Math 6170 – Class Notes

Math 6170 Fall '01

Problem 1

Problem. *Find all continuous functions, $f\colon \mathbb{R} \to \mathbb{R}$, with the property that $f(x+y) = f(x) + f(y)$.*

Background. Some hints for **Problem 1** are:

- $f(0) = f(0)f(0)$, so $f(0) = 0$ or $f(0) = 1$. If $f(0) = 0$ it's easy to show that $f(x) = 0$ for all x, so assume $f(0) = 1$.
- $f(2x) = f(x)f(x)$, so $f(2x) \geq 0$, so $f(x) \geq 0$ for all x.
- f is continuous, so $\int_s^t f(r)\,dr$ exists for all real numbers s and t. Consider $f(x)\int_1^t f(r)\,dr\ldots$

Solution. [Ioana] Set $f(1) = b$. Then $f(1)^n = f(n)$, so $f(m/n)^n = f(m) = b^m$ for all integers m and n. Thus, $f(m/n) = b^{m/n}$ for all rational numbers, m/n. Since the rationals are dense in the reals and f is continuous, it follows that $f(x) = b^x$ for all x.

Comments. Suppose f is as in the problem and $f(1) = f'(0) = b$. Define $T\colon [0,\infty) \to L(\mathbb{R})$ by $T(t) = \ell_{e^{bt}}$ where $\ell_a\colon \mathbb{R} \to \mathbb{R}$ by $\ell_a(x) = ax$. Then using the terminology to be introduced in **Problem 11**, T is a semigroup on $\mathbb{R}$ and D_T is the set of all x in $\mathbb{R}$ for which $\lim_{t\to 0}(1/t)(e^{bt}-1)x$ exists. But $\lim_{t\to 0}(1/t)(e^{bt}-1)x = bx$ for every x in $\mathbb{R}$, so in this example, $D_T = \mathbb{R}$ and $A_T(x) = bx$ for x in $\mathbb{R}$, that is, $A_T = \ell_b = \ell_{f'(0)}$.

Problem 2

Problem. *Find all continuous functions, $T\colon [0,\infty) \to M_2(\mathbb{R})$, with the property that $T(x+y) = T(x)T(y)$.*

Background: Quick Review of some Banach Space Theory. Suppose X is denote a normed linear space (either real or complex) and $|\cdot|$ is a norm on X. Recall that $|\cdot|$ is *subadditive:* $|x+y| \leq |x| + |y|$ for x and y in X; *multiplicative:* $|\alpha x| = |\alpha|\,|x|$ for x in X and a scalar, α; and *positive definite:* $|x| \geq 0$, and $|x| = 0$ if and only if $x = 0$ for x in X.

The norm on X defines a metric on X and hence a topology on X. With this topology, the vector space operations on X are continuous.

Definition 2.1. A subset, A, of X is *bounded* if for every neighborhood, U of 0, $A \subseteq tU$ for all sufficiently large t in $\mathbb{R}$.

Proposition 2.2. *Suppose X is a normed linear space.*

(a) *There is a neighborhood basis of* 0 *in X consisting of convex sets, so X is* locally convex.
(b) *There is a neighborhood basis of* 0 *in X consisting of bounded sets, so X is* locally bounded.
(c) *There is a countable neighborhood basis of* 0 *in X, so X is* first countable.

Proof. These are all clear from the definitions. □

Theorem 2.3. *X is locally compact if and only if X is finite dimensional.*

Proof. This follows from Theorems 1.21 and 1.22 in [Rud73]. □

Definition 2.4. Suppose Y is a topological vector space and $\{y_n\}$ is a sequence in Y.

- The sequence $\{y_n\}$ *converges to* y if every neighborhood of y contains all but finitely many of the y_n's. (This definition makes sense in any topological space.)
- The sequence $\{y_n\}$ is a *Cauchy sequence* if for every neighborhood, U of 0, there is an N so that $y_m - y_n$ are in U for all $m, n \geq N$.

Definition 2.5. Suppose X and Y are topological vector spaces. A linear transformation, $T\colon X \to Y$ is *bounded* if T carries bounded sets to bounded sets, that is, if A is a bounded subset of X, then $T(A)$ is a bounded subset of Y.

Theorem 2.6. *The following are equivalent for a linear transformation, $T\colon X \to Y$, when X is a normed linear space and Y is a topological vector space:*

(a) *T is bounded,*
(b) *T is continuous,*
(c) *if $\{x_n\}$ is a sequence that converges to 0 in X, then $\{Tx_n\}$ is a bounded subset of Y, and*
(d) *if $\{x_n\}$ is a sequence that converges to 0 in X, the $\{Tx_n\}$ converges to 0 in Y.*

Proof. This is Theorem 1.32 in [Rud73]. □

Example 2.7. Consider the Banach space C([0, 1]) of continuous functions on $[0,1]$ with the sup-norm. The functions on $[0,1]$ that are differentiable form a subspace, say $\mathcal{D}([0,1])$. With the subspace topology, differentiation is a map from $\mathcal{D}([0,1])$ to C([0, 1]) that's linear, but not continuous.

Notation 2.8. If X is a normed linear space and Y is a topological vector space, then $L(X, Y)$ will denote the set of all continuous linear transformations from X to Y. The special cases when $Y = X$ and $Y = \mathbb{R}$ will be denoted by $L(X)$ and X^* respectively.

It's easy to see that if T_1 and T_2 are in $L(X, Y)$ and α is a scalar, then $T_1 + T_2$ and αT_1 are in $L(X, Y)$, so $L(X, Y)$ is a subspace of the vector space of all linear transformations from X to Y.

Definition 2.9. Suppose X and Y are normed linear spaces and T is in $L(X, Y)$. Then T is bounded so the image of the unit ball in X is a bounded subset of Y. Define

$$|T| = \sup\{\, |Tx| \mid x \in X, |x| \leq 1 \,\} = \inf\{\, M \mid |Tx| \leq M\,|x|\,, \forall x \in X \,\}.$$

Proposition 2.10. *If X is a normed linear space and A and B are in $L(X)$, then $|AB| \leq |A|\,|B|$.*

Proof. By definition, for x in X, $|ABx| \leq |A|\,|Bx| \leq |A|\,|B|\,|x|$ so $|AB| \leq |A|\,|B|$. □

Theorem 2.11. *Suppose X and Y are normed linear spaces. Then $|\cdot|$ is a norm on $L(X, Y)$ and if Y is a Banach space, so is $(L(X, Y), |\cdot|)$.*

Proof. This is Theorem 4.1 in [Rud73]. □

Solution. [henry, Ioana]

Claim 2.12. [Henry] $(T(t) - I) \int_0^s T(r)\, dr = (T(s) - I) \int_0^t T(r)\, dr$

Proof. Under the change of variable $u = t + r$ the integral $\int_0^s T(t + r)\, dr$ becomes $\int_t^{s+t} T(r)\, dr$ and similarly $\int_0^t T(s + r)\, dr = \int_s^{t+s} T(r)\, dr$. Also,

$$(T(t) - I) \int_0^s T(r)\, dr = \int_0^s T(t + r)\, dr - \int_0^s T(r)\, dr = \int_t^{s+t} T(r)\, dr - \int_0^s T(r)\, dr$$

and

$$(T(s) - I) \int_0^t T(r)\, dr = \int_0^t T(s + r)\, dr - \int_0^s T(r)\, dr = \int_s^{s+t} T(r)\, dr - \int_0^s T(r)\, dr.$$

The difference between these last two equations is,

$$\int_t^{t+s} T(r)\, dr - \int_0^s T(r)\, dr - \int_s^{t+s} T(r)\, dr + \int_0^t T(r)\, dr = 0$$

□

Claim 2.13. [Henry] *If t is sufficiently small, then $\int_0^t T(r)\, dr$ is invertible in $M_2(\mathbb{R})$.*

Proof. Define $V(t) = \int_0^t T(r)\, dr$ for $t \geq 0$. Then for $t > 0$, V is continuously differentiable at t and $V'(t) = T(t)$ by the continuity of T. Using again that T is continuous, taking the limit as t decreases to 0 it follows that V is differentiable from the right at 0 and $V'(0) = T(0) = I$. Thus $\lim_{t\to 0+} \frac{1}{t}(V(t) - V(0)) = I$ and so $\lim_{t\to 0+} \frac{1}{t} \int_0^t T(r)\, dr = I$. The determinant function from $M_2(R)$ to $\mathbb{R}$ is continuous, so the invertible matrices form an open subset of $M_2(\mathbb{R})$. Therefore, if t is sufficiently close to zero, then $\frac{1}{t} \int_0^t T(r)\, dr$ is invertible and hence $\int_0^t T(r)\, dr$ is invertible. □

Claim 2.14. [Henry] *The function T has a right derivative at 0 and*

$$T'(0) = \left(\int_0^s T(r)\, dr\right)^{-1} (T(s) - I)$$

for all sufficiently small s.

Proof. Notice that $\int_0^s T(r)\, dr$ and $T(t) - I$ commute since $T(r + t) = T(t + r)$. Therefore, it follows from Claim 2.12 that

$$\left(\int_0^s T(r)\, dr\right) (T(t) - I) = (T(s) - I) \int_0^t T(r)\, dr,$$

so it follows from Claim 2.13 that for sufficiently small s's we have

$$T(t) - I = \left(\int_0^s T(r)\, dr\right)^{-1} (T(s) - I) \left(\int_0^t T(r)\, dr\right). \tag{2.15}$$

Now multiply by $\frac{1}{t}$ and take the limit as t decreases to 0 to get

$$\begin{aligned}\lim_{t\to 0+} \frac{1}{t}(T(t) - T(0)) &= \left(\int_0^s T(r)\, dr\right)^{-1} (T(s) - I) \left(\lim_{t\to 0+} \frac{1}{t} \int_0^t T(r)\, dr\right) \\ &= \left(\int_0^s T(r)\, dr\right)^{-1} (T(s) - I)V'(0) \\ &= \left(\int_0^s T(r)\, dr\right)^{-1} (T(s) - I).\end{aligned}$$

This proves the claim. □

Now set $A = T'(0) = \left(\int_0^s T(r)\,dr\right)^{-1}(T(s) - I)$ (for sufficiently small s). Then Equation 2.15 becomes $T(t) - I = A\int_0^t T(r)\,dr$ and so $T(t) = I + A\int_0^t T(r)\,dr$. It follows that T is differentiable and that $T'(t) = AT(t)$. It will be shown in Remark 8.1 that for every X in $M_2(\mathbb{R})$, the initial value problem:

IVP:$M_2(\mathbb{R})$ $\qquad$ $G\colon [0,\infty) \to M_2(\mathbb{R}),\ G(0) = X,\ \text{and } G'(t) = AG(t)$

has a unique solution, so since $T(0) = I$ and $T'(t) = AT(t)$ for t in $[0,\infty)$, it follows that T may be characterized as the unique solution to IVP:$M_2(\mathbb{R})$ with $T(0) = I$.

Claim 2.16. *If $|\cdot|$ denotes the 1-norm on $\mathbb{R}^2$, A is in $M_2(\mathbb{R})$, and $a_{i,j}$ is the (i,j)-entry of A, then $|a_{i,j}| \leq |A|$.*

Proof. If e_j is the j^{th} standard basis vector of $\mathbb{R}^2$ for $j = 1, 2$, then $|e_j| = 1$ and so

$$|a_{i,j}| \leq \sum_i |a_{i,j}| = |Ae_j| \leq |A|\,|e_j| = |A|\,.$$

□

Claim 2.17. [Ioana] *Suppose B is a matrix in $M_2(\mathbb{R})$. Then the series $\sum_{k=0}^{\infty}(1/k!)B^k$ converges in $M_2(\mathbb{R})$.*

Proof. It will be shown in the comments after **Problem 3** that the product topology on $M_2(\mathbb{R})$ is the same as the topology induced by the operator norm, so it's enough to show that the series converges in the product topology on $M_2(\mathbb{R})$. Therefore, it's enough to show that the (i,j)-entry of $\sum_k(1/k!)B^k$ converges for every $1 \leq i \leq n$ and $1 \leq j \leq n$.

Fix i and j with $1 \leq i \leq n$ and $1 \leq j \leq n$ and let $b_{i,j}^{(k)}$ denote the (i,j)-entry of B^k for $k \geq 0$. Then using Claim 2.16 and Proposition 2.10 we have $\left|b_{i,j}^(k)\right| \leq \left|B^k\right| \leq |B|^k$, so $\left|b_{i,j}^(k)\right|/k! \leq |B|^k/k!$. Therefore, the series $\sum \left|b_{i,j}^(k)\right|/k!$ converges by the Comparison Test for series of real numbers since the series $\sum |B|^k/k!$ converges. □

Denote the sum $\sum_0^{\infty}(1/k!)B^k$ by e^B.

Recall that we've set $A = T(0)$ and define $E\colon [0,\infty) \to M_2(\mathbb{R})$ by $E(t) = e^{tA}$. We'll show that E is differentiable and $E'(t) = AE(t)$. Since $E(0) = I$ it will then follow from the uniqueness statement in Remark 8.1 that $T = E$, so $T(t) = e^{tA}$ for t in $[0,\infty)$.

Fix $\alpha > 0$ and let $a_{i,j}^{(k)}$ denote the (i,j)-entry of A^k for $k \geq 0$. Then as in the proof of Claim 2.17, for t in $[0,\alpha]$, we have $t^k\left|a_{i,j}^{(k)}/k!\right| \leq t^k|A|^k/k! \leq \alpha^k|A|^k/k!$, so the series $\sum_k t^k a_{i,j}^{(k)}/k!$ converges uniformly on $[0,\alpha]$. Therefore, the series $\sum_k t^kA^k/k!$ converges uniformly on $[0,\alpha]$ and so we can compute its derivative as the sum $\sum_k kt^{k-1}A^k/k! = Ae^{tA}$. Thus, $E'(t) = AE(t)$ for $t \geq 0$ as claimed.

PROBLEM 3

Problem. *Suppose* $T\colon [0,\infty) \to M_2(\mathbb{R})$*. Write* $T(t) = \begin{bmatrix} a(t) & b(t) \\ c(t) & d(t) \end{bmatrix}$ *where* a*,* b*,* c*, and* d*, are real-valued functions. Show that* a*,* b*,* c*, and* d *are all continuous if and only if* T *is continuous when* $M_2(\mathbb{R})$ *is given the topology determined by the operator norm that's induced by the 2-norm on* $\mathbb{R}^2$*.*

Background. For $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$, define $\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_1 = |x| + |y|$, this is the 1-norm on $\mathbb{R}^2$ and $\left\| \begin{bmatrix} x \\ y \end{bmatrix} \right\|_2 = \sqrt{x^2 + y^2}$, this is the 2-norm on $\mathbb{R}^2$.

Solution. [Deana]

Assume first that T is continuous at t with respect to the 2-norm topology on $M_2(\mathbb{R})$ and fix $\epsilon > 0$. Choose δ so that $|T(t) - T(s)| < \epsilon$ whenever $|t - s| < \delta$ and s is in $[0, \infty)$. Recall that $T(x)$ is defined by $|T(x)| = \inf\{\, M \mid |T(x)v| \leq M\,|v|\, \forall v \in \mathbb{R}^2 \,\}$. Thus, taking $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ it follows that

$$\left\| \begin{bmatrix} a(t) - a(s) & b(t) - b(s) \\ c(t) - c(s) & d(t) - d(s) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|_2 < \epsilon \left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|_2 .$$

for s in $[0, \infty)$ with $|t - s| < \delta$. Since $\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|_2 = 1$, it follows that $\left\| \begin{bmatrix} a(t) - a(s)) \\ c(t) - c(s) \end{bmatrix} \right\|_2 < \epsilon$. By definition, the left hand side of the last equality is $\sqrt{(a(t) - a(s))^2 + (c(t) - c(s))^2}$. Then

$$|a(t) - a(s)| \leq \sqrt{(a(t) - a(s))^2 + (c(t) - c(s))^2} < \epsilon,$$

and so a is continuous at t.

The same argument shows that c is continuous at t, and an analogous argument with $w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ replacing $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ shows that b and d are continuous at t.

Claim 3.1. *There are constants,* c_1 *and* c_2 *in* $\mathbb{R}$ *so that* $c_1\,|v|_2 \leq |v|_1 \leq c_2\,|v|_2$ *for every* v *in* $\mathbb{R}^2$*.*

Proof. The first inequality is valid with $c_1 = 1$ by the triangle inequality. We'll show that $c_2 = \sqrt{3}$ works for the second inequality. This is clear if $x = y = 0$.

If $(a, b) \neq (0, 0)$ and $|a| \leq |b|$, then $\frac{|ab|}{a^2+b^2} = \frac{|a/b|}{(a/b)^2+1} \leq \frac{|a/b|}{1} \leq 1$. It follows that $2\frac{|xy|}{x^2+y^2} \leq 2$ and so $1 + 2\frac{|xy|}{x^2+y^2} \leq 3$. therefore, $x^2 + y^2 + 2\,|xy| \leq 3(x^2 + y^2)$, and it follows that $(|x| + |y|)^2 \leq 3(x^2 + y^2)$. This implies the result. □

It follows from the claim that $|x| + |y| \leq \sqrt{3}\sqrt{x^2 + y^2}$ for all real numbers x and y.

Now, suppose a, b, c, and d are continuous at t and $\epsilon > 0$ is given. Choose $\delta > 0$ so that $|a(t) - a(s)| < \epsilon/2$, $|b(t) - b(s)| < \epsilon/2$, $|c(t) - c(s)| < \epsilon/2$, and $|d(t) - d(s)| < \epsilon/2$ whenever s is in $[0, \infty)$ and $|t - s| < \delta$.

Then as above

$$\left|(T(t)-T(s))\begin{bmatrix}1\\0\end{bmatrix}\right|_2 = \sqrt{(a(t)-a(s))^2+(c(t)-c(s))^2} \le \epsilon/\sqrt{2}$$
$$\left|(T(t)-T(s))\begin{bmatrix}0\\1\end{bmatrix}\right|_2 = \sqrt{(b(t)-b(s))^2+(b(t)-b(s))^2} \le \epsilon/\sqrt{2}$$

Now for any $\begin{bmatrix}x\\y\end{bmatrix}$ in $\mathbb{R}^2$,

$$\begin{aligned}
\left|(T(s)-T(t))\begin{bmatrix}x\\y\end{bmatrix}\right|_2 &= \left|(T(s)-T(t))\left(\begin{bmatrix}x\\0\end{bmatrix}+\begin{bmatrix}0\\y\end{bmatrix}\right)\right|_2 \\
&\le \left|(T(s)-T(t))\begin{bmatrix}x\\0\end{bmatrix}\right|_2 + \left|(T(s)-T(t))\begin{bmatrix}0\\y\end{bmatrix}\right|_2 \\
&= |x|\left|(T(s)-T(t))\begin{bmatrix}1\\0\end{bmatrix}\right|_2 + |y|\left|(T(s)-T(t))\begin{bmatrix}0\\1\end{bmatrix}\right|_2 \\
&\le (|x|+|y|)\frac{\epsilon}{\sqrt{2}} \\
&\le \epsilon\sqrt{3}/\sqrt{2}.
\end{aligned}$$

This shows that $|T(t)-T(s)| \le \epsilon\sqrt{3}/\sqrt{2}$ whenever $|t-s|<\delta$ and hence that T is continuous at t.

Comments. A solution to **Problem 3** follows trivially from the apparently more general fact that the product topology on $M_2(\mathbb{R})$ is the same as the operator topology on $M_2(\mathbb{R})$ induced by either the 1-norm or the 2-norm on $\mathbb{R}^2$. It follows from Claim 3.1 that the two operator topologies coincide and the first argument above essentially shows that all three topologies coincide. We'll show that the product topology and the operator topology induce by the 1-norm on $\mathbb{R}^2$ are equal.

For a matrix, A, in $M_2(\mathbb{R})$ and $\epsilon > 0$, let $B_\epsilon(A)$ denote the open ϵ-ball around A in the operator topology induced by the 1-norm on $\mathbb{R}^2$. Suppose $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$ is in $M_2(\mathbb{R})$, fix $\epsilon > 0$, and consider $B_\epsilon(A)$. If $X = \begin{bmatrix}x & y\\z & w\end{bmatrix}$ is in $B_\epsilon(A)$, then $|A-X| < \epsilon$. Applying $A-X$ to the standard basis vectors of $\mathbb{R}^2$, which have 1-norm equal 1, gives that $\left\|\begin{bmatrix}a-x\\c-z\end{bmatrix}\right\| < \epsilon$ and $\left\|\begin{bmatrix}b-y\\d-w\end{bmatrix}\right\| < \epsilon$, so $|a-x|<\epsilon$, $|c-z|<\epsilon$, $|b-y|<\epsilon$, and $|d-w|<\epsilon$.

Now suppose (a_1,a_2), (b_1,b_2), (c_1,c_2), and (d_1,d_2) are four open intervals in $\mathbb{R}$ and let R denote the set of all $X = \begin{bmatrix}x & y\\z & w\end{bmatrix}$ with the property that x is in (a_1,a_2), y is in (b_1,b_2), z is in (c_1,c_2), and w is in

(d_1, d_2), so R is a basic open set in the product topology. Fix $A = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ in R, set

$$\epsilon = \min\{\,|p - a_1|, |p - a_2|, |q - b_1|, |q - b_2|, |r - c_1|, |r - c_2|, |s - d_1|, |s - d_2|\,\},$$

and consider $B_\epsilon(A)$. If $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ is in $B_\epsilon(A)$, then it was shown in the last paragraph that $|a - x| < \epsilon$, $|c - z| < \epsilon$, $|b - y| < \epsilon$, and $|d - w| < \epsilon$. Therefore, X is in R. Since X was arbitrary, $B_\epsilon(A) \subseteq R$ and it follows that R is open in the norm topology on $M_2(\mathbb{R})$. Therefore, the norm topology contains the product topology.

Conversely, suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in is $M_2(\mathbb{R})$ and $\epsilon > 0$. We'll show that A is contained in an open subset in the product topology that's totally contained in $B_\epsilon(A)$. Let R be the open subset in the product topology consisting of all matrices, $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$, with the property that $|a - x| < \epsilon/2$, $|b - y| < \epsilon/2$, $|c - z| < \epsilon/2$, and $|d - w| < \epsilon/2$. For such an X,

$$\begin{aligned} \left|(A - X)\begin{bmatrix} r \\ s \end{bmatrix}\right| &\le |r|(|a - x| + |b - y|) + |s|(|c - z| + |d - w|) \\ &\le \epsilon(|r| + |s|)/2 \\ &= \epsilon/2 \left|\begin{bmatrix} r \\ s \end{bmatrix}\right|, \end{aligned}$$

So X is in $B_\epsilon(A)$. It follows that R is contained in $B_\epsilon(A)$ and hence that the two topologies on $M_2(\mathbb{R})$ are equal.

Problem 4

Problem. *Suppose $y_0 : \mathbb{R} \to \mathbb{R}$ is continuous and $y_i : \mathbb{R} \to \mathbb{R}$ is defined by $y_i(t) = 1 + \int_0^t y_{i-1}(r)\,dr$ for $i \ge 1$. Show that the sequence $\{y_n\}$ converges uniformly on any finite interval. If y is the limit, show that y is the unique function with the property $y(t) = 1 + \int_0^t y(r)\,dr$ for all t.*

Solution. [Henry for existence, Ahmed for uniqueness.] It follows from the Fundamental Theorem of Calculus, the assumption that y_0 is continuous, and recursion, that y_n is continuously differentiable and $y_n' = y_{n-1}$ for $n \ge 1$.

Set $f_n = y_n - y_{n-1}$ for $n \ge 1$. Then $f_n(t) = \int_0^t (y_{n-1} - y_{n-2}) = \int_0^t f_{n-1}$ for $n > 1$. Fix t in $\mathbb{R}$. Since f_1 is continuous there is an M so that $|f_1(x)| \le M$ for all x between 0 and t. Then $|f_2(x)| = \left|\int_0^x f_1\right| \le \left|\int_0^x |f_1|\right| \le M|x|$ for x between 0 and t. Assume that $|f_n(x)| \le M|x|^{n-1}/(n-1)!$. Then

$$|f_{n+1}(x)| = \left|\int_0^x f_n\right| \le \left|\int_0^x (M|r|^{n-1}/(n-1)!)\,dr\right| = M|x|^n/n!.$$

Therefore, $|f_n(x)| \le M|x|^{n-1}/(n-1)!$ for $n \ge 2$ and x between 0 and t.

Now let $s_n = f_1 + \cdots + f_n$ be the n^{th} partial sum of the sequence $\{f_n\}$. Notice that $s_n = y_n - y_0$. We'll show that the sequence of s_n's converges uniformly on the interval with endpoints 0 and t and it will then follow that the y_n's converge uniformly on interval with endpoints 0 and t.

To show that the sequence of s_n's converges uniformly, it's enough to show that it's a uniformly Cauchy sequence. That is, given $\epsilon > 0$ there is an N so that $|s_n(x) - s_m(x)| \le \epsilon$ for all $m, n \ge N$ and for all x between 0 and t. So fix $\epsilon > 0$. The series $\sum t^n/n!$ converges absolutely by the Ratio Test, so there is an N so that $\sum_{i=n+1}^{\infty} t^i/i! \le \epsilon$ when $n \ge N$. But then if $m > n \ge N$ and x is between 0 and t we have

$$\begin{aligned} |s_n(x) - s_m(x)| &\le \sum_{i=n+1}^{m} |f_i(x)| \\ &\le \sum_{i=n+1}^{m} M\,|x|^{i-1}/(i-1)! \\ &\le M \sum_{i=N+1}^{\infty} |x|^{i-1}/(i-1)! \\ &\le M \sum_{i=N+1}^{\infty} |t|^{i-1}/(i-1)! \\ &\le M\epsilon. \end{aligned}$$

It follows that $\{s_n\}$ is uniformly Cauchy on the interval with endpoints 0 and t and hence converges uniformly on the interval with endpoints 0 and t. Therefore, the series converges uniformly on any finite interval.

Define $y = \lim_{n\to\infty} y_n$. Now

$$y_n(t) = 1 + \int_0^t y_{n-1} \tag{4.1}$$

for $n \ge 1$ by definition and the y_n's converge uniformly to y on the interval with endpoints 0 and t, so applying $\lim_{n\to\infty}$ to both sides of equation (4.1) gives

$$y(t) = 1 + \lim_{n\to\infty} \int_0^t y_{n-1} = 1 + \int_0^t \lim_{n\to\infty} y_{n-1} = 1 + \int_0^t y. \tag{4.2}$$

It remains to show that y is the unique function satisfying equation (4.2). So suppose g also satisfies $g(t) = 1 + \int_0^t g$. Then $y - g$ is continuous on the interval with endpoints 0 and t. Say $|y(x) - g(x)| \le M$ for x in the interval with endpoints 0 and t. Then

$$|y(x) - g(x)| = \left| \int_0^x y - g \right| \le \left| \int_0^x |y - g| \right| \le M\,|x|\,.$$

Assume that $|f(x) - g(x)| \le M|x|^n/n!$ for x in the interval with endpoints 0 and t. Then

$$|y(x) - g(x)| = \left| \int_0^x y - g \right| \le \left| \int_0^x |y - g| \right| \le \left| \int_0^x M|r|^n/n!\,dr \right| = M|x|^{n+1}/(n+1)!.$$

It follows that $|f(x) - g(x)| \le M|x|^n/n!$ for all x and for all $n \ge 1$. Therefore $y(x) = g(x)$ for all x and so $y = g$.

Problem 5

Problem. *For $s \ge 0$, define $T(s)\colon [0,\infty) \to [0,\infty)$ by $(T(s))(x) = y(s)$ where y is the unique solution to the initial value problem IVP. Show that $T(s+t) = T(s)T(t)$ for s and t in $[0,\infty)$.*

Background. Consider the initial value problem:

IVP $$y\colon [0,\infty) \to \mathbb{R}, \quad y(0) = x, \quad \text{and} \quad y' = -y^2$$

where $x \geq 0$.

To see that this initial value problem has a unique solution, suppose that y is a solution. Then $-y'/y^2$ is constantly one, so $\int_0^t -y'/y^2 = t$. But $\int_0^t -y'/y^2 = (1/y(t)) - (1/y(0))$, so $y(t) = x/(1+tx)$ for t and x in $[0,\infty)$. Since this formula for y defines a function that is easily seen to be a solution to the initial value problem IVP, it must be the unique solution.

Solution. [Ioana] Fix t, s, and x in $[0,\infty)$. By definition, if y is the unique solution to the initial value problem: $y\colon \mathbb{R} \to \mathbb{R}$, $y(0) = x$ and $y' = -y^2$, then $T(s)(x) = y(s)$ and $T(t+s)(x) = y(t+s)$.

Define f by $f(r) = y(r+s)$. Then clearly $f(0) = y(s)$ and $f'(r) = y'(r+s) = -y(r+s)^2 = -f(r)^2$. Therefore, f is the unique solution to IVP with $f(0) = y(s)$, and so $T(t)(y(s)) = f(t)$. But $f(t) = y(t+s) = T(t+s)(x)$ and $T(t)(y(s)) = T(t)(T(s)(x))$, so $T(t+s)(x) = T(t)(T(s)(x))$.

Comments.

Remark 5.1. An example of an initial value problem without a unique solution is $y\colon \mathbb{R} \to \mathbb{R}$, $y(0) = 0$, and $y'(t) = y(t)^{2/3}$. For $a \geq 0$, define $y_a(t) = 0$ for $t \leq a$ and $y_a(t) = (t-a)^3/27$ for $t \geq a$. Then y_a is a solution for every $a \geq 0$. Similarly, if $b \leq 0$ and $a \geq 0$, define $y_{a,b}(t) = (t-b)^3/27$ for $t \leq b$, $y_{a,b}(t) = 0$ for $b \leq t \leq a$, and $y_{a,b}(t) = (t-a)^3/27$ for $t \geq a$. Then $y_{a,b}$ is a solution for all a and b.

Remark 5.2. Consider the heat equation with "c=1:"

BVP:HE $$u\colon [0,\infty) \times [0,1] \to \mathbb{R},\ u(t,0) = u(t,1) = 0 \text{ for all } t,\ u(0,\,\cdot\,) = h,\ \text{and } u_t(t,x) = u_{xx}(t,x)$$
$$\text{for all } t \text{ in } (0,\infty) \text{ and all } x \text{ in } (0,1).$$

Assume that X is a space of functions on $[0,1]$ with the properties: 1) for every h in X, there is a unique u satisfying the initial value problem BVP:HE; 2) if u^h denotes the solution to BVP:HE with initial condition h, then $u^h(t,\,\cdot\,)$ is in X for t in $[0,\infty)$; and 3) X is a subspace of the vector space of all functions on $[0,1]$. It can be shown that it's possible to take $X = L^2([0,1])$.

For t in $[0,\infty)$, define $\ell_t\colon [0,\infty) \to [0,\infty)$ by $\ell_t(s) = t+s$ and define $\iota_t\colon [0,1] \to [0,\infty) \times [0,1]$ by $\iota_t(x) = (t,x)$. Notice that for s and t in $[0,\infty)$ we have $(\ell_s \times \mathrm{id}) \circ \iota_t = \iota_{s+t}$.

Define $T\colon [0,\infty) \to \mathrm{Map}(X,X)$ by $T(t)h = u^h \circ (\ell_t \times \mathrm{id}) = u^h(t,\,\cdot\,)$. The argument in the solution to **Problem 5** shows that T defines a "semigroup" on X (in the extended sense, as defined in **Problem 11**) as follows.

Claim 5.3. $u^{T(s)h} = u^h \circ (\ell_s \times \mathrm{id})$.

Proof. For s is in $[0,\infty)$ and h in X, define $\tilde{u}\colon [0,\infty) \times [0,1] \to \mathbb{R}$ by $\tilde{u}(t,x) = u^h(s+t,x)$, so with the preceding notation, $\tilde{u} = u^h \circ (\ell_s \times \mathrm{id})$.

We need to show that $\tilde{u}$ satisfies BVP:HE with the initial condition $T(s)h$. Since the derivative of $t \mapsto s+t$ with respect to t is 1 (this fact is the crucial ingredient, the rest is elementary and formal) it follows from the chain rule that $\tilde{u}_t(t,x) = u_t(s+t,x)$, $\tilde{u}_x(t,x) = u_x(s+t,x)$, and $\tilde{u}_{xx}(t,x) = u_{xx}(s+t,x)$, so $\tilde{u}$ satisfies the differential equation. Clearly $\tilde{u}(t,0) = \tilde{u}(t,1) = 0$ for all t. Finally, by definition, $\tilde{u}(0,x) = u^h(s,x) = T(s)h(x)$. ☐

To complete the argument that T is a semigroup, suppose h is in X. Then

$$\begin{aligned} T(t)T(s)h &= T(t)\,(T(s)h) \\ &= u^{T(s)h} \circ \iota_t \\ &= u^h \circ (\ell_s \times \mathrm{id}) \circ \iota_t \\ &= u^h \circ \iota_{s+t} \\ &= T(s+t)h. \end{aligned}$$

Problem 6

Problem. *If P and Q are two inner products for a normed linear space $(X, |\cdot|)$, then $P = Q$.*

Background. Recall that an *inner product* for a normed linear space, $(X, |\cdot|)$, is a bilinear function $P\colon X \times X \to \mathbb{R}$ with the property that $P(x,x) = |x|^2$ for every x in X.

Solution. [Ahmed] By definition, $P(x,x) = |x|^2 = Q(x,x)$ for all x in X. Thus, for x and y in X,

$$\begin{aligned} P(x,x) + 2P(x,y) + P(y,y) &= P(x+y, x+y) \\ &= |x+y|^2 \\ &= Q(x+y, x+y) \\ &= Q(x,x) + 2Q(x,y) + Q(y,y). \end{aligned}$$

The result now follows using the fact that $P(x,x) = |x|^2 = Q(x,x)$.

Problem 7

Problem. *Suppose a, b, c, d, r, and s are real numbers. Show there exists a unique pair of differentiable functions, $u, v\colon [0,\infty) \to \mathbb{R}$ with the properties*

$$\begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} u(t) \\ v(t) \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix} \tag{7.1}$$

for all t in $[0,\infty)$.

Background. Set $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $Y = \begin{bmatrix} u \\ v \end{bmatrix}$, and $X = \begin{bmatrix} r \\ s \end{bmatrix}$. Then the problem is equivalent to showing that the initial value problem

IVP:$\mathbb{R}^2$ $\qquad Y\colon [0,\infty) \to \mathbb{R}^2,\ Y(0) = X,\ \text{and}\ Y'(t) = AY(t),\ \text{for all } t \text{ in } [0,\infty)$

has a unique solution.

Solution. [Ioana] In this argument, $|\cdot|$ will denote the 1-norm on $\mathbb{R}^2$ and also the induced operator norm on $M_2(\mathbb{R})$.

Define $Y_0\colon [0,\infty) \to \mathbb{R}^2$ by $Y_0(t) = X$ for all t, and for $n \geq 1$, define $Y_n\colon [0,\infty) \to \mathbb{R}^2$ by $Y_n(t) = X + \int_0^t AY_{n-1}$. Also, define $F_n = Y_n - Y_{n-1}$ for $n \geq 1$, so $F_n(t) = \int_0^t A\,(Y_{n-1}(s) - Y_{n-2}(s))\; ds$ for $n \geq 2$.

Claim 7.2. *If $G\colon [0,\infty) \to \mathbb{R}^2$ is continuous, then $|AG(t)| \leq |A|\,|G(t)|$ and $\left|\int_0^t AG(s)\,ds\right| \leq \int_0^t |AG(s)|\; ds$ for all t in $[0,\infty)$.*

Proof. It follows from the definition of $|A|$ that $|AG(t)| \leq |A|\,|G(t)|$. Suppose $G(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$. Then

$$\begin{aligned}
\left|\int_0^t AG(s)\,ds\right| &= \left|\begin{bmatrix} \int_0^t af + bg \\ \int_0^t cf + dg \end{bmatrix}\right| \\
&= \left|\int_0^t af + bg\right| + \left|\int_0^t cf + dg\right| \\
&\leq \int_0^t |af + bg| + \int_0^t |cf + dg| \\
&= \int_0^t |AG(s)|\; ds.
\end{aligned}$$

(See also Theorem 16.3 and Proposition 16.9.) □

Now fix $\alpha > 0$ and suppose t is in $[0,\alpha]$. Then $Y_1(t) = X + \int_0^t AX = X + tAX$ and so

$$|F_1(t)| = |X + tAX - X| = |tAX| \leq t\,|A|\,|X|\,.$$

Thus,

$$|F_2(t)| \leq \int_0^t |A|\,|F_1(s)|\; ds \leq \int_0^t |A|\,|A|\,|X|\,s\,ds = \frac{t^2}{2}\,|A|^2\,|X|\,.$$

Inductively, assume $|F_n(t)| \leq (|X|\,t^n\,|A|^n)\,/n!$. Then

$$|F_{n+1}(t)| \leq \int_0^t |A|\,|F_n(s)|\; ds \leq \int_0^t |A|\,\frac{|X|\,s^n\,|A|^n}{n!}\; ds = \frac{t^{n+1}}{(n+1)!}\,|A|^{n+1}\,|X|\,.$$

Since $t \leq \alpha$, it follows that $|F_n(t)| \leq |X|\,|A|^n\,\alpha^n/\,n!$ for $n \geq 1$. Therefore, the series $\sum |F_n(t)|$ converges and so the series $\sum F_n$ converges uniformly on $[0,\alpha]$ by the Weierstrass M-test. Let F denote the sum of the series, so $F(t) = \sum_{n=1}^{\infty} F_n(t)$.

Now $Y_n(t) = Y_0(T) + \sum_{i=1}^{n} F_i(t)$, and so the sequence $\{Y_n\}$ converges uniformly on $[0,\alpha]$ to $Y_0 + F = X + F$. Define $Y = X + F$, so the Y_n's converge uniformly to Y.

Claim 7.3. *$Y'(t) = AY(t)$*

Proof. Using that the Y_n's converge uniformly to Y on any interval $[0,\alpha]$ we have

$$\begin{aligned}
Y(t) &= \lim_{n\to\infty} Y_n(t) \\
&= \lim_{n\to\infty} \left(X + \int_0^t AY_{n-1}(s)\,ds\right) \\
&= X + \int_0^t A \lim_{n\to\infty} Y_{n-1}(s)\,ds
\end{aligned}$$

$$= X + \int_0^t AY(s)\, ds.$$

Now it follows from the Fundamental Theorem of Calculus that $Y'(t) = AY(t)$ for t in $[0, \infty)$. □

Clearly $Y_n(0) = X$ for all n, so $Y(0) = X$ and hence it follows from Claim 7.3 that Y is a solution to the matrix initial value problem.

To show that Y is the unique solution, suppose G is any solution. Then, using the Fundamental theorem of Calculus, we have

$$G(t) = X + \int_0^t AG(s)\, ds \quad \text{and} \quad Y(t) = X + \int_0^t AY(s)\, ds.$$

Fix $\alpha > 0$ and let $M = \sup\{\, |Y(t) - G(t)| \mid t \in [0, \alpha]\,\}$. Then for t in $[0, \alpha]$,

$$\begin{aligned} |Y(t) - G(t)| &= \left| \int_0^t A(Y(s) - G(s))\, ds \right| \\ &\le \int_0^t |A|\, |Y(s) - G(s)|\, ds \\ &\le |A|\, Mt. \end{aligned}$$

Assume $|Y(t) - G(t)| \le M\, |A|^n\, t^n/n!$. Then

$$\begin{aligned} |Y(t) - G(t)| &= \left| \int_0^t A(Y(s) - G(s))\, ds \right| \\ &\le \int_0^t |A|\, |Y(s) - G(s)|\, ds \\ &\le \int_0^t |A|\, M\, |A|^n\, \frac{t^n}{n!}\, ds \\ &= M\, |A|^{n+1}\, \frac{t^{n+1}}{(n+1)!}. \end{aligned}$$

Therefore, $|Y(t) - G(t)| \le M\, |A|^n\, t^n/n!$ for all $n \ge 1$ and so $Y(t) = G(t)$ for t in $[0, \alpha]$. Since α was arbitrary, $Y = G$.

Comments. The preceding argument will be generalized in **Problem 13** to show that if X is a Banach space, $[a, b]$ is a closed interval, $g\colon [a, b] \to X$ is continuous, and $A\colon [a, b] \to L(X, X)$ is continuous, then there is a unique $y\colon [a, b] \to X$ so that $y'(t) = A(t)y(t) + g(t)$ for all t in $[a, b]$.

Problem 8

Problem. *If $T\colon [0, \infty) \to M_2(\mathbb{R})$ satisfies $T(0) = I$ and there is a matrix, A, in $M_2(\mathbb{R})$ so that $T'(t) = AT(t)$ for all t, show that $T(s + t) = T(s)T(t)$ for all s and t in $[0, \infty)$.*

Background.

Remark 8.1. Suppose A and X are in $M_2(\mathbb{R})$ and consider the initial value problem:

IVP:$M_2(\mathbb{R})$ $\qquad G\colon [0,\infty) \to M_2(\mathbb{R}),\ G(0) = X,$ and $G'(t) = AG(t)$ for t in $[0,\infty)$.

Write $G(t) = \begin{bmatrix} G_1(t) & G_2(t) \end{bmatrix}$ and $X = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$. Then $G(0) = \begin{bmatrix} G_1(0) & G_2(0) \end{bmatrix}$ and

$$AG(t) = \begin{bmatrix} AG_1(t) & AG_2(t) \end{bmatrix} = \begin{bmatrix} G_1'(t) & G_2'(t) \end{bmatrix}.$$

Thus, the given initial value problem is equivalent to the two initial value problems: $G_i\colon [0,\infty) \to \mathbb{R}^2$, $G_i(0) = X_i$, $G_i'(t) = AG_i(t)$ for t in $[0,\infty)$, for $i = 1, 2$. It was shown in **Problem 7** that these last two initial value problems have unique solutions, and so IVP:$M_2(\mathbb{R})$ has a unique solution.

Solution. [Ioana] With the notation of the problem, fix s and define $F\colon [0,\infty) \to M_2(\mathbb{R})$ by $F(t) = T(t+s) - T(t)T(s)$. Then clearly $F(0) = 0$ and using the assumption on T we have

$$F'(t) = T'(t+s) - T'(t)T(s) = AT(t+s) - AT(t)T(s) = AF(t).$$

Therefore, F satisfies IVP:$M_2(\mathbb{R})$ with $X = 0$. But $G(t) = 0$ also satisfies IVP:$M_2(\mathbb{R})$ with $X = 0$, so by uniqueness, $F = G$ and hence $F(t) = 0$ for all t. It follows that $T(t+s) = T(t)T(s)$ for all t and s.

Comments. Combining the results in **Problems 7** and **8**, it follows that for a function, $T\colon [0,\infty) \to M_2(\mathbb{R})$, with $T(0) = I$, the following statements are equivalent:

- T is continuous and $T(s+t) = T(s)T(t)$ for all s and t in $[0,\infty)$.
- T is differentiable and there is a matrix, A, in $M_2(\mathbb{R})$ so that $T'(t) = AT(t)$ for all t in $[0,\infty)$.
- T is smooth and there is a matrix, A, in $M_2(\mathbb{R})$ so that $T'(t) = AT(t)$ for all t in $[0,\infty)$.

If T satisfies the conditions in the first statement, then T is differentiable and the matrix "A" in the second statement is $T'(0)$.

If T satisfies any of the conditions above, then $T(t) = e^{tA}$ where $A = T'(0)$ and so T can be extended to a smooth function on all of $\mathbb{R}$.

It follows (using the comment after **Problem 3**, the definitions in **Problem 11**, and some linear algebra) that if T is a continuous, linear semigroup on $\mathbb{R}^2$, then $T(t)x = e^{tA}x$ for all x in $\mathbb{R}^2$, where $A = T'(0)$. Moreover, it's easy to see that (with the notation of **Problem 11**) $D_T = \mathbb{R}^2$ and the generator of T is the linear transformation $T'(0)$, that is $A_T x = T'(0)x$ for x in $\mathbb{R}^2$.

Also, there is a canonical bijection between semigroups on $\mathbb{R}^2$ and initial value problems IVP:$M_2(\mathbb{R})$ with the fixed initial condition $G(0) = I$. If T is a semigroup on $\mathbb{R}^2$, then T is the unique solution to IVP:$M_2(\mathbb{R})$ with initial condition $G(0) = I$ and with $A = T'(0)$. Conversely, if T is the solution to IVP:$M_2(\mathbb{R})$ with initial condition $G(0) = I$, then T is in fact a semigroup and $T'(0) = A$.

Problem 9

Problem. *Show that ℓ^2 is closed under addition.*

Background. Recall ℓ^2 denote the set of all sequences, $\{x_n\} = (x_0, x_1, x_2, \dots)$ in $\mathbb{R}^{\mathbb{N}}$ with the property that the series $\sum x_n^2$ converges. Sequences are added and multiplied by scalars pointwise, and $|\{x_n\}| = \sqrt{\sum_{n=0}^{\infty} x_n^2}$. Then ℓ^2 is complete and an inner product space, so it's a *Hilbert space.*

Equivalently, ℓ^2 is the set of all functions, $x\colon \mathbb{N} \to \mathbb{R}$ with the property that $\sum_n x(n)^2$ converges. Functions are added and multiplied by scalars pointwise.

Solution. [Deana]

Claim 9.1. *Suppose s and t are in $[0, \infty)$. Then $st \le s^2/2 + t^2/2$.*

Proof. Consider the line whose equation is $y = x$ in the plane and the rectangle in the first quadrant with one side $[0, s]$ on the x-axis and an adjacent side $[0, t]$ on the y-axis. Then the area of the rectangle is st and is at most $\int_0^s x\,dx + \int_0^t y\,dy = s^2/2 + t^2/2$. Notice that equality holds if and only if $s = t$. ☐

Now let $\{x_n\}$ and $\{y_n\}$ be in ℓ^2. Then by Claim 9.1,

$$\sum_n |x_n y_n| \le \sum_n \frac{x_n^2}{2} + \frac{y_n^2}{2},$$

and so $\sum_n x_n y_n$ converges absolutely and hence

$$\begin{aligned}\sum_n |x_n + y_n|^2 &= \sum_n x_n^2 + 2\,|x_n y_n| + y_n^2 \\ &\le \sum_n x_n^2 + \sum_n 2\left(\frac{x_n^2}{2} + \frac{y_n^2}{2}\right) + \sum_n y_n^2 \\ &= 2\sum_n x_n^2 + 2\sum_n y_n^2.\end{aligned}$$

The result follows by taking square roots of both sides and using that $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for positive real numbers a and b.

Comments. Notice that the argument shows that $|\{x_n\} + \{y_n\}| \le \sqrt{2}(|\{x_n\}| + |\{y_n\}|)$. However, it's not hard to show that $|\cdot|$ is a norm on ℓ^2 (see **Problem 10**), so in fact $|\{x_n\} + \{y_n\}| \le |\{x_n\}| + |\{y_n\}|$.

Problem 10

Problem. *Show that ℓ^2 is an inner product space.*

Background. Recall that $(X, \langle \cdot\,, \cdot \rangle)$ is an *inner product space* if $\langle \cdot\,, \cdot \rangle$ is *bilinear*: $\langle \alpha x + y, z \rangle = \alpha\langle x, z\rangle + \langle y, z\rangle$; $\langle \cdot\,, \cdot \rangle$ is *symmetric*: $\langle x, y\rangle = \langle y, x\rangle$; and $\langle \cdot\,, \cdot \rangle$ is *positive definite*: $\langle x, x\rangle \ge 0$, and $\langle x, x\rangle = 0$ if and only if $x = 0$.

It's not too hard to show that the Cauchy-Schwarz inequality is true in any inner product space. It follows easily that if $(X, \langle \cdot\,, \cdot \rangle)$ is an inner product space and $|\cdot|$ is defined by $|x| = \sqrt{\langle x, x\rangle}$, then $|\cdot|$ is a norm on X.

Solution. [Deana] It was shown in the solution to **Problem 9** that if $\{x_n\}$ and $\{y_n\}$ are in ℓ^2, then $\sum_n x_n y_n$ converges absolutely. Define

$$\langle \{x_n\}, \{y_n\}\rangle = \sum_n x_n y_n \text{ and } |\{x_n\}| = \sqrt{\langle \{x_n\}, \{x_n\}\rangle} = \sum_n x_n^2.$$

It follows immediately from standard properties of absolutely convergent infinite series of real numbers that $\langle\,\cdot\,,\,\cdot\,\rangle$ is symmetric and bilinear. Also $\langle \{x_n\}, \{x_n\}\rangle = \sum_n x_n^2 = |\{x_n\}|^2$. Therefore, $\langle\,\cdot\,,\,\cdot\,\rangle$ is positive definite and it follows from the remark above that ℓ^2 is an inner product space.

Problem 11

Problem. *For t in $[0,\infty)$, define $T(t)\colon \ell^2 \to \ell^2$ by $T(t)\{x_n\} = \{e^{-nt}x_n\}$. Check that $T(t)$ is linear and continuous, so $T(t)$ is in $L(\ell_2)$.*

(a) *Show that T is a linear semigroup on ℓ^2.*
(b) *Compute $|T(t) - I|$ for $t > 0$.*
(c) *Show that T is strongly continuous but not continuous.*
(d) *Compute D_T and the generator, A_T, of T.*

Background.

Definition 11.1. If X is a topological space, a *semigroup on* X is a function, $T\colon [0,\infty) \to \mathrm{C}(X,X)$, where $\mathrm{C}(X,X)$ denotes the set of continuous functions $X \to X$, with the properties that $T(0)$ is the identity mapping and $T(s+t) = T(s)T(t)$ for all s and t in $[0,\infty)$. If X is a topological vector space and $T(t)$ is linear for every t, so $T\colon [0,\infty) \to L(X)$, then T is a *linear* semigroup on X. If for each x in X the *trajectory*, $t \mapsto T(t)x$ defines a continuous function from $[0,\infty)$ to X, then T is *strongly continuous.*

The conditions that X be a toploogical space and that the image of T is a subset of $\mathrm{C}(X,X)$ can be relaxed. A more general definition of a "semigroup of transformations on X" would be just that T is a representation of the monoid $[0,\infty)$ in the category of sets. In other words, X is a set and $T\colon [0,\infty) \to \mathrm{Map}(X,X)$ is a homomorphism of monoids, where the monoid structure on $\mathrm{Map}(X,X)$ is the categorical one given by composition.

The examples above and below will all be semigroups in the sense of Definition 11.1:

- **Problem 1:** $T_b\colon [0,\infty) \to M_1(\mathbb{R})$ by $T_b(t)(s) = e^{bt}s$.
- **Problem 2, 7, 8:** $T_A\colon [0,\infty) \to M_2(\mathbb{R})$ by $T_A(t)(x) = e^{tA}x$.
- **Problem 5:** $T\colon [0,\infty) \to \mathrm{C}([0,\infty),[0,\infty))$ by $T(t)(x) = \frac{x}{1+tx}$.
- **Problem 5:** $T\colon [0,\infty) \to \mathrm{C}(X,X)$ by $T(t)(h) = u^h(t,\cdot)$.
- **Problem 11:** $T\colon [0,\infty) \to L(\ell^2)$ by $T(t)\{x_n\} = \{e^{-nt}x_n\}$.
- **Problem 12:** $T\colon [0,\infty) \to L(\mathrm{CB}([0,\infty)))$ by $T(t) = \ell_t^{\#}$ where $\ell_t\colon [0,\infty) \to [0,\infty)$ by $\ell_t(x) = t+x$. Notice that this is a special case of the construction in the example in **Problem 18** with $X = [0,\infty)$ and $T(t) = \ell_t$.
- **Problem 12:** $T\colon [0,\infty) \to L(\mathrm{UCB}([0,\infty)))$ by $T(t) = \ell_t^{\#}$ where $\ell_t\colon [0,\infty) \to [0,\infty)$ by $\ell_t(x) = t+x$.
- **Problem 18:** If $T\colon [0,\infty) \to \mathrm{C}(X,X)$ is a semigroup on a complete metric space, X. Then $S\colon [0,\infty) \to L(\mathrm{CB}(X))$ by $S(t) = T(t)^{\#}$ is a linear semigroup.

Notation 11.2. Suppose X is a topological space and T is a semigroup on X. For x in X, the trajectory of x will be denoted by g_x, so $g_x\colon [0,\infty) \to X$ by $g_x(t) = T(t)x$.

Definition 11.3. If X is a normed linear space space and T is a semigroup on X, define $D_T = \{\, x \in X \mid \lim_{t\to 0}(1/t)\,(T(t)-I)\,x \text{ exists}\,\}$. In other words, x is in D_T if and only if there is a y in X with the property that for every $\epsilon > 0$, there is a $\delta > 0$ so that $|(1/t)\,(T(t)-I)\,x - y| < \epsilon$ whenever $0 < t < \delta$. Define $A_T\colon D_T \to X$ by $A_T x = \lim_{t\to 0}(1/t)\,(T(t)-I)\,x$. The function, A_T, is called the *generator of* T.

Remark 11.4. There are variants of this definition, see Remark 11.13.

Solution. [Deana, Ioana] To show that T is a semigroup on ℓ^2, notice first that clearly $T(0)\{x_n\} = \{x_n\}$ and so $T(0)$ is the identity transformation. Now fix s and t in $[0,\infty)$. Then

$$\begin{aligned} T(t)T(s)(\{x_n\}) &= T(t)(\{e^{-ns}x_n\}) \\ &= \{e^{-nt}e^{-ns}x_n\} \\ &= \{e^{-n(t+s)}x_n\} \\ &= T(t+s)(\{x_n\}). \end{aligned}$$

Therefore, T is a semigroup on ℓ^2. Since the vector space operations on ℓ^2 are pointwise it follows trivially that $T(t)$ is a linear transformation for t in $[0,\infty)$.

Remark 11.5. To complete the argument that T is a linear semigroup on ℓ^2 it remains to show that $T(t)$ is in fact defined, that is, $T(t)(\{x_n\})$ is in ℓ^2, and that $T(t)$ is continuous for t in $[0,\infty)$.

The fact that $T(t)(\{x_n\})$ is in ℓ^2 when $\{x_n\}$ is in ℓ^2 follows from the fact that $\sum_n \left(e^{-nt}x_n\right)^2 \le \sum_n x_n^2$ since $0 < e^{-nt} \le 1$ for $n \ge 0$. To show that $T(t)$ is continuous, it's enough to show $|T(t)| = 1$. We just saw that

$$|T(t)\{x_n\}| = \left|\{e^{-nt}x_n\}\right| = \sum_n \left(e^{-nt}x_n\right)^2 \le \sum_n x_n^2 = |\{x_n\}|$$

so $|T(t)| \le 1$. However, if $x_0 = 1$ and $x_n = 0$ for $n > 0$, then $|\{x_n\}| = 1$ and $T(t)\{x_n\} = \{x_n\}$, so $|T(t)| = 1$.

This completes the argument for part (a).

For part (b), we'll show that $|T(t) - I| = 1$ for $t > 0$. So fix $t > 0$. Then $e^{-nt} \le 1$ for all $n \ge 0$, so $|e^{-nt}x_n| \le |x_n|$ and $e^{-2nt}x_n^2 \le x_n^2$ for all $n \ge 0$. Now

$$\begin{aligned} |(T(t)-I)\{x_n\}|^2 &= |T(t)\{x_n\} - \{x_n\}|^2 \\ &= \sum_n \left(e^{-nt}x_n - x_n\right)^2 \\ &= \sum_n \left(1 - e^{-nt}\right)^2 x_n^2 \\ &\le \sum_n x_n^2 \\ &= |\{x_n\}|\,. \end{aligned}$$

It follows that $|T(t) - I| \le 1$.

To show that $|T(t) - I| \geq 1$ it's enough to show that $|T(t) - I| \geq 1 - \epsilon$ for all $\epsilon > 0$. So fix $\epsilon > 0$. It's enough to find $\{x_n\}$ in ℓ^2 with $|(T(t) - I)\{x_n\}| > (1 - \epsilon)\{x_n\}$. Choose N so that $e^{-Nt} < \epsilon$ and define $x_n = \delta_{N,n}$ (kronecker delta) for $n \geq 0$. Then $\sum_n x_n^2 = 1$ and so $\{x_n\}$ is in ℓ^2 and $|\{x_n\}| = 1$. Now

$$|(T(t) - I)\{x_n\}|^2 = \left(1 - e^{-Nt}\right)^2 > (1 - \epsilon)^2 = (1 - \epsilon)^2 \left|\{x_n\}\right|^2.$$

Taking square roots of both ends of the chain of inequalities gives the result. This completes the argument for part (b).

For the first part of part (b), fix $x = \{x_n\}$ in ℓ^2 and consider the trajectory g_x. We'll show first that g_x is continuous for $t > 0$ and then that g_x is continuous from the right at 0.

Claim 11.6. *If $t > 0$, then g_x is continuous at t.*

Proof. It follows from the Root Test that if $t > 0$, then the series $\sum_k e^{-kt}$ converges, so we can define a function, $p\colon (0, \infty) \to \mathbb{R}$ by $p(t) = \sum_{k=0}^{\infty} e^{-kt}$. To show that p is continuous it's enough to show that the series $\sum_k e^{-kt}$ converges uniformly on any interval $[\epsilon, \infty)$ where $\epsilon > 0$.

So fix $\epsilon > 0$. Recall that the sequence $\{(lnk)/k\}$ is a decreasing sequence that converges to 0. Choose N so that $\epsilon > (2 \ln N)/N$, so $\epsilon > (2 \ln k)/k$ for $k \geq N$. Now define $M_k = 1$ for $0 \leq k \leq N$ and $M_k = 1/k^2$ for $k > N$, so the series $\sum_k M_k$ converges. If $t > 0$, then $e^{-kt} \leq 1$, so $e^{-kt} \leq M_k$ for $0 \leq k \leq N$. If $t \geq \epsilon$ and $k > N$, then $t \geq (2 \ln k)/k$, so $e^{-kt} \leq 1/k^2 = M_k$. Therefore, it follows from the Weierstrass M-test that the series $\sum_k e^{-kt}$ converges uniformly on the interval $[\epsilon, \infty)$.

Now fix $t > 0$ and choose M so that $M \geq x_k^2$ for all k. Then for $s > 0$ we have

$$\begin{aligned}
|g_x(s) - g_x(t)|^2 &= \left|\{e^{-ks}x_k - e^{-kt}x_k\}\right|^2 \\
&= \sum_k \left|e^{-ks}x_k - e^{-kt}x_k\right|^2 \\
&= \sum_k x_k^2 \left(e^{-ks} - e^{-kt}\right)^2 \\
&\leq M \sum_k \left(e^{-ks} - e^{-kt}\right)^2 \\
&= M \sum_k e^{-2ks} - 2e^{-ks}e^{-kt} + e^{-2kt} \\
&= M(p(2s) - 2p(s + t) + p(2t)).
\end{aligned}$$

Since p is continuous on $(0, \infty)$, it follows that $p(2s) - 2p(s + t) + p(2t)$ approaches 0 as s approaches t and hence that g_x is continuous at t for $t > 0$. ☐

Claim 11.7 (Ioana)**.** *g_x is continuous at 0.*

Proof. Suppose $t \geq 0$. Then

$$\begin{aligned}
|g_x(t) - g_x(0)|^2 &= \left|\{e^{-kt}x_k - x_k\}\right|^2 \\
&= \sum_k \left|e^{-kt}x_k - x_k\right|^2 \\
&= \sum_k x_k^2 \left(e^{-kt} - 1\right)^2.
\end{aligned}$$

Thus, if we define $s(t) = \sum_{k=0}^{\infty} x_k^2 \left(e^{-kt} - 1\right)^2$ for $t \geq 0$, it's enough to show that $s(t)$ approaches 0 as t approaches 0.

Fix $\epsilon > 0$. Since $\{x_k\}$ is in ℓ^2, we can choose N so that $\sum_{k=N+1}^{\infty} x_k^2 \leq \epsilon/2$. Then

$$\sum_{k=N+1}^{\infty} x_k^2 \left(e^{-kt} - 1\right)^2 \leq \sum_{k=N+1}^{\infty} x_k^2 \leq \epsilon/2.$$

For any n,

$$\sum_{k=0}^{n} x_k^2 \left(e^{-kt} - 1\right)^2 \leq \sum_{k=0}^{n} x_k^2 \left(1 - e^{-nt}\right)^2 \leq \left(1 - e^{-nt}\right)^2 |x|.$$

Finally, $\left(1 - e^{-Nt}\right)^2$ decreases to 0 as t decreases to 0, so there is a δ so that $\left(1 - e^{-Nt}\right)^2 \leq \epsilon/(2|x|)$ whenever $t < \delta$. But now, if $t < \delta$, then

$$s(t) = \sum_{k=0}^{N} x_k^2 \left(e^{-kt} - 1\right)^2 + \sum_{k=N+1}^{\infty} x_k^2 \left(e^{-kt} - 1\right)^2 \leq \left(1 - e^{-Nt}\right)^2 |x| + \sum_{k=N+1}^{\infty} x_k^2 \leq \epsilon/2 + \epsilon/2 = \epsilon.$$

Therefore, $\lim_{t \to 0^+} s(t) = 0$ as desired. □

The second part of (c) follows immediately from (b) since $L(\ell^2)$ is a metric space and by (b), $|T(t) - I| = |T(t) - T(0)| = 1$ for $t > 0$ and so does not approach 0 as t approaches 0. Therefore T is not continuous at 0.

Claim 11.8. [Ioana] *The domain of the generator of T is $D_T = \{\, \{x_n\} \mid \sum_n n^2 x_n^2 < \infty \,\}$ and for $\{x_n\}$ in D_T, $A_T\{x_n\} = \{-nx_n\}$.*

Proof. Suppose $x = \{x_n\}$ is in D_T, so $\lim_{t \to 0+}(1/t)(T(t)x - x)$ exists. Set $y = \lim_{t \to 0+}(1/t)(T(t)x - x)$ so

$$y_n = \lim_{t \to 0+} \frac{1}{t}(e^{-nt}x_n - x_n) = x_n \lim_{t \to 0+} \frac{e^{-nt} - 1}{t} = -nx_n.$$

But y is in ℓ^2, so $\sum_n y_n = \sum_n n^2 x_n^2$ converges. Thus $D_T \subseteq \{\, \{x_n\} \mid \sum_n n^2 x_n^2 < \infty \,\}$ and if $\sum_n n^2 x_n^2$ converges, then $A_T\{x_n\} = \{-nx_n\}$.

To complete the proof, it remains to show that $\{\, \{x_n\} \mid \sum_n n^2 x_n^2 < \infty \,\} \subseteq D_T$, so suppose $\{x_n\}$ is in ℓ^2 and $\sum_n n^2 x_n^2$ converges. Define $y_n = -nx_n$ for all n. Then it suffices to show that $(1/t)(T(t) - I)\{x_n\}$ approaches $\{y_n\}$ in ℓ^2 as t approaches 0. Set $s(t) = |\{y_n\} - (1/t)(T(t) - I)\{x_n\}|^2$, we need to show that $s(t)$ approaches 0 as t approaches 0.

For $n \geq 0$ and $t \geq 0$, apply the Mean Value Theorem to the function $h(t) = e^{-nx}$ on the interval $[0, t]$ to find $c_{n,t}$ with $c_{n,t}$ in $[0, t]$ and $(1/t)(e^{-nt} - 1) = -ne^{-nc_{n,t}}$. Then

$$\begin{aligned} s(t) &= |\{y_n\} - (1/t)(T(t) - I)\{x_n\}|^2 \\ &= \sum_n \left(-nx_n - (1/t)(e^{-nt}x_n - x_n)\right)^2 \\ &= \sum_n x_n^2 \left(-n - (1/t)(e^{-nt} - 1)\right)^2 \\ &= \sum_n x_n^2 \left(-n + ne^{-nc_{n,t}}\right)^2 \end{aligned}$$

$$= \sum_n x_n^2 n^2 \left(1 - e^{-nc_{n},t}\right)^2.$$

Now fix $\epsilon > 0$ and $\alpha > 0$ and suppose t in $[0, \alpha]$. Since $\sum_n n^2 x_n^2$ converges, there is an N so that $\sum_{k=n+1}^{\infty} k^2 x_k^2 < \epsilon/2$ when $n > N$. Then

$$\sum_{k=N+1}^{\infty} x_k^2 k^2 \left(1 - e^{-kc_{k,t}}\right)^2 \leq \sum_{k=N+1}^{\infty} k^2 x_k^2 < \epsilon/2. \tag{11.9}$$

Set $M = \sum_k x_k^2 k^2$. Then,

$$\sum_{k=0}^{N} x_k^2 k^2 \left(1 - e^{-kc_{k,t}}\right)^2 \leq \sum_{k=0}^{N} x_k^2 k^2 \left(1 - e^{-kt}\right)^2 \leq \left(1 - e^{-Nt}\right)^2 \sum_{k=0}^{N} x_k^2 k^2 \leq M \left(1 - e^{-Nt}\right)^2. \tag{11.10}$$

Since $\left(1 - e^{-Nt}\right)^2$ approaches 0 as t approaches 0 we can choose $\delta > 0$ so that $M \left(1 - e^{-Nt}\right)^2 < \epsilon/2$ when $0 < t < \delta$. Thus, when $0 < t < \delta$ we have $\sum_{k=0}^{N} x_k^2 k^2 \left(1 - e^{-kc_{k,t}}\right)^2 \leq \epsilon/2$, so using Equations 11.9 and 11.10, we have

$$s(t) = \sum_{k=0}^{N} x_k^2 k^2 \left(1 - e^{-kc_{k,t}}\right)^2 + \sum_{k=N+1}^{\infty} x_k^2 k^2 \left(1 - e^{-kc_{k,t}}\right)^2 \leq \epsilon/2 + \epsilon/2 = \epsilon$$

Therefore $(1/t)(T(t) - I)\{x_n\}$ approaches $\{y_n\}$ in ℓ^2 as t approaches 0 and so $\{x_n\}$ is in D_T. □

Comments.

Remark 11.11. In general, a continuous semigroup is strongly continuous.

Remark 11.12. We'll see in **Problem 17** that D_T is dense in X when X is a Banach space. It's also true that $D_T = X$ if and only if T is continuous.

Remark 11.13. Suppose Y is a Banach space, T is a semigroup on Y, and X is a T-stable subspace of Y, so $T(t)X \subseteq X$ for all t in $[0, \infty)$. Then T restricts to s semigroup on X, say $T_1 \colon [0, \infty) \to \mathrm{C}(X, X)$. If X is not closed in Y, then it's possible that given x in X, the limit $\lim_{t \to 0^+} (1/t)(T_1(t) - I)x$ defines element in Y that's not in X. In this case, one could define $D_{T_1} = \{\, x \in X \mid \lim_{t\to 0} (1/t)\,(T(t) - I)\,x \text{ exists in } Y \,\}$. In other words, x is in D_{T_1} if and only if there is a y in Y with the property that for every $\epsilon > 0$, there is a $\delta > 0$ so that $|(1/t)\,(T(t) - I)\,x - y| < \epsilon$ whenever $0 < t < \delta$. In this case, A_{T_1} would be the function from D_{T_1}, a subset of X, to Y defined by $A_{T_1} x = \lim_{t\to 0} (1/t)\,(T(t) - I)\,x$.

Problem 12

Problem. *Define* $T \colon [0, \infty) \to L(\mathrm{CB}([0, \infty)))$ *by* $(T(t)(f))(x) = f(t+x)$. *In other words, if* $l_t \colon [0, \infty) \to [0, \infty)$ *is translation by* t *(*$l_t(x) = t + x$ *for* $t \geq 0$*) then* $T(t) = l_t^{\#}$ *(that is,* $T(t)(f) = f l_t$*).*

(a) *Show that* T *is a linear semigroup on* $\mathrm{CB}([0, \infty))$.
(b) *Compute* $|T(t) - I|$ *for* $t > 0$.
(c) *Show that* T *is not strongly continuous but that the "restriction" of* T *to* $\mathrm{UCB}([0, \infty))$ *is strongly continuous.*
(d) *Compute* D_T *and the generator,* A_T*, of the restriction of* T *to* $\mathrm{UCB}([0, \infty))$.
(e) *Suppose* f *is in* $\mathrm{CB}([0, \infty))$ *and* $\{t_n\}$ *is a sequence in* $[0, \infty)$ *converging to* t. *Show that the sequence* $\{T(t_n)f\}$ *is* β*-convergent to* $T(t)f$.

(f) *Consider the restriction of* T *to* UCB$([0,\infty))$. *For* $\lambda > 0$, *compute* $(I - \lambda A_T)^{-1}$ *when* $A_T f = f'$ *(find a "nice" form of the answer). Show that* A_T *is closed, densely defined, and linear. Show also that for* $\lambda > 0$, $(I - \lambda A_T)^{-1}$ *exists, has domain all of* UCB$([0,\infty))$, *is non-expansive (that is,* $\left|(I - \lambda A_T)^{-1}\right| \leq 1$*), and that for* f *in* UCB$([0,\infty))$, $\lim_{n\to\infty}(I - \frac{\lambda}{n} A_T)^{-1} x = T(\lambda)x$.

Background. Recall that if X is a complete metric space. Then CB(X) is the vector space of all continuous, bounded, real-valued functions on X. For f in CB(X), $|f| = \sup\{\, f(x) \mid x \in X \,\}$. With this norm, CB$(X)$ is a Banach space. The subset of CB(X) consisting of uniformly continuous functions is a closed, and hence complete, subspace denoted by UCB(X).

In particular, CB$([0,\infty))$ and UCB$([0,\infty))$ are Banach spaces.

Definition 12.1. A sequence of functions, $\{f_n\}$, in CB(X) is said to be *β-convergent* if it's bounded (that is, there is an M so that $|f_n| < M$ for all n) and it converges uniformly on compact sets (that is, if K is a compact subset of X and $\epsilon > 0$, there is an N so that $n > N$ implies $|f_n(x) - f(x)| < \epsilon$ for all x in K).

Definition 12.2. If T is a linear semi-group on a normed linear space, then T is *non-expansive* or a *contraction semigroup* if $|T(t)| \leq 1$ for all t in $[0,\infty)$.

Remark 12.3. For part (f), notice that $(I - \lambda A)^{-1} f = g$ is equivalent to $f = g - \lambda g'$ and this ODE has a unique *bounded* solution.

Solution. [Ioana] It's obvious that $T(0) = I$. For $t \geq 0$ define $\ell_t\colon [0,\infty) \to [0,\infty)$ by $\ell_t(x) = t + x$ so $T(t)f = f\ell_t$. Now

$$\begin{aligned} T(t)T(s)(f) &= T(t)(T(s)(f)) \\ &= T(t)(f\ell_s) \\ &= f\ell_s\ell_t \\ &= f\ell_{s+t} \\ &= T(s+t)(f). \end{aligned}$$

Therefore, $T(s+t) = T(s)T(t)$ for s and t in $[0,\infty)$. It follows from the continuity of composition that $T(t)$ is continuous for all t in $[0,\infty)$. Finally, $T(t)(f+g) = (f+g)\ell_t = f\ell_t + g\ell_t = T(t)f + T(t)g$ and $T(t)(\alpha f) = \alpha f \ell_t = \alpha T(t)(f)$, so T is a linear semigroup on CB$([0,\infty))$. This proves part (a).

Claim 12.4. $|T(t) - I| = 2$ *for* $t > 0$.

Proof.

$$\begin{aligned} |T(t) - I| &= \sup\{\, |T(t)f - f| \mid f \in \mathrm{CB}([0,\infty)),\ |f| \leq 1 \,\} \\ &= \sup_{|f|\leq 1} \sup_{x \geq 0} |T(t)f(x) - f(x)| \\ &= \sup_{|f|\leq 1} \sup_{x \geq 0} |f(t+x) - f(x)| \\ &\leq 2. \end{aligned}$$

Therefore $|T(t) - I| \leq 2$. To show that $|T(t) - I| = 2$ it suffices to show that for $\epsilon > 0$ there is an f in CB$([0,\infty))$ with $|f| \leq 1$ and $|T(t)f - f| > 2 - \epsilon$.

So, fix $t \geq 0$ and $\epsilon > 0$. Choose α and β with $|\alpha| \leq 1$, $|\beta| \leq 1$, and $|\alpha - \beta| \geq 2 - \epsilon$. Define $f\colon [0,\infty) \to \mathbb{R}$ by: 1) on the interval $[0, t/2]$, f is the line joining $(0,\alpha)$ and $(t/2, \beta)$; and 2) for $x \geq t/2$, $f(x) = \beta$. Then $|f| \leq 1$ and $\sup_{x \geq 0} |f(t+x) - f(x)| = |\alpha - \beta| \geq 2 - \epsilon$.

Alternately [Deana] it's easy to check that if $f(x) = \cos(\pi x/t)$, then $|f| = 1$ and $|T(t)f - f| = 2$, so $|T(t)f - f| = 2$. □

This completes the proof of part (b).

[Ioana] To show that the restriction of T to UCB($[0,\infty)$) is strongly continuous, notice first that if f is uniformly continuous, then so is $T(t)f$ for fixed t, so $T(t)$ maps UCB($[0,\infty)$) to itself and so T restricts to a semigroup on UCB($[0,\infty)$).

Now fix $t \geq 0$ and f in UCB($[0,\infty)$) and consider $g\colon [0,\infty) \to$ UCB($[0,\infty)$) by $g(t) = T(t)f$. We need to show that g is continuous. Since L(UCB($[0,\infty)$)) is a normed linear space (it's a Banach space), it's enough to show that if $\{t_n\}$ is a sequence in $[0,\infty)$ that converges to t, then $\lim_{n\to\infty} |g(t_n) - g(t)| = 0$. Notice that

$$\begin{aligned} |g(t_n) - g(t)| &= |T(t_n)f - T(t)f| \\ &= \sup_{x \geq 0} |T(t_n)f(x) - T(t)f(x)| \\ &= \sup_{x \geq 0} |f(t_n + x) - f(t+x)|\,. \end{aligned}$$

Now suppose $\epsilon > 0$ is given. Since f is uniformly continuous, we can choose δ so that $|f(x_2) - f(x_1)| < \epsilon$ whenever $|x_2 - x_1| < \delta$. Since the t_n's converge to t, we can choose an N so that $|t_n - t| < \delta$ whenever $n > N$. But then if $n > N$, $|(t_n + x) - (t + x)| = |t_n - t| < \delta$, so $|f(t_n + x) - f(t + x)| < \epsilon$. It follows that $|g(t_n) - g(t)| < \epsilon$ and so $\lim_{n\to\infty} |g(t_n) - g(t)| = 0$. This proves the second statement in (c).

For part (d), let T_1 denote the restriction of T to UCB($[0,\infty)$). We need to describe the domain, D_{T_1} of the generator, A_{T_1}, and find a formula for A_{T_1}. We'll show that D_{T_1} consists precisly of those functions in UCB($[0,\infty)$) that are differentiable and whose derivatives are uniformly continuous and bounded. Let B denote this set, so B is the set of functions in UCB($[0,\infty)$) that are differentiable and whose derivatives are uniformly continuous and bounded.

For $f\colon [0,\infty) \to \mathbb{R}$ and x in $[0,\infty)$, define the *right derivative of f at x* by $f_r'(x) = \lim_{t\to 0^+}(f(t+x) - f(x))/t$ if the limit exists. If $f_r'(x)$ exists for all x in $[0,\infty)$, then $x \mapsto f_r'(x)$ defines a function from $[0,\infty)$ to $\mathbb{R}$.

Proposition 12.5 (Henry). *Assume that if f is in* C($[0,\infty)$), *$f_r'(x)$ exists for all x, and f_r' is continuous, then $f_r' = f'$. Then $D_{T_1} = B$ and $A_{T_1}(f) = f'$ for f in* UCB($[0,\infty)$).

Proof. By definition, if f is in UCB($[0,\infty)$), then f is in D_{T_1} if and only if $\lim_{t\to 0^+}(T_1(t)f - f)$ exists. Recall that "exists" means that there is a function, g, in UCB($[0,\infty)$), so that for every $\epsilon > 0$, there is $\delta > 0$ so that $|(1/t)(T_1(t) - I)f - g| < \epsilon$ whenever $0 < t < \delta$. But

$$|(1/t)(T_1(t) - I)f - g| = \sup\{\, |(1/t)(f(t+x) - f(x)) - g(x)| \mid x \in [0,\infty), \}.$$

Therefore, given x_0 in $[0,\infty)$, the limit $\lim_{t\to 0^+}(1/t)(f(t+x_0) - f(x_0))$ exists and is equal $g(x_0)$. It follows that $D_{T_1} \subseteq B$ and that $A_{T_1}(f) = f_r'$ for f in D_{T_1}.

It remains to show that $B \subseteq D_{T_1}$. So suppose f and f'_r are both in $\mathrm{UCB}([0,\infty))$. Then by assumption, f is differentiable and $f' = f'_r$ is uniformly continuous and bounded. We'll show that $\lim_{t\to 0^+}(1/t)(T_1(t) - I)f = f'$.

Suppose $\epsilon > 0$. We need to find $\delta > 0$ so that

$$|(1/t)(T(t) - I)f - f'| \leq \epsilon \quad \text{whenever} \quad 0 < t < \delta.$$

Since the norm on $\mathrm{UCB}([0,\infty))$ is the sup-norm, the left hand side of the first inequality is

$$\sup\{\, |(1/t)(f(x+t) - f(x)) - f'(x)| \mid x \in [0,\infty)\,\}.$$

Since f' is uniformly continuous, there is a δ so that $|f'(y) - f'(x)| \leq \epsilon$ whenever $|y - x| \leq \delta$.

Fix t with $0 < t < \delta$ and suppose x s in $[0,\infty)$. Since f is continuously differentiable, by the Mean Value Theorem applied to the restriction of f to $[x, x+t]$, we can choose a number depending on x and t, say $c_{x,t}$, so that $(1/t)(f(x+t) - f(x)) = f'(c_{x,t})$. Then for any x in $[0,\infty)$ we have

$$\left|\frac{f(x+t) - f(x)}{t} - f'(x)\right| = |f'(c_{x,t}) - f'(x)| \leq \epsilon$$

since $|c_{x,t} - x| \leq |(x+t) - x| \leq t \leq \delta$. It follows that $|(1/t)(T(t) - I)f - f'| \leq \epsilon$ whenever $0 < t < \delta$. Therefore, $\lim_{t\to 0^+}(1/t)(T(t) - I)f = f'$ and so f is in D_{T_1}. □

To complete the description of the generator of T_1, we need to show that the assumption in the preceding proposition holds. That is, if f'_r exists and is continuous, then f is differentiable and $f' = f'_r$.

Proposition 12.6 (Ioana). *Suppose f is in $\mathrm{C}([0,\infty))$. If $f'_r(x)$ exists for all x in $[0,\infty)$ and the restriction of f'_r to any closed interval is bounded, then the restriction of f to any closed interval, say $[a,b]$, is Lipschitz, with Lipschitz constant at most $1 + M$ where $M = \sup\{\, f'_r(x) \mid x \in [a,b]\,\}$.*

Proof. Fix a closed interval, $[a,b]$ and set $M = \sup\{\, f'_r(x) \mid x \in [a,b]\,\}$ and $M' = 1 + M$. To show that the restriction of f to $[a,b]$ is Lipschitz with Lipschitz constant at most M', it's enough to show that $|f(y) - f(x)| \leq M'\,|y - x|$ for all x and y in $[a,b]$ with $a \leq x < y \leq b$.

Suppose x is in $[a,b)$. Then $f'_r(x)$ exists, so there is a $\delta_x > 0$ so that

$$\left|\frac{f(y) - f(x)}{y - x} - f'_r(x)\right| \leq 1 \quad \text{whenever} \quad x < y \leq x + \delta_x.$$

Then

$$\left|\frac{f(y) - f(x)}{y - x}\right| = \left|\frac{f(y) - f(x)}{y - x} - f'_r(x) + f'_r(x)\right| \leq \left|\frac{f(y) - f(x)}{y - x} - f'_r(x)\right| + |f'_r(x)| \leq M',$$

so $|f(y) - f(x)| \leq M'\,|y - x|$ for y in $(x, x + \delta_x]$. Define

$$E_x = \{\, y \in (x,b] \mid |f(z) - f(x)| \leq M'\,|z - x|\,, \forall z \in [x,y]\,\}.$$

If $x < y \leq x + \delta_x$, then y is in E_x, so E_x is non-empty. To complete the proof, it's enough to show that $E_x = (x,b]$ for every x in $[a,b)$.

Set $b_0 = \mathrm{lub} E_x$. Then $|f(z) - f(x)| \leq M'\,|z - x|$ for all z with $x < z < b_0$. Therefore,

$$\lim_{z\to b_0^-} |f(z) - f(x)| \leq \lim_{z\to b_0^-} M'\,|z - x|\,.$$

Since f is continuous it follows that $|f(b_0) - f(x)| \leq M'\,|b_0 - x|$, and then that b_0 is in E_x, so $E_x = (x, b_0]$.

To get a contradiction, just suppose that $b_0 < b$. As above, there is a $\delta_0 > 0$ so that $|f(y) - f(b_0)| \leq M' |y - b_0|$ for y in $(b_0, b_0 + \delta_0]$. We may assume that $b_0 + \delta_0 \leq b$. Now consider $y = b_0 + (1/2)\delta_0$. If z is in $(x, b_0]$, then $|f(z) - f(x)| \leq M' |z - x|$ since b_0 is in E_x. If z is in $(b_0, y]$, then

- $|f(z) - f(x)| \leq |f(z) - f(b_0)| + |f(b_0) - f(x)|$ by the triangle inequality,
- $|f(z) - f(b_0)| \leq M' |z - b_0|$ since z is in $(b_0, b_0 + \delta_0]$, and
- $|f(x) - f(b_0)| \leq M' |x - b_0|$ since b_0 is in E_x.

Thus, $|f(z) - f(x)| \leq M' |z - x|$ when z is in $(b_0, y]$ since $|z - b_0| + |x - b_0| = |z - x|$. Therefore, $|f(z) - f(x)| \leq M' |z - x|$ for all z in $[x, y]$, so so y is in E_x. This contradicts the assumption that $\text{lub} E_x = b_0 < b$, and so it must be the case that $\text{lub} E_x = b$ and hence $E_x = (x, b]$ as desired. □

Proposition 12.7 (Ioana). *Suppose f is in* $C([0, \infty))$, *that $f_r'(x)$ exists for very x in* $[0, \infty)$, *and that f_r' is continuous. Then f is differentiable and $f' = f_r'$.*

Proof. It suffices to show that

$$\lim_{a \to c^-} \frac{f(a) - f(c)}{a - c} = f_r'(c) \quad \text{for } c \text{ in } (0, \infty).$$

Fix c in $(0, \infty)$ and $\epsilon > 0$. Since f_r' is continuous, we can choose a $\delta > 0$ so that $|f_r'(y) - f_r'(c)| < \epsilon/3$ whenever $|y - c| < \delta$. Fix a in $(c - \delta, c)$.

By Proposition 12.6, the restriction of f to $[a, c]$ is Lipschitz, so we can choose an M so that $|f(y) - f(x)| \leq M |y - x|$ for all x and y in $[a, c]$.

If x is in $[a, c)$, then as in the proof of Proposition 12.6 we can choose $\delta_x > 0$ so that

$$\left| \frac{f(y) - f(x)}{y - x} - f_r'(x) \right| < \epsilon/3 \quad \text{whenever} \quad x < y \leq x + \delta_x.$$

It follows that the collection of subintervals,

$$\{ [x, y] \mid \left| \frac{f(y) - f(x)}{y - x} - f_r'(x) \right| < \epsilon/3 \}, \quad \text{of } [a, c]$$

is a Vitali cover of $[a, c]$.

Set $\eta = \epsilon/(3(f_r'(c) + M))$, so $(f_r'(c) + M)\eta = \epsilon/3$. It follows from the Vitali Lemma that there is a finite collection of disjoint subintervals, $\{ [x_i, y_i] \mid 1 \leq i \leq n \}$, of $[a, c]$, so that the complement of $\cup_{i=1}^n [x_i, y_i]$ in $[a, c]$ has Lebesgue measure less than $\eta(c - a)$ and $|((f(y_i) - f(x_i))/(y_i - x_i)) - f_r'(x_i)| \leq \epsilon$ for $1 \leq i \leq n$. Thus, if we set $y_0 = a$ and $x_{n+1} = c$, then $\sum_{i=1}^{n+1} x_i - y_{i-1} = x_1 - a + \sum_{i=2}^n (x_i - y_{i-1}) + c - y_n < \eta(c - a)$.

Now consider $|((f(c) - f(a))/(c - a)) - f_r'(c)|$. We have

$$\begin{aligned} \left| \frac{f(c) - f(a)}{c - a} - f_r'(c) \right| &= \left| \frac{\sum_{i=1}^n f(y_i) - f(x_i) + \sum_{i=1}^{n+1} f(x_i) - f(y_{i-1})}{c - a} - f_r'(c) \right| \\ &\leq \left| \sum_{i=1}^n \frac{f(y_i) - f(x_i)}{c - a} - f_r'(c) \right| + \left| \frac{\sum_{i=1}^{n+1} f(x_i) - f(y_{i-1})}{c - a} \right|. \end{aligned} \tag{12.8}$$

The second term in the last sum is bounded as follows, using the Lipschitz property and that $\sum_{i=1}^{n+1} x_i - y_{i-1} < \eta(c-a)$:

$$\begin{aligned}\left|\frac{\sum_{i=1}^{n+1} f(x_i) - f(y_{i-1})}{c-a}\right| &= \left|\sum_{i=1}^{n+1} \frac{f(x_i) - f(y_{i-1})}{c-a}\right| \\ &\leq M \left|\sum_{i=1}^{n+1} \frac{x_i - y_{i-1}}{c-a}\right| \\ &\leq M\eta. \end{aligned} \tag{12.9}$$

Next, consider the first summand in the last line of Equation 12.8:

$$\begin{aligned}&\left|\sum_{i=1}^{n} \frac{f(y_i) - f(x_i)}{c-a} - f_r'(c)\right| \\ &\qquad = \left|\sum_{i=1}^{n} \frac{f(y_i) - f(x_i)}{y_i - x_i}\left(\frac{y_i - x_i}{c-a}\right) - \sum_{i=1}^{n} f_r'(x_i)\frac{y_i - x_i}{c-a} + \sum_{i=1}^{n} f_r'(x_i)\frac{y_i - x_i}{c-a} - f_r'(c)\right|,\end{aligned}$$

so

$$\left|\sum_{i=1}^{n} \frac{f(y_i) - f(x_i)}{c-a} - f_r'(c)\right| \leq \left|\sum_{i=1}^{n}\left(\frac{f(y_i) - f(x_i)}{y_i - x_i} - f_r'(x_i)\right)\frac{y_i - x_i}{c-a}\right| + \left|\sum_{i=1}^{n} f_r'(x_i)\frac{y_i - x_i}{c-a} - f_r'(c)\right|. \tag{12.10}$$

The first term in the last sum is bounded as follows, using that $|((f(y_i) - f(x_i))/(y_i - x_i)) - f_r'(x_i)| \leq \epsilon/3$ for $1 \leq i \leq n$ and that $\sum_{i=1}^{n} y_i - x_i \leq c - a$:

$$\begin{aligned}\left|\sum_{i=1}^{n}\left(\frac{f(y_i) - f(x_i)}{y_i - x_i} - f_r'(x_i)\right)\frac{y_i - x_i}{c-a}\right| &\leq \sum_{i=1}^{n}\left|\frac{f(y_i) - f(x_i)}{y_i - x_i} - f_r'(x_i)\right|\left|\frac{y_i - x_i}{c-a}\right| \\ &\leq \epsilon/3 \sum_{i=1}^{n} \frac{y_i - x_i}{c-a} \\ &\leq \epsilon/3.\end{aligned} \tag{12.11}$$

Next, consider the second summand on the right-hand side of Equation 12.10:

$$\begin{aligned}\left|\sum_{i=1}^{n} f_r'(x_i)\frac{y_i - x_i}{c-a} - f_r'(c)\right| &= \left|\sum_{i=1}^{n} f_r'(x_i)\frac{y_i - x_i}{c-a} - f_r'(c)\frac{\sum_{i=1}^{n} y_i - x_i + \sum_{i=1}^{n+1} x_i - y_{i-1}}{c-a}\right| \\ &\leq \left|\sum_{i=1}^{n} f_r'(x_i)\frac{y_i - x_i}{c-a} - f_r'(c)\frac{y_i - x_i}{c-a}\right| + \left|f_r'(c)\frac{\sum_{i=1}^{n+1} x_i - y_{i-1}}{c-a}\right| \\ &\leq \sum_{i=1}^{n} |f_r'(x_i) - f_r'(c)|\,\frac{y_i - x_i}{c-a} + |f_r'(c)|\,\frac{\sum_{i=1}^{n+1} x_i - y_{i-1}}{c-a} \\ &\leq \frac{\epsilon}{3}\sum_{i=1}^{n} \frac{y_i - x_i}{c-a} + |f_r'(c)|\,\eta \\ &\leq \frac{\epsilon}{3} + |f_r'(c)|\,\eta.\end{aligned} \tag{12.12}$$

Finally, combining Equations 12.8, 12.9, 12.10, 12.11, and 12.12 we have

$$\begin{aligned}\left|\frac{f(c)-f(a)}{c-a}-f_r'(c)\right| &\le \frac{\epsilon}{3}+\frac{\epsilon}{3}+|f_r'(c)|\,\eta+M\eta\\ &=\frac{2\epsilon}{3}+(|f_r'(c)|+M)\,\eta\\ &=\frac{2\epsilon}{3}+\frac{\epsilon}{3}\\ &=\epsilon.\end{aligned}$$

Therefore,

$$\left|\frac{f(c)-f(a)}{c-a}-f_r'(c)\right|\le\epsilon \quad \text{whenever } c-\delta<a<c,$$

so $\lim_{a\to c^-}(f(a)-f(c))/(a-c)=f_r'(c)$ for every c in $[0,\infty)$. □

[Deana] For part (e), suppose $\{t_n\}$ converges to t and f is in CB($[0,\infty)$). We need to show that the sequence $\{T(t_n)f\}$ is uniformly bounded and that it converges on compact subsets of $[0,\infty)$. Since f is bounded and $T(s)f(x)=f(s+x)$, it's clear that the sequence $\{T(t_n)f\}$ is uniformly bounded. Suppose K is a compact subset of $[0,\infty)$ and fix $\epsilon>0$. Say $K\subseteq[0,M_1]$. Since the sequence, $\{t_n\}$ converges, it's bounded, so there is an M_2 with $t_n\le M_2$ for all n. Set $M_3=M_1+M_2$. Since f is continuous, it's uniformly continuous on the compact set $[0,M_3]$. Choose $\delta>0$ so that if x and y are in $[0,M_3]$ and $|x-y|<\delta$, then $|f(x)-f(y)|\le\epsilon$. Finally, choose N so that $|t_n-t|<\delta$ when $n>N$. Notice that if y is in K, then $y-t$ and $y-t_n$ are in $[0,M_3]$ for all n. Therefore, for $n>N$ and y in K, $|(y+t_n)-(y+t)|=|t_n-t|<\delta$ and so $|T(t_n)f(y)-T(t)f(y)|=|f(y+t_n)-f(y+t)|<\epsilon$. Thus, $\{T(t_n)f\}$ converges uniformly to $T(t)f$ in K. This proves part (e).

For part (f), recall that the generator of the restriction of T to UCB($[0,\infty)$) is defined by $Af=f'$ when f and f' are in UCB($[0,\infty)$).

Claim 12.13 (JWN). *If f is in* UCB($[0,\infty)$) *and $\lambda>0$, then*

$$((I-\lambda A)^{-1}f)(t)=\frac{1}{\lambda}\int_0^\infty e^{-s/\lambda}f(s+t)\,ds$$

for t in $[0,\infty)$.

Proof. Notice that since f is bounded, the integral converges. Suppose $(I-\lambda A)^{-1}f=g$. Then $(I-\lambda A)g=f$ and so $g-\lambda g'=f$ and hence g is a solution to the first order, inhomogeneous differential equation $g'-\frac{1}{\lambda}g=-\frac{1}{\lambda}f$. An integrating factor for this equation is $e^{-t/\lambda}$ and it follows that

$$g(t)=e^{t/\lambda}\left(C-\frac{1}{\lambda}\int_0^t e^{-s/\lambda}f(s)\,ds\right)$$

for some constant, C. Since $e^{t/\lambda}$ approaches infinity as t approaches ∞, in order for g to be bounded it must be the case that

$$\lim_{t\to\infty}\left(C-\frac{1}{\lambda}\int_0^t e^{-s/\lambda}f(s)\,ds\right)=0,\ \text{so } C=\frac{1}{\lambda}\int_0^\infty e^{-s/\lambda}f(s)\,ds.$$

Therefore,

$$g(t)=e^{t/\lambda}\frac{1}{\lambda}\int_0^\infty e^{-s/\lambda}f(s)\,ds-e^{t/\lambda}\frac{1}{\lambda}\int_0^t e^{-s/\lambda}f(s)\,ds$$

$$
\begin{aligned}
&= e^{t/\lambda}\frac{1}{\lambda}\int_t^\infty e^{-s/\lambda} f(s)\,ds \\
&= \frac{1}{\lambda}\int_t^\infty e^{(t-s)/\lambda} f(s)\,ds \\
&= \frac{1}{\lambda}\int_0^\infty e^{-s/\lambda} f(s+t)\,ds.
\end{aligned}
$$

□

Comments.

Remark 12.14. In general, if A_T is the generator of a strongly continuous semigroup, then $I + \alpha A_T$ is not continuous for small α in $\mathbb{R}$. However, if T is linear, then there is a $\delta > 0$ so that if $0 \le \alpha \le \delta$, then $(I - \alpha A_T)^{-1}$ is continuous and globally defined, even though A_T may only be densely defined.

Remark 12.15. Notice that if f_1 and f_2 are functions in $\mathrm{CB}([0,\infty))$ and there is a real number, λ, so that $f_1(t) = f_2(t)$ for $t \ge \lambda$, then $g_{f_1}(t) = g_{f_2}(t)$ for $t \ge \lambda$, so the trajectories g_{f_1} and g_{f_2} are eventually equal. In particular, it's possible for distinct trajectories to intersect, so for a general semigroup of transformations, unlike the case of a group of transformations, the trajectories don't partition the space.

Remark 12.16. There's a much shorter proof of Proposition 12.7 that avoids the use of the Vitali Lemma, and instead uses the following theorem characterizing absolutely continuous functions on an interval.

Theorem 12.17. *Suppose $f\colon [a,b] \to \mathbb{R}$. Then f is absolutely continuous if and only if f' exists almost everywhere (with respect to Lebesgue measure) on (a,b), f' is in $L^1((a,b))$, and*

$$f(x) - f(a) = \int_a^x f' \quad \textit{for all } x \textit{ in } [a,b].$$

Proof. This is Theorem 7.29 on p. 118 in [WZ77]. □

So assume that $f\colon [0,\infty) \to \mathbb{R}$ is continuous, that $f_r'(x)$ exists for all x in $[0,\infty)$, and that f_r' is continuous. We'll show that f_r' satisfies the conclusion of the Mean Value Theorem.

So fix a closed interval, $[a,b]$. Since f_r' is continuous, it satisfies the hypothesis of the "Integral Mean Value Theorem," so there is a c in $[a,b]$ so that $\int_a^b f_r' = (b-a)f_r'(c)$. By Proposition 12.6 the restriction of f to $[a,b]$ satisfies a Lipschitz condition and hence is absolutely continuous. It follows from Theorem 12.17 that f' exists almost everywhere on $[a,b]$. Clearly $f' = f_r'$ when f' exists and so $\int_a^b f_r' = \int_a^b f'$. Again using Theorem 12.17 we have $\int_a^b f' = f(b) - f(a)$. Therefore, we've found a c in $[a,b]$ so that $f(b) - f(a) = (b-a)f_r'(c)$.

The proof of Proposition 12.7 can now be completed as follows. Fix c in $[0,\infty)$ and $\epsilon > 0$. Choose $\delta > 0$ so that $|f_r'(y) - f_r'(c)| < \epsilon$ whenever $|y - c| < \delta$. Then if a is in $(c-\delta, c)$, there is a number depending on a, say y_a, in $[a,c]$ so that $f(c) - f(a) = (c-a)f_r'(y_a)$. But then

$$\left| \frac{f(a) - f(c)}{a - c} - f_r'(c) \right| = |f_r'(y_a) - f_r'(c)| \le \epsilon$$

since $|y_a - c| < |a - c| \le \delta$.

Problem 13

Problem. *Suppose X is a Banach space, x is in X, $[a, b]$ is a closed interval, c is in $[a, b]$, $g\colon [a, b] \to X$ is continuous, and $A\colon [a, b] \to L(X)$ is continuous. Show that the initial value problem*

IVP:B $\qquad y\colon [a, b] \to X,\ y(c) = x,$ *and* $y'(t) = A(t)y(t) + g(t)$ *for all* t

has a unique solution.

Background. Notice first that the problem is a general existence and uniqueness theorem for *linear* systems of (possible infinitely many) ordinary differential equations. As long as the "coefficients" of the system (the function A) are continuous as a function of t in $[a, b]$, and the inhomogeneous terms (the function g) are continuous as a function of t in $[a, b]$, then given any possible initial condition (a number, c, in $[a, b]$ and an element, x, in X), the system has a unique solution satisfying the initial condition and this solution is defined on the entire interval $[a, b]$.

A very special case is when $X = M_n(\mathbb{R})$, then in terms of coordinates, the initial value problem IVP:B is equivalent to the system

$$\begin{aligned} y_1' &= a_{1,1}(t)y_1 + \cdots + a_{1,n}(t)y_n + g_1(t) \\ y_2' &= a_{2,1}(t)y_1 + \cdots + a_{2,n}(t)y_n + g_2(t) \\ &\ \vdots \\ y_n' &= a_{n,1}(t)y_1 + \cdots + a_{n,n}(t)y_n + g_(t) \end{aligned}$$

with initial conditions

$$y_1(c) = x_1, \quad y_2(c) = x_2, \quad \ldots, \quad y_n(c) = x_n,$$

where y_i, $a_{i,j}$, g_i, and x_i are the coordinates of y, A, g, and x respectively.

Since every solution to $y' = Ay + g$ satisfies some initial condition, it follows that if y_x denotes the solution with $y(c) = x$, then $x \mapsto y_x$ defines a bijection betwee X and the solution set of the equation $y' = Ay + g$. The inverse function is defined by $y \mapsto y(c)$. If $g = 0$, then these maps are vectorspace isomorphisms.

Definition 13.1. Suppose X is a Banach space and $f\colon [a, b] \to X$ is a function. If t is in (a, b) and the limit, $\lim_{h\to 0}(1/h)\,(f(t + h) - f(t))$, exists, then the limit is denoted by $f'(t)$ and f is said to be *differentiable at t*. Say f is *differentiable at a* if the limit $\lim_{h\to 0^+}(1/h)\,(f(a + h) - f(a))$ exists and f is *differentiable at b* if the limit $\lim_{h\to 0^-}(1/h)\,(f(b + h) - f(b))$ exists. If f is differentiable at t for all t in $[a, b]$, then say f is *differentiable.*

Proposition 13.2. *[Properties of the derivative.]*

(a) *If $f\colon [a, b] \to X$ is a constant function, then f is differentiable and $f'(t) = 0$ for all t in $[a, b]$.*
(b) *(Differentiation is linear.) If f and g are differentiable and r is in $\mathbb{R}$, then $rf + g$ is also differentiable and $(rf + g)' = rf' + g'$.*
(c) *(Product Rule) Suppose Y is a Banach space and $A\colon [a, b] \to L(X, Y)$ and $f\colon [a, b] \to X$ are differentiable. The Af, defined by $(Af)(t) = A(t)f(t)$ is differentiable map from $[0, \infty)$ to Y, and $(Af)' = Af' + A'f$.*

Proof. Parts (a), (b), and (c) follow from the definition using the standard arguments from Calculus. Part (d) follows using the standard argument (subtract and add $A(t)f(t+h)$ in the difference quotient) and the trivial fact that composition distributes over addition (that is, function spaces are always modules for spaces of operators). □

Two definitions of the integral of a function $f\colon [a,b] \to X$ when X is a Banach space will be given in the background for **Problem 16**.

Proposition 13.3. *Suppose $f\colon [a,b] \to X$ is a continuous function, c is in $[a,b]$, and $F\colon [a,b] \to X$ is defined by $F(t) = \int_c^t f(r)\,dr$. Then F is differentiable and $F'(t) = f(t)$ for t in $[a,b]$.*

Proof. This argument is taken from [Rud76]. Suppose $\epsilon > 0$. Since f is continuous, we can choose $\delta > 0$ so that $|f(t) - f(s)| < \epsilon$ when $|t - s| < \delta$. Notice that for $s_1 < s_2$ in $[a,b]$, $(1/(s_2 - s_1)) \int_{s_1}^{s_2} f(t)\,dr = f(t)$. Therefore, using either Theorem 16.3 or Proposition 16.9, if $s_1 < t < s_2$ and $s_2 - s_1 < \delta$ we have

$$\begin{aligned}\left|\frac{1}{s_2 - s_1}\int_{s_1}^{s_2} f(r)\,dr - f(t)\right| &= \left|\frac{1}{s_2 - s_1}\int_{s_1}^{s_2} (f(r) - f(t))\,dr\right| \\ &\le \frac{1}{s_2 - s_1}\int_{s_1}^{s_2} |f(r) - f(t)|\,dr \\ &\le \epsilon.\end{aligned}$$

It follows that $\left|(1/h)\left(\int_c^{t+h} f(r)\,dr - \int_c^t f(r)\,dr\right) - f(t)\right|$ approaches 0 as h approaches 0, so $F'(t) = f(t)$. □

Proposition 13.4. *If f and g are differentiable functions from $[a,b]$ to X and $f'(t) = g'(t)$ for all t, then there is an x in X so that $g(t) = f(t) + x$ for all t.*

Proof. Differentiation is linear and so $(f - g)'$ is a constant function. Now apply Proposition 13.2(a). □

Proposition 13.5. *If $\{f_n\}$ is a sequence of continuous functions from $[a,b]$ to X that converges uniformly to a function, f, then the sequence $\{\int_a^b f_n\}$ converges in X to $\int_a^b f$.*

Proof. Consider first the measure-theoretic definition of integration. If (Q,μ) is a measure space where Q is a compact, Hausdorff space, μ is a Borel measure, and $\mu(Q) < \infty$, then it's straightforward to show that for λ in X^*, the λf_n's converge uniformly to λf (since λ is continuous, given $\epsilon > 0$, choose $\delta > 0$ so that $|\lambda(x) - \lambda(y)| < \epsilon$ when $|x - y| < \delta$, then choose N so that $|f_n(t) - f(t)| < \delta$ for $n > N$ ...). It follows that the sequence, $\{\int_Q f_n\,d\mu\}$, converges to $\int_Q f\,d\mu$.

Now consider the case of an interval, $[a,b]$, and the Calculus definition of integration. Fix $\epsilon > 0$. Choose $\delta > 0$ so that $\left|\int_a^b f - \sum_{i=1}^m (t_i - t_{i-1}) f(t_i^*)\right| < \epsilon/3$ for all partitions, $(t_0, \ldots, t_m)$ of $[a,b]$ with $t_i - t_{i-1} < \delta$ and for all choices of t_i^* in $[t_i, t_{i-1}]$ for $1 \le i \le m$. Choose N so that $|f(t) - f_n(t)| < \epsilon/(3(b-a))$ for $n > N$. Then for $n > N$, a partition, $(t_0, \ldots, t_m)$ of $[a,b]$ with $t_i - t_{i-1} < \delta$, and any choice of t_i^* in $[t_i, t_{i-1}]$ for $1 \le i \le m$, we have

$$\begin{aligned}\left|\int_a^b f - \sum_{i=1}^m (t_i - t_{i-1}) f_n(t_i^*)\right| &\le \left|\int_a^b f - \sum_{i=1}^m (t_i - t_{i-1}) f(t_i^*)\right| + \left|\sum_{i=1}^m (t_i - t_{i-1})(f(t_i^*) - f_n(t_i^*))\right| \\ &\le \frac{\epsilon}{3} + (b-a)\frac{\epsilon}{3(b-a)}\end{aligned}$$

$$\leq \frac{2\epsilon}{3}.$$

For fixed n with $n > N$ we have $\left|\int_a^b f - \sum_{i=1}^m (t_i - t_{i-1}) f_n(t_i^*)\right| \leq 2\epsilon/3$ for every partition, $(t_0, \ldots, t_m)$ of $[a, b]$ with $t_i - t_{i-1} < \delta$ and every choice of t_i^* in $[t_i, t_{i-1}]$ for $1 \leq i \leq m$, so $\left|\int_a^b f - \int_a^b f_n\right| \leq \epsilon$ for $n > N$. It follows that the $\int_a^b f_n$'s converge to $\int_a^b f$. □

Remark 13.6. Notice that if a series, $\sum_n x_n$, in X is *absolutely convergent,* that is, the series of real numbers, $\sum_n |x_n|$ converges, then it follows from the triangle inequality that the partial sums of the series $\sum_n x_n$ form a Cauchy sequence in X and hence converge. Therefore, a series in X that converges absolutely must converge.

Also, if the series $\sum_n x_n$ is absolutely convergent, then $|\sum_{k=0}^n x_k| \leq \sum_{k=0}^m |x_k|$ for all n and so $|\sum_n x_n| \leq \sum_n |x_n|$.

Proposition 13.7. *If $\{f_n\}$ is a sequence of continuous functions from $[a, b]$ to X and there is a sequence of real numbers, $\{r_n\}$, so that $|f_n(t)| \leq r_n$ for all t in $[a, b]$, and so that the series $\sum_n r_n$ converges, then the series $\sum_n f_n$ converges uniformly on $[a, b]$.*

Proof. Since $|f_n(t)| \leq r_n$ for all t in $[a, b]$ and the series $\sum_n r_n$ converges, it follows from the preceding remark that the series $\sum_n f_n(t)$ converges, so the series $\sum_n f_n$ converges pointwise to a function, say $s\colon [a, b] \to X$.

Suppose $\epsilon > 0$. Choose N so that $\left|\sum_{k=n+1}^{\infty} r_k\right| < \epsilon$ for $n > N$. Then if $n > N$ we have

$$\left|s(t) - \sum_{k=0}^{n} f_k(t)\right| = \left|\sum_{k=n+1}^{\infty} f_k(t)\right| \leq \sum_{k=n+1}^{\infty} |f_k(t)| \leq \sum_{k=n+1}^{\infty} |r_k| \leq \epsilon$$

for all t in $[a, b]$, so $\sum_n f_n$ converges uniformly to s. □

Solution. The argument will be modeled on the argument in **Problem 7**, with some modifications to take into account the different domain and the slightly different initial condition.

So, define $y_0\colon [a, b] \to X$ by $y_0(t) = x$ for all t and for $n \geq 1$ define $y_n\colon [a, b] \to X$ and $f_n\colon [a, b] \to X$ by

$$y_n(t) = x + \int_c^t (A(r)y_{n-1}(r) - g(r))\, dr = x + \int_c^t (Ay_{n-1} + g) \text{ and } f_n = y_n - y_{n-1}.$$

Notice that $f_n = y_0 + \sum_{k=1}^n f_k$ for $n \geq 1$ and that $f_n(t) = \int_c^t A(r) f_{n-1}(r)\, dr$ for $n \geq 2$.

Since A and g are continuous, we can choose M_A and M_g so that $|A(t)| < M_A$ and $|g(t)| < M_g$ for all t in $[a, b]$. Then using Theorem 16.3 or Proposition 16.9 we have,

$$\begin{aligned} |f_1(t)| &= \left|\int_c^t Ay_0 + g\right| \\ &\leq \left|\int_c^t |Ay_0 + g|\right| \\ &\leq \left|\int_c^t (M_A |x| + M_g)\, dr\right| \\ &= (M_A |x| + M_g) |t - c|. \end{aligned}$$

Similarly,

$$\begin{aligned} |f_2(t)| &= \left| \int_c^t A f_1 \right| \\ &\le \left| \int_c^t |A(r)|\,|f_1(r)|\ dr \right| \\ &\le \left| \int_c^t (M_A(M_A\,|x| + M_g)\,|r-c|)\ dr \right| \\ &= M_A(M_A\,|x| + M_g)\frac{1}{2}\,|t-c|^2 . \end{aligned}$$

Assume that $|f_n(t)| \le M_A^{n-1}(M_A\,|x| + M_g)\frac{1}{n!}\,|t-c|^n$. Then

$$\begin{aligned} |f_{n+1}(t)| &= \left| \int_c^t A f_n \right| \\ &\le \left| \int_c^t |A(r)|\,|f_n(r)|\ dr \right| \\ &\le \left| \int_c^t \left(M_A M_A^{n-1}(M_A\,|x| + M_g)\frac{1}{n!}\,|r-c|^n \right)\ dr \right| \\ &= M_A^n(M_A\,|x| + M_g)\frac{1}{(n+1)!}\,|t-c|^{n+1} . \end{aligned}$$

It follows that $|f_n(t)| \le M_A^{n-1}(M_A\,|x| + M_g)\frac{1}{n!}\,|t-c|^n$ for $n \ge 2$. Therefore $|f_n(t)| \le M_A^{n-1}(M_A\,|x| + M_g)\frac{1}{n!}(b-a)^n$ for $n \ge 2$. Hence, it follows from Proposition 13.7 that the series $\sum_{n=1}^{\infty} f_n$ converges uniformly on $[a,b]$ to a function $f\colon [a,b] \to X$. Since $y_n = y_0 + \sum_{k=1}^{n} f_k$, it follows that the sequence of y_n's converges uniformly on $[a,b]$ to a function $y\colon [a,b] \to X$.

Claim 13.8. *$y'(t) = A(t)y(t)$ for all t in $[a,b]$.*

Proof. Since the y_n's converge uniformly to y, using Proposition 13.5, we have

$$y(t) = \lim_{n\to\infty} \left(x + \int_c^t (Ay_{n-1} + g) \right) = x + \int_c^t (Ay + g).$$

Therefore, it follows from the "Fundamental theorem of Calculus" (Proposition 13.3) that $y'(t) = A(t)y(t)$ for all t in $[a,b]$. □

Since $y_n(c) = x$ for all n, it follows that $y(c) = x$ and hence that y is indeed a solution to IVP:B.

To prove that y is the unique solution, suppose that $w\colon [a,b] \to X$ is also a solution.

Claim 13.9. *$w(t) = x + \int_c^t (Aw + g)$ for all t in $[a,b]$.*

Proof. By assumption, $w' = Aw + g$. It follows from the "Fundamental Theorem of Calculus" (Proposition 13.3) that if $w_1(t) = \int_c^t (Aw + g)$, then $w_1'(t) = w'(t)$ for all t and so $w(t) = x_0 + w_1(t)$ for some x_0 in X by Proposition 13.4. Since $w(c) = x$, it follows that $x_0 = x$ and so $w(t) = x + \int_c^t (Aw + g)$. □

It follows from the claim that $y(t) - w(t) = \int_c^t A(r)(y(r) - w(r))\,dr$ for t in $[a,b]$. Set

$$M = \max\{\, |y(t) - w(t)| \mid t \in [a,b] \,\}.$$

Then

$$
\begin{aligned}
|y(t)-w(t)| &= \left|\int_c^t A(y-w)\right| \\
&\le \left|\int_c^t M_A M\, dr\right| \\
&= M_A M\,|t-c|\,.
\end{aligned}
$$

Assume that $|y(t)-w(t)| \le M_A^n M \frac{1}{n!}|t-c|^n$ for all t in $[a,b]$. Then

$$
\begin{aligned}
|y(t)-w(t)| &= \left|\int_c^t A(r)(y(r)-w(r))\, dr\right| \\
&\le \left|\int_c^t M_A M_A^n M \frac{1}{n!}|r-c|^n\, dr\right| \\
&= M_A^{n+1} M \frac{1}{(n+1)!}|t-c|^{n+1}\,.
\end{aligned}
$$

It follows that $|y(t)-w(t)| \le M_A^n M \frac{1}{n!}|t-c|^n$ for all t in $[a,b]$. Since the right hand side of the previous inequality approaches 0 as n approaches infinity, it follows that $y(t)=w(t)$ for all t in $[a,b]$ and so $w=y$.

Problem 14

Problem. *Suppose $g\colon [0,1]\to\mathbb{R}$ is a continuous function.*

(a) *Find all $f\colon [0,1]\to\mathbb{R}$ with $-f''=g$ and $f(0)=0=f(1)$.*
(b) *Find all real numbers, λ, for which there is a non-zero function, $f\colon [0,1]\to\mathbb{R}$ satisfying*

$$
-f''=\lambda f \quad \textit{and} \quad f(0)=0=f(1).
$$

Solution. [Mario] By the Fundamental Theorem of Calculus (twice) there is a function, G, so that $G''=g$. Then $f(x)=-G(x)+C_1x+C_2$ for some constants C_1 and C_2. Using that $f(0)=f(1)=0$ gives: $0=-G(0)+C_2$ and $0=-G(1)+C_1+C_2$. Therefore, $C_2=G(0)$ and $C_1=G(1)-G(0)$. Thus, $f(x)=G(x)+(G(1)-G(0))x+G(0)$ where G is arbitrary with $G''=g$.

If G_1 is another function with $G_1''=g$, then $f(x)=G_1(x)+(G_1(1)-G_1(0))x+G_1(0)$ and $G_1(x)=G(x)+D$. Clearly $G_1(1)-G_1(0)=G(1)-G(0)$ and so $G_1(x)+(G_1(1)-G_1(0))x+G_1(0)=G(x)+(G(1)-G_0))x+G(0)$. It follows that the boundary value problem in (a) has a unique solution.

To prove (b), assume that $f(t)=e^{\gamma t}$. Then f is a solution if and only if $\gamma=\pm\sqrt{-\lambda}$. There are three cases: $\lambda=0$, $\lambda>0$ and $\lambda<0$.

If $\lambda=0$, then $f(x)=c_1+c_2x$ for some constants c_1 and c_2. To satisfy the initial conditions, $c_1=c_2=0$, so f is identically 0.

If $\lambda<0$, then $f(x)=c_1e^{\sqrt{-\lambda}x}+c_2e^{\sqrt{-\lambda}x}$ for some constants c_1 and c_2. To satisfy the initial conditions, $c_1=c_2=0$, so f is identically 0.

If $\lambda>0$, then $f(x)=c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x$ for some constants c_1 and c_2. The initial condition $f(0)=0$ forces $c_1=0$ and then the initial condition $f(1)=0$ forces either $c_2=0$ or $\sin\sqrt{\lambda}=0$. If f is not identically 0, then it must be that $\sqrt{\lambda}=n\pi$ for some positive integer, n, and then $f(x)=c_2\sin n\pi x$.

Comments.

Remark 14.1. An argument that shows that the preceding argument for part (b) actually produces all possible solutions is as follows.

Take $X = \mathbb{R}^2$, $A(t) = A = \begin{bmatrix} 0 & 1 \\ -\lambda & 0 \end{bmatrix}$, and $g(t) = 0$ in **Problem 13**. Then given x in $\mathbb{R}^2$, there is a unique function, depending on x, say $y_x\colon [0,\infty) \to \mathbb{R}^2$, so that $y_x(0) = x$ and $y_x'(t) = Ay_x(t)$ for all t in $[0,\infty)$. It follows that the mapping $x \mapsto y_x$ is a bijection between $\mathbb{R}^2$ and the set of all solutions to the equation $y' = Ay$. The inverse mapping is $y \mapsto y(0)$ and it's easy to see that the mappings are in fact linear transformations.

It's also straightforward to check, using the differentiation formulas in Proposition 13.2, that for x in $\mathbb{R}^2$, the function whose rule is $t \mapsto e^{tA}x$ is a solution to $y' = Ay$ whose value at $t = 0$ is x Therefore, $y_x(t) = e^{tA}x$ for t in $[0,\infty)$.

Notice that if $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ satisfies $y' = Ay$, then $y_1' = y_2$ and $y_2' = -\lambda y_1$, so $y_1'' = -\lambda y_1$. It's easy to check that the mapping $f \mapsto y_f = \begin{bmatrix} f \\ f' \end{bmatrix}$ defines a bijection between the set of all solutions to the second order equation $f'' = -\lambda f$ and the set of all solutions to $y' = Ay$.

Combining the arguments in the preceding paragraphs, it follows that if f satisfies $f'' = -\lambda f$, then $f(t)$ is the first component of the vector valued function whose rule is $e^{tA}x$ for some unique x in $\mathbb{R}^2$ and that the correspondence $f \mapsto x$ is bijective.

Computing the first component of the vector valued function whose rule is $e^{tA}x$ is a straightforward computation. Set $D = \begin{bmatrix} i\sqrt{\lambda} & 0 \\ 0 & -i\sqrt{\lambda} \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ i\sqrt{\lambda} & -i\sqrt{\lambda} \end{bmatrix}$, and $x = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, where i is the complex number "i." Then

$$\begin{aligned} e^{tA}x &= e^{tPDP^{-1}}x \\ &= Pe^{tD}P^{-1}x \\ &= P\begin{bmatrix} e^{it\sqrt{\lambda}} & 0 \\ 0 & e^{-it\sqrt{\lambda}} \end{bmatrix} P^{-1}x \\ &= \begin{bmatrix} c_1 \cos t\sqrt{\lambda} + (c_2/\sqrt{\lambda}) \sin t\sqrt{\lambda} \\ -c_1\sqrt{\lambda} \sin t\sqrt{\lambda} + c_2 \cos t\sqrt{\lambda} \end{bmatrix}. \end{aligned}$$

Therefore, f satisfies $f'' = -\lambda f$ if and only if $f(t) = c_1 \cos t\sqrt{\lambda} + (c_2/\sqrt{\lambda}) \sin t\sqrt{\lambda}$ for some (unique) pair of real numbers c_1 and c_2.

Finally, an argument similar to that given above, shows that if $f(t) = c_1 \cos t\sqrt{\lambda} + (c_2/\sqrt{\lambda}) \sin t\sqrt{\lambda}$ satisfies the boundary conditions $f(0) = 0 = f(1)$, then $c_1 = 0$ and either $c_2 = 0$ or $\lambda = n^2\pi^2$ for some integer n.

It follows that the set of real numbers satisfying the conditions in part (b) of the problem is $\{\, n^2\pi^2 \mid n \in \mathbb{Z}\,\}$. If $\lambda = n^2\pi^2$, $-f'' = \lambda f$, and $f(0) = 0 = f(1)$, then $f(t) = c \sin n\pi t$ for some real number c.

Remark 14.2 (JWN)**.** The expression $f(x) = G(x) + (G(1) - G(0))x + G(0)$ can be rewritten without reference to G as follows. First, G can be defined by the formula $G(x) = \int_0^x \int_0^s g(r)\,dr\,ds$, Then if h satisfies $h'(s) = 1$ for all s, using integration by parts we have

$$\begin{aligned} G(x) &= \int_0^x h'(s) \left(\int_0^s g(r)\,dr \right) ds \\ &= h(s) \int_0^s g(r)\,dr \Big|_{s=0}^{s=x} - \int_0^x g(s)h(s)\,ds \\ &= -\int_0^x (s - x)g(s)\,ds \end{aligned}$$

if we take $h(s) = s - x$. (Since g is continuous, this formula for G also follows immediately using Fubini's Theorem.)

Now taking $G(x) = -\int_0^x (s - x)g(s)\,ds$ in the formula for f we get

$$\begin{aligned} f(x) &= \int_0^x (s - x)g(s)\,ds + x \int_0^1 (1 - s)g(s)\,ds \\ &= \int_0^x (s - x)g(s)\,ds + x \int_0^x (1 - s)g(s)\,ds + x \int_x^1 (1 - s)g(s)\,ds \\ &= \int_0^x (s - x + x - xs)g(s)\,ds + \int_x^1 (x - xs)g(s)\,ds \\ &= \int_0^x (s - xs)g(s)\,ds + \int_x^1 (x - xs)g(s)\,ds. \end{aligned}$$

Define $k\colon [0,1] \times [0,1] \to \mathbb{R}$ by

$$k(x, s) = \begin{cases} s(1 - x) & 0 \le s \le x \\ x(1 - s) & x \le s \le 1, \end{cases}$$

then $f(x) = \int_0^1 k(x, s)g(s)\,ds$.

We can conclude that for g in $\mathrm{C}([0, 1])$, the boundary value problem

(BVP:[0, 1]) $$-y'' = g, \quad y(0) = 0 = y(1)$$

has a unique solution and that this solution is given by $y(x) = \int_0^1 k(x, s)g(s)\,ds$.

Notice that

- k is continuous,
- k is symmetric: $k(x, s) = k(s, x)$,
- $k(x, \cdot)$ is a solution to the homogeneous boundary value problem: $-y'' = 0$, $y(0) = 0 = y(1)$, and
- $k_s(x, s) = \begin{cases} 1 - x & 0 \le x < s \\ -x & s < x \le 1 \end{cases}$, so $k_s(x, x-) - k_s(x, x+) = 1$, where $k_s(x, x-) = \lim_{s \to x-} k(x, s)$ and $k_s(x, x+) = \lim_{s \to x+} k(x, s)$.

It can be shown that these conditions uniquely determine k. The function, k, is a *kernel* or *Green's function.*

Problem 15

Problem. *Prove* Schwarz's inequality: *if H is an inner product space, then $\langle x, y\rangle \leq |x|^2 |y|^2$ for all x and y in H.*

Solution. [Deana] Suppose first that $|x| = |y| = 1$. Then

$$|x-y|^2 = \langle x-y, x-y\rangle = |x|^2 - 2\langle x, y\rangle + |y|^2 \geq 0$$

so $\langle x, y\rangle \leq 1$. Also,

$$|x+y|^2 = \langle x+y, x+y\rangle = |x|^2 + 2\langle x, y\rangle + |y|^2 \geq 0$$

so $\langle x, y\rangle \geq -1$. Therefore, $0 \leq |\langle x, y\rangle| \leq 1 = |x|\,|y|$ and squaring both sides gives the result.

Next suppose that $x = 0$ or $y = 0$, then the result is obvious.

Finally, assume that neither x nor y is 0. Then by the first case above, $\langle x/|x|, y/|y|\rangle^2 \leq 1$ and this implies the result.

Comments. Notice that if $x \neq 0$ and $y \neq 0$, then $\dfrac{\langle x, y\rangle}{|x|\,|y|}$ may be interpreted as $\cos\theta$ where θ is the *angle* between x and y. In other words, Scwartz's inequality can be used to *define* the notion of the angle between two "vectors" in an arbitrary inner product space.

Notice also that if x and y are random variables, then $\langle x/|x|, y/|y|\rangle$ is the correlation between x and y.

Remark 15.1. The preceding argument applies when H is a normed linear space, but in fact, as noted in the Background for **Problem 10**, the axioms for an inner product space alone (see the Background for **Problem 10**) are enough to prove Schwarz's inequality, as well as the fact that the "norm" induced by an inner product is indeed a norm in the sense defined in the Background for **Problem 2**.

Problem 16

Problem. *Suppose X is a Banach space and T is a strongly continuous linear semigroup on X. Recall that for x in X, g_x denotes the trajectory of x, so $g_x\colon [0,\infty) \to X$. Show that*

$$(T(t) - I)\int_0^s g_x = (T(s) - I)\int_0^t g_x$$

for every x in X and s and t in $[0,\infty)$.

Background. Recall that the integral $\int_a^b f$, where $f\colon [a,b] \to X$ is a continuous function and X is a Banach space has two equivalent definitions. We'll give the measure-theoretic one first and then a Calculus definition.

Suppose Q is a topological space and μ is a finite, real-valued, Borel measure on Q. For example, μ could be a probability measure on Q. If $f\colon Q \to X$ is continuous, then Tf is continuous for all T in X^* and so the mapping $T \mapsto \int_Q Tf\,d\mu$ defines a linear transformation from X^* to $\mathbb{R}$. Recall that if x is in X, then x defines an element in X^{**} by evaluation. Precisely, define $\hat{x}\colon X^{**} \to \mathbb{R}$ by $\hat{x}(T) = T(x)$. It's easy to see that the mapping $x \mapsto \hat{x}$ defines an injective linear transformation from X to X^{**}, this is the *canonical embedding of X in X^{**}.*

Definition 16.1. If the mapping $T \mapsto \int_Q Tf\,d\mu$ is an element in X^{**} that lies in the image of X under the canonical embedding of X in X^{**}, then the mapping is given by evaluation at x for some unique x in X. This element will be denoted by $\int_Q f\,d\mu$. Thus, if it exists, $\int_Q f\,d\mu = x$ is the unique element in X with the property that $T(x) = \int_Q Tf\,d\mu$ for all T in X^*.

Notice that if $\int_Q f\,d\mu$ exists, then by definition we have $T\left(\int_Q f\,d\mu\right) = \int_Q Tf\,d\mu$.

If $f\colon Q \to X$ is continuous, then $q \mapsto |f(q)|$ defines a continuous function from Q to $\mathbb{R}$ also denoted by $|f|$, and so we may consider $\int_Q |f|\,d\mu$.

Proposition 16.2. *Suppose $c_x\colon Q \to X$ is the constant function with $c_x(q) = x$ for all q in Q. Then c_x is integrable and $\int_Q c_x\,d\mu = \mu(Q)x$.*

Proof. If T is in X^*, then $Tc_x(q) = T(x)$ is a constant function, so $\int_Q Tc_x\,d\mu = \mu(Q)T(x) = T(\mu(Q)x)$, so $\int_Q c_x\,d\mu = T(x)$. □

Theorem 16.3. *If Q is a compact Hausdorff space and X is a Banach space, then $\int_Q f\,d\mu$ exists for every continuous function $f\colon Q \to X$. Moreover,*

$$\left|\int_Q f\,d\mu\right| \leq \int_Q |f|\,d\mu.$$

Proof. This follows from Theorems 3.27 and 3.29 in [Rud73], since if X is a Banach space, then X^* separates points by the Corollary to Theorem 3.4 in [Rud73], and the convex hull of $f(Q)$ has compact closure by Theorem 3.25 in [Rud73], which applies since Banach spaces are Fréchet spaces. □

Claim 16.4. *Suppose Q is a compact Hausdorff space, μ is a Borel measure on Q, and $g\colon [a,b] \to X$ is a continuous function. If Y is a Banach space and A is in $L(X,Y)$, then $A\left(\int_Q g\,d\mu\right) = \int_Q A \circ g\,d\mu$.*

Proof. It follows from Theorem 16.3 and the definitions, that $\int_Q Ag\,d\mu$ is the unique element in Y with the property that $S\left(\int_Q Ag\,d\mu\right) = \int_Q SAg\,d\mu$ for all S in Y^*. So suppose S is in Y^* and consider $S\left(A\left(\int_Q g\,d\mu\right)\right)$. Now SA is in X^*, so $SA\left(\int_Q g\,d\mu\right) = \int_Q SAg\,du$ by the defining property of $\int_Q g\,d\mu$. Therefore, $S\left(A\left(\int_Q g\,d\mu\right)\right) = \int_Q SAg\,du$ for all S in Y^*, so $A\left(\int_Q g\,d\mu\right) = \int_Q A \circ g\,d\mu$. □

The second approach to defining $\int_a^b f$ is the obvious generalization of the usual Riemann integral of a real-valued function defined on $[a,b]$. Define $f\colon [a,b] \to X$ to be *integrable* if there is an x in X so that for every $\epsilon > 0$, there is a $\delta > 0$ so that if $(t_0, \ldots, t_n)$ is a partition of $[a,b]$ with $t_i - t_{i-1} \leq \delta$ for $1 \leq i \leq n$, then $|x - \sum_{i=1}^n (t_i - t_{i-1})f(t_i^*)| \leq \epsilon$ for every choice of $(t_1^*, \ldots, t_n^*)$ with t_i^* in $[t_{i-1}, t_i]$ for $1 \leq i \leq n$.

Proposition 16.5. *If f is integrable, then there is at most one x in X with the property that for every $\epsilon > 0$, there is a $\delta > 0$ so that if $(t_0, \ldots, t_n)$ is a partition of $[a,b]$ with $t_i - t_{i-1} \leq \delta$ for $1 \leq i \leq n$, then $|x - \sum_{i=1}^n (t_i - t_{i-1})f(t_i^*)| \leq \epsilon$ for every choice of $(t_1^*, \ldots, t_n^*)$ with t_i^* in $[t_{i-1}, t_i]$ for $1 \leq i \leq n$.*

Proof. Suppose x_1 and x_2 satisfy the condition. Fix $\epsilon > 0$ and choose δ_1 and δ_2 so that if $(t_0, \ldots, t_n)$ is a partition of $[a, b]$ with $t_i - t_{i-1} \leq \min\{\delta_1, \delta_2\}$, then

$$\left| x_1 - \sum_{i=1}^{n} (t_i - t_{i-1}) f(t_i^*) \right| \leq \epsilon/2 \text{ and } \left| x_2 - \sum_{i=1}^{n} (t_i - t_{i-1}) f(t_i^*) \right| \leq \epsilon/2$$

for every choice of $t_1^*, \ldots, t_n^*$ with t_i^* in $[t_{i-1}, t_i]$ for $1 \leq i \leq n$. Then $|x_1 - x_2| \leq \epsilon$. Since ϵ was arbitrary, $x_1 = x_2$. □

Proposition 16.6. *Suppose $c_x \colon [a, b] \to X$ is the constant function with $c_x(t) = x$ for all t in $[a, b]$. Then c_x is integrable and $\int_a^b c_x = (b - a)x$.*

Proof. This follows easily from the fact that $\sum_{i=1}^{n} (t_i - t_{i-1}) c_x(t_i^*) = (b - a)x$ for all choices of $(t_0, \ldots, t_n)$ and $t_1^*, \ldots, t_n^*$. □

Proposition 16.7. *If $f \colon [a, b] \to X$ and $g \colon [a, b] \to X$ are integrable and r is a real number, then $rf + g$ is integrable and $\int_a^b (rf + g) = r \int_a^b f + \int_a^b g$.*

Proof. Fix $\epsilon > 0$ and choose δ_1 and δ_2 so that if $(t_0, \ldots, t_n)$ is a partition of $[a, b]$ with $t_i - t_{i-1} \leq \delta_1$ and $(t_0', \ldots, t_m')$ is a partition of $[a, b]$ with $t_i' - t_{i-1}' \leq \delta_2$, then

$$\left| \int_a^b f - \sum_{i=1}^{n} (t_i - t_{i-1}) f(s_i) \right| \leq \epsilon/2r \text{ and } \left| \int_a^b g - \sum_{i=1}^{m} (t_i' - t_{i-1'}) g(s_i') \right| \leq \epsilon/2$$

for every choice of $s_1, \ldots, s_n$ with s_i in $[t_{i-1}, t_i]$ for $1 \leq i \leq n$ and for every choice of $s_1', \ldots, s_m'$ with s_i' in $[t_{i-1}', t_i']$ for $1 \leq i \leq m$.

Set $\delta = \min\{\delta_1, \delta_2\}$. Then if $(t_0, \ldots, t_n)$ is a partition of $[a, b]$ and t_i^* is in $[t_{i-1}, t_i]$ for $1 \leq i \leq n$, then

$$\left| r \int_a^b f + \int_a^b g - \sum_{i=1}^{n} (t_i - t_{i-1})(rf(t_i^*) + g(t_i^*)) \right| \leq (r\epsilon/2r) + (\epsilon/r) = \epsilon.$$

It follows that $rf + g$ is integrable and that $\int_a^b (rf + g) = r \int_a^b f + \int_a^b g$. □

Proposition 16.8. *If $f \colon [a, b] \to X$ is integrable, Y is a Banach space, and A is in $L(X, Y)$, then Af is integrable and $\int_a^b Af = A\left(\int_a^b f\right)$. In particular, taking $A = \mathbb{R}$, it follows that $\int_a^b f = \int_{[a,b]} f \, d\lambda$ where μ is Lebesgue measure in $[a, b]$*

Proof. Use that

$$A\left(\sum_{i=1}^{n} (t_i - t_{i-1}) f(s_i) \right) = \sum_{i=1}^{n} (t_i - t_{i-1}) Af(s_i)$$

whenever $(t_0, \ldots, t_n)$ is a partition of $[a, b]$ and t_i^* is in $[t_{i-1}, t_i]$ for $1 \leq i \leq n$ □

Proposition 16.9. *If $f \colon [a, b] \to X$ is continuous, then f and $|f|$ are integrable and $\left| \int_a^b f \right| \leq \int_a^b |f|$.*

Sketch of proof. For $n \geq 1$, set $d_n = (b - a)/n$ and define a step function, $f_n \colon [a, b] \to X$, by $f_n(a) = f(a)$ and $f_n(t) = f(a + (k + \frac{1}{2}) d_n$ for t in $(a + (k - 1)d_n, a + kd_n]$, $1 \leq k \leq n$. It follows from Propositions 16.6 and 16.7 that each f_n is integrable and that $\int_a^b f_n = \sum_{k=1}^{n} d_n f(a + (k + \frac{1}{2}) d_n)$.

Now show that the sequence $\{\int_a^b f_n\}$ is a Cauchy sequence using the continuity of f and let x denote the limit. Then show that x satisfies the defining property of $\int_a^b f$, so f is integrable and $x = \int_a^b f$.

Since f is continuous, so is $|f|$, and hence $|f|$ is integrable.

To show that $\left|\int_a^b f\right| \leq \int_a^b |f|$, use the triangle inequality in X: $|\sum_{i=0}^n (t_i - t_{i-1}) f(t_i^*)| \leq \sum_{i=0}^n (t_i - t_{i-1}) |f(t_i^*)|$. ☐

Proposition 16.10. *Suppose $f\colon [a,c] \to X$ is integrable. Then*

(a) *for b in $[a,c]$, the restrictions of f to $[a,b]$ and $[b,c]$ are integrable and $\int_a^c f = \int_a^b f + \int_b^c f$, and*
(b) *if r_1, r_2, $s+r_1$, and $s+r^2$ are all in $[a,c]$ and $r_1 \leq r_2$, then $\int_{r_1}^{r_2} f \circ \ell_s = \int_{s+r_1}^{s+r_2} f$ (recall that $\ell_s\colon \mathbb{R} \to \mathbb{R}$ by $\ell_s(t) = s+t$).*

Solution. [Ahmed] The argument proceeds exactly as in Claim 2.12, using Propositions 16.7, 16.8, and 16.10.

Problem 17

Problem. *Suppose T is a strongly continuous semigroup on a Banach space, X, with generator A_T.*

(a) *Show that if T is linear, then $(1/t)\int_0^t g_x$ is in D_T for all x in X and t in $[0,\infty)$.*
(b) *Show that $\lim_{t\to 0}(1/t)\int_0^t g_x = x$ for every x in X.*

Conclude that if T is linear, then D_T is dense in X.

Solution. [Ioana] We'll prove (b) first. Fix x in X. We'll show that $\lim_{t\to 0}\left|\left(\frac{1}{t}\int_0^t g_x\right) - x\right| = 0$. So suppose $\epsilon > 0$. Since T is strongly continuous, g_x is continuous, and since $g_x(0) = x$ we can choose $\delta > 0$ so that if $0 \leq t < \delta$, then $|g_x(t) - x| = |T(t)x - x| < \epsilon$. Notice that with the notation of Proposition 16.6 we have $x = \frac{1}{t}\int_0^t c_x$. Therefore, using Propositions 16.7 and 16.9 it follows that

$$\begin{aligned}\left|\frac{1}{t}\int_0^t g_x - x\right| &= \left|\frac{1}{t}\int_0^t g_x - \int_0^t c_x\right| \\ &= \left|\frac{1}{t}\int_0^t (g_x - c_x)\right| \\ &\leq \frac{1}{t}\int_0^t |g_x - c_x| \\ &\leq \frac{1}{t}\int_0^t \epsilon \\ &= \epsilon.\end{aligned}$$

This proves part (b).

To prove (a), recall first that since T is linear, we have

$$(T(t) - I)\int_0^s g_x = (T(s) - I)\int_0^t g_x$$

for x in X by **Problem 16**. Next, by the discussion in Remark 17.1 we have $T(t)g_x - g_x = g_{(T(t)-I)x}$. Finally, using (b) for the last equality, it follows that

$$\lim_{s\to 0^+}\left(\frac{1}{s}\right)\left((T(s) - I)\left(\frac{1}{t}\int_0^t g_x\right)\right) = \frac{1}{t}\lim_{s\to 0^+}\frac{1}{s}(T(t) - I)\left(\int_0^s g_x\right)$$

$$= \frac{1}{t} \lim_{s\to 0+} \frac{1}{s} \int_0^s (T(t)g_x - g_x)$$
$$= \frac{1}{t} \lim_{s\to 0+} \frac{1}{s} \int_0^s g_{(T(t)-I)x}$$
$$= \frac{1}{t}(T(t) - I)x.$$

Therefore $(1/t)\int_0^t g_x$ is in D_T for all t and x and moreover $A_T\left((1/t)\int_0^t g_x\right) = \frac{1}{t}(T(t) - I)x$.

When T is linear, given x in X we have $x = \lim_{t\to 0}(1/t)\int_0^t g_x$ and $(1/t)\int_0^t g_x$ is in D_T for every t in $[0, \infty)$, so x is in the closure of D_T. Therefore, D_T is dense in X.

Comments.

Remark 17.1. If x and y are in X then it's easy to see that $g_x + g_y = g_{x+y}$. If $A\colon X \to X$ is a continuous function that centralizes the image of T (that is, $AT(t) = T(t)A$ for all t in $[0, \infty)$), then it's easy to see that $Ag_x = g_{Ax}$ for x in X. In particular, taking A to be multiplication by a real number, r, we get $rg_x = g_{rx}$ and taking A to be $T(t)$ we get $T(t)g_x = g_{T(t)x}$.

Remark 17.2. A more intuitive, and slightly shorter proof of part (b), using basically the same ideas is as follows.

Notice that $x = g_x(0) = (1/t)\int_0^t g_x(0)\,dr$ for x in X and $t > 0$. Thus

$$\left|\frac{1}{t}\int_0^t g_x(r)\,dr - x\right| = \left|\frac{1}{t}\int_0^t (g_x(r) - g_x(0))\,dr\right| \le \frac{1}{t}\int_0^t |g_x(r) - g_x(0)|\,dr.$$

The assumption that T is strongly continuous means that g_x is continuous for all x. Given $\epsilon > 0$, choose $\delta > 0$ so that $g_x(r) - g_x(0) \le \epsilon$ when $t \le \delta$. Then clearly $\frac{1}{t}\int_0^t |g_x(r) - g_x(0)|\,dr \le \epsilon$ when $t \le \delta$. It follows that $\left|\frac{1}{t}\int_0^t g_x(r)\,dr - x\right|$ approaches 0 as t approaches 0.

Remark 17.3. Suppose T is a strongly continuous, linear semigroup. Then part (a) says that for x in X, even though x itself may not be in D_T, when x is averaged over any interval, $(1/t)\int_0^t T(r)x\,dr$, the average is in D_T and

$$A_T\left((1/t)\int_0^t g_x(r)\,dr\right) = \frac{1}{t}(T(t) - I)x = \frac{g_x(t) - g_x(0)}{t - 0}.$$

We'll see later, in **Problem 20**, that for $\lambda > 0$, the "average" $(1/\lambda)\int_0^\infty e^{-s/\lambda}g_x(r)\,dr$ is in D_T for all x in X and we'll find a formula for the image under A of this average.

Part (a) says that averaging x using the uniform distribution gives elements in D_T and **Problem 20** will show that averaging x using the exponential distribution gives elements in D_T. In general, it can be shown that averaging x using any "reasonable" distribution gives elements in D_T.

Remark 17.4. As in the previous remark, suppose that T is a strongly continuous, linear semigroup on a Banach space, X. Then for x in X, the trajetory, g_x, is continuous, and x is in D_T if and only if

$$\lim_{t\to 0+} \frac{1}{t}(T(t) - I)\,x = \lim_{t\to 0+} \frac{g_x(t) - g_x(0)}{t - 0}$$

exists. In other words, D_T is the set of x in X with the property that g_x is right differentiable at 0. We'll see below that the existence of a right derivative at 0 for g_x is enough to guarantee that g_x is a differentiable on all of $[0, \infty)$. Therefore, if T is a strongly continuous semigroup, then D_T can be characterized as the set of all x in X with the property that the trajectory, g_x, is differentiable.

We'll need a boundedness property of strongly continuous, linear semigroups.

Proposition 17.5. *Suppose that T is a strongly continuous, linear semigroup on a Banach space, X. Then $\{\,|T(t)| \mid t \in [a,b]\,\}$ is bounded for every interval, $[a,b]$, contained in $[0,\infty)$.*

Proof. Fix a closed interval $[a,b]$ in $[0,\infty)$ and set $\Gamma = \{\,T(t) \mid t \in [a,b]\,\}$, so Γ is a subset of $L(X)$. For x in X, let Γ_x denote the image of the interval $[a,b]$ under the trajectory g_x, so $\Gamma_x = \{\,g_x(t) \mid t \in [a,b]\,\}$. Since the trajectory g_x is continuous and the interval $[a,b]$ is compact, it follows that Γ_x is compact, and hence bounded. Therefore, by Theorem 2.6 in [Rud73], the set, Γ, is equicontinuous and hence by Theorem 2.4 in [Rud73], if E is any bounded subset of X, there is a bounded subset, F, of X so that $T(t)E \subseteq F$ for all $T(t)$ in Γ. Take E to be the closed unit ball in X and choose M so that $|y| \le M$ for all y in the corresponding set F. Then if t is in $[a,b]$, we have $|T(t)x| \le M$ whenever $|x| \le 1$, so $|T(t)| \le M$. □

Proposition 17.6. *Suppose that T is a strongly continuous, linear semigroup on a Banach space, X. Then for x in D_T, the trajectory, g_x, is a differentiable function from $[0,\infty)$ to X and for t in $[0,\infty)$, $g_x'(t) = T(t)A_T(x)$.*

Proof. (Ahmed). Fix x in D_T. Then $\lim_{t\to 0^+}(1/t)(T(t)-I)x = \lim_{t\to 0^+}(1/t)(g_x(t)-g_x(0))$ exists and is equal $A_T(x)$, so g_x is differentiable from the right at 0 and $g_x'(0) = A_T(x) = T(0)A_T(x)$. Suppose $t > 0$. Then

$$\begin{aligned}
\lim_{h\to 0^+} \frac{g_x(t+h)-g_x(t)}{h} &= \lim_{h\to 0^+} \frac{T(t+h)x - T(t)x}{h} \\
&= \lim_{h\to 0^+} T(t)\left(\frac{T(h)x - x}{h}\right) \\
&= T(t)\left(\lim_{h\to 0^+} \frac{T(h)x - x}{h}\right) \\
&= T(t)A_T(x).
\end{aligned}$$

To complete the proof, it suffices to show that $\lim_{h\to 0^-}(1/h)(g_x(t+h)-g_x(t)) = T(t)A_T(x)$, or equivalently, that $(1/h)(g_x(t+h)-g_x(t)) - T(t)A_T(x)$ approaches 0 as h increases 0.

Consider h's with $-t \le h < 0$, so $-h > 0$ and $t+h \ge 0$. Using Proposition 17.5, we can choose an M so that $|T(t+h)| < M$ for h in $[-t, 0]$. Then

$$\begin{aligned}
\left|\frac{g_x(t+h)-g_x(t)}{h} - T(t)A_T(x)\right| &= \left|\frac{T(t+h)x - T(t)x}{h} - T(t)A_T(x)\right| \\
&= \left|T(t+h)\left(\frac{x - T(-h)x}{h} - A_T(x)\right) + T(t+h)A_T(x) - T(t)A_T(x)\right| \\
&\le \left|T(t+h)\left(\frac{x - T(-h)x}{h} - A_T(x)\right)\right| + |T(t+h)A_T(x) - T(t)A_T(x)| \\
&= \left|T(t+h)\left(\frac{T(-h)x - x}{-h} - A_T(x)\right)\right| + \left|g_{A_T(x)}(t+h) - g_{A_T(x)}(t)\right| \\
&\le |T(t+h)|\left|\frac{T(-h)x - x}{-h} - A_T(x)\right| + \left|g_{A_T(x)}(t+h) - g_{A_T(x)}(t)\right|
\end{aligned}$$

$$\leq M \left| \frac{T(-h)x - x}{-h} - A_T(x) \right| + \left| g_{A_T(x)}(t+h) - g_{A_T(x)}(t) \right|$$

Since x is in D_T, it follows that $((T(-h)x - x)/(-h)) - A_T(x)$ approaches 0 as h increases to 0 and since T is strongly continuous, $g_{A_T(x)}$ is continuous, so $g_{A_T(x)}(t+h) - g_{A_T(x)}(t)$ approaches 0 as h increases to 0. It follows that if x is in D_T, then g_x is differentiable and $g_x'(t) = T(t)A_T(x)$. □

Problem 18

Problem. *Suppose X is a metric space and T is a semigroup on X. Show that the induced mapping, $S\colon [0,\infty) \to \mathrm{L}(\mathrm{CB}(X))$, is a linear semigroup on $\mathrm{CB}(X)$.*

Background.

Definition 18.1. Suppose X is a topological space and $T\colon [0,\infty) \to \mathrm{C}(X,X)$ is a semigroup of X. Define $S\colon [0,\infty) \to \mathrm{C}(\mathrm{CB}(X), \mathrm{C}(X))$ by $S(t) = T(t)^{\#}$, so $S(t)(f) = fT(t)$ and $(S(t)f)(x) = (f \circ T(t))(x)$ for t in $[0,\infty)$, f in $\mathrm{CB}(X)$, and x in X.

Recall that since X is complete, $\mathrm{CB}(X)$ with the sup-norm is a Banach space, and so $L(\mathrm{CB}(X))$ is a Banach space with the operator-norm.

Solution. [Deana] By definition, $T(t)$ is continuous and f is bounded (that is, the image of f is bounded) and continuous, so $f \circ T(t)$ is in $\mathrm{CB}(X)$.

Since $T(0) = I$, it follows immediately that $S(0) = I$. Since T is a semigroup,

$$S(s)(S(t)(f)) = (S(t)(f))T(s) = fT(t)T(s) = fT(s+t) = S(s+t)(f).$$

Therefore S is a semigroup on $\mathrm{CB}(X)$.

Finally, the vector space operations on $\mathrm{CB}(X)$ are pointwise, so

$$S(t)(f+g) = (f+g)T(t) = fT(t) + gT(t) = S(t)(f) + S(t)(g) \text{ and } S(t)(\alpha f) = \alpha f T(t) = \alpha S(t)(f).$$

Therefore, S is a linear semigroup.

Comments. In general, even if T is strongly continuous, it's not necessarily the case that S is strongly continuous in the operator topology on $L(\mathrm{CB}(X))$.

Notice that $\mathrm{CB}(X)$ is an $\mathbb{R}$-algebra with pointwise multiplication of functions and that for t in $[0,\infty)$, $S(t)$ is in fact an algebra homomorphism. It follows that for x in X, the evaluation map $\eta_{t,x}\colon \mathrm{CB}(X) \to \mathbb{R}$ by $\eta_{t,x}(f) = (S(t)f)(x) = f(T(t)(x))$ is a multiplicative linear functional, hence, when it's not identically zero, its kernel is a maximal ideal in $\mathrm{CB}(X)$.

Problem 19

Problem. *Suppose a, b, α, β, and M are real numbers with $\alpha > 0$, $\beta > 0$, and $M > 0$, and f is a continuous, real-valued function with domain $\Omega = [a-\alpha, a+\alpha] \times [b-\beta, b+\beta]$ with the property that*

$$|f(t,x) - f(t,y)| \leq M\,|x-y| \quad \text{for } (t,x) \text{ and } (t,y) \text{ in } \Omega.$$

Show there is a number, d, in $(0, \alpha]$ *and a unique real-valued function, y, with domain* $[a-d, a+d]$ *satisfying the conditions* $y(a) = b$ *and* $y'(t) = f(t, y(t))$ *for all t in* $[a-d, a+d]$.

Background. Notice first that the version of the problem given in Theorem 19.1 below is a general local existence and uniqueness theorem for arbitrary systems of (possible infinitely many) ordinary differential equations. As long as the system (the function f) is uniformly Lipschitz as a function of t in $[a, b]$, then given an initial condition in the domain of f, (a number, c, in $[a, b]$ and an element, x, in U), there is an interval, say J, containing c so that the system has a unique solution satisfying the initial condition that's defined on J.

A very special case is when $X = M_n(\mathbb{R})$, then in terms of coordinates, the initial value problem is equivalent to the system

$$\begin{aligned} y_1' &= f_1(t, y_1, \ldots, y_n) \\ y_2' &= f_2(t, y_1, \ldots, y_n) \\ &\vdots \\ y_n' &= f_n(t, y_1, \ldots, y_n) \end{aligned}$$

with initial conditions

$$y_1(c) = x_1, \quad y_2(c) = x_2, \quad \ldots \quad y_n(c) = x_n,$$

where y_i, f_i, and x_i are the coordinates of y, f, and x respectively.

The solution might not be defined on the whole interval I. However, a minimum length for the interval on which the solution is guaranteed to be defined, that depends on the distances from a and x to the boundary of I and U respectively, and on the maximum value of f on an open set containing (a, x), is given in the statement of the theorem.

Notice that when $f(t, y) = A(t)y + g(t)$, then the system is linear and we've already proved a global existence and uniqueness theorem in **Problem 13**.

Solution. [Henry] The result in the problem follows immediately from the following more general theorem in the special case when $X = \mathbb{R}$.

Theorem 19.1. *Suppose I is a closed interval with non-empty interior, X is a Banach space, U is an open subset of X. Fix a in the interior of I and x in U. Assume that* $\alpha > 0$, $\beta > 0$, $[a-\alpha, a+\alpha]$ *is contained in I, and the closed ball about x with radius* β, $\overline{B_\beta(x)}$, *is contained in U. Suppose that* $f\colon I \times U \to X$ *is a function satisfying the Lipschitz condition*

$$|f(t, y_1) - f(t, y_2)| \le M\,|y_1 - y_2| \quad \text{for } (t, y_1) \text{ and } (t, y_2) \text{ in } I \times U,$$

so f is continuous. Set $S = \sup\{\, |f(t, y)| \mid (t, y) \in [a-\alpha, a+\alpha] \times \overline{B_\beta(x)} \,\}$, *and choose d in* $(0, \alpha]$ *with* $Sd \le \beta$.

Then there is a unique function, $y\colon [a-\alpha, a+\alpha] \to X$, *with the properties: 1)* $y(a) = x$ *and 2)* $y'(t) = f(t, y(t))$ *for all t in* $[a-d, a+d]$.

Proof. Define $y_0\colon [a-d, a+d] \to X$ by $y_0(t) = x$ and for $n \geq 1$ define $y_n\colon [a-d, a+d] \to X$ by $y_n(t) = x + \int_a^t f(s, y_{n-1}(s))\, ds$.

Claim 19.2. *If t is in $[a-d, a+d]$ and $n \geq 0$, then $y_n(t)$ is in $\overline{B_\beta(x)}$.*

Proof. Use induction on n. The result is clear for $n = 0$. Assume that $y_{n-1}(t)$ is in $\overline{B_\beta(x)}$ for all t in $[a-d, a+d]$. Then if s is in $[a-d, a+d]$, $(s, y_{n-1}(s))$ is in $I \times \overline{B_\beta(x)}$ and so $|f(s, y_{n-1}(s))| \leq S$. Therefore, for t in $[a-d, a+d]$ we have

$$\begin{aligned}
|y_n(t) - x| &= \left| \int_a^t f(s, y_{n-1}(s))\, ds \right| \\
&\leq \left| \int_a^t |f(s, y_{n-1}(s))|\, ds \right| \\
&\leq \left| \int_a^t S\, ds \right| \\
&= S\, |a - t| \\
&\leq Sd \\
&\leq \beta
\end{aligned}$$

□

Now, for $n \geq 1$, define $Y_n\colon [a-d, a+d] \to X$ by $Y_n = y_n - y_{n-1}$. Then clearly,

$$|Y_1(t)| = |y_1(t) - y_0(t)| = |y_1(t) - x| \leq \beta$$

for t in $[a-d, a+d]$. Next, using this bound on Y_1 and the Lipschitz condition on f, we have

$$\begin{aligned}
|Y_2(t)| &= |y_2(t) - y_1(t)| \\
&= \left| \int_a^t (f(s, y_1(s)) - f(s, y_0(s)))\, ds \right| \\
&\leq \left| \int_a^t |f(s, y_1(s)) - f(s, y_0(s))|\, ds \right| \\
&\leq M \left| \int_a^t |y_1(s) - y_0(s)|\, ds \right| \\
&= M \left| \int_a^t |Y_1(s)|\, ds \right| \\
&\leq M\beta\, |t - a|\, .
\end{aligned}$$

Assume that $|Y_n(t)| \leq \beta M^{n-1} |t-a|^{n-1} / (n-1)!$ for t in $[a-d, a+d]$. Then, using the Lipschitz condition on f, we have

$$\begin{aligned}
|Y_{n+1}(t)| &= |y_{n+1}(t) - y_n(t)| \\
&= \left| \int_a^t (f(s, y_n(s)) - f(s, y_{n-1}(s)))\, ds \right| \\
&\leq \left| \int_a^t |f(s, y_n(s)) - f(s, y_{n-1}(s))|\, ds \right|
\end{aligned}$$

$$\leq M \left| \int_a^t |y_n(s) - y_{n-1}(s)| \, ds \right|$$
$$= M \left| \int_a^t |Y_n(s)| \, ds \right|$$
$$\leq M \left| \int_a^t \frac{\beta M^{n-1} |s-a|^{n-1}}{(n-1)!} \, ds \right|$$
$$= \frac{M^n \beta |t-a|^n}{n!}.$$

Therefore, $|Y_n(t)| \leq \beta M^{n-1} |t-a|^{n-1} / (n-1)!$ for t in $[a-d, a+d]$ and for all $n \geq 1$.

Clearly $\beta M^{n-1} |t-a|^{n-1} / (n-1)! \leq \beta M^{n-1} d^{n-1}/(n-1)!$. The series $\sum M^{n-1} d^{n-1}/(n-1)!$ converges, so the series $\sum Y_n(t)$ converges absolutely for all t and the series of functions $\sum Y_n$ converges uniformly on $[a-d, a+d]$. Since $\sum_{n=1}^N y_n = y_N - y_0$, the sequence $\{y_n\}$ converges uniformly. Define $y = \lim_{n\to\infty} y_n$.

If s in $[a-d, a+d]$, then $y_n(s)$ is in $\overline{B_\beta(x)}$ for all n and so it follows that $y(s)$ is in $\overline{B_\beta(x)}$. Now, using again the Lipschitz condition on f, we have

$$\left| \int_a^t f(s, y_{n-1}(s)) \, ds - \int_a^t f(s, y(s)) \, ds \right| \leq \left| \int_a^t |f(s, y_{n-1}(s)) - f(s, y(s))| \, ds \right|$$
$$\leq M \left| \int_a^t |y_{n-1}(s) - y(s)| \, ds \right|.$$

It follows, using that the y_n's converge uniformly to y, that $y(t) = x + \int_a^t f(s, y(s)) \, ds$ for all t in $[a-d, a+d]$. Therefore, it follows from the Fundamental Theorem of Calculus (Proposition 13.3) that $y'(t) = f(t, y(t))$ for all t in $[a-d, a+d]$. Since $y_n(a) = x$ for all n, it follows that $y(a) = x$ and so y satisfies the conditions in the statement of the Theorem.

To show that y is unique, argue as in **Problems 7** and **13**, using the Lipschitz condition on f as in the preceding argument. □

Comments.

Remark 19.3. Take $a = b = 0$ and define $f(t, x) = x^{2/3}$. Then f is continuous on all of $\mathbb{R}^2$, but the equation $y'(t) = f(t, y(t))$ has infinitely many solutions by the calculation in Remark 5.1, so f cannot satisfy the Lipschitz condition.

Problem 20

Problem. *Suppose T is a strongly continuous, linear semigroup on a Banach space X with generator $A = A_T$.*

(a) *Show that $I_\lambda x$ is in D_T for all $\lambda > 0$ and for all x in X.*
(b) *Show that $(I - \lambda A) I_\lambda x = x$ for all x in X.*
(c) *Show that $I_\lambda (I - \lambda A) x = x$ for all x in D_T.*
(d) *Show that if T is non-expansive, then $|I_\lambda| \leq 1$.*

Conclude that I_λ defines a bounded, bijective linear transformation from X to D_T.

Background.

Definition 20.1. Suppose T is a strongly continuous, linear semigroup on a Banach space, X. For $\lambda > 0$ define $I_\lambda\colon X \to X$ by

$$I_\lambda x = \frac{1}{\lambda}\int_0^\infty e^{-s/\lambda}T(s)x\,ds = \frac{1}{\lambda}\int_0^\infty e^{-s/\lambda}g_x(s)\,ds.$$

Remark 20.2. Notice that $I - \lambda A_T$ is a map from D_T to X. In the problem it will be shown that this map is a continuous isomorphism between D_T and X with inverse I_λ. The map $I_\lambda = (I-\lambda A)^{-1}$ is the *resolvent of* A_T.

Remark 20.3. For part (c), we'll use the fact that for x in X, the trajectory, g_x, is differentiable and that $g_x' = Ag_x$.

Solution. [Henry and Ioana] [Henry] To prove parts (a) and (b), fix x in X and $\lambda > 0$. Then the following statements are equivalent:

- $(I-\lambda A)I_\lambda x = x$,
- $\lambda A I_\lambda x = I_\lambda x - x$, and
- $AI_\lambda x = \frac{1}{\lambda}I_\lambda x - \frac{1}{\lambda}x$.

We'll show that the third statement is true.

Fix $r > 0$ and recall that $Ax = \lim_{t\to 0^+}(1/t)(T(t)-I)x$ for x in D_T. Using the change of variable $u = r+s$ to get the third equality, we have

$$\begin{aligned}
\frac{1}{r}\left(T(r)-I\right)I_\lambda x &= \frac{1}{r}\left(T(r)-I\right)\frac{1}{\lambda}\int_0^\infty e^{-s/\lambda}T(s)x\,ds \\
&= \frac{1}{r\lambda}\int_0^\infty e^{-s/\lambda}T(r+s)x\,ds - \frac{1}{r\lambda}\int_0^\infty e^{-s/\lambda}T(s)x\,ds \\
&= \frac{1}{r\lambda}\int_r^\infty e^{-(s-r)/\lambda}T(s)x\,ds - \frac{1}{r\lambda}\int_0^\infty e^{-s/\lambda}T(s)x\,ds \\
&= \frac{1}{r\lambda}\int_r^\infty \left(e^{(r-s)/\lambda} - e^{-s/\lambda}\right)T(s)x\,ds - \frac{1}{r\lambda}\int_0^r e^{-s/\lambda}T(s)x\,ds \\
&= \frac{e^{r/\lambda}-1}{r\lambda}\int_r^\infty e^{-s/\lambda}T(s)x\,ds - \frac{1}{r\lambda}\int_0^r e^{-s/\lambda}T(s)x\,ds \\
&= \left(\frac{1}{\lambda}\right)\left(\frac{e^{r/\lambda}-1}{r}\right)\int_r^\infty e^{-s/\lambda}T(s)x\,ds - \left(\frac{1}{\lambda}\right)\left(\frac{1}{r}\right)\int_0^r e^{-s/\lambda}T(s)x\,ds.
\end{aligned}$$

Clearly,

$$\lim_{r\to 0^+}(e^{r/\lambda}-1)/r = 1/\lambda \quad\text{and}\quad \lim_{r\to 0^+}\int_r^\infty e^{-s/\lambda}T(s)x\,ds = \int_0^\infty e^{-s/\lambda}T(s)x\,ds = \lambda I_\lambda x.$$

An argument similar to the one given in part (a) of **Problem 17** shows that

$$\lim_{r\to 0^+}\frac{1}{r}\int_0^r e^{-s/\lambda}T(s)x\,ds = x.$$

Therefore, it follows that

$$\lim_{r\to 0^+}\frac{1}{r}\left(T(r)-I\right)I_\lambda x = \frac{1}{\lambda}I_\lambda x - \frac{1}{\lambda}x.$$

This shows that $I_\lambda x$ is in D_T for all x in X and that $AI_\lambda x = (1/\lambda)I_\lambda x - (1/\lambda)x$, and so $(I - \lambda A)I_\lambda x = x$ for all x in X. This completes the argument for parts (a) and (b).

[Ioana] For part (c), fix $\lambda > 0$ and x in D_T, so $\lim_{t\to 0+}(1/t)\,(T(t) - I)\,x$ exists and is equal Ax. Then

$$\begin{aligned} I_\lambda\,(I - \lambda A)\,x &= I_\lambda\,(x - \lambda Ax) \\ &= \frac{1}{\lambda}\int_0^\infty e^{-s/\lambda}T(s)(x - \lambda Ax)\,ds \\ &= \frac{1}{\lambda}\int_0^\infty e^{-s/\lambda}T(s)x\,ds - \int_0^\infty e^{-s/\lambda}T(s)Ax\,ds \\ &= I_\lambda x - \int_0^\infty e^{-s/\lambda}T(s)Ax\,ds. \end{aligned}$$

To complete the proof of part (c), it suffices to show that

$$\int_0^\infty e^{-s/\lambda}T(s)Ax\,ds = -x + I_\lambda x.$$

Since T is a semigroup and addition of real numbers is commutative, we clearly have that $AT(s) = T(s)A$ for all s in $[0,\infty)$. Therefore, $T(s)Ax = AT(s)x = Ag_x(s)$ for s in $[0,\infty)$. By proposition 17.6, the trajectory, g_x, is differentiable and $g_x' = Ag_x$. Also, the product rule is valid for functions $[0,\infty) \to X$ by Proposition 13.2, so the usual integration by parts formula from calculus is valid. Therefore,

$$\begin{aligned} \int_0^\infty e^{-s/\lambda}T(s)Ax\,ds &= \int_0^\infty e^{-s/\lambda}Ag_x\,ds \\ &= \int_0^\infty e^{-s/\lambda}\frac{d}{ds}\left(g_x(s)\right)\,ds \\ &= e^{-s/\lambda}g_x(s)\Big|_{s=0}^{\infty} - \int_0^\infty \frac{-1}{\lambda}e^{-s/\lambda}g_x(s)\,ds \\ &= -x + \frac{1}{\lambda}\int_0^\infty e^{-s/\lambda}T(s)x\,ds \\ &= -x + I_\lambda x. \end{aligned}$$

This completes the argument for part (c).

To prove part (d), suppose T is non-expansive and $\lambda > 0$. Using the hypothesis on T, for x in X we have

$$\begin{aligned} |I_\lambda x| &= \left|\frac{1}{\lambda}\int_0^\lambda e^{-s/\lambda}T(s)x\,ds\right| \\ &\le \frac{1}{\lambda}\int_0^\infty e^{-s/\lambda}\,|T(s)x|\,ds \\ &\le \frac{1}{\lambda}\int_0^\infty e^{-s/\lambda}\,|T(s)|\,|x|\,ds \\ &\le \frac{|x|}{\lambda}\int_0^\infty e^{-s/\lambda}\,ds \\ &= |x|\left(e^{-s/\lambda}\Big|_0^\infty\right) \\ &= |x|\,. \end{aligned}$$

Therefore, $|I_\lambda| \le 1$.

Comments. Notice that it may be the case that $I-\lambda A$ is only densely defined and nowhere continuous, but for a strongly continuous, linear semigroup it's always the case$(I-\lambda A)^{-1}=I_\lambda$ is globally defined and continuous.

PROBLEM 21

Problem. *Suppose $G\colon[a,b]\to\mathbb{R}^2$ and $Q\colon[a,b]\to L(\mathbb{R}^2)$ are continuous and A and B are in $L(\mathbb{R}^2)$. Consider the initial value problem:*

BVP:$\mathbb{R}^2$ $\qquad Y\colon[a,b]\to\mathbb{R}^2,\ AY(a)+BY(b)=0,$ *and* $Y'=QY+G.$

Show that the equation $-y''=g$, $y(0)=y(1)=0$ in **Problem 14***(a) is a special case of BVP:$\mathbb{R}^2$ for a suitable choice of G, Q, A, and B.*

Solution. [Ahmed] Set $G=\begin{bmatrix}0\\-g\end{bmatrix}$, $Q(t)=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ for all t, $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, and $B=\begin{bmatrix}0&0\\1&0\end{bmatrix}$. Then if $x_1=y$, $x_2=x_1'$, and $Y=\begin{bmatrix}x_1\\x_2\end{bmatrix}$, the equation $-y''=g$ becomes $Y'=QY+G$ and $y(0)=0=y(1)$ is equivalent to $AY(a)+BY(b)=0$. More generally, any matrices $A=\begin{bmatrix}a&0\\c&0\end{bmatrix}$, and $B=\begin{bmatrix}b&0\\d&0\end{bmatrix}$ with $\begin{bmatrix}a&b\\c&d\end{bmatrix}$ invertible will work.

PROBLEM 22

Problem. *Supose $[a,b]$ is an interval, $Q\colon[a,b]\to L(\mathbb{R}^2)$ is continuous, and W is in $L(\mathbb{R}^2)$. Show that there is a unique $M\colon[a,b]\to L(\mathbb{R}^2)$ with the properties that $M(a)=W$ and $M'=-MQ$. Moreover, if W is invertible, show that $M(t)$ is invertible for all t.*

Background. An "integrating factor" for the equation

$$Y\colon[a,b]\to\mathbb{R}^2,\quad Y'-QY=G,$$

where $Q\colon[a,b]\to L(\mathbb{R}^2)$ and $G\colon[a,b]\to\mathbb{R}^2$, is a function $M\colon[a,b]\to L(\mathbb{R}^2)$ with the property that $MY'-MQY=(MY)'$. If this is the case, then applying M to both sides of $Y'-QY=G$ gives $(MY)'=MG$, so $M(t)Y(t)=C+\int_a^t MG$ for some constant vector, C, in $\mathbb{R}^2$. Since $(MY)'=MY'+M'Y$, we need to choose M so that $M'=-MQ$.

Assuming we can find M with $M'=-MQ$ and with $M(t)$ invertible for all t, it follows that the solutions to $Y'-QY=G$ all have the form

$$Y(t)=M(t)^{-1}C+M(t)^{-1}\int_a^t MG \tag{22.1}$$

for some C in $\mathbb{R}^2$. The formula (22.1) is called the *variation of parameters formula.*

Remark 22.2. A hint for the second part of the problem is to define $h(t)=\det M(t)$ and find a formula for $h'(t)$. Then show that either h is identically 0 or that $h(t)\neq 0$ for all t.

Solution. [Ahmed, Ioana] The argument in the Background for **Problem 8** that the initial value problem IVP:$M_2(\mathbb{R})$ has a unique solution, together with **Problem 13**, shows that there is a unique M_1 with $M_1' = (-{}^tQ)M_1$ and $M_1(a) = {}^tW$. By taking transposes and noticing the taking transposes commutes with differentiation, it follows that there is a unique $M\colon [a,b] \to L(\mathbb{R}^2)$ with $M' = -MQ$ and $M(a) = W$. This proves the first statement in the problem.

Comments. Notice that it's likely that $Q(s)Q(t) \neq Q(t)Q(s)$ for s and t in $[a,b]$, so the obvious generalization of the "general solution" in the case of a real-valued function, namely $M(t) = Ae^{-\int_a^t Q}$, may not have the property that $M'(t) = -Q(t)M(t)$. As an example, suppose A and B are in $M_2(\mathbb{R})$ and $AB \neq BA$. Define $Q\colon [0,1] \to M_2(\mathbb{R})$ by $Q(t) = tA + (1-t)B$. Then $\int_0^t Q = (1/2)t^2A - (1/2)B + (1/2)(1-t)^2B$ and so $e^{-\int_0^t Q} = e^{-(1/2)(t^2A+(1-t)^2B}e^{B/2} = \ldots$

Problem 23

Problem. *It was shown in* **Problem 14** *that for g in* $\mathrm{C}([0,1])$*, the boundary value problem:*

$$\text{(BVP:}[0,1]) \qquad -y'' = g, \quad y(0) = 0 = y(1)$$

has a unique solution. Therefore, the rule that assigns to a function, g, in $\mathrm{C}([0,1])$ *the unique solution to BVP:*$[0,1]$ *defines a function, $T\colon \mathrm{C}([0,1]) \to \mathrm{C}([0,1])$. Show that $T\colon \mathrm{C}([0,1]) \to \mathrm{C}([0,1])$ is a self-adjoint operator when* $\mathrm{C}([0,1])$ *is given the inner product it inherits as a subspace of $L^2([0,1])$.*

Background. Recall that we saw in the comments after **Problem 14** that the unique solution to BVP:$[0,1]$ is given by $x \mapsto \int_0^1 k(x,s)g(s)$, where the kernel, k, is defined by

$$k\colon [0,1] \times [0,1] \to \mathbb{R} \quad \text{by} \quad k(x,s) = \begin{cases} s(1-x) & 0 \le s \le x \\ x(1-s) & x \le s \le 1. \end{cases}$$

Thus, $T(g)(x) = \int_0^1 k(x,s)g(s)$, for x in $[0,1]$. It follows immediately that T is a linear transformation from $\mathrm{C}([0,1])$ to itself. (This fact also follows easily from the definition of T using that differentiation is linear.)

Recall that $L^2([0,1])$ is an inner product space with inner product

$$\langle f, g \rangle = \int_0^1 f(s)g(s)\,ds,$$

for f and g in $L^2([0,1])$. The restriction of this bilinear form to $\mathrm{C}([0,1])$ is an inner product on $\mathrm{C}([0,1])$, but $\mathrm{C}([0,1])$ is not complete with respect to the corresponding norm. In fact, since $\mathrm{C}([0,1])$ is dense in $L^2([0,1])$, the completion of $\mathrm{C}([0,1])$ with is $L^2([0,1])$.

Finally, recall that if $(X, \langle\,\cdot\,,\,\cdot\,\rangle)$ is an inner product space and $T\colon X \to X$ is a linear transformation, then the *adjoint* or *transpose* of T is the mapping $T^*\colon X \to X$ defined as follows. Given y in X, the mapping $e_y\colon X \to \mathbb{R}$ by $e_y(x) = \langle Tx, y\rangle$ is an element in X^*, so since $\langle\,\cdot\,,\,\cdot\,\rangle$ is non-singular (because it's definite), there is a unique element, denoted by T^*y, in X so that $e_y(x) = \langle x, T^*y\rangle$. Thus, $\langle Tx, y\rangle = e_y(x) = \langle x, T^*y\rangle$. It's easy to check that T^* is a linear transformation from X to itself.

A linear transformation, $T\colon X \to X$, is *self-adjoint* if $T = T^*$.

Solution. Coming soon. . . ?

Comments. The solution to part (b) of **Problem 14** is essentially the computation of the eigenvalues and eigenfunctions (that is, the eigenvectors) of T. If $Tg = \alpha g$, then by the definition of T we have that $-(\alpha g)'' = g$, so g is twice continuously differentiable and $-g'' = (1/\alpha)g$. Therefore, $\alpha = 1/n^2\pi^2$ for some integer, n, which might as well be assumed to be positive, and $g(t) = c\sin n\pi t$ for some real number c.

It follows that the eigenvalues of T are $\{\, 1/n^2\pi^2 \mid n \in \mathbb{N} \}$ and that the $(1/n^2\pi^2)$-eigenspace of T is the one dimensional vector space $\{\, s_c \mid c \in \mathbb{R}\,\}$ where $s_c\colon [0,1] \to \mathbb{R}$ by $s_c(t) = c\sin n\pi t$.

Problem 24

Problem. *Suppose $Q\colon [a,b] \to L(\mathbb{R}^2)$ is continuous and A and B are in $L(\mathbb{R}^2)$. Suppose also that for every continuous function $G\colon [a,b] \to \mathbb{R}^2$, the boundary value problem*

$$BVP:\mathbb{R}^2 \qquad Y\colon [a,b] \to \mathbb{R}^2,\ AY(a) + BY(b) = 0, \text{ and } Y'(t) = Q(t)Y(t) + G(t) \text{ for all } t$$

has a unique solution. Show that there is a function, $K\colon [a,b] \times [a,b] \to L(\mathbb{R}^2)$ with the property that $Y(t) = \int_a^b K(t,s)G(s)\,ds$ for all t in $[a,b]$.

Problem 25

Problem. *Suppose $q\colon [a,b] \to \mathbb{R}$ is continuous and A and B are in $L(\mathbb{R}^2)$. Suppose also that for every continuous function $g\colon [a,b] \to \mathbb{R}$, the boundary value problem*

$$BVP:\mathbb{R} \qquad y\colon [a,b] \to \mathbb{R},\ A\begin{bmatrix} y(a) \\ y'(a) \end{bmatrix} + B\begin{bmatrix} y(a) \\ y'(a) \end{bmatrix} = 0, \text{ and } y''(t) - q(t)y(t) = g(t) \text{ for all } t$$

has a unique solution. Show that there is a unique continuous function, $k\colon [a,b] \times [a,b] \to \mathbb{R}$ with the property that $y(t) = \int_a^b k(t,s)g(s)\,ds$ for all t in $[a,b]$.

Comments. It can be shown that with the notation of the problem, k is symmetric if and only if $\det A = \det B$.

References

[Rud73] W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973.

[Rud76] W. Rudin, *Principles of mathematical analysis*, International Series in Pure and Applied Mathematics, McGraw-Hill, New York, 1976.

[WZ77] R.L. Wheeden and A. Zygmund, *Measure and integral*, Pure and Applied Mathematics, Marcel Dekker, New York, 1977.