender over $\mathrm{L}_\alpha[\vec{E}{\restriction}\alpha]$ satisfying the **coherence property**: if $j : \mathrm{L}_\alpha[\vec{E}{\restriction}\alpha] \longrightarrow \text{ult}(\mathrm{L}_\alpha[\vec{E}{\restriction}\alpha], \vec{E}_\alpha)$ is the ultrapower map, then $j(\vec{E}) \restriction \alpha + 1 = \vec{E} \restriction \alpha$. A **potential premouse** is a structure $\mathcal{M} = \langle \mathrm{L}_\alpha[\vec{E} \restriction \alpha], E_\alpha \rangle$ where $\vec{E}$ is a coherent extender sequence. The ordinal α is called the **height of** $\mathcal{M}$ and is denoted $\text{ht}(\mathcal{M})$, $\vec{E} \restriction \alpha$ is denoted $E^{\mathcal{M}}$, and E_α is denoted $F^{\mathcal{M}}$. If $E_\alpha \neq \emptyset$, then $\mathcal{M}$ is said to be **active** and E_α is called the **top extender** of $\mathcal{M}$, otherwise $\mathcal{M}$ is called **passive**. To make notation simpler let $\mathcal{M}\|\beta = \langle L_\beta[\vec{E} \restriction \beta], E_\beta \rangle$ and $\mathcal{M}|\beta = L_\beta[\vec{E}{\restriction}\beta]$. Thus if $\mathcal{M}\|\beta$ is passive, then there is no real difference between $\mathcal{M}\|\beta$ and $\mathcal{M}|\beta$. A **premouse** is a potential premouse which satisfies an additional condition known as *the initial segment condition*. Two premice $\mathcal{M}$ and $\mathcal{N}$ are said to be **lined up** if there is an β such that $\mathcal{M} = \mathcal{N}\|\beta$ or $\mathcal{M}\|\beta = \mathcal{N}$.

There is a method for *comparing* premice $\mathcal{M}$ and $\mathcal{N}$ by *iterating* them to premice $\mathcal{M}^*$ and $\mathcal{N}^*$ such that $\mathcal{M}^*$ and $\mathcal{N}^*$ are lined up. *Iterability* is best described via a game aptly called the **iteration game** and denoted $\mathcal{G}(\mathcal{M}, \theta)$. This game has θ many rounds. At the beginning of round α players I and II have constructed an (normal and maximal) **iteration tree** $\mathcal{T} \restriction \alpha = \langle T \restriction \alpha, \langle E_\beta : \beta + 1 < \alpha \rangle, \langle \mathcal{M}_\beta : \beta < \alpha \rangle, \langle j_{\gamma,\beta} : \gamma T \beta \rangle \rangle$ satisfying:

- $T \restriction \alpha$ is a *tree ordering* on α meaning $\gamma T \beta \Rightarrow \gamma < \beta < \alpha$, if $\gamma = \beta + 1$ is a successor ordinal, then γ is also a successor in the tree order with T-predecessor β^*, and if $\beta < \alpha$ is a limit, then $[0, \beta)_T = \{\gamma : \gamma T \beta\}$ is cofinal in β.
- For $\beta + 1 < \alpha$, E_β is a $(\kappa_\beta, \lambda_\beta)$-extender from the sequence of M_β, such that $V^{\mathcal{M}_{\beta^*}}_{\kappa_\beta+1} = V^{\mathcal{M}_\beta}_{\kappa_\beta+1}$ hence E_β is a pre-extender over M_{β^*}. Moreover, β^* is the least ordinal such that $V^{\mathcal{M}_{\beta^*}}_{\kappa_\beta+1} = V^{\mathcal{M}_\beta}_{\kappa_\beta+1}$ (*maximality*) and $\lambda_\beta > \lambda_\gamma$ for $\gamma < \beta$ (*normality*). $\mathcal{M}_{\beta+1}$ is the ultrapower of $\mathcal{M}_{\beta^*}$ by E_β and $j_{\beta^*,\beta+1} : \mathcal{M}_{\beta^*} \to \mathcal{M}_{\beta+1}$ is the ultrapower embedding and $\mathcal{M}_{\beta+1}$ is wellfounded.
- At limit $\beta < \alpha$, $\mathcal{M}_\beta$ is wellfounded and the direct limit $\mathcal{M}_0 \xrightarrow{j_{0,\beta_1}} \mathcal{M}_{\beta_1} \xrightarrow{j_{\beta_1,\beta_2}} \mathcal{M}_{\beta_2} \cdots \mathcal{M}_\beta$ where $\langle 0, \beta_1, \beta_2, \cdots \rangle = [0, \beta)_T$. The maps $j_{\gamma,\beta} : \mathcal{M}_\gamma \longrightarrow \mathcal{M}_\beta$ are the corresponding direct limit maps.

Player I plays at successor stages $\alpha = \gamma + 1$ by choosing an extender E_γ on the $\mathcal{M}_\gamma$ sequence. Player II plays at limit stages by choosing a cofinal branch through $\mathcal{T}{\restriction}\alpha$. In each case the goal is to extend to an iteration tree of length $\alpha + 1$. Player I needs only to hope that $\text{ult}(\mathcal{M}_{\gamma^*}, E_\gamma)$ is wellfounded, whereas, player II has to hope both that $\mathcal{T} \restriction \alpha$ has a cofinal branch and that the direct limit model is wellfounded. If the game lasts θ rounds, then player II wins by default. The first player who can't play looses.

Suppose $\mathcal{N}$ and $\mathcal{M}$ are two iterable premice. Play the iteration game where at a successor stage $\alpha+1$ player I chooses extenders based on the *least disagreement* between $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$. For example, if λ_0 is the first place where $\mathcal{M}\|\lambda_0 \neq \mathcal{N}\|\lambda_0$, then at least one of $\vec{E}^{\mathcal{M}}_{\lambda_0}$ or $\vec{E}^{\mathcal{N}}_{\lambda_0}$ is non-empty. Suppose both $\vec{E}^{\mathcal{M}}_{\lambda_0}$ or $\vec{E}^{\mathcal{N}}_{\lambda_0}$ are non-empty, the other cases being the similar. Form $\mathcal{M} = \mathcal{M}_0 \xrightarrow{j_{0,1}} \mathcal{M}_1 = \text{ult}(\mathcal{M}_0, E^{\mathcal{M}}_\lambda)$ and $\mathcal{N} = \mathcal{N}_0 \xrightarrow{\pi_{0,1}} \mathcal{N}_1 = \text{ult}(\mathcal{N}_0, E^{\mathcal{N}}_\lambda)$. By the coherence property on the sequences of extenders $\mathcal{M}_1$

and $\mathcal{N}_1$ agree past λ_0. Let $\lambda_1 > \lambda_0$ be the first disagreement between $\mathcal{M}_1$ and $\mathcal{N}_1$. Again assume both $\vec{E}^{\mathcal{M}_1}_{\lambda_1}$ and $\vec{E}^{\mathcal{N}_1}_{\lambda_1}$ are non-empty and apply these extenders to the first model where it makes sense to apply them. Continuing in this manner players I and II construct iteration trees $\mathcal{T}$ and $\mathcal{U}$ with models $\mathcal{M}_\alpha$ and $\mathcal{N}_\alpha$ respectively along with an increasing sequence of ordinals λ_α such that for all $\alpha < \beta$, $M_\beta \| \lambda_\alpha = \mathcal{N}_\beta \| \lambda_\alpha$. Intuitively, the goal of this game is to *iterate out* the least disagreement. The initial segment property guarantees that a $\theta < \max\{|M|^+, |N|^+\}$ is reached such that $\mathcal{M}_\theta$ and $\mathcal{N}_\theta$ are lined up. Hence $\mathcal{M}$ and $\mathcal{N}$ have been compared. In order for comparison to succeed player II must have winning strategies in $\mathcal{G}(\mathcal{M}, \kappa+1)$ and $\mathcal{G}(\mathcal{N}, \kappa+1)$ where $\kappa = \max\{|M|^+, |N|^+\}$. A **mouse** is a sufficiently iterable premouse, where sufficiently iterable means that player II wins $\mathcal{G}(\mathcal{M}, \theta+1)$ where θ is big enough to do all relevant comparisons. For example if we are only concerned with countable structures, then *sufficiently iterable* would mean $\omega_1 + 1$-iterable. There is one final notion to address. $\mathcal{M}$ is a **collapsing mouse** if every element of $\mathcal{M}$ is Σ_1-definable over $\mathcal{M}$ from no parameters. If $\mathcal{M}$ and $\mathcal{N}$ are collapsing mice, then they are already lined up, although the iterability of the mice is required still to prove this.

How are these related? Initially Martin [Mar70] showed that the existence of a measurable cardinal implies $\underset{\sim}{\Sigma}^1_1$-determinacy, then Martin and Steel [MS89] showed that the existence of n-Woodin cardinals and a measurable above implies $\underset{\sim}{\Sigma}^1_{n+1}$-determinacy. Woodin showed that ω-Woodin cardinals and a measurable above yields $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ and beyond. In the other direction, Woodin showed that "ZF+DC+AD" implies the existence of an inner model of ω-Woodin cardinals. Thus $\mathrm{Con}(\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}) \implies \mathrm{Con}(\mathrm{ZFC} +$ *"there are ω-Woodin cardinal"*). Determinacy asserts the existence of certain reals, namely iteration strategies, and this could not possibly imply the existence large cardinals outright. Yet inner models (mice) for large cardinal axioms are countable objects and hence essentially reals; it turns out that varying amounts of determinacy corresponds exactly to the existence of certain mice.

The **sharp** of a real x is simply the unique active collapsing mouse $\mathcal{M}^\sharp_0(x)$ *over* x with $E^{\mathcal{M}^\sharp_0(x)} = \emptyset$. Sharps resemble inner models for measurable cardinals, except that by doing one more level of construction the universe is collapsed to ω. In other words if $\mathcal{M}^\sharp_0(x) = \langle L_\alpha, F\rangle$, then $\mathcal{M}^\sharp_0(x) \models$ "F is a measure on $\kappa = \mathrm{crit}(F^{\mathcal{M}})$", but $L_{\alpha+1}[F] \models$ "$|\mathcal{M}_0(x)| = \omega$". Harrington [Har78] showed $\underset{\sim}{\Sigma}^1_1$-determinacy is equivalent to the existence of $\mathcal{M}^\sharp_0(x)$ for all $x \in \mathbb{R}$. Woodin [Woo99] and recently Neeman [Nee95] show that $\underset{\sim}{\Sigma}^1_{n+1}$-determinacy is equivalent to the existence of $\mathcal{M}^\sharp_n(x)$ where $\mathcal{M}^\sharp_n(x)$ is the *unique* active collapsing mouse $\mathcal{N}$ over x satisfying "*there are n-Woodin cardinals*"; again this resembles an inner model with n-Woodin cardinals and a measurable above, except that everything collapses as soon as one more level of construction is done.

Essentially Neeman shows that **if** there is a *description* $\dot{A}$ for a set of reals A **and if** this description can be incorporated into a new hierarchy of *hybrid mice* of the form $\langle\rangle \mathrm{L}_\alpha[\dot{A} \restriction \alpha, \vec{E} \restriction \alpha], \dot{A}_\alpha E_\alpha$ **and if** for each real x, the sharp for a Woodin cardinal is reached in this hierarchy, i.e. $\mathcal{M}^{h,\sharp}_1(x)$ exists, **then** A is determined. This is at the heart of a technique invented by Woodin to prove determinacy from combinatorial principles. Steel's core model theory allows one to produce inner models with Woodin cardinals from combinatorial principles, e.g. an ω_1-dense ideal on ω_1, Martin's Maximum, PFA, generic absoluteness, failure of $\square_\kappa$ for κ a strong limit, ... Combined with Neeman's method of proving determinacy it is possible to show that $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ holds in a generic extension of V. The idea is to work inductively through more and more complicated sets of reals showing that there are very good descriptions for these sets and then using core model techniques to get mice with Woodin cardinals with respect to this description. This process is known as *the core model induction.* The notion of *more complicated set of reals* is directly related to iteration strategies for mice. In particular suppose Γ determinacy has been shown. There is a closure notion $C_\Gamma : \mathrm{HC} \to \mathrm{HC}$, moreover this closure operation is given by stacking mice that have already been shown to exist. Some technique is employed to produce a unique mouse $\mathcal{M}_\Gamma$ and a canonical ω_1+1-iteration strategy S_Γ on $\mathcal{M}_\Gamma$. It is at this step that Woodin's analysis of HOD is used to produce the pair $\langle \mathcal{M}_\Gamma, S_\Gamma\rangle$. Let $S^*_\Gamma \subseteq \mathbb{R}$ be the coded version of S_Γ. It is always going to be the case that $S^*_\Gamma \notin \Gamma$ and the definition of S^*_Γ as "*the set of reals coding* S_Γ" turns out to be a sort of name for S^*_Γ that can be used in Neeman's result. Steel's core model theory is employed to produce an S_Γ-hybrid mouse reaching the sharp of a Woodin cardinal. By Neeman's result S^*_Γ-determinacy follows. Since $S^*_\Gamma \notin \Gamma$ the amount of determinacy has been we now have more determinacy that we started with. To date not much has been published concerning this method, however see [Ket00, Ste03]. Remedying this situation is part of the first project.

The goal of a core model induction is always to produce a model of "$\mathrm{ZF} + \mathrm{AD}^+ + \mathrm{V} = \mathrm{L}(\mathcal{P}(\mathbb{R}))$". AD^+ is a strengthening of AD designed to lift the structure theory for $\mathrm{L}(\mathbb{R})$ under AD to much richer models of the form $\mathrm{L}(\mathcal{P}(\mathbb{R}))$. The existence of *strong models of determinacy* yield inner models for *strong large cardinals*. The strength of a model of AD^+ depends on the properties of two ordinals Θ and Ω. Under AD, sets of reals come in a wellordered hierarchy called the Wadge hierarchy. A course version is given by $A \leq_w B$ if $P_A \subseteq P_B$ where $P_A = \{C \subseteq \mathbb{R} : C \text{ is projective in } A\}$. Martin showed that this forms a well ordered hierarchy. The rank of $A \subseteq \mathbb{R}$ in this hierarchy is called the **Wadge rank** of A (denoted $w(A)$). The ordinal Θ is defined to be the height of this hierarchy. It is not hard to show that Θ can equivalently be described as the supremum of all α such that there is a function $f : \mathbb{R} \xrightarrow{\text{onto}} \alpha$ (this is essentially $\boldsymbol{c}^+$ in a choiceless setting). The Wadge hierarchy naturally defines pointclasses $\mathcal{P}_\alpha(\mathbb{R}) = \{A \subseteq \mathbb{R} : w(A) < \alpha\}$. Under "$\mathrm{AD}^+ + \mathrm{V} = \mathrm{L}(\mathcal{P}(\mathbb{R}))$" define for each set $\beta < \Theta$ an ordinal $\Theta_+(\beta) = \sup\{\alpha : \text{there is a real } z \text{ and a set } A \text{ of Wadge rank } \beta \text{ and an } \mathrm{OD}_{A,z} \text{ function } f : \mathbb{R} \xrightarrow{\text{onto}} \alpha\}$. Inductively, $\Theta_0 = \Theta_+(0)$; if Θ_α is defined and $\Theta_\alpha < \Theta$, then $\Theta_{\alpha+1} = \Theta_+(\Theta_\alpha)$; for limit α, $\Theta_\alpha = \sup_{\beta<\alpha}\Theta_\beta$. The pointclasses $\mathcal{P}_{\Theta_\alpha}(\mathbb{R})$ have very strong closure properties, for example, $\mathrm{L}(\mathcal{P}_{\Theta_\alpha}(\mathbb{R})) \cap \mathcal{P}(\mathbb{R}) = \mathcal{P}_{\Theta_\alpha}(\mathbb{R})$. The ordinal Ω is the length of this sequence of Θ_α's. At he bottom of the AD^+ models are those satisfying "$\Theta = \Theta_0$", for example $\mathrm{L}(\mathbb{R})$ is in this group, then come those satisfying "$\Theta_0 < \Theta$", etc. $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ is just strong enough to yield inner models with ω-Woodin cardinals, while "$\mathrm{AD}^+ + \Theta_0 < \Theta$" yields a non-tame mouse (an active mouse $\mathcal{M}$ with a Woodin cardinal δ such $\mathrm{crit}(F^{\mathcal{M}}) < \delta$), this is past "ZFC + *there is a strong cardinal that is also a limit of Woodin cardinals*" (this is due to Woodin, but see [Ste99b]). If Ω is a limit ordinal, then $\mathrm{AD}_\mathbb{R}$ holds (all infinite two player games where each player plays reals are determined), this yields a mouse for the "$\mathrm{AD}_\mathbb{R}$-*hypothesis*" which asserts "there is a cardinal δ which is simultaneously a limit of Woodin cardinals and cardinals strong to δ". Woodin has shown both that a model of "$\mathrm{ZF} + \mathrm{DC}_\mathbb{R} + \mathrm{AD}_\mathbb{R}$" follows from the $\mathrm{AD}_\mathbb{R}$-hypothesis and that a model of "$\mathrm{ZF} + \mathrm{DC}_\mathbb{R} + \mathrm{AD}_\mathbb{R}$" yields a model of the $\mathrm{AD}_\mathbb{R}$-hypothesis. If $\mathrm{cf}(\Omega) > \omega$, then "$\mathrm{AD}_\mathbb{R} + \mathrm{DC}$" must hold [KS85]. Etc...

2. Publication of results from ongoing projects

2.1. **Strength from ideals on ω_1.** This part of the project is largely pedantic, but nevertheless important and addresses one of the concerns raised toward the end of the introduction. The author has yet to produce a write up of his results relating determinacy and the existence of certain ideals on $\mathcal{P}(\omega_1)$. Part of the difficulty is *non-existence* of any sort of write up of a large body of work that is used by the author, which is due to Woodin. This situation is changing, Steel has now written and circulated several notes with much of the requisite background material. The material to be written up includes a presentation of Woodin's core model induction deriving $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ from an ω_1-dense ideal on ω_1. This is an optimal result in that from $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ there is an inner model N of a forcing extension of $\mathrm{L}(\mathbb{R})$ such that N satisfies "ZFC + there is an ω_1-*dense ideal on* ω_1"[Woo99]. If one adds *Cantor's Continuum Hypothesis* (CH) however, then much more than $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ can be derived. At this time it is not at all clear precisely what can be obtained. It is known that

$$(*) \qquad \mathrm{Con}(\text{“ZFC} + \text{CH} + \textit{there is an } \omega_1\textit{-dense ideal on } \omega_1\text{”})$$

follows from

$$\mathrm{Con}(\text{“}\textit{there is a supercompact cardinal with a Woodin cardinal above}\text{“})$$

This is a recent unpublished result due to Woodin, although much earlier Woodin had shown that $(*)$ follows from

$$(**) \qquad \mathrm{Con}(\text{“ZF} + \text{DC} + \text{AD}_{\mathbb{R}} + \Theta \text{ is regular”})$$

Unfortunately, mice for "there is a supercompact cardinal with a Woodin cardinal above" are no where in sight so for now our sites must be set a bit lower. Attempting to derive $(**)$ from $(*)$ also appears out of reach at the moment. In an effort to get as much strength as currently possible from $(*)$ the author has shown from a hypothesis stronger than $(*)$ there is a pointclass Γ such that $L(\Gamma, \mathbb{R}) \models$ "ZF + $\text{AD}^+ + \Theta_0 < \Theta$". This in turn would yield Con("ZFC+*there is a strong cardinal which is a limit of Woodin cardinals*"). Unfortunately the current hypotheses are a bit *ad-hoc* being designed to make a critical argument work (†):

- ZFC + CH
- For each stationary $S \subseteq \omega_1$, there is a stationary $S' \subseteq S$ such that $\mathcal{I}_{NS} \restriction S'$ is ω_1-dense
- There is a map $j : \text{OR} \to \text{OR}$ (in V) such that whenever $G \subseteq \mathcal{P}(\omega_1)/\mathcal{I}_{NS}$ is generic over V with associated generic embedding j_G, then $j = j_G \restriction \text{OR}$

Matt Foreman has suggested that the addition of this last clause might genuinely add some strength. It is known, again due to Woodin, that $\mathrm{Con}(**) \to \mathrm{Con}(\dagger)$. No new ideas are required to improve the authors current results to:

$$(\dagger) \implies \text{there is a pointclass } \Gamma\text{, such that } L(\Gamma, \mathbb{R}) \models \text{ZF} + \text{DC}_{\mathbb{R}} + \text{AD}_{\mathbb{R}}$$

This would then yield Con("$\text{AD}_{\mathbb{R}}$-*hypothesis*"). A bit more work should yield significant improvements, although this still must be verified,

$$(\dagger) \implies \text{there is a pointclass } \Gamma\text{, such that } L(\Gamma, \mathbb{R}) \models \text{ZF} + \text{DC}_{\mathbb{R}} + \Theta = \Theta_{\omega_1}$$

The goal would to have a complete write up of this material submitted within a year.

2.2. **Complexity among simple cardinals in the absence of choice.** There is a second project the author needs to write up and published. This project is completely unrelated to the above. The work falls under the category of Borel equivalence relations or more generally under the study of cardinals in the absence of choice. The author has shown assuming "ZF + AD^+" that there is a collection of very-simple Borel equivalence relations E_x for $x \in \mathbb{R}$ such that the question "*is there an* $f : \mathbb{R} \to \mathbb{R}$ *such that* $\forall u, v \in \mathbb{R}\, u\, E_0\, v \iff f(u)\, E_x\, f(v)$" is complete $\underset{\sim}{\Sigma}^2_1$. There is a little work yet to be done, by restricting the the complexity allowed f, more local, results should be obtained which require less determinacy. For example, by restricting to Borel f the question should be complete $\underset{\sim}{\Sigma}^1_2$. This would yield a new proof of a result by Kechris and Adams [AK00] which asserts that the reducibility relation among Borel equivalence relations is complete $\underset{\sim}{\Sigma}^1_2$; of course much more is shown in [AK00]. The result itself is probably of less interests than the techniques employed in its proof. Questions such as "*Given an* ω*-model* M *of set theory such that* ω_1^M *is uncountable, when does* $\mathbb{R}^M = \mathbb{R}$*?*" are addressed. Similar questions were addressed in [VW98].

3. MORE ON STRENGTH FROM AN ω_1-DENSE IDEAL ON ω_1

This is a major portion of the project and again relates to work already done and ongoing by the author. Unlike the write up and relatively simple improvements mentioned above of the some of the authors earlier results, there is a genuine hurdle to be overcome. The only method known to get strength at the level of a non-tame mouse from combinatorial principles low in the cumulative hierarchy is Woodin's core model induction. Thus one must produce from the property a model of "ZF + AD^+ + V = $\mathrm{L}(\mathcal{P}(\mathbb{R}))$ + $\Theta_0 < \Theta$". There are two big steps to be taken. Initially take $(*)$ as the hypothesis from which we are trying to derive strength, $(*)$ is a statement about the structure $\mathrm{H}(\omega_2)$ (this is what is meant by *low in the cumulative hierarchy*). One argues that there is a *maximal model* $M_0 = \mathrm{L}(\Gamma_0, \mathbb{R})$ of "AD^+ + V = $\mathrm{L}(\mathcal{P}(\mathbb{R}))$ + $\Theta = \Theta_0$". Thus one needs only produce an $S^* \subseteq \mathbb{R}$ such that $S^* \notin M_0$ and $\mathrm{L}(S^*, \mathbb{R}) \models \mathrm{AD}^+$. A natural candidate for S^* would be the coded version an iteration strategy S on a mouse $\mathcal{M}$ satisfying (‡):

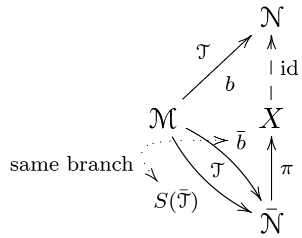
- $\mathcal{M}$ closed under the operation $a \to \mathrm{HOD}_a^{M_0} \cap \mathcal{P}(a)$
- $\mathcal{M}$ has a Woodin cardinal δ^M, hence $\mathrm{HOD}_{\mathcal{M}}^{M_0} \models$ "$\delta^{\mathcal{M}}$ *is Woodin*"
- $\mathcal{M} \models$ "$\delta^{+\omega} = \mathrm{OR}$"
- for all $\bar{\delta} < \delta$, $\mathrm{HOD}^{M_0} \models$ "$\bar{\delta}$ *is not Woodin*

S should additionally satisfy that all S-iterates continue to satisfy (‡). It is known that S^* is not in M_0, since if S^* were in M_o, then S^* would have Wadge rank Θ_0, hence M_0 would satisfy $\Theta_0 < \Theta$. This is a typical application, in which a strategy for a certain *full* mouse yields a set of reals of high complexity.

The first big step is to produce a pair $\langle \mathcal{M}, S \rangle$ as described. Moreover, S will have the additional property that there is a sequence $\mathcal{A} = \langle A_i : i \in \omega \rangle$ of OD^{M_0} sets of reals so that $\mathcal{M}$ has names $\tau_{i,j}^{\mathcal{M}}$ for each A_i such that $\tau_{i,j}^{\mathcal{M}}[g] = A_i \cap \mathcal{M}[g]$ whenever g is generic for collapsing $\delta_j^{\mathcal{M}}$, moreover, S is *uniquely determined* by the fact that it preserve (‡) and *correctly* moves each $\tau_{i,j}^{\mathcal{M}}$. This step uses Woodin's calculation of HOD^{M_0} as a fine structural model. The point is that if G is generic for $\mathcal{P}(\omega_1)/\mathcal{I}$ and $j_G : V \to N$ is the generic embedding, then for $\theta = \Theta^{M_0}$, $\mathcal{M}^* = \mathrm{HOD}^{M_0} \restriction (\theta^{+\omega})^{\mathrm{HOD}^{M_0}}$ is a mouse satisfying the above properties from the point of view of $j(M_0)$. Moreover, CH implies $\Theta^{M_0} < \omega_2 = \omega_1^N$ so N sees $\mathcal{A} = \langle A_i : i \in \omega \rangle$ enumerating j_G"$\mathrm{OD}^{M_0} \cap \mathcal{P}(\mathbb{R})$. The calculation of HOD^{M_0} shows that there is an iteration strategy S^* *guided* by preserving (‡) and correctly moving names for the sets in $\mathcal{A}$. Using j_G the pair $\langle \mathcal{M}^*, S^* \rangle$ can be pulled back to the desired pair in $\langle \mathcal{M}, S \rangle$ in V.

The next step is to show that S satisfies a condensation property called *branch condensation* this yields a fine structure hierarchy relative to S, i.e. a hierarchy of hybrid mice of the form $\langle \mathrm{L}_\alpha[\vec{E} \restriction \alpha, S \restriction \alpha], E_\alpha, S_\alpha \rangle$. The condensation property states:

> For any iteration tree $\mathcal{T}$ of length $\gamma + 1$ on $\mathcal{M}$ according to the strategy S with final model $\mathcal{N} = \mathcal{M}_\gamma$ and branch $b = [0, \gamma)_{\mathcal{T}}$ and for any $X \prec \mathcal{N}$ such that $j_{0,\gamma}$"$\mathcal{M} \subseteq X$, letting $\bar{\mathcal{N}}$, $\bar{\mathcal{T}}$, and $\bar{b}$ be the collapse of $\mathcal{N}$, $\mathcal{T}$, and b respectively, then $\bar{\mathcal{T}}$ is according to S and $b = S(\bar{\mathcal{T}})$.



At the moment the current proof uses the extra hypotheses of (†) to prove this condensation.

Proving condensation is the main obstacle to applying this technique to several other hypotheses. Once the hybrid hierarchy is shown to exist and have the desired properties it is routine exercise to use Steel's core model theory to get a Woodin cardinal relative to S. Neeman's method for proving determinacy [Nee95] then yields the determinacy of S^* (the set of reals coding S on countable trees). Inductively $\mathrm{L}(S^*, \mathbb{R}) \models \mathrm{AD}^+$ is shown. Voila! Discovering a new proof of condensation that avoids the additional assumptions of (†) is the goal of this project. Success would entail the moving forward of several related projects by the author and others.

4. Isolating the cardinals of $\mathrm{L}(\mathbb{R})$ by their large cardinal properties in $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}$

This part of the project complements the work of Steve Jackson, UNT, on the study of the cardinal structure of $\mathrm{L}(\mathbb{R})$ under determinacy assumptions (see [Jac99a, Jac99b]). It was in order to collaborate with Jackson on this project that I came as a visitor to UNT for two years, 2003 – 2005. Jackson has produced a very precise level-by-level inductive analysis of an initial segment of the regular cardinals in $\mathrm{L}(\mathbb{R})$. Just to illustrate Jackson's work, he has successfully computed the regular cardinals $\underset{\sim}{\delta}^1_n$'s showing that $\underset{\sim}{\delta}^1_{2n+1} = \aleph^+_{\omega(2n-1)}$ where $\omega(1) = \omega$ and $\omega(n+1) = \omega^{\omega(n)}$. While it had been known for a long time that $\underset{\sim}{\delta}^1_{2n+2} = \left(\underset{\sim}{\delta}^1_{2n+1}\right)^+$ and $\underset{\sim}{\delta}^1_{2n+1} = \left(\underset{\sim}{\kappa}_{2n+1}\right)^+$ where $\underset{\sim}{\kappa}_{2n+1}$ is a Suslin cardinal of cofinality ω, Jackson's breakthrough was in calculating $\underset{\sim}{\kappa}_{2n+1} = \aleph_{\omega(2n-1)}$. Part of this calculation shows that there are exactly $2^{n+1} - 1$ regular cardinals between $\underset{\sim}{\delta}^1_{2n+1}$ and $\underset{\sim}{\delta}^1_{2n+3}$ thus the picture of the first few regular cardinals (together with the $\underset{\sim}{\kappa}_i$'s which have cofinality ω) are

$$\begin{aligned}\underset{\sim}{\kappa}_1 = &\aleph_0 < \underset{\sim}{\delta}^1_1 = \aleph_1 < \underset{\sim}{\delta}^1_2 = \aleph_2 < \underset{\sim}{\kappa}_3 = \aleph_\omega < \underset{\sim}{\delta}^1_3 = \aleph_{\omega+1} < \\ &\underset{\sim}{\delta}^1_4 = \aleph_{\omega+2} < \aleph_{\omega 2+1} < \aleph_{\omega^\omega+1} < \underset{\sim}{\kappa}_5 = \aleph_{\omega^{\omega^\omega}} < \underset{\sim}{\delta}^1_5 = \aleph_{\omega^{\omega^\omega}+1} < \underset{\sim}{\delta}^1_6 = \aleph_{\omega^{\omega^\omega}+2} \cdots\end{aligned}$$

Jackson's techniques currently break down at the first $\underset{\sim}{\delta}^1_\kappa$ for κ weakly inaccessible. He has been searching for global structural principles that would allow his methods to proceed through all of the ordinals below $\Theta^{\mathrm{L}(\mathbb{R})}$, with some success [Jac01].

The proposed project aims directly at a global "top-down" characterization of all regular cardinals of $\mathrm{L}(\mathbb{R})$ via their large cardinal properties in the model $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}$. The idea is to use the representation of $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}$ as a limit of a directed system of mice initially discovered by John Steel and Hugh Woodin at UC, Berkeley. Steel showed assuming the existence of $\mathcal{M}_\omega$ (the minimal mouse for ω-Woodin cardinals) that $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}$ is a fine structural model up to the ordinal $\Theta^{\mathrm{L}(\mathbb{R})}$ ([Ste95, Ste99a]) and used this calculation to prove that all regular cardinals of $\mathrm{L}(\mathbb{R})$ are measurable. Woodin extended this analysis to show that $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}$ is a fine structural inner model of a new type by showing that $\mathrm{HOD}^{\mathrm{L}(\mathbb{R})}$ is the direct limit of a directed system of mice [Ste96] together with information about how to iterate the resulting direct limit model at it's first Woodin cardinal. Woodin also reduced the hypothesis showing that the assumption of $\mathrm{AD}^{\mathrm{L}(\mathbb{R})}$ is sufficient for this analysis. By investigating related systems of inner models Woodin was able to characterize the

Suslin cardinals of $L(\mathbb{R})$ via their large cardinal properties in $HOD^{L(\mathbb{R})}$. For example at the level of the projective ordinals Woodin showed

$$\underset{\sim}{\kappa}_3 = \text{the least } \alpha \text{ such that } L(V_\alpha^{HOD^{L(\mathbb{R})}}) \models \text{“}\alpha \textit{ is Woodin}\text{”}$$

This analysis also shows that

$$\underset{\sim}{\delta}^1_1 = \text{the least cardinal of } HOD^{L(\mathbb{R})} \text{ strong to } \underset{\sim}{\kappa}_3$$

Continuing upward let $\pi(\alpha)$ be the least $\beta > \alpha$ such that $L(V_\beta^{HOD^{L(\mathbb{R})}}) \models$ “β is Woodin”

$$\underset{\sim}{\kappa}_5 = \text{the least } \alpha \text{ such that } L(V_\alpha^{HOD^{L(\mathbb{R})}}) \models \text{“}\alpha \textit{ is Woodin}\text{” and } \alpha \text{ is closed under } \pi$$

$$\underset{\sim}{\delta}^1_3 = \text{the least cardinal of } HOD^{L(\mathbb{R})} \text{ strong to } \underset{\sim}{\kappa}_5$$

$$\vdots$$

Using a finer analysis of the same directed system used by Steel and Woodin to calculate $HOD^{L(\mathbb{R})}$ Greg Hjorth, UCLA, showed how to characterize the first ω many $L(\mathbb{R})$ cardinals by their corresponding *large cardinal* properties in $HOD^{L(\mathbb{R})}$ [Hjo01]. It is this line of research that we wish to continue. The goal is to calculate all regular $L(\mathbb{R})$ cardinals via their large cardinal properties in $HOD^{L(\mathbb{R})}$. An initial step will be to provide characterizations of the three regular cardinals $\aleph_{\omega+2}$, $\aleph_{\omega 2+1}$, and $\aleph_\omega^\omega + 1$ appearing between $\underset{\sim}{\delta}^1_3$ and $\underset{\sim}{\delta}^1_5$ in terms of their large cardinal properties in $HOD^{L(\mathbb{R})}$.

Failing a complete analysis of all regular cardinals via their large cardinal properties $HOD^{L(\mathbb{R})}$. Jackson's analysis of the regular projective cardinals should be applicable in extending Hjorth's analysis so that a characterization of all *projective* regular cardinals via their large cardinal properties in $HOD^{L(\mathbb{R})}$ might at least be obtained. At the same time the analysis of $HOD^{L(\mathbb{R})}$ might lead to additional structural properties that would allow Jackson's inductive analysis to proceed through all ordinals below $\Theta^{L(\mathbb{R})}$. Examples of structural properties arising from this analysis are Woodin's proof that ω_1 is supercompact to $\Theta^{L(\mathbb{R})}$ as well as Steel's original result that all regular cardinals are measurable.

Looking forward, there is no reason to stop at $L(\mathbb{R})$. Woodin has shown that on a cone of x, the calculation of HOD_x succeeds just assuming “$ZF + AD^+ + V = L(\mathcal{P}(\mathbb{R}))$”. This allows one to derive the *boldface* consequences of the analysis of $HOD^{L(\mathbb{R})}$, e.g. “ω_1 is Θ-supercompact” and “all regular cardinals measurable in HOD”. Thus a general characterization via their large cardinal properties in HOD_x on a cone of x of the regular cardinals in a model satisfying “$ZF + AD^+ + V = L(\mathcal{P}(\mathbb{R}))$” would be the ultimate goal. Similarly very general structural properties for models of “$ZF + AD^+$” might be obtained so that Jackson's inductive approach might made to work through all models of “$ZF + AD^+ + V = L(\mathcal{P}(\mathbb{R}))$”. On the other hand, recently Steel has been investigating models of determinacy that are derived from iterable inner models. These determinacy models satisfy some extra properties which might make the resolution of the project described more likely.

Initially this project will require some time for the author to become familiar with Jackson's analysis and at the same time the author needs to convey to Jackson the analysis of HOD as well as the various consequences of this analysis. We will attempt to use this time to bring graduate students into this exchange of ideas by running a series of seminars in tandem on the various topics which need to be covered. This initial transfer of knowledge will take some time and is the reason two years of support are being requested. The first challenge will be to characterize the regular cardinals below $\underset{\sim}{\delta}^1_5$. *Any* results from this collaboration will surely be of value to those set theorists interested the relationship between inner model theory and models of determinacy.

5. Impact

There will be opportunities for graduate students at UNT to benefit from this project, particularly the part that involves Steve Jackson. While the author is just visiting and too junior for graduate students of his own, Jackson has graduate students that will have an opportunity both to learn and perhaps even begin projects of their own based on seminars that we offer.

This work directly builds on work that the author has done in collaboration with Woodin and Steel at UC, Berkeley, and also builds on work of Hjorth and Neeman with whom the author has worked while at UCLA. The collaborations will continue to play a major roll in the completion of the projects outlined above thus strengthening ties in the set theory communities at UNT, UCLA, and Berkeley.

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