Many problems are taken from the following sources:

- Richard J. Crouse and Clifford W. Sloyer, *Mathematical Questions from the Classroom, Parts I and II*, Janson Publications, Providence, 1987.
- Ira J. Papick, "Strengthening the Mathematical Content Knowledge of Middle and Secondary Mathematics Teachers," *Notices of the American Mathematical Society*, Vol. 58, No. 3 (March 2011), pp. 389–392.

1. A seventh-grade student asks you, "Why when multiplying or dividing decimals do we put the decimal point where we do?"

2. A bright student asks, "I noticed the following pattern:

$$\begin{aligned} 256 \times 256 &= 65,5\underline{36}, \\ 257 \times 257 &= 66,0\underline{49}, \\ 258 \times 258 &= 66,5\underline{64}. \end{aligned}$$

   Is there a reason why the last two digits are always perfect squares? I know it usually doesn't work out this way."

3. A student asks, "Why does the finger trick for multiplying by 9 work?"

4. A student asks, "Why does the telephone trick for multiplying by 7 work?"

5. A student asks, "Can you give me a real-world example for why adding and multiplying fractions are important?"

6. A student asks, "Can you give a real-world example to explain the relative sizes of a thousand, a million, a billion, and a trillion?"

7. The homework assignment asks to find the next term in the list of numbers $3, 5, 7, \ldots$ John said the answer is 9 (he was thinking of odd numbers). Sally said the answer is 11 (she was thinking odd prime numbers). Mary said the answer is 3 (she was thinking of a periodic pattern). Who is right?

8. A student asks, "Why is $x^0 = 1$?"

9. A student asks, "Why is $x^{-4} = 1/x^4$?"

10. A student asks, "What is 0^{-7}?"

11. A student asks, "You said $|a| = -a$, but how can the absolute value of a number be negative?

12. A student asks, "My father was helping me with my homework last night and he said the book is wrong. He said that $\sqrt{4} = 2$ and $\sqrt{4} = -2$, because $2^2 = 4$ and $(-2)^2 = 4$, but the book says that $\sqrt{4} \neq -2$. He wants to know why we are using a book that has mistakes."

13. A sixth-grader asks why $3\frac{1}{3} \times \frac{3}{5} = 2$. What the student does not understand is that the answer, 2, is smaller than one of the factors since multiplication is just repeated addition.

14. A student explains that $\frac{1}{5} + \frac{2}{3} = \frac{3}{8}$ as follows: "If a baseball player is up at bat 5 times on the first day and gets one hit, and on the second day he gets 2 hits out of 3 at-bats, then altogether he has 3 hits out of 8 at-bats."

15. A student asks, "Can you divide fractional numbers by separately dividing the numerators and the denominators?" For example,

$$\frac{14}{9} \div \frac{2}{3} = \frac{14 \div 2}{9 \div 3} = \frac{7}{3}.$$

16. An eighth-grade student hands in the following solution:

$$\frac{64}{16} = \frac{\not{6}4}{1\not{6}} = \frac{4}{1} = 4.$$

17. There is $\frac{3}{4}$ of a pie left. Jane and Sue want to divide what is left in half so they each get an equal part. How much do each of them receive. An answer:

$$\frac{3}{4} \div \frac{1}{2} = \frac{3}{4} \times \frac{2}{1} = \frac{6}{4} = 1\frac{2}{4} = 1\frac{1}{2}.$$

18. A student asks, "My teacher from last year told me that whatever I do to one side of an equation, I must do the same thing to the other side to keep the equality true. I can't figure out what I'm doing wrong by adding 1 to the numerator of both fractions in the equality $\frac{1}{2} = \frac{2}{4}$ and getting $\frac{2}{2} = \frac{3}{4}$.

19. You are teaching division of fractions. A seventh-grader asks, "Why does multiplying by the reciprocal of the divisor give the correct answer?"

20. A student in your class asks, "Why don't we need to get common denominators when multiplying fractions?"

21. A teacher asks a student the following question. "Which would you rather have, 20/35 of a cake or 4/7 of a cake?" The student's answer: "20/35 because there are more pieces."

22. You have given your class the following problem: "A 20-foot board is to be cut into shorter pieces. If each piece is $4\frac{3}{4}$ feet long, how many pieces will there be?" One student hands in the following work:

$$\text{“}20 \div 4\frac{3}{4} = 20 \div \frac{19}{4} = 20\frac{4}{19} = \frac{80}{19} = 4\frac{4}{19}.$$

    Thus there are 4 pieces and 4/19 of a foot left over."

23. A student asks, "You always ask us to explain our thinking. I know that two fractions can be equal, but their numerators and denominators don't have to be equal. What about if $\frac{a}{b} = \frac{c}{d}$, and they are both reduced to simplest form. Does $a = c$ and $b = d$, and how should we explain this?

24. A student asks, "Can you give a real-world examples when linear functions would be used?"

25. You are teaching your students the idea of the degree measure of an angle and you have related it to the circle. One student makes the assertion that the degree will vary because the angle intercepts arc of different lengths depending on the size of the circle.

26. A student asks, "Is a straight angle of 180° really an angle?"

27. You ask your eighth-grade students to find the area of a circle with a radius of 5 inches. One of your students answers, "A circle is defined to be the set of all points a fixed distance away from a given point. So we cannot find the area of a circle; we can only find its circumference." Is the student right?

28. While you are teaching your class the formulas for simple polygons (triangle, square, etc.), a student excitedly says that she has a formula that does it all: $A = \frac{1}{2}h(t+b)$ where

    - A is the area of the polygon,
    - h is the height,
    - t is the length of the top segment,
    - b is the length of the bottom segment.

29. A student asks, "If two figures have the same area and the same perimeter, are the figures congruent?"

30. A student is asked the definition of congruence and responds, "Congruence means equal."

31. A student in your eighth-grade class is trying to solve the following problem: "A man drives a car for one mile at 60 miles per hour and for another mile at 120 miles per hour. What is his average speed?" The student reasons that the answer should be $(60 + 120)/2 = 90$ miles per hour.

32. A junior-high student is excited: "Miss Jones, I've found a new way to get the average of two numbers. Take 12 and 18, for example. Subtract 12 from 18; that's 6. Divide by 2, you get 3. Add 3 to 12, and you get the average, 15." If you were Miss Jones, what would you tell the student?

33. A student provides the following work:

$$\frac{7.2 \times 10^8}{1.2 \times 10^5} = (7.2 - 1.2) \times 10^{8-5} = 6 \times 10^3.$$

34. A student asks, "When you subtract two negatives, why do you have to change the sign of the subtrahend?"

35. A student says, "It doesn't make any sense to me that when you divide a number like -10 into -200 you get 20 because that means when you divided a small number into an even smaller number you get a great big number."

36. A student says, "$6 \div 0$ equals nothing. Since you are dividing by nothing, you get nothing as a result."

37. You have explained that $0 \div 0$ is undefined because the value could have too many answers. A student asks, "Why can't we just define $0 \div 0$ to be equal to one and always use it that way?"

38. While studying a unit on the properties of 0 and 1, a student says that not only is zero an additive identity but it is also an identity with respect to subtraction.

39. While working on a unit covering negative integers, you make the statement that $-3 < -1$. However, a student doesn't believe it. He says if you have a debt of three dollars then that is larger than a debt of one dollar. How do you convince him that $-3 < -1$?

40. You ask a student to find 28% of 50. A student's solution: "$28 \times 0.5 = 14$." Is this student correct?

41. A student asks, "My father bought a house for \$200,000 at 12% interest. He told me that by the time he finishes paying for the house, it will have cost him more than \$500,000. How is that possible? 12% of \$200,000 is only \$24,000."

42. A student asks, "If $5^2 = 25$ and $(-5)^2 = 25$, then why doesn't $\sqrt{25} = \pm 5$?"

43. A student asks, "Why are trigonometric functions important?"

44. A student asks, "Why are inverse trigonometric functions important?"

45. A teacher of eighth-grade math writes on the board a right triangle $\triangle ABC$ with the following labels: B is the right angle, $AB = x$, $AC = 20$, $BC = 12$, $m\angle BAC = 20°$, $m\angle ACB = 70°$. The students were told to solve for x. However:

    - $\tan 20° = 12/x$; so $x \approx 32.6$
    - $\sin 70° = x/20$; so $x \approx 18.8$
    - From the Pythagorean theorem, $x = 16$

    What is wrong?

46. A student asks, "The carpenter who is remodeling our kitchen told me that geometry is important. He said he uses his tape measure and the Pythagorean theorem to tell if a corner is square. He marks off 3 inches on one edge of the corner, 4 inches on the other edge, and then connects the marks. If the line connecting them is 5 inches long, he knows by the Pythagorean theorem that the corner is square. This seems different from the way we learned the Pythagorean theorem."

47. A student asks, "Why aren't '$|x + y| = 5$' and '$|x| + |y| = 5$' the same?"

48. A student asks, "Why does the book say that a polynomial $a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = 0$ if and only if each $a_i = 0$, and then later says that $2x^2 + 5x + 3 = 0$?

49. A student asks, "Can you give me a real-world examples for when quadratic functions would be used?"

50. A student hands in the following work:

    $$\frac{x+6}{2} = \frac{x+\cancel{6}^{3}}{\cancel{2}} = x+3$$

    When the student asks why he can't cancel the 2 into the 6, how would you answer?

51. A tenth-grade student hands in the following work:

    $$\frac{x^{\cancel{2}} - y^{\cancel{2}}}{\cancel{x} + \cancel{y}} = x - y$$

52. A student asks, "My algebra teacher said

    $$\frac{x^2+x-6}{x-2} = \frac{(x+3)(x-2)}{x-2} = x+3,$$

    but my sister's boyfriend (who is in college) says that they are not equal, because the original expression is not defined at 2, but the other expression equals 5 when evaluated at 2."

53. A student asks you if $\sqrt{a^2 + b^2} = a + b$.

54. A student who was told that $0.999\ldots = 1$ says, "Then you should be able to change 9/9 to $0.999\ldots$ by dividing the denominator into the numerator. But $9/9 = 1$ and not $9/9 = 0.999\ldots$"

55. While solving the equation

$$\frac{x}{x-1} = 1 + \frac{1}{x-1}$$

a student gets down to the part where $x = x$ and $x \neq 1$. The student says that she doesn't know where to go from here. How would you help this student?

56. A student presents the following work:

$$\begin{aligned} \frac{3}{8} &= \frac{x+2}{16} \\ \frac{3-2}{8-2} &= \frac{x+2-2}{16-2} \\ \frac{1}{6} &= \frac{x}{14} \\ 6x &= 14 \\ x &= \frac{14}{6} \end{aligned}$$

57. A student presents the following work:

$$\begin{aligned} 4x^2 - 23x + 15 &= 0 \\ t^2 - 23t + (15)(4) &= 0 \\ t^2 - 23t + 60 &= 0 \\ (t-20)(t-3) &= 0 \\ t = 20 \quad &or \quad t = 3 \end{aligned}$$

Therefore, $x = 20/4 = 5$ or $x = 3/4$.

58. A student asks you if x/x and $(x-3)/(x-3)$ are equivalent expressions.

59. A student presents the following solution:

$$\begin{aligned} x^3 &= 2x^2 + 8x \\ x^2 &= 2x + 8 \\ x^2 - 2x - 8 &= 0 \\ (x-4)(x+2) &= 0 \\ x = 4 \quad &or \quad x = -2 \end{aligned}$$

60. A ninth-grade student says, "Since $(a \cdot b)^2 = a^2 \cdot b^2$, we can follow the pattern so that $(a+b)^2 = a^2 + b^2$."

61. A student asks, "I understand why you would want to square or cube a number. But why would I want to raise a number to a fractional exponent?"

62. A student asks, "Can you change a positive exponent to a negative one by taking the reciprocal of the base? Does $2^{2/3} = (1/2)^{-2/3}$?"

63. A student asks, "Can you give a real-world examples explaining when exponential functions would be used?"

64. You are asked, "How would you evaluate 2^{3^2}? Is it $8^2 = 64$ or $2^9 = 512$?"

65. A student presents the following work:

$$y^2 - 2xy + x^2 - w^2 = (y-x)^2 - w^2 = (y-x+w)(y-x-w).$$

Another student presents the following:

$$y^2 - 2xy + x^2 - w^2 = x^2 - 2xy + y^2 - w^2 = (x-y)^2 - w^2 = (x-y+w)(x-y-w).$$

Which way is correct?

66. You are teaching factoring of quadratic polynomials. A student asks, "Is there any other way, other than trial and error?"

67. You have given your student a quiz on factoring polynomials. A student presents the following solution:

$$3x + 8 = 3\left(x + \frac{8}{3}\right)$$

Do you take off points or not?

68. A student presents the following work:

$$\begin{array}{rcl} x^2 - 14x + 24 & = & 3 \\ (x-12)(x-2) & = & 3 \\ (x-12)(x-2) & = & 3 \cdot 1 \\ x - 12 = 3 & \textit{or} & x - 2 = 1 \\ x = 15 & \textit{or} & x = 3 \end{array}$$

69. A student suddenly smiles and makes the following assertion: "Addition, subtraction, multiplication and division are all examples of functions, aren't they?" What response do you give?

70. You give your students the following problem: "Graph $x = 3$." One student shows an $xy-$graph with a vertical line with $x-$intercept 3. Another student shows a number line with a closed circle on the number 3. Which student is correct?

71. A student asks you if a line having zero slope means the same thing as having no slope.

72. You solve $3x + 3 < x + 5$ over the set of integers and come out with $x < 1$. A student asks, "How would you check this problem as you would with an equation?"

73. A student asks, "How do I know whether the solution of an inequality, such as $|x - 3| > 1$, is the union or the intersection of the individual solution sets of each part?"

74. A student asks, "When we write $a \leq b$, we mean that a is less than b or a is equal to b. But when we write $a \not\leq b$, you say that this means that a is not less than b <u>and</u> a is not equal to b. I don't understand why it changes from <u>or</u> to <u>and</u>."

75. While studying a unit on inequalities, a student asks you if it's true that if $a > b$ and $c > d$, then $a - c > b - d$.

76. In your ninth-grade algebra class, you are asked, "Does $(a + b)(c + d) = ac + bd$?"

77. In a discussion on rates, a student says, "If I walk 4 mph for one mile, I'll have to run 8 mph the second mile in order to average 6 mph for 2 miles. Right?"

78. A student asks you if $\dfrac{1}{a+b} = \dfrac{1}{a} + \dfrac{1}{b}$.

79. A student hands in the following work: $(6t)\sqrt{3} = (6\sqrt{3})(t\sqrt{3})$.

80. Your students have just been told that they can't add 5 days and 2 hours which changing one of the units. When asked what to do with $2x + 5y$, one student wants to know how he can change x and y so that they can be added.

81. A student provides the following work: $(3 + x) - (x - 5) = 3 + (x - x) - 5 = 3 - 5 = -2$.

82. You make the statement that the polynomial $x^2 - 2$ cannot be factored. One student says, "Yes it can, because $x^2 - 2 = (x - \sqrt{2})(x + \sqrt{2})$. What reply would you make?

83. A student asks, "Is it correct to write $2^{1/16}$ as $\sqrt{\sqrt{\sqrt{\sqrt{2}}}}$?

84. In response to the question, "What is $\sqrt{x^2}$?", a student says the answer is x.

85. A student asks, "Why isn't $\sqrt{3} + \sqrt{12} = \sqrt{15}$?"

86. A student is trying to solve the set of simultaneous linear equations:

$$\begin{aligned} 2x + 3y &= 6 \\ -6x - 9y &= -14 \end{aligned}$$

He comes up with $0 = 4$ but doesn't know what it means.

87. A student is trying to solve the set of simultaneous linear equations:

$$\begin{aligned} 2x + 3y &= 6 \\ -4x - 6y &= -12 \end{aligned}$$

A student ends up with $0 = 0$ and doesn't know what to do from here.

88. A student hands in the following work:

$$\begin{aligned} 2x + 3y &= 4 \\ x - y &= 5 \\ \\ -x - \frac{3}{2}y &= -2 \\ x - y &= 5 \\ \\ -\frac{5}{2}y &= 3 \\ \\ y &= -\frac{6}{5}. \end{aligned}$$

Is the student's work correct?

89. A student asks, "Can you give me a real-world examples explaining when a rational function would be used?"

90. The problem is: *John rows up the river at 6 mph and down the river at 8 mph. What would John's rate be in still water and what is the rate of the current of the river?* A student says, "To solve this, I add 6 mph and 8 mph and divide by 2. So John's rate in still water is 7 mph. To find the rate of the current, I subtract 7 from 8 or else subtract 6 from 7. The rate of the current is 1 mph." Is the student correct?

91. The problem is: *A person invests half a sum of 3.5% interest and half a sum at 4.5% interest. At the end of a year, the person received $300 in interest. How much did the person invest at each sum?* The solution is done in class:

$$\begin{aligned} \frac{1}{2}(0.035)x + \frac{1}{2}(0.045)x &= 300 \\ 0.04x &= 300 \\ x &= \$7,500 \end{aligned}$$

A student asks, "Why can't you always average the two interest rates and then divide that percent rate into the interest to find the principal?"

92. The problem is: *A pump can fill a tank in 3 hours. With the drain open, it will empty in 5 hours. With the pump working and then drain open, how long will it take to fill the tank?* The response:

$$\begin{aligned} \left(\frac{1}{3} - \frac{1}{5}\right)x &= 1 \\ \left(\frac{5}{15} - \frac{3}{15}\right)x &= \frac{15}{15} \\ \frac{2}{15}x &= \frac{15}{15} \\ x &= \frac{15}{2} \end{aligned}$$

Is the solution correct?

93. The problem is: *In an election, one candidate received 96 votes more than his opponent. Since there were 804 votes cast, the candidates received _______ votes and _______ votes.* A student says, "Divide 804 by 2 since there were 2 candidates. $804/2 = 402$. One candidate got $402+96 = 498$ votes. The other got $804-498 = 306$ votes." Another student asks, "Why doesn't this work?"

94. The problem is: *A boat travels downstream 24 miles in two hours and comes back upstream in four hours. Find the rate of the boat in still water.* A ninth-grade student reasons that the current going one way cancels it going the other way. Therefore, the boat goes a distance of 48 miles in six hours, or 8 mph. Is the student correct?

95. Your class is doing a word problem concerning three consecutive even integers. You ask how these numbers can be represented, and a student answers, "x, $x + 2$, $x + 4$." Is the student correct?

96. After showing students how to bisect an angle, how do you answer the student who asks, "Is it possible to divide an angle into three equal parts?"

97. A textbook makes the statement that an acute angle is an angle whose measure is less than 90°. A student then asks you if an angle whose measure is zero is an acute angle.

98. A student asks, "If the word *point* is undefined, how is it possible to use something that is not really there?"

99. A student asks, "What is the difference between *necessary* and *sufficient* in the statement and proof of a theorem?"

100. A student asks you why axioms are not proved as theorems are.

101. While studying circles in geometry class, a student asks you what is the relationship between a tangent to a circle and the tangent of an angle.

102. You ask your students to draw two circles such that they have three common tangents. One student says it can't be done. Is the student right?

103. A student asks you if all circles are similar.

104. A student asks, "Can circles be circumscribed only about regular polygons?"

105. A student asks, "Why is there no SSA or AAA postulate?"

106. A student asks, "In graphing an inequality of the form $ax + by < c$, why is it sufficient to test only one point, not on $ax + by = c$, to find which region to shade?"

107. You ask your class if the following statement is true or false: *If not a square, then not a rhombus.* One student says that "if not a square" implies that it could be a rectangle and a rectangle is not a rhombus. Therefore the statement is true. Do you agree?

108. During a lesson on indirect proofs, a student says, "Why do we have to assume the opposite of the conclusion? Why can't we assume the opposite of the given and try to contradict the conclusion?"

109. You ask a student to give you the negation of "All men are good." The student says the negation would be "All men are not good."

110. You ask your students to put the following statement into "if-then" form: *Two planes intersect provided that they are not parallel.* One student writes, "If two planes intersect, then they are not parallel." Is the student right?

111. A student asks, "Is it ever possible to prove a statement or theorem by proving a special case of the statement or theorem?"

112. The problem: *What is the negation of the statement "Some men are not wise?"* A student replies that the negation is "Some men are wise?" Do you agree?

113. Your class asks the difference between the original statement and the contrapositive.

114. A student asks, "If a regular polyhedron is made up of regular polygons and there are an infinite number of regular polygons, then why are there only five regular polyhedrons?"

115. A student asks you if parallel lines meet at infinity.

116. One student says that segments which have no points in common are either parallel or skew segments. Do you agree?

117. A student asks, "If a line l is perpendicular to a line in the plane, is l perpendicular to the plane?"

118. A student asks, "If two polygons have the corresponding angles equal, are their sides proportional?"

119. A student makes the following statement: "It looks like if you join the midpoints of the sides of a quadrilateral, the resulting inscribed figure is always a parallelogram."

120. A student asks, "Is the area of a geometric (two-dimensional) figure always smaller than its perimeter? Likewise, is the volume of a solid always smaller than its surface area?"

121. A student asks you if it is possible for a figure to be symmetric about a point but not symmetric about any line.

122. A student asks, "If a square $ABCD$ is folded on either of its diagonals, it is mapped onto itself. Does this occur for a rhombus which is not a square?"

123. A student asks you if there is such a thing as equivalent triangles.

124. A student asks, "Is is possible for one triangle to have five parts (angles, sides) congruent to five parts of another triangle and still not have the two triangles congruent?"

125. A student asks you if there are any triangles in which the perimeter equals the area.

126. A student asks, "If a leg and an acute angle of one right triangle are congruent to a leg and an acute angle of another right triangle, then the triangles are congruent." Do you agree?

127. A student asks, "If $\triangle ABC \cong \triangle DEF$, is $\triangle ABC \cong \triangle DFE$?

128. A student asks, "If $\triangle ABC \sim \triangle DEF$, is $\triangle ABC \sim \triangle DFE$?

129. A student asks, "Why is the triangle inequality so named?"

130. A student asks, "Does the bisector of an angle in a triangle also bisect the opposite side?"

131. A student asks, "If we solve $|x+2| \geq 7$, we get $x \geq 5$ or $x \leq -9$. If we solve $|x+2| \leq 7$, we get $x \leq 5$ and $x \geq -9$. How do we account for the use of *and* and *or*?"

132. A student solves the problem $|x-5| \geq 6$ as follows:

$$\begin{array}{rcl} |x-5| & \geq & 6 \\ x-5 \geq 6 & & -x+5 \geq 6 \\ x \geq 11 & & -1 \geq x \end{array}$$

Therefore, $-1 \geq x \geq 11$.

133. A student asks you if binomial coefficients are defined for fractions and negative numbers and, if so, what are the definitions?

134. The problem is, *Find the sum of the coefficients of* $(x-2y)^{18}$. A student's answer, "All you need to do is compute $(1-2)^{18} = (-1)^{18} = 1$." Is the method correct? Is the solution correct?

135. You have just shown your class Pascal's triangle. A student then asks, "Do the diagonals of Pascal's triangle have any interpretation?"

136. While studying Pascal's triangle, a student asks you if there is a corresponding configuration for figuring out the coefficients of a trinomial.

137. A student asks you if the binomial theorem can be extended for negative exponents.

138. A student asks you if the binomial theorem can be extended for fractional exponents.

139. A student asks, "Why is $0! = 1$?"

140. A student asks if factorial can be defined for fractions or for negative numbers.

141. A student asks, "We can use the quadratic formula to find the roots of any quadratic equation. Is there a similar formula for cubic equations?"

142. You ask your class to solve the following equation: $3^{2x+1} + 3^2 = 3^{x+3} + 3^x$. A student hands in the following work, "Since we are adding numbers of the same base, we multiply the exponents.

$$\begin{aligned} (2x+1)2 &= (x+3)x \\ 4x+2 &= x^2+3x \\ 0 &= x^2-x-2 \\ 0 &= (x-2)(x+1) \end{aligned}$$

Therefore, $x = 2$ or $x = -1$." Is the student correct?

143. A student's solution:

$$\begin{aligned} \sqrt{2m+1} &= \sqrt{m}+3 \\ 2m+1 &= m+9 \\ m &= 8 \end{aligned}$$

144. A student solves $x^2 + ix + 2 = 0$ and finds that the roots are i and $-2i$. He then says that this example contradicts the theorem which says that if $b^2 - 4ac < 0$, then the equation has two complex conjugate roots.

145. A student asks, "What is the difference between the equations

$$\{(x, y) \mid y = x^2 - x - 6\}$$

and

$$\{(x, y) \mid x^2 - x - 6 = 0\}?$$

They look the same to me."

146. A student asks, "What are complex numbers good for?"

147. A student makes the following observation:

$$(-8)^{1/3} = \sqrt[3]{-8} = -2,$$

but

$$(-8)^{2/6} = \sqrt[6]{(-8)^2} = 2.$$

This student then reasons that $(-8)^{1/3} \neq (-8)^{2/6}$ even though $1/3 = 2/6$.

148. A student asks if the following is correct:

$$\begin{aligned} \sqrt{(-3)^2} &= [(-3)^2]^{1/2} \\ &= (-3)^{2 \cdot 1/2} \\ \sqrt{9} &= -3. \end{aligned}$$

How would you respond?

149. A student hands in the following: $(x^{-1} + y^{-1})^{-1} = x + y$.

150. A student asks you if $(\sqrt[n]{b})^n = \sqrt[n]{b^n}$ when $b \geq 0$.

151. A student hands in the following:

$$\frac{2^{\sqrt{27}}}{8^{\sqrt{3}}} = \frac{1^{\sqrt{27}}}{4^{\sqrt{3}}} = \frac{1}{4^{\sqrt{3}}}.$$

Is the student correct?

152. A student asks you if there is a function that is equal to its inverse.

153. A student asks, "If f has an inverse function, does that mean f is either strictly increasing or strictly decreasing?"

154. A student asks, "How can $f(x) = x^2$ and $f(t) = t^2$ be the same function if different letters are used?"

155. A student asks, "What is the difference between $f = 0$ and $f(x) = 0$?"

156. A student asks, "If f and g are periodic functions, is $f + g$ is also periodic?"

157. A student asks, "If two functions, f and g, have the exact same range and domain, are the two functions identical?"

158. A student asks, "Explain again the difference between fg and $f \circ g$. How do you determine the domain of each?"

159. A student asks, "Why is an inverse function a reflection about the line $y = x$?"

160. A student asks, "Why is it important to determine zeroes of a function?"

161. A student asks, "How can you tell if a function is neither odd nor even?"

162. A student asks, "What does the graph of $x^2 + y^2 = -9$ look like?"

163. A student asks, "Isn't a circle just a special case of an ellipse?"

164. A student asks you if the graph of $y = (x^2)^2 = x^4$ is also a parabola.

165. A student asks, "If $y = ax^2$ is the equation for the family of parabolas in standard position, is the equation $y = 0x^2$ considered a parabola?"

166. A student asks, "Is it possible to define the ellipse and the hyperbola in terms of a focus and directrix as one does with a parabola?"

167. A student comments, "A hyperbola is really just two parabolas put together."

168. A student asks you for some applications of the conic sections.

169. A student asks, "If $0.\overline{9} = 1$, then shouldn't it follow that eventually a hyperbola should meet its asymptote?"

170. A student asks, "How do you know for such that some irrational number does not have a repeating decimal eventually?"

171. A student asks, "How can one prove that π or e are irrational?"

172. A student asks, "Are all nonterminating decimals irrational numbers?"

173. A student asks, "What is e? How does it arise in mathematics? Why does such a complicated number become so important?"

174. A student asks, “Why are logarithms are important?”

175. A student asks you why logarithms in base e are called natural logarithms and why they are so important.

176. A student asks, “I have worked this problem four times, but I don’t get the answer that is in the book:

$$\frac{\log_{10} 4}{\log_{10} 2} = \log_{10} 4 - \log_{10} 2 = 0.602 - 0.301 = 0.301.$$

The answer in the book is 2. Is the book wrong?”

177. A student asks, “Why doesn’t $\log_b(x + y) = \log_b x + \log_b y$?”

178. A student asks you why you can’t find the log of a negative number, like $\ln(-1)$.

179. The problem: *Prove that if a and b are real numbers such that $ab = 0$, then $a = 0$ or $b = 0$.* The student said, “I will restate the conclusion of the theorem. What I shall prove is that if $a \neq 0$, then $b = 0$.” However, classmates were not convinced. Some said that one should also prove that if $b \neq 0$, then $a = 0$.

180. A student asks you how one proves that there is no largest integer.

181. You ask a student to explain why zero is the geometric mean of zero and any real number. One student says that zero cannot be the geometric mean, or else that would mean that $0/0 = 0/r$, and $0/0$ is undefined.

182. A student asks you why $\sqrt{ab}$ is called the geometric mean of a and b.

183. A seventh-grade student asks, “What is the difference between 3 and +3?”

184. A student comments, “$C = \pi D$, so $\pi = C/D$, and therefore π is a rational number.”

185. A student says, “$3 + 4i = 4 + 3i$ because numbers are commutative.”

186. A student asks, “Are there numbers other than those in the complex number system?”

187. A student asks you if imaginary numbers are only numbers in our imagination.

188. A student asks you why they call complex number *complex*.

189. A student presents the following work: $\sqrt{-4}\sqrt{-9} = \sqrt{36} = 6$. Is the student correct?

190. A student asks, “If $\sqrt{-1} = i$, then what is $\sqrt{i}$?”

191. A student says that since $-1 > -4$, then $\sqrt{-1} > \sqrt{-4}$, or $i > 2i$.

192. A student asks you for some applications of complex numbers.

193. A student asks, “Is $\sqrt{-4}$ equal to $2i$ or $-2i$?”

194. A student submits the following work:

$$\frac{\sqrt{-4}}{\sqrt{-9}} = \sqrt{\frac{4}{9}} = \frac{2}{3}.$$

195. A student submits the following work:

$$\frac{\sqrt{4}}{\sqrt{-9}} = \sqrt{\frac{-4}{9}} = i\frac{2}{3}.$$

196. A student asks, "Can we define $<$ for complex numbers? For example, is it true that $i < 2i$?"

197. A student asks you, "What is the difference between a vector an a ray?"

198. A student asks you why the constant polynomial zero does not have a degree.

199. A student asks, "When we divide $4x^2-5x+9$ by $x-1$, is the quotient $4x-1$ or is it $4x-1+\frac{8}{x-1}$?"

200. One student asks you the rationale behind synthetic division.

201. A student asks, "Why can a rational function cross its horizontal asymptote but not its vertical asymptote(s)?"

202. A student asks, "I know how an asymptote behaves, but what is a rigorous definition?"

203. A student asks, "Aren't the functions $f(x) = \dfrac{x^2-9}{x-3}$ and $g(x) = x+3$ really the same function?

204. You have just shown your class how to use the roots of a polynomial to provide a sketch of its graph. A student asks, "How can we be sure of the precise locations of the peaks and valleys in the graph?"

205. A student asks, "You say that a polynomial has a unique factorization of linear terms. But for ordinary factoring, there are a lot of ways that it can be done, like $12 = 2 \times 6 = 3 \times 4$. Why is this different for polynomials?"

206. You ask your class to simplify

$$\frac{1+1/\sqrt{3}}{1-1/\sqrt{3}}$$

One student says all you have to do is invert each part of the fraction; i.e.,

$$\frac{1+1/\sqrt{3}}{1-1/\sqrt{3}} = \frac{1/1+1/(1/\sqrt{3})}{1/1-1/(1/\sqrt{3})} = \frac{1+\sqrt{3}}{1-\sqrt{3}}.$$

207. A student asks, "Why should we rationalize the denominator? Isn't $1/\sqrt{2}$ just as good as $\sqrt{2}/2$?"

208. A student asks if $\sqrt[n]{a^m} = \sqrt[m/n]{a}$.

209. A student asks, "Is there such a thing as $\sqrt[1/2]{x}$?"

210. A student asks, "Is $\sqrt[-3]{7}$ defined?"

211. The problem is, *Find the real roots of* $x + \sqrt{x-2} = 4$. Using standard methods, this students finds that 3 and 6 are possible solutions, but only 3 is a solution. He asks you why it is that 6 is not a solution (i.e., does not work).

212. A student asks, "You said that 3 was a zero of $x^2 - 4x + 3 = 0$, but 3 is not zero. I don't understand."

213. A teacher presents the following solution:

$$\begin{aligned} 2-y &= 2\sqrt{y+1} \\ (2-y)^2 &= [2\sqrt{y+1}]^2 \\ 4-4y+y^2 &= 4y+4 \\ y^2-8y &= 0 \\ y(y-8) &= 0 \\ y=0 \quad &\textit{or} \quad y=8 \end{aligned}$$

A student then asks why $[2\sqrt{y+1}]^2$ wasn't written as $4|y+1|$ because squaring makes a number positive, but $4y+4$ could be negative.

214. A student asks, "We have that the system of equations

$$3x + 4y = 18 \qquad \text{and} \qquad 2x - y = 1$$

is equivalent to the system

$$x = 2 \qquad \text{and} \qquad y = 3.$$

However, $x = 2$ is not equivalent to either $3x + 4y = 18$ or $2x - y = 1$. Why?"

215. A student asks, "Isn't a determinant really just a function?"

216. A student asks, "What good are determinants?"

217. A student asks, "Why do we write $\sin^2 x$ instead of $\sin x^2$?"

218. A student asks, "Why is true that $\sin^2\theta + \cos^2\theta = 1$?"

219. A student asks, "What is the difference between deriving an identity and verifying an identity?"

220. The problem is: Prove that $\sec\theta - \cos\theta = \tan\theta\sin\theta$. A student's solution:

$$\begin{aligned} \frac{1-\cos^2\theta}{\cos\theta} &= \tan\theta\sin\theta \\ 1-\cos^2\theta &= \cos\theta\tan\theta\sin\theta \\ 1-\cos^2\theta &= \sin^2\theta, \end{aligned}$$

which is a known identity.

221. A student writes that $\cos(a-b) = \cos a - \cos b$. When you tell him that he is incorrect, he replies that he is using the distributive property.

222. A student asks, "Why does $\sin^{-1} x$ and $\tan^{-1} x$ take values between $-\pi/2$ and $\pi/2$ but $\cos^{-1} x$ take values between 0 and π?

223. A student asks, "Is there any relationship between the secant line and the secant function?"

224. A student presents the following solution:

$$\begin{aligned} \tan x \cos x &= 1 \\ \frac{\sin x}{\cos x}\cos x &= 1 \\ \sin x &= 1 \\ x &= \frac{\pi}{2} + \pi n \end{aligned}$$

225. A student asks, "We have defined the trigonometric functions three ways: with right triangles, with an angle in standard position, and with the unit circle. Are all of these definitions the same? Why do we have three different definitions of the same concept?"

226. A student asks, "In Algebra I, we learned that $a^{-1} = 1/a$. How come $\sin^{-1} x$ is not equal to $\csc x$?"

227. A student asks you if it is possible to write any equation of the form

$$y = A\cos(bx + c_1) + B\sin(bx + c_2)$$

in terms of either a cosine function or a sine function of some kind?

228. A student writes "$\sec 38^\circ = \cos \frac{1}{38^\circ}$."

229. A student asks, "We know the SOHCAHTOA rule. But what do you do with this rule if the angle is 90°? Which side is adjacent?"

230. A student asks, "When graphing $y = \sin(x + \pi/3)$, you said to shift the sine curve $\pi/3$ to the left. Why don't we shift it to the right, since $\pi/3$ is positive?"

231. A student comments, "$\sin x$ and $\sin^{-1} x$ are inverse functions. Therefore, $\arcsin(\sin x) = x$."

232. A student asks why it's important to use radians instead of degrees.

233. A student asks you if the hyperbolic functions are related to the hyperbola as the circular functions are related to the circle.

234. A student asks, "Why are polar coordinates important?"

235. The problem is: *Find the coordinates of the point of intersection of the polar graphs of the equations $r = 2 + 4\sin\theta$ and $r = 2$.* A student asks, "When I solve the equation simultaneously, I get $(2, 0)$ and $(2, \pi)$ as the two solutions. But when I graph the equations, it seems as if the point $(2, \pi/2)$ is also a point of intersection. Why didn't I get this point by solving the equations?"

236. A student asks, "Can mathematical induction be used with negative integers?"

237. A student asks, "Can mathematical induction be used with only even integers?"

238. A student asks, "Is there mathematical induction for the entire set of real numbers?"

239. A student asks, "Isn't the principle behind mathematical induction just circular logic? After all, you're assuming the thing that you're trying to prove in the first place."

240. A student asks you if there is a definition of multiplication of vectors that results in a vector, and if so, what is it used for?

241. A student asks, "If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are nonzero vectors and $\mathbf{u} \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{v}$, is it true that $\mathbf{u} = \mathbf{w}$?"

242. A twelfth-grade student asks, "We have been working with vectors and you keep telling us that the arrows we see in the book are not vectors. I don't quite understand. Could you explain the difference between a vector an an arrow? What exactly is a vector?"

243. A twelfth-grade student asks, "What is the difference between permutations and combinations?"

244. A student asks you what the difference is between independent and mutually exclusive events.

245. You ask your twelfth-grade students to find the probability of both children being boys in a two-child family. One student says the probability is 1/3; another says it's 1/4. Is either student correct?

246. A student asks, "Why are sequences and series important?"

247. A student asks you if it is possible for a series to converge and yet the associated sequence be a divergent sequence.

248. A student asks, "How can the sum of an infinite number of terms be finite?"

249. A student asks you if a sequence can diverge even though it is bounded.

250. A student asks, "Are diverging and ∞ synonymous?"

251. A student asks, "Can a recursively defined sequence always be written in closed-form?"

252. A student asks, "Why doesn't

$$\sum_{i=1}^{n} a_i b_i = \left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right)?''$$

253. A student asks, "How do we know that we can use the formula for an arithmetic series to evaluate

$$\sum_{n=4}^{123} \frac{1}{2}(n+4)?''$$

254. A student asks, "What is meaning of ∞?"

255. A student asks, "Why is $0 \cdot \infty$ an indeterminate form since zero times anything is zero?"

256. A student asks, "When can one evaluate $\lim_{x\to a} f(x)$ simply by computing $f(a)$?

257. A student asks, "If $f(a)$ is defined, must $\lim_{x\to a} f(x)$ exist?

258. A student asks, "Why isn't the limit of the general term of an infinite series equal to zero sufficient for convergence of the series?"

259. A student asks, "In order to find the limit on some problems as the variable tends to infinity, why do you divide through by the highest power in the denominator?"

260. A student asks, "Why is the cancellation of common factors, such as

$$\lim_{x\to 4} \frac{x-4}{(x-4)(x+3)} = \lim_{x\to 4} \frac{1}{x+3} = \frac{1}{7},$$

logically justified?"

261. A student asks, "The text talks a lot about limits. What is a limit, really?"

262. A student asks, "What happens if ϵ and δ are reversed in the definition of limit; i.e., you are given a horizontal tolerance δ and you need to find the vertical tolerance ϵ?"

263. A student asks, "What graphical meaning does a limit have if $f(x)$ is undefined at x?"

264. A student asks, "How does one decide what value to choose for ϵ? In practice, what are some values of ϵ that are chosen by engineers, etc.?"

265. A student asks, "In the definition of a limit, is the $\epsilon - \delta$ method or a more informal method more meaningful?"

266. A student asks, "What is an example in which the limit from the right is not the same as the limit from the left?"

267. A student asks, "Why does one consider the limit of a function as x *approaches* a, but not actually *at* $x = a$?"

268. A student asks, "Why is the study of limits important to calculus?"

269. A student asks, "How is possible that

$$\lim_{x\to 3} \frac{x^2-5x+6}{x^2-2x-3} = \frac{1}{4}?$$

If we simply plug in 3, we get 0/0, which is undefined. If this is the case, why don't we say that the limit doesn't exist, instead of saying that it's 1/4?"

270. A student asks, "Does $\lim_{x\to 0} \frac{1}{x} = \infty$, or does this limit not exist?"

271. A student asks, "What do you mean when you say that a limit does not exist?"

272. A student asks, "In a form such as $\lim_{x\to 2}(x^2-4x)$, why are you justified in finding the limit by substituting $x=2$ in $f(x)$ when you have said time and time again that x only approaches but never equals 2?"

273. A student presents the following solution:

$$\begin{aligned}
\lim_{n\to\infty} \frac{(n-1)!}{n!} &= \lim_{n\to\infty} \frac{(n-1)!}{n!} \cdot \frac{1/n}{1/n} \\
&= \lim_{n\to\infty} \frac{(1-1/n)!}{1!} \\
&= \frac{\left(\lim_{n\to\infty} 1 - \lim_{n\to\infty} \frac{1}{n}\right)!}{\left(\lim_{n\to\infty} 1\right)!} \\
&= 1.
\end{aligned}$$

Is this student correct?

274. A student asks you if the derivative and the integral of a function are both functions.

275. A student gives you the following argument: "The derivative of any function is defined at a value of x. Therefore, the function is constant at the value of x. Therefore, the derivative is zero because the derivative of a constant is zero."

276. A student asks, "Why can't we differentiate $y = x^x$ as $x \cdot x^{x-1} = x^x$?"

277. A student has just been told that the absolute value function does not have a derivative at zero. He wants to know if the following function has a derivative at zero:

$$f(x) = \begin{cases} x^3, & x < 0 \\ x^2, & x \geq 0 \end{cases}$$

278. A student gives the following definition of a tangent to a curve: "It is a line that intersects the curve in only one point."

279. The class is asked to find dy/dx if $4/y = (x-8)/x$. One student presents the following solution:

$$\begin{aligned}
4y^{-1} &= 1 - 8x^{-1} \\
-4y^{-2}\frac{dy}{dx} &= 8x^{-2} \\
\frac{dy}{dx} &= -\frac{2y^2}{x^2}
\end{aligned}$$

Another student presents the following solution:

$$\begin{aligned} 4x &= yx - 8y \\ 4 &= \frac{dy}{dx}x + y - 8\frac{dy}{dx} \\ 4 - y &= \frac{dy}{dx}(x-8) \\ \frac{4-y}{x-8} &= \frac{dy}{dx} \end{aligned}$$

Which student has made a mistake?

280. A student asks, "If $f(x) = x^1$, then $f'(x) = 1 \cdot x^0$, and so $f'(0) = 1 \cdot 0^0$, which is undefined."

281. A student asks, "Why isn't $\frac{d}{dx}(uv) = \frac{du}{dx}\frac{dv}{dx}$?"

282. A student presents the following solution:

$$\frac{d}{dx}(\sin 2x + x^2)^4 = 4(\sin 2x + x^2)^3(\cos 2x + 2x)(2) = 8(\sin 2x + x^2)(\cos 2x + 2x).$$

283. A student asks, "If a function is continuous on a closed interval, then it is bounded on that interval. Is the converse true?"

284. A student asks, "If a function is differentiable, must it also be continuous?"

285. A student asks, "If a function is continuous, must it also be differentiable?"

286. A student asks, "You say that there any functions that can be differentiated can also be integrated, but not necessarily vice versa. However, I can find the derivative of $\frac{\sin x}{x}$, but I can't compute its integral."

287. A student asks, "Do the absolute extrema of a continuous function on a closed interval always occur at points where $f'(c) = 0$?"

288. A student asks, "If $f'(c) = 0$, does this guarantee that the point is an relative extremum?"

289. A student asks, "If a function has an extremum at an interior point, must $f'(c) = 0$?"

290. A student asks, "If a function has a point of inflection at c, must $f''(c) = 0$?"

291. A student asks, "If $f''(c) = 0$, must c be a point of inflection?"

292. A student asks, "Can you explain why increasing/decreasing and concave up/concave down are different concepts?"

293. A student asks, "What's the difference between definite and indefinite integrals?"

294. You're starting a section on definite integrals. A student asks, "What does these weird-looking Riemann sums have to do with anything we've seen so far in calculus?"

295. A student asks, "If $\int \tan\theta d\theta = -\ln|\cos\theta| + C$, why isn't

$$\int \tan^6 \theta d\theta = [-\ln|\cos\theta| + C]^6?''$$

296. A student asks, "What's the use of the trapezoid rule or Simpson's rule?"

297. A student asks, "To find the volume of a solid of revolution, how do I tell when to use disks/washers and when to use shells?"