Math 4050 Review #4

1. If $x > 0$, explain why $\ln x = \displaystyle\int_1^x \frac{dt}{t}$.

2. State three definitions of the number e, and explain why they work.

3. State the definition of e^z if $z \in \mathbb{C}$.

4. What is the definition of e^{ix}? Explain the relationship of this definition to the usual Taylor series expansions of e^x, $\cos x$ and $\sin x$.

5. Suppose that $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Prove that
   - $z_1 z_2 = r_1 r_2 \left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$
   - $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$
   - $z_1^n = r_1^n\left[\cos(n\theta_1) + i\sin(n\theta_1)\right]$ if n is a nonnegative integer
   - $z_1^n = r_1^n\left[\cos(n\theta_1) + i\sin(n\theta_1)\right]$ if n is a negative integer

6. Let $|a| < 1$. Prove that

$$\begin{aligned} 1 + a\cos\theta + a^2\cos 2\theta + a^3\cos 3\theta + \dots &= \frac{1 - a\cos\theta}{1 - 2a\cos\theta + a^2}, \\ a\sin\theta + a^2\sin 2\theta + a^3\sin 3\theta + \dots &= \frac{a\sin\theta}{1 - 2a\cos\theta + a^2}. \end{aligned}$$

7. Simplify $(1+i)^{2012} + (1-i)^{2012}$.

8. Use DeMoivre's theorem to derive identities for $\cos 4\theta$ and $\sin 4\theta$.

9. Find all fourth roots of $-8 + 8i\sqrt{3}$. Then find $(-8+8i\sqrt{3})^{1/4}$.

10. Find all fifth roots of $-12 + 5i$. Then find $(-12+5i)^{1/5}$.

11. Prove that $\dfrac{d^n}{dx^n}\left(e^x\cos x\right) = 2^{n/2} e^x \cos\left(x + \dfrac{n\pi}{4}\right)$. *Hint:* $e^x\cos x = \operatorname{Re}\left(e^x e^{ix}\right) = \operatorname{Re}\left(e^{(1+i)x}\right)$.

12. Discuss how a calculator computes e^x and $\ln x$.

13. Compute the following quantities using the definitions given in this class, or explain why the number does not exist.
   - $\sqrt[3]{64}$
   - $64^{1/3}$
   - $\sqrt{64}$
   - $64^{1/2}$
   - $\sqrt[3]{-64}$
   - $(-64)^{1/3}$
   - $\sqrt{-64}$
   - $64^{1/2}$
   - $\sqrt{-2-2i\sqrt{3}}$
   - $(-2-2i\sqrt{3})^{1/2}$

14. Compute the following quantities. Avoid using a calculator until absolutely necessary.
   - $e^{i\pi/3}$
   - e^{2+5i}
   - $\log(4-4i)$.
   - $\log(-5+12i)$
   - $(3-3i\sqrt{3})^{2i}$
   - $(3-4i)^{2+i}$

15. Prove that $e^{\log z} = z$.
16. Give an example of z so that $\log(e^z) \neq z$.
17. If $z \in \mathbb{C}$, prove that $e^z \neq 0$, so that the range of e^z is $\mathbb{C} \setminus \{0\}$.
18. Let $z = x + iy$, where $x, y \in \mathbb{R}$. Simplify $|e^z|$.
19. Prove that e^z is periodic with period $2\pi i$.
20. Find conditions on z, w_1 and w_2 for the following to be true, and give a counterexample when the conditions are not satisfied.
    - $z^{w_1} z^{w_2} = z^{w_1 + w_2}$
    - $\left(z^{w_1}\right)^{w_2} = z^{w_1 w_2}$
    - ${w_1}^z {w_2}^z = (w_1 w_2)^z$
21. If $x, y \in \mathbb{R}$, prove that $e^{x+iy} = e^x (\cos y + i \sin y)$.
22. State the definition of z^w under the following conditions:
    - $z \in \mathbb{C}$, $w \in \mathbb{Z}^+$
    - $z \in \mathbb{C} \setminus \{0\}$, $w \in \mathbb{Z}$
    - $z \in \mathbb{R}$, $w = 1/n$, $n \in \mathbb{Z}^+$
    - $z > 0$, $w \in \mathbb{Q}$
    - $z > 0$, $w \in \mathbb{R}$
    - $z \in \mathbb{C} \setminus \{0\}$, $w \in \mathbb{Q}$
    - $z \in \mathbb{C} \setminus \{0\}$, $w \in \mathbb{C}$
23. State the definition of $\ln z = \log z$ if (a) $z \in \mathbb{R}$, (b) $z \in \mathbb{C}$.
24. Consider $(2 - 2i)^z$.
    - For what values of z is $|(2 - 2i)^z| = 1$?
    - For what values of z is $(2 - 2i)^z$ a positive real number?
    - For what values of z is $(2 - 2i)^z$ a pure imaginary number?
25. Let $z \in \mathbb{C} \setminus \{0\}$ and $w \in \mathbb{Q}$. Show that the two possible definitions of z^w are equivalent.
26. For $z \in \mathbb{C}$, define $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$.
    - Prove that this makes sense if $z \in \mathbb{R}$.
    - Show that the only roots of $\cos z = 0$ are $z = \left(k + \frac{1}{2}\right)\pi$, $k \in \mathbb{Z}$.
    - Find all solutions of $\cos z = 2$ in the complex plane.
27. For $z \in \mathbb{C}$, define $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$.
    - Prove that this makes sense if $z \in \mathbb{R}$.
    - Show that the only roots of $\sin z = 0$ are $z = k\pi$, $k \in \mathbb{Z}$.
    - Find all solutions of $\sin z = 2$ in the complex plane.
28. Prove that $\overline{e^z} = e^{\overline{z}}$.
29. Prove that $\overline{\sin z} = \sin \overline{z}$.
30. Prove that $\overline{\cos z} = \cos \overline{z}$.