Math 4050 Review #1

1. State the definition of *prime number*.
2. Prove that there are an infinite number of prime numbers.
3. State (without proof) the *division algorithm*.
4. Find the prime factorization of $4,700,619$.
5. How many digits are there in the representation of 2^{100} in base 2? base 8? base 16?
6. State (without proof) the *fundamental theorem of arithmetic*.
7. Suppose that $a, b, q, r \in \mathbb{Z}$ so that $a = bq + r$. Prove that $\gcd(a, b) = \gcd(r, b)$.
8. Express the number $5,197$ in (a) base 5; (b) base 16.
9. Express the number 123_{six} in base 10.
10. State and prove the rule for checking if a number is divisible by 3.
11. State and prove the rule for checking if a number is divisible by 4.
12.
    - If $k \geq 0$, prove that $11 \mid 10^{2k} - 1$. (This can be done either with or without induction.)
    - If $k \geq 0$, prove that $11 \mid 10^{2k+1} - 10$.
    - State and prove the rule for checking if a number is divisible by 11.
13. Let $n \in \mathbb{Z}$, and suppose that $2^n - 1 = p$ is prime. (For example, if $n = 3$, then $p = 7$.)
    - List all factors of $N = 2^{n-1}p$. (For example, for $n = 3$, the factors are $1, 2, 4, 7, 14$ and 28.)
    - Prove that the sum of all the factors of N is equal to $2N$. In other words, prove that N is a *perfect* number.
14. Let $a, b, d, m, n \in \mathbb{Z}$. Prove: If $d \mid a$, then $d \mid ma + nd$.
15. Let pqr_b be a 3-digit number in base b. Prove that $pqr_b - rqp_b$ is a multiple of $b - 1$ and $b + 1$.
16. Prove that $\sqrt{7}$ is irrational.
17. Let p and q be distinct primes and $m, n \in \mathbb{Z}^+$ with $m > n$. Find $\gcd(p^m q^n, p^n q^m)$ and $\operatorname{lcm}(p^m q^n, p^n q^m)$.
18. Find $\gcd(4912, 6860)$ and $\operatorname{lcm}(4912, 6860)$.
19. Describe the sieve of Erastonthenes and what it is used for.
20. Give an example of integers a, b and c so that $a \mid bc$ but $a \nmid b$ and $a \nmid c$.
21. Use lattice multiplication to find 12×27, and explain why this technique works.
22. Without a calculator, find $\sqrt{691}$ to one decimal place, and explain why your technique works.
23. Find a number a with exactly 12 positive divisors.
24. Suppose that $\gcd(b, d) = \gcd(a, b) = \gcd(c, d) = 1$. Prove that $\dfrac{ad + bc}{bd}$ is in simplest form.
25. For homework, you analyzed a magic trick which used binary numbers to identify any number from 1 to 63 using only six cards. Make a similar magic trick using base-3 arithmetic to identify any number from 1 to 26 using six cards.