INTEGRATED FORM OF CONTINUOUS NEWTON'S METHOD

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Abstract. An integrated form of continuous Newton's method is defined. Under rather minimal conditions the method is shown to lead to a zero of the given function. The result is applied to recover a recent Nash-Moser type inverse function theorem.

1. Introduction

Many problems in PDE may be expressed as a problem of finding u so that

$$F(u) \ = \ 0 \tag{1}$$

where F is a function from a Banach space H to a Banach space K. A prospective method for solving (1) is called here Integrated Continuous Newton's Method: Given $z_0 \in H$, find a continuous function $z : [0,\infty) \to H$ such that

$$z(0) = z_0, \ (F(z))'(t) \ = \ -F(z(t)), \ t \geq 0. \tag{2}$$

If F is C^1, $F'(x)^{-1}$ exists for $x \in H$ and z is differentiable, then (2) takes the form

$$z(0) = z_0, \ z'(t) \ = -F'(z(t))^{-1}F(z(t)), \ t \geq 0. \tag{3}$$

If we interpret invertibility in the set theoretic sense

$$F'(x)^{-1}y \ = \ \{w \in H : F'(x)w = y\}$$

then a substitute for (3) takes the form

$$z(0) = z_0, \ z'(t) \ \in \ -F'(z(t))^{-1}F(z(t)), \ t \geq 0, \tag{4}$$

what may be called a differential inclusion.

However, even in the case of ordinary complex polynomials [5], (2) may have meaning where neither (3),(4) does. Accordingly we concentrate on (2).

We attempt some perspective on PDE and the present work. Denote $[0,1]^2$ by Ω, $H = H^{1,2}(\Omega)$, $K = L_2(\Omega)$ and define F so that

$$F(u) \ = \ u_1, \ u \in H,$$

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where u_1 indicates the derivative of u in its first argument. Given $g \in K$, we have the problem of finding $u \in H$ such that

$$F(u) \ = \ g. \tag{5}$$

Some reflection yields that this problem has no solution if g lacks a certain amount of smoothness in its second argument. It is this lack of proper inverses that makes discrete Newton's method difficult (see [3]). Earlier related work on continuous Newton's method is found in [4]

Recent papers [1],[2],[6] have used continuous methods to overcome loss of derivative problems which seem unavoidable in conventional Newton's method. The present note seeks to relate this line of development to (2).

2. Statement of Main Result

Suppose that each of H, J, K is a Banach space with H compactly embedded in J. By this we mean that the points of H form a dense linear subspace of J, $\|x\|_H \geq \|x\|_J,\ x \in H$, and that every bounded sequence in H has a convergent subsequence in J to an element of H. If $x \in H, s > 0$, then

$$B_s(x) \ = \ \{y \in H : \|y - x\|_H \leq s\}$$

and

$$b_s(x) \ = \ \{y \in H : \|y - x\|_H < s\}.$$

Fix $v \in H,\ r > 0$. Suppose that F is a function with range in K so that the domain of F contains $B_r(v)$. Suppose also that F is continuous as a function on J into K.

Theorem 1. *Suppose that for each $x \in b_r(v)$ there is $h \in B_r(0)$ so that*

$$F'(x)h \ = \ -F(v). \tag{6}$$

Then there is a function $z : [0, \infty) \to H$, continuous as a function into J so that

$$z(0) = v,\ (F(z))'(t) \ = \ -F(z(t)),\ t \geq 0.$$

Moreover,

$$u \ = \ J - \lim_{t \to \infty} z(t) \text{ exists and } F(u) \ = \ 0.$$

The derivative in (6) is defined simply as

$$F'(x)h \ = \ \lim_{t \to 0+} \frac{1}{t}(F(x + th) - F(x)).$$

No Fréchet differentiability is assumed.

An argument is based on the following:

Lemma 1. *Under the hypothesis of Theorem 1, suppose that*

$$t \in (0, 1),\ y \in b_{tr}(v),\ s \in (0, 1 - t].$$

Then if $\epsilon > 0$, there is $w \in B_{sr}(y)$ so that

$$\|F(w) - F(y) \ + \ sF(v)\|_K \ \leq \ \epsilon s.$$

3. Proofs

Proof. [Lemma 1] Suppose $0 < t < 1,\ y \in b_{tr}(v), \epsilon > 0$. Pick $h \in B_r(0)$ so that

$$F'(y)h \ = \ -F(v).$$

There is $\delta \in (0, 1-t]$ such that if $0 < s \le \delta$, then

$$\|F(y+sh) - F(y) + sF(v)\|_K \ \le \ s\epsilon.$$

Note that $y + sh \in B_{sr}(y)$. Denote by $M_{y,\epsilon}$ the collection of all $\delta \in (0, 1-t]$ such that if $0 < s \le \delta$ then there is $w \in B_{sr}(y)$ so that

$$\|F(w) - F(y) + sF(v)\|_K \ \le \ s\epsilon.$$

Clearly $M_{y,\epsilon}$ is connected. The lemma follows if it can be shown that $M_{y,\epsilon}$ contains $1-t$.

Denote $\sup M_{y,\epsilon}$ by q. Denote by $\{w_k\}_{k=1}^\infty$ a sequence in H and $\{q_k\}_{k=1}^\infty$ an increasing sequence of positive numbers convergent to q such that

$$w_k \in B_{rq_k}(y) \text{ and } \|F(w_k) - F(y) + q_kF(v)\|_K \ \le \ q_k\epsilon.\ k = 1, 2, \ldots.$$

Choose an increasing sequence of positive integers $\{k_i\}_{i=1}^\infty$ so that

$$\{w_{k_i}\}_{i=1}^\infty$$

converges in $\|\cdot\|_J$ to some $w \in H$. Note that $w \in B_{qr}(y)$ and

$$\|F(w) - F(y) + qF(v)\|_K \ \le \ q\epsilon. \tag{7}$$

Hence $q \in M_{y,\epsilon}$. If $q = 1-t$ the argument is finished.

Suppose then that $q < 1-t$. Define $h_0 \in B_r(0)$ so that

$$F'(w)h_0 \ = \ -F(v)$$

Denote by d a positive number so that $d < 1-(t+q)$ and so that if $0 < s \le d$ then

$$\|F(w+sh_0) \ - \ F(w) + sF(v)\|_K \ \le \ s\epsilon. \tag{8}$$

Note that for such a choice of s, $w + sh_0 \in B_{(q+s)r}(y)$. Combining (7), (8), if $0 < s \le d$, then

$$\|F(w+sh_0) - F(y) + (s+q)F(v)\|_K \ \le \ (s+q)\epsilon.$$

This places $d + q \in M_{y,\epsilon}$, contradicting the proposition that $q = \sup M_{y,\epsilon}$. Thus the assumption that $q < 1-t$ is false and the argument is finished. □

Proof. [Theorem 1] Denote by $\{p^{(k)}\}_{k=1}^\infty$ a sequence of partitions of $[0,1]$ so that $p^{(k+1)}$ is a refinement of $p^{(k)},\ k = 1, 2, \ldots$, and so that the mesh of $p^{(k)} \to 0$ as $k \to \infty$. Suppose that k is a positive integer and that

$$p^{(k)} \ = \ t_0^{(k)}, \ldots, t_{n_k}^{(k)}.$$

Denote by z_k a continuous function on $[0,1]$ which is linear on each of $[t_{i-1}^k, t_i^{(k)}]$, $i = 1, \ldots, n_k$, and so that $z_k(0) = v$,

$$\|z_k(t_i^k) - z_k(t_{i-1}^k)\|_H \ \le \ (t_i^k - t_{i-1}^k)r,$$

and

$$\|F(z_k(t_i^{(k)})) - F(z_k(t_{i-1}^{(k)})) + (t_i^{(k)} - t_{i-1}^{(k)})F(v)\|_K \leq (t_i^{(k)} - t_{i-1}^{(k)})/k,$$

$i = 1, \ldots, n_k,\ \ k = 1, 2, \ldots$. Summing the above inequalities from $i = 1$ to $i = j \leq n_k$ and using the triangle inequality we see that

$$\|F(z_k(t_j^{(k)}) - F(v) + t_j^{(k)}F(v)\|_K \ \leq\ t_j^{(k)}/k \ \leq\ 1/k. \tag{9}$$

(The preceding lemma permits this construction.)

Note that $\{z_k\}_{k=1}^{\infty}$ is uniformly bounded relative to $\|\cdot\|_H$ and equicontinuous (actually uniformly lipschitz with constant r). Hence there is an increasing sequence $\{k_j\}_{j=1}^{\infty}$ of positive integers so that

$$\{z_{k_j}\}_{j=1}^{\infty}$$

converges uniformly using $\|\cdot\|_J$ to a J continuous $H-$ valued function z on $[0,1]$. Since F is continuous as a function on J, $\{F(z_{k_j})\}_{j=1}^{\infty}$ converges uniformly to $F(z)$ on $[0,1]$.

Suppose now that $s \in [0,1]$. Then s is a sequential limit of a sequence $\{s_k\}_{k=1}^{\infty}$ where $s_k \in p^{(k)},\ k = 1, 2, \ldots$. Using (9) it follows that

$$F(z(s)) \ =\ (1-s)F(v). \tag{10}$$

Hence

$$(F(z))'(s) \ = -F(v),\ s \in [0,1]. \tag{11}$$

It is clear that $F(z(1)) = 0$. Conclusions (10), (11) are at least as significant as the stated conclusion to the theorem.

To reach our final conclusion, define $\alpha : [0,\infty) \to [0,1)$ by

$$\alpha(t) \ =\ 1 - \exp(-t),\ t \geq 0.$$

Define $w : [0, \infty \to H$ by

$$w(t) \ =\ z(\alpha(t)),\ t \geq 0$$

and note that

$$(F(w))'(t) \ =\ ((F(z))(\alpha))'(t) \ =\ (F(z))'(\alpha(t))\alpha'(t) \ =$$
$$- \exp(-t)F(v) \ =\ -F(w(t)),\ t \geq 0$$

since

$$F(w(t)) \ =\ F(z(\alpha(t))) \ =\ (1-\alpha(t))F(v) \ =\ \exp(-t)F(v).$$

As noted above, $F(z(1)) = 0$ so it follows that

$$u \ =\ J - \lim_{t\to\infty} w(t) \ =\ z(1),$$

and so

$$F(u) \ =\ 0.$$

□

4. Application to a Nash-Moser Inverse Function Theorem

The above theorem can be used to recover an inverse function theorem in [6]:

Theorem 2. *Suppose H, J, K are as in Theorem 1, $r, M > 0$ and G is a function with domain $B_r(0)$ so that $G(0) = 0$ and G is continuous as a function on J. Suppose furthermore that $g \in K$ and if $y \in b_r(0)$, there is $h \in B_M(0)$ so that*

$$G'(y)h \ = \ g.$$

Then there is $\lambda > 0$ so that

$$G(u) \ = \ \lambda g \text{ for some } u \in B_r(0).$$

Proof. Pick $\lambda > 0$ so that $\lambda M \leq r$. Take $v = 0$ in Theorem 1. Define F on $B_r(0)$ so that

$$F(x) \ = \ G(x) - \lambda g, \ x \in B_r(0).$$

Suppose $x \in b_r(0)$. Then there is $h_0 \in B_M(0)$ so that

$$G'(x)h_0 \ = \ g$$

and so with $h = \lambda h_0$,

$$G'(x)h \ = \ \lambda g,$$

i.e.,

$$F'(x)h \ = \ G'(x)h \ = \ \lambda g \ = \ -F(0), \ h \in B_r(0).$$

Applying Theorem 1 we get the existence of $u \in B_r(0)$ so that $F(u) = 0$, i.e.,

$$G(u) \ = \ \lambda g.$$

□

References

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