INTERNALLY CLUB AND APPROACHABLE FOR LARGER STRUCTURES

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ABSTRACT. We generalize the notion of a fat subset of a regular cardinal κ to a fat subset of $P_\kappa(X)$, where $\kappa \subseteq X$. Suppose $\mu < \kappa$, $\mu^{<\mu} = \mu$, and κ is supercompact. Then there is a generic extension in which $\kappa = \mu^{++}$, and for all regular $\lambda \geq \mu^{++}$, there are stationarily many N in $[H(\lambda)]^{\mu^+}$ which are internally club but not internally approachable.

Suppose μ is an infinite cardinal. A set N is *internally approachable with length* μ^+ if N is the union of an increasing and continuous sequence $\langle N_i : i < \mu^+ \rangle$ of sets with size μ such that for all $\alpha < \mu^+$, $\langle N_i : i < \alpha \rangle$ is in N. A related idea is that of an internally club set. A set N with size μ^+ is *internally club* if $N \cap [N]^\mu$ contains a club subset of $[N]^\mu$. In other words, N is the union of an increasing and continuous sequence $\langle N_i : i < \mu^+ \rangle$ of sets with size μ such that each N_i is in N.

Foreman and Todorčević [3] asked whether the properties of being internally approachable and internally club are equivalent. In [5] we proved that under PFA, for all regular $\lambda \geq \omega_2$ there are stationarily many structures $N \prec H(\lambda)$ with size $\aleph_1$ such that N is internally club but not internally approachable. In this paper we generalize this result to larger structures.

Theorem 1. *Suppose $\mu < \kappa$, $\mu^{<\mu} = \mu$, and κ is supercompact. Then there is a μ-closed, μ^+-proper forcing poset which collapses κ to become μ^{++}, and forces that for all regular $\lambda \geq \mu^{++}$, there are stationarily many N in $[H(\lambda)]^{\mu^+}$ which are internally club but not internally approachable.*

In the model we construct to prove Theorem 1, we have that $2^\mu = \mu^{++}$. In fact, if $2^\mu = \mu^+$, then any elementary substructure $N \prec H(\lambda)$ with size μ^+ and which contains μ^+ is internally club iff it is internally approachable; this is shown at the end of the paper.

In Section 1 we review notation and some background material. Section 2 generalizes the idea of a fat subset of a regular cardinal κ to a fat subset of $P_\kappa(X)$, where $\kappa \subseteq X$. Section 3 presents the basic forcing poset we use in our consistency result, and in Section 4 we describe how to iterate this poset with a mixed support forcing iteration. In Section 5 we prove Theorem 1.

1. PRELIMINARIES

If κ is regular and $\kappa \subseteq X$, we say $C \subseteq P_\kappa(X)$ is *club* if it is closed under unions of increasing sequences of length less than κ, and is cofinal. A set $S \subseteq P_\kappa(X)$ is *stationary* if it has non-empty intersection with every club. We will use the fact that if $C \subseteq P_\kappa(X)$ is club, and $A \subseteq C$ is a directed set with size less than κ, then

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$\bigcup A \in C$ (see Lemma 8.25 of [4] for a proof). By *directed* we mean that if a and b are in A, then there is c in A such that $a \cup b \subseteq c$.

If N is a set, $\mathbb{P}$ is a forcing poset, and G is a filter on $\mathbb{P}$, then $N[G]$ denotes the set $\{\dot{a}^G : \dot{a} \in N \cap V^{\mathbb{P}}\}$. A filter G on $\mathbb{P}$ is *N-generic* if for every dense set $D \subseteq \mathbb{P}$ in N, $N \cap D \cap G$ is non-empty. A condition q in $\mathbb{P}$ is *N-generic* if q forces $\dot{G}$ is N-generic, where $\dot{G}$ is a name for the generic filter. Suppose λ is regular with $\mathbb{P} \in H(\lambda)$, and $N \prec \langle H(\lambda), \in, \mathbb{P}\rangle$. Then for any condition q in $\mathbb{P}$, the following are equivalent: (1) q is N-generic, (2) for every dense set $D \subseteq \mathbb{P}$ in N, $N \cap D$ is predense below q, (3) q forces $N[\dot{G}] \cap On = N \cap On$, and (4) q forces $N[\dot{G}] \cap V = N$. Note that if q is N-generic, then for any set X, q forces $N[\dot{G}] \cap \check{X} = N \cap \check{X}$.

Suppose $\mathbb{P}$ is a forcing poset and λ is regular with $\mathbb{P} \in H(\lambda)$. If G is generic for $\mathbb{P}$ over V, then $H(\lambda)^{V[G]} = H(\lambda)^V[G]$. Suppose $N \prec \langle H(\theta), \in, \mathbb{P}\rangle$ in V. If G is generic for $\mathbb{P}$ over V, then $N[G] \prec H(\theta)^{V[G]}$.

Let $\mathbb{P}$ be a forcing poset and μ a regular cardinal with $\mu^{<\mu} = \mu$. Then $\mathbb{P}$ is *μ^+-proper* if for any regular cardinal $\theta > 2^{|\mathbb{P}|}$ with $\mathbb{P}$ in $H(\theta)$, if N is an elementary substructure of $\langle H(\theta), \in, \mathbb{P}\rangle$, N has size μ, and $N^{<\mu} \subseteq N$, then for all p in $N \cap \mathbb{P}$, there is $q \leq p$ which is N-generic. Any μ^+-proper forcing poset preserves μ^+. Note that if $\mathbb{P}$ is μ^+-c.c. then any condition in $\mathbb{P}$ is N-generic, since every maximal antichain of $\mathbb{P}$ in N is actually a subset of N.

If μ is a regular cardinal and $\mathbb{P}$ is a forcing poset, we say $\mathbb{P}$ is *μ-distributive* if for any collection $\mathcal{D}$ of less than or equal to μ many dense open subsets of $\mathbb{P}$, $\bigcap \mathcal{D}$ is dense open. This property is equivalent to $\mathbb{P}$ not adding any new sequences of ordinals with order type less than or equal to μ. If κ is a cardinal we say $\mathbb{P}$ is *$< \kappa$-distributive* if $\mathbb{P}$ is μ-distributive for all regular $\mu < \kappa$.

Let $\mathbb{P}$ be a forcing poset and μ a regular cardinal. We say $\mathbb{P}$ is *μ-closed* if whenever $\langle p_i : i < \xi\rangle$ is a descending sequence of conditions in $\mathbb{P}$ with $\xi < \mu$, there is q in $\mathbb{P}$ such that $q \leq p_i$ for all $i < \xi$. If $A \subseteq \mathbb{P}$, a *greatest lower bound* of A, or *glb* of A, is a condition q such that $q \leq p$ for all p in A, and whenever $r \leq p$ for all p in A, $r \leq q$. We say $\mathbb{P}$ is *μ-glb closed* if whenever $\langle p_i : i < \xi\rangle$ is a descending sequence of conditions in $\mathbb{P}$ with $\xi < \mu$, there exists a greatest lower bound for the set $\{p_i : i < \xi\}$.

2. Generalized Fat Sets

Let κ be a regular uncountable cardinal. Recall that a set $A \subseteq \kappa$ is *fat* if for any club set $C \subseteq \kappa$ and $\xi < \kappa$, $A \cap C$ contains a closed subset with order type at least ξ.

Fact 2.1 (Abraham and Shelah [1])**.** *Suppose κ is strongly inaccessible or $\kappa = \mu^+$ where $\mu^{<\mu} = \mu$. Then the following are equivalent for a set $A \subseteq \kappa$:*

(1) A is fat.

(2) There is a $< \kappa$-distributive forcing poset $\mathbb{P}$ which forces that A contains a club set.

Suppose κ is a regular uncountable cardinal and $\kappa \subseteq X$. We generalize the idea of fatness to subsets of $P_\kappa(X)$ with the following definition.

Definition 2.2. *Suppose κ is a regular uncountable cardinal and $\kappa \subseteq X$. A set $A \subseteq P_\kappa(X)$ is* fat *if for all regular $\theta \geq \kappa$ with $X \subseteq H(\theta)$, for any club $C \subseteq P_\kappa(H(\theta))$ and $\xi < \kappa$, there is an increasing and continuous sequence $\langle N_i : i < \xi\rangle$ such that for all $i < \xi$, $N_i \in C$, $N_i \cap X \in A$, and $N_i \in N_{i+1}$ when $i + 1 < \xi$.*

Lemma 2.3. *Suppose $\kappa = \mu^+$. Then $A \subseteq P_\kappa(X)$ is fat iff for all regular $\theta \geq \kappa$ with $X \subseteq H(\theta)$, for any club $C \subseteq P_\kappa(H(\theta))$, and for any regular cardinal $\lambda \leq \mu$, there is an increasing and continuous sequence $\langle N_i : i \leq \lambda \rangle$ such that for $i \leq \lambda$, $N_i \in C$, $N_i \cap X \in A$, and $N_i \in N_{i+1}$ when $i < \lambda$.*

Proof. Suppose A satisfies the second property. Then clearly A is stationary in $P_\kappa(X)$. Fix $\theta \geq \kappa$ regular with $X \subseteq H(\theta)$. We prove by induction on $\xi < \mu^+$ that for any club set $C \subseteq P_\kappa(H(\theta))$, there is an increasing and continuous sequence $\langle N_i : i < \xi \rangle$ such that for all $i < \xi$, $N_i \in C$, $N_i \cap X \in A$, and $N_i \in N_{i+1}$ when $i + 1 < \xi$. The successor step of the induction follows from the fact that A is stationary.

Suppose $\delta < \mu^+$ is a limit ordinal and the claim holds for all $\delta' < \delta$. Let $\langle \delta_i : i < \mathrm{cf}(\delta) \rangle$ be increasing and cofinal in δ. Note that $\mathrm{cf}(\delta) \leq \mu$. Let

$$\mathcal{A} = \langle H(\theta), \in, <, X, A, \delta, \langle \delta_i : i < \mathrm{cf}(\delta) \rangle \rangle,$$

where $<$ is a well-ordering of $H(\theta)$. Fix an increasing and continuous sequence $\langle N_i : i \leq \mathrm{cf}(\delta) \rangle$ of sets such that for $i \leq \mathrm{cf}(\delta)$, $N_i \in C$, $N_i \prec \mathcal{A}$, $\mu \subseteq N_i$, $N_i \cap X \in A$, and $N_i \in N_{i+1}$ when $i < \mathrm{cf}(\delta)$.

Fix $i < \mathrm{cf}(\delta)$. By the induction hypothesis, let $\langle M^i_j : j \leq \delta_i \rangle$ be the $<$-least increasing and continuous sequence with length $\delta_i + 1$ such that $\mu \cup \{N_i\} \subseteq M^i_0$, and for $j \leq \delta_i$, $M^i_j \in C$, $N_i \prec M^i_j$, $M^i_j \cap X \in A$, and $M^i_j \in M^i_{j+1}$ when $j < \delta_i$. By elementarity, this sequence is in N_{i+1}. Then the set

$$\{N_i : i \leq \mathrm{cf}(\delta)\} \cup \{M^i_j : i < \mathrm{cf}(\delta), j \leq \delta_i\},$$

well-ordered by $\in$, is increasing and continuous with order type at least δ, and for all N in this set, $N \in C$ and $N \cap X \in A$. $\square$

We will now show that our definition of fatness generalizes the classical notion. Indeed, let A be a fat subset of a regular cardinal κ. We show A is a fat subset of $P_\kappa(X)$, where $X = \kappa$, according to Definition 2.2. So let $\theta \geq \kappa$ be regular, and let $C \subseteq P_\kappa(H(\theta))$ be club. Fix $\xi < \kappa$. Define by induction an increasing and continuous sequence $\langle M_i : i < \kappa \rangle$ such that for $i < \kappa$, $M_i \cap \kappa \in \kappa$, $M_i \in C$, and $M_i \in M_{i+1}$. Then $\langle M_i \cap \kappa : i < \kappa \rangle$ is a club subset of κ. Since A is fat, there is a closed set $a \subseteq \kappa$ with order type at least ξ such that $\{M_i \cap \kappa : i \in a\} \subseteq A$. Then $\langle M_i : i \in a \rangle$ is as required.

Suppose on the other hand that $A \subseteq \kappa$ is fat as a subset of $P_\kappa(\kappa)$ by Definition 2.2, and we show A is fat as a subset of κ. Let $C \subseteq \kappa$ be club and fix $\xi < \kappa$. Let $\langle N_i : i \leq \xi \rangle$ be an increasing and continuous sequence of sets in $P_\kappa(H(\kappa))$ such that for $i \leq \xi$, $N_i \prec \langle H(\kappa), \in, C \rangle$, $N_i \cap \kappa \in \kappa$, $N_i \cap \kappa \in A$, and $N_i \in N_{i+1}$ when $i < \xi$. Then $\{N_i \cap \kappa : i \leq \xi\}$ is a closed set contained in $A \cap C$.

The next theorem generalizes Fact 2.1.

Theorem 2.4. *Suppose κ is strongly inaccessible or $\kappa = \mu^+$ where $\mu^{<\mu} = \mu$. Let X be a set containing κ. Then the following are equivalent for a set $A \subseteq P_\kappa(X)$:*

(1) A is fat.

(2) There is a $< \kappa$-distributive forcing poset which forces there is an increasing and continuous sequence $\langle a_i : i < \kappa \rangle$ which is cofinal in $P_\kappa(X)$ such that $a_i \in A$ for $i < \kappa$.

Proof. Suppose $A \subseteq P_\kappa(X)$ and $\mathbb{P}$ is a $< \kappa$-distributive forcing poset which forces that $\langle \dot{a}_i : i < \kappa \rangle$ is increasing, continuous, and cofinal in $P_\kappa(X)$ such that $\dot{a}_i \in A$

for $i < \kappa$. We prove that A is fat. So let $\theta \geq \kappa$ be regular with $X \subseteq H(\theta)$. Suppose $C \subseteq P_\kappa(H(\theta))$ is club. Let G be generic for $\mathbb{P}$ over V, and let $a_i = \dot{a}_i^G$ for $i < \kappa$. Since $\mathbb{P}$ is $< \kappa$-distributive, in $V[G]$ the set C is still a club subset of $P_\kappa(H(\theta)^V)$.

We work in $V[G]$. Since $X = \bigcup\{a_i : i < \kappa\}$ and $|a_i| < \kappa$ for all $i < \kappa$, X has size κ in the extension. So let $\langle x_i : i < \kappa\rangle$ enumerate X. We define by induction an increasing and continuous sequence $\langle N_i : i < \kappa\rangle$ such that for all $i < \kappa$, $N_i \in N_{i+1}$ and $N_i \in C$. Choose N_0 in C arbitrarily. At limits take unions. Suppose N_i is defined. Then N_i is in $H(\theta)^V$, so choose N_{i+1} in C such that $N_i \cup \{N_i\} \cup \{x_i\} \subseteq N_{i+1}$. This completes the definition. Now $\langle a_i : i < \kappa\rangle$ and $\langle N_i \cap X : i < \kappa\rangle$ are both club in $P_\kappa(X)$. So there is a club $D \subseteq \kappa$ such that for all $i \in D$, $a_i = N_i \cap X$. Then $\langle N_i : i \in D\rangle$ is an increasing and continuous sequence such that for all $i \in D$, $N_i \in C$, $N_i \in N_{i+1}$, and $N_i \cap X \in A$. But every initial segment of this sequence is in V since $\mathbb{P}$ is $< \kappa$-distributive. So A is fat.

In the other direction, suppose $A \subseteq P_\kappa(X)$ is fat. Define a forcing poset $\mathbb{P}(A)$ as follows. A condition in $\mathbb{P}(A)$ is an increasing and continuous sequence $\langle a_i : i \leq \gamma\rangle$, where $\gamma < \kappa$, such that $a_i \in A$ for all $i \leq \gamma$. The ordering is by extension of sequences. We claim that $\mathbb{P}(A)$ is $< \kappa$-distributive and $\mathbb{P}(A)$ forces that the union of the generic filter is an increasing and continuous sequence cofinal in $P_\kappa(X)$ with order type κ whose elements are in A.

Suppose $\langle D_i : i < \xi\rangle$ is a sequence of dense open subsets of $\mathbb{P}(A)$, where $\xi < \kappa$ is a cardinal. Let p be in $\mathbb{P}(A)$, and we find $q \leq p$ which is in $\bigcap\{D_i : i < \xi\}$. Fix a regular cardinal $\theta \gg \kappa$ with $X \in H(\theta)$, and let

$$\mathcal{A} = \langle H(\theta), \in, X, A, \mathbb{P}(A), p, \langle D_i : i < \xi\rangle\rangle.$$

Since A is fat we can find an increasing and continuous sequence $\langle N_i : i \leq \xi\rangle$ such that for all $i \leq \xi$, $N_i \prec \mathcal{A}$, $N_i \cap \kappa \in \kappa$, $\xi \subseteq N_i$, $N_i \cap X \in A$, and when $i < \xi$, $N_i \in N_{i+1}$.

We define by induction a descending sequence of conditions $\langle p_i : i \leq \xi\rangle$ in $\mathbb{P}(A)$. Our induction hypothesis is that p_i is in N_{i+1} and the maximum element of p_i is $N_i \cap X$. Let $p_0 = p^\frown(N_0 \cap X)$. Then p_0 is a condition, because $p \in N_0$ and thus all the elements of p are subsets of $N_0 \cap X$. Suppose $i < \xi$, and for all $j \leq i$, p_j is defined, p_j is a member of N_{j+1}, and the maximum element of p_j is $N_j \cap X$. Since $\xi \subseteq N_{i+1}$, D_i is in N_{i+1}. Fix $p_i^* \leq p_i$ in $D_i \cap N_{i+1}$. Since p_i^* has size less than κ and $N_{i+1} \cap \kappa \in \kappa$, $p_i^* \subseteq N_{i+1}$, and so every element of p_i^* is a subset of N_{i+1} as well. Therefore if we let $p_{i+1} = p_i^{*\frown}(N_{i+1} \cap X)$, p_{i+1} is a condition in $N_{i+2} \cap D_i$ below p_i.

Suppose $\delta \leq \xi$ is a limit ordinal and $p_i \in N_{i+1}$ is defined for all $i < \delta$. Let

$$p_\delta = \bigcup\{p_i : i < \delta\}^\frown(N_\delta \cap X),$$

which is a condition since $N_\delta \cap X \in A$ and $N_\delta = \bigcup\{N_i : i < \delta\}$. We need to show that p_δ is in $N_{\delta+1}$ when $\delta < \xi$. The sequence $\langle p_i : i < \delta\rangle$ is in $N_\delta^{<\xi}$. Since κ is either strongly inaccessible or $\kappa = \mu^+$ where $\mu^{<\mu} = \mu$, $N_\delta^{<\xi}$ has size less than κ. But $N_\delta^{<\xi} \in N_{\delta+1}$. Since $N_{\delta+1} \cap \kappa \in \kappa$, $N_\delta^{<\xi} \subseteq N_{\delta+1}$. So the sequence $\langle p_i : i < \delta\rangle$ is in $N_{\delta+1}$. Clearly then p_δ is in $N_{\delta+1}$ as well.

This completes the construction of $\langle p_i : i \leq \xi\rangle$. The condition p_ξ is below p and is in $\bigcap\{D_i : i < \xi\}$. So $\mathbb{P}(A)$ is $< \kappa$-distributive.

For each $\alpha < \kappa$ let D_α be the set of conditions in $\mathbb{P}(A)$ with length at least α. Clearly D_0 is dense open, and if D_i is dense open, D_{i+1} is dense open as well.

Assume $\delta < \kappa$ is a limit ordinal and D_i is dense open for all $i < \delta$. Since $\mathbb{P}(A)$ is $< \kappa$-distributive, $\bigcap\{D_i : i < \delta\}$ is dense open. But if p is in this intersection, p has length at least δ. So $\mathbb{P}(A)$ forces the union of the generic filter has length κ. By an easy density argument, $\mathbb{P}(A)$ forces the union of the generic filter is cofinal in $P_\kappa(X)$. $\square$

Since we will use the forcing poset from the last theorem in our consistency proof, we describe it explicitly in the following definition.

Definition 2.5. *Suppose κ is regular, $\kappa \subseteq X$, and $A \subseteq P_\kappa(X)$ is fat. Let $\mathbb{P}(A)$ be the forcing poset consisting of increasing and continuous sequences $\langle a_i : i \leq \gamma \rangle$, where $\gamma < \kappa$ and $a_i \in A$ for $i \leq \gamma$, ordered by extension of sequences.*

The forcing poset $\mathbb{P}(A)$ is $< \kappa$-distributive and adds an increasing, continuous, and cofinal sequence $\langle a_i : i < \kappa \rangle$ through $P_\kappa(X)$ such that $a_i \in A$ for $i < \kappa$. In particular, $\mathbb{P}(A)$ collapses the size of X to be κ.

If $\kappa = \omega_1$ and $\omega_1 \subseteq X$, one can show using Lemma 2.3 that any stationary set $A \subseteq P_{\omega_1}(X)$ is fat. Thus $\mathbb{P}(A)$ is ω-distributive for any stationary set $A \subseteq P_{\omega_1}(X)$.

3. The Basic Forcing Poset

We now describe the forcing poset which we will use in our consistency proof.

Suppose $\mu^{<\mu} = \mu$ and $\mu^+ \subseteq X$. The basic forcing poset we will use is $\textsc{Add}(\mu) * \mathbb{P}(\dot{S})$, where $\textsc{Add}(\mu)$ adds a Cohen subset to μ, $\textsc{Add}(\mu)$ forces $\dot{S} = [X]^\mu \cap V$, and $\mathbb{P}(\dot{S})$ is the forcing poset from Definition 2.5. Thus we need to know that $\textsc{Add}(\mu)$ forces $\dot{S}$ is fat. If $\mu^{<\mu} = \mu$ then $\textsc{Add}(\mu)$ is μ^+-c.c., so this follows from the next proposition.

Proposition 3.1. *Suppose κ is regular and $\kappa \subseteq X$. Let $\mathbb{P}$ be a κ-c.c. forcing poset. Then $\mathbb{P}$ forces $P_\kappa(X) \cap V$ is fat.*

Proof. Let G be generic for $\mathbb{P}$ over V. Working in $V[G]$, fix $\theta \geq \kappa$ regular with $X \subseteq H(\theta)$, and let $C \subseteq P_\kappa(H(\theta))$ be club. Fix $\chi \gg \theta$ regular such that $H(\chi)$ contains C and $\mathbb{P}$ as members. Recall that $H(\chi)^{V[G]} = H(\chi)^V[G]$. Let $\dot{C}$ be a name for C in $H(\chi)^V$.

Now back in V, define by induction an increasing and continuous sequence $\langle N_i : i < \kappa \rangle$ of elementary substructures of $\langle H(\chi), \in, X, \dot{C}, \mathbb{P} \rangle$ such that for all $i < \kappa$, $|N_i| < \kappa$, $N_i \cap \kappa \in \kappa$, and $N_i \in N_{i+1}$. Then in $V[G]$, for all $i < \kappa$, $N_i[G] \prec \langle H(\chi)^{V[G]}, \in, C \rangle$. By elementarity, $N_i[G] \cap C$ is a directed subset of C with size less than κ whose union is equal to $N_i[G] \cap H(\theta)$. So $N_i[G] \cap H(\theta)$ is in C. Since $N_i \in N_{i+1}$, $N_i[G] \in N_{i+1}[G]$, and therefore $N_i[G] \cap H(\theta) \in N_{i+1}[G] \cap H(\theta)$. But $\mathbb{P}$ is κ-c.c., so $N_i[G] \cap V = N \cap V$. For if $x \in N_i[G] \cap V$, there is a name $\dot{x}$ for x in N_i. The maximal antichain of conditions deciding $\dot{x}$ is in N_i, and has size less than κ, so is a subset of N_i. But then x is in N_i. In particular, $N_i[G] \cap X = N_i \cap X$, which is in $P_\kappa(X) \cap V$. $\square$

The forcing poset $\textsc{Add}(\mu)$ is μ-glb closed. Indeed, if $\langle p_i : i < \xi \rangle$ is decreasing in $\textsc{Add}(\mu)$ where $\xi < \mu$, then $\bigcup\{p_i : i < \xi\}$ is the greatest lower bound. Note that any two step forcing iteration of μ-glb closed forcing posets is μ-glb closed.

Lemma 3.2. *Suppose $\mu^{<\mu} = \mu$, $\mu^+ \subseteq X$, and $\dot{S}$ is an $\textsc{Add}(\mu)$-name for $[X]^\mu \cap V$. Then $\textsc{Add}(\mu)$ forces that $\mathbb{P}(\dot{S})$ is μ-glb closed. Hence $\textsc{Add}(\mu) * \mathbb{P}(\dot{S})$ is μ-glb closed.*

Proof. Let G be generic for $\textsc{Add}(\mu)$. In $V[G]$, suppose $\langle p_i : i < \xi\rangle$ is a descending sequence of conditions in $\mathbb{P}(S)$ where $\xi < \mu$ is a limit ordinal. For each i write $p_i = \langle a_j : j \le \gamma_i\rangle$. Let $\gamma = \sup(\{\gamma_i : i < \xi\})$ and $a = \bigcup\{a_i : i < \gamma\}$. Let

$$q = \bigcup\{p_i : i < \xi\} \cup \{\langle \gamma, a\rangle\}.$$

Then q is a condition in $\mathbb{P}(S)$ iff a is in V. But since $\textsc{Add}(\mu)$ is μ-closed, the sequence $\langle a_{\gamma_i} : i < \xi\rangle$ is in V, and hence its union a is in V. Clearly any condition which extends each p_i must extend q, so q is the greatest lower bound of the sequence. □

4. Iterating the Basic Forcing Poset

We now describe a mixed support iteration of the forcing poset introduced in the last section.

Fix a cardinal μ such that $\mu^{<\mu} = \mu$. We consider a forcing iteration

$$\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \le \alpha, j < \alpha\rangle,$$

satisfying the following recursive definition:

(1) If $i < \alpha$ is even, $\mathbb{P}_i$ forces $\dot{\mathbb{Q}}_i = \textsc{Add}(\mu)$, and $\mathbb{P}_i$ forces $\dot{X}_i$ is a set containing μ^+.

(2) If $i = j+1 < \alpha$ is odd, $\mathbb{P}_i$ forces $\dot{S}_i = [\dot{X}_j]^\mu \cap V[\dot{G}_j]$, where $\dot{G}_j$ is a name for the generic filter for $\mathbb{P}_j$, and $\dot{\mathbb{Q}}_i = \mathbb{P}(\dot{S}_i)$ is the poset from Definition 2.5.

(3) If $i \le \alpha$ is a limit ordinal, $\mathbb{P}_i$ is the poset consisting of partial functions $p : i \to V$ such that $p \restriction j \in \mathbb{P}_j$ for $j < i$, $|\mathrm{dom}(p) \cap \{j < i : j \text{ even}\}| < \mu$, and $|\mathrm{dom}(p) \cap \{j < i : j \text{ odd}\}| \le \mu$.

We assume the following recursion hypotheses for all $\beta < \alpha$, which guarantee that the definition above makes sense.

(4) $\mathbb{P}_\beta$ is μ-glb closed and μ^+-proper, and so preserves cardinals and cofinalities less than or equal to μ^+.

(5) Let $\mathbb{P}^*_\beta$ be the set of p in $\mathbb{P}_\beta$ such that for all even j in $\mathrm{dom}(p)$, there is x in $\textsc{Add}(\mu)$ such that $p(j) = \check{x}$. Then $\mathbb{P}^*_\beta$ is dense in $\mathbb{P}_\beta$.

(6) If $\langle p_i : i < \xi\rangle$ is a descending sequence of conditions in $\mathbb{P}^*_\beta$ with $\xi < \mu$, then the greatest lower bound of this sequence is in $\mathbb{P}^*_\beta$.

We prove that properties (4), (5), and (6) above also hold for $\mathbb{P}_\alpha$.

Case 1: $\alpha = \beta + 1$ is a successor ordinal.

We show that $\mathbb{P}_\alpha$ is μ-glb closed. This will follow from the fact that a two-step iteration of μ-glb closed forcing posets is μ-glb closed. If β is even, then $\mathbb{P}_\alpha = \mathbb{P}_\beta * \textsc{Add}(\mu)$. Since $\mathbb{P}_\beta$ is μ-glb closed by recursion, clearly $\mathbb{P}_\alpha$ is μ-glb closed as well. Suppose $\beta = \gamma + 1$ is odd. Then $\mathbb{P}_\alpha = \mathbb{P}_\gamma * \textsc{Add}(\mu) * \mathbb{P}(\dot{S}_\beta)$. By recursion, $\mathbb{P}_\gamma$ is μ-glb closed, and by Lemma 3.2, $\mathbb{P}_\gamma$ forces that $\textsc{Add}(\mu) * \mathbb{P}(\dot{S}_\beta)$ is μ-glb closed. So $\mathbb{P}_\alpha$ is μ-glb closed. We prove in Proposition 4.2 below that $\mathbb{P}_\alpha$ is μ^+-proper.

Now we prove that $\mathbb{P}^*_\alpha$ is dense in $\mathbb{P}_\alpha$. Consider a condition p in $\mathbb{P}_\alpha$. If β is not in the domain of p or if β is odd, fix $q \le p \restriction \beta$ in $\mathbb{P}^*_\beta$. Then $q \le p$ is in $\mathbb{P}^*_\alpha$ if β is not in $\mathrm{dom}(p)$, and $q^\frown p(\beta) \le p$ is in $\mathbb{P}^*_\alpha$ otherwise. Assume β is in $\mathrm{dom}(p)$ and β is even. Since $\mathbb{P}_\beta$ is μ-closed, it forces that $p(\beta)$ is an element of $\textsc{Add}(\mu)$ in the ground model. So choose $r \leq p \restriction \beta$ in $\mathbb{P}^*_\beta$ and x in $\textsc{Add}(\mu)$ such that r forces $p(\beta) = \check{x}$. Then $r^\frown\check{x}$ is as required.

Suppose $\langle p_i : i < \xi \rangle$ is a descending sequence of conditions in $\mathbb{P}^*_\alpha$ with $\xi < \mu$. We show that the greatest lower bound of this sequence is in $\mathbb{P}^*_\alpha$. Now $\langle p_i \restriction \beta : i < \xi \rangle$ is a descending sequence in $\mathbb{P}^*_\beta$. By induction the greatest lower bound q of this sequence is in $\mathbb{P}^*_\beta$. If β is not in $\mathrm{dom}(p_i)$ for all $i < \xi$, then q is the greatest lower bound of $\langle p_i : i < \xi \rangle$ in $\mathbb{P}^*_\alpha$. Otherwise let $\gamma < \xi$ be the least ordinal such that β is in $\mathrm{dom}(p_\gamma)$. If β is odd, let $\dot{u}$ be a $\mathbb{P}_\beta$-name for the greatest lower bound of $\{p_i(\beta) : \gamma \leq i < \xi\}$. Then $q^\frown\dot{u}$ is as required. If β is even, then fix for each $\gamma \leq i < \xi$ a condition x_i in $\textsc{Add}(\mu)$ such that $p_i(\beta) = \check{x}_i$. Let $x = \bigcup\{x_i : \gamma \leq i < \xi\}$. Then $q^\frown\check{x}$ is as required.

Case 2: α is a limit ordinal.

We show that $\mathbb{P}_\alpha$ is μ-glb closed. Suppose $\langle p_i : i < \xi \rangle$ is a descending sequence of conditions in $\mathbb{P}_\alpha$, with $\xi < \mu$. For each $i < \alpha$, $\mathbb{P}_i$ forces $\dot{\mathbb{Q}}_i$ is μ-glb closed. Define q with support equal to $\bigcup\{\mathrm{dom}(p_i) : i < \xi\}$, so that for each β in this support, $q \restriction \beta$ forces $q(\beta)$ is the greatest lower bound of $\langle p_i(\beta) : \gamma_\beta \leq i < \xi \rangle$, where γ_β is the least $i < \xi$ with β in $\mathrm{dom}(p_i)$. Clearly then q is the greatest lower bound of $\{p_i : i < \xi\}$ in $\mathbb{P}_\alpha$. Suppose moreover that $p_i \in \mathbb{P}^*_\alpha$ for all $i < \xi$. Then q can be chosen to be in $\mathbb{P}^*_\alpha$ as well. Namely, for each even β in $\mathrm{dom}(q)$, and for $\gamma_\beta \leq i < \xi$, choose x^β_i in $\textsc{Add}(\mu)$ such that $p_i(\beta) = \check{x}^\beta_i$. Then let $q(\beta)$ be a name for $\bigcup\{x^\beta_i : \gamma_\beta \leq i < \xi\}$.

Now we show $\mathbb{P}^*_\alpha$ is dense in $\mathbb{P}_\alpha$. First assume $\mathrm{cf}(\alpha) \geq \mu$, and let p be in $\mathbb{P}_\alpha$. Then there is $\xi < \alpha$ such that $\mathrm{dom}(p) \cap \{i < \alpha : i \text{ even}\} \subseteq \xi$. By induction we can choose $q \leq p \restriction \xi$ in $\mathbb{P}^*_\xi$. Then $q^\frown p \restriction [\xi, \alpha)$ is in $\mathbb{P}^*_\alpha$ and is below p.

Suppose $\mathrm{cf}(\alpha) < \mu$ and let p be in $\mathbb{P}_\alpha$. Fix an increasing and continuous sequence $\langle \xi_i : i < \mathrm{cf}(\alpha) \rangle$ cofinal in α with $\xi_0 = 0$, and let $\xi_{\mathrm{cf}(\alpha)} = \alpha$. We define by induction a descending sequence $\langle p_i : i \leq \mathrm{cf}(\alpha) \rangle$ so that $p_i \restriction \xi_i$ is in $\mathbb{P}^*_{\xi_i}$. Let $p_0 = p$. Given p_i, apply the recursion hypotheses to choose $q \leq p_i \restriction \xi_{i+1}$ in $\mathbb{P}^*_{\xi_{i+1}}$, and let $p_{i+1} = q^\frown p \restriction [\xi_{i+1}, \alpha)$. Suppose $\delta \leq \mathrm{cf}(\alpha)$ is a limit ordinal and p_i is defined for all $i < \delta$. Let q be the greatest lower bound of the sequence $\langle p_i \restriction \xi_i : i < \delta \rangle$. Since each $p_i \restriction \xi_i$ is in $\mathbb{P}^*_{\xi_i} \subseteq \mathbb{P}^*_{\xi_\delta}$, q is in $\mathbb{P}^*_{\xi_\delta}$. Now define $p_\delta = q^\frown p \restriction [\xi_\delta, \alpha)$. This completes the definition. The condition $p_{\mathrm{cf}(\alpha)}$ is below p and is in $\mathbb{P}^*_\alpha$.

Now we prove that $\mathbb{P}_\alpha$ is μ^+-proper. The proof is the same whether α is a successor or a limit ordinal.

We will use the following basic observation.

Lemma 4.1. *Suppose p and q are conditions in $\mathbb{P}_\alpha$ such that for all γ in $\mathrm{dom}(p) \cap \mathrm{dom}(q)$, either $p \restriction \gamma$ or $q \restriction \gamma$ forces $p(\gamma)$ and $q(\gamma)$ are compatible in $\dot{\mathbb{Q}}_\gamma$. Then p and q are compatible.*

Proposition 4.2. *The poset $\mathbb{P}_\alpha$ is μ^+-proper.*

Proof. Fix a regular cardinal $\theta > 2^{|\mathbb{P}_\alpha|}$ such that $\mathbb{P}_\alpha$ is in $H(\theta)$. Let $N \prec \langle H(\theta), \in, \mathbb{P}_\alpha \rangle$ be a set with size μ with $N^{<\mu} \subseteq N$. We would like to show that for every p in $N \cap \mathbb{P}_\alpha$, there is $q \leq p$ which is N-generic. In Proposition 4.5 we need q to satisfy a slightly stronger property, which we describe in the following claim.

Claim 4.3. *For all p in $N \cap \mathbb{P}_\alpha$, there is $q \leq p$ which satisfies the property that for all $r \leq q$, and for any dense set $D \subseteq \mathbb{P}_\alpha$ in N, there is q' in $D \cap N$ compatible with r such that for all odd γ in $\mathrm{dom}(q')$, $\gamma \in \mathrm{dom}(r)$ and $r \restriction \gamma$ forces $r(\gamma) \leq q'(\gamma)$.*

Let $\langle \langle D_i, f_i \rangle : i < \mu \rangle$ be an enumeration of all pairs $\langle D, f \rangle$ in N such that $D \subseteq \mathbb{P}_\alpha$ is dense and $f : \{\beta < \alpha : \beta \text{ even}\} \to \textsc{Add}(\mu)$ is a partial function with $|\mathrm{dom}(f)| < \mu$.

We define by induction a descending sequence $\langle p_i : i < \mu \rangle$ of conditions in $N \cap \mathbb{P}^*_\alpha$ and a sequence $\langle q_i : i < \mu \rangle$ of conditions in $N \cap \mathbb{P}^*_\alpha$ such that:

(1) for $i < \mu$, $\mathrm{dom}(p_i) \cap \{\beta < \alpha : \beta \text{ even}\} = \mathrm{dom}(p_0) \cap \{\beta < \alpha : \beta \text{ even}\}$,

(2) for $i < \mu$, for all even β in $\mathrm{dom}(p_i)$, $p_i(\beta) = p_0(\beta)$.

Fix $p_0 \leq p$ in $N \cap \mathbb{P}^*_\alpha$. If $\delta < \mu$ is a limit ordinal and p_i is defined for all $i < \delta$, let p_δ be the greatest lower bound of $\{p_i : i < \delta\}$. Since $N^{<\mu} \subseteq N$, $\langle p_i : i < \delta \rangle$ is in N, and therefore p_δ is in $N \cap \mathbb{P}^*_\alpha$.

Suppose p_i is defined for a fixed $i < \mu$. Consider the pair $\langle D_i, f_i \rangle$. If there is q in $N \cap D_i$ below p_i such that $\mathrm{dom}(f_i) \subseteq \mathrm{dom}(q)$, and for all β in $\mathrm{dom}(f_i)$, $q(\beta)$ is a name for $f_i(\beta)$, then choose q_i as such a q. Otherwise just pick $q_i \leq p_i$ in $N \cap D_i$. Now define p_{i+1} with support equal to

$$(\mathrm{dom}(p_i) \cap \{\beta < \alpha : \beta \text{ even}\}) \cup (\mathrm{dom}(q_i) \cap \{\gamma < \alpha : \gamma \text{ odd }\})$$

so that $p_{i+1}(\beta) = p_i(\beta)$ for even β, and $p_{i+1}(\gamma) = q_i(\gamma)$ for odd γ.

We define a lower bound q for $\langle p_i : i < \mu \rangle$, and prove that q satisfies the requirements of Claim 4.3. Clearly then q is N-generic. The domain of q is $\bigcup\{\mathrm{dom}(p_i) : i < \mu\}$. In particular, $\mathrm{dom}(q) \cap \{\beta < \alpha : \beta \text{ even}\} = \mathrm{dom}(p_0) \cap \{\beta < \alpha : \beta \text{ even}\}$, which has size less than μ. For even β in $\mathrm{dom}(q)$, let $q(\beta) = p_0(\beta)$.

Suppose $\gamma = \beta + 1$ is an odd ordinal in $\mathrm{dom}(q)$. Let $i_\gamma < \mu$ be the least i such that γ is in $\mathrm{dom}(p_i)$. For $i_\gamma \leq i < \mu$, fix a name $\dot{\sigma}^\gamma_i$ so that $\mathbb{P}_\gamma$ forces $p_i(\gamma)$ has domain $\dot{\sigma}^\gamma_i + 1$. Let $\dot{\sigma}_\gamma$ be a $\mathbb{P}_\gamma$-name for $\sup(\{\dot{\sigma}^\gamma_i + 1 : i_\gamma \leq i < \mu\})$. Then $\mathbb{P}_\gamma$ forces that the union of the conditions in $\{p_i(\gamma) : i_\gamma \leq i < \mu\}$ is a sequence of length $\dot{\sigma}_\gamma$. Let $\langle \dot{a}^\gamma_i : i < \dot{\sigma}_\gamma \rangle$ be a sequence of names such that

$$\mathbb{P}_\gamma \Vdash \bigcup\{p_i(\gamma) : i_\gamma \leq i < \mu\} = \langle \dot{a}^\gamma_i : i < \dot{\sigma}_\gamma \rangle.$$

Let $\dot{a}^\gamma_{\dot{\sigma}_\gamma}$ be a name for $\bigcup\{\dot{a}^\gamma_i : i < \dot{\sigma}_\gamma\}$. Finally, let $q(\gamma)$ be a name for the sequence $\langle \dot{a}^\gamma_i : i \leq \dot{\sigma}_\gamma \rangle$.

We prove by induction that for all $\gamma \leq \alpha$, $q \restriction \gamma$ is a condition in $\mathbb{P}_\gamma$ and is below $p_i \restriction \gamma$ for all $i < \mu$. Limit stages are clear. Suppose $q \restriction \gamma$ satisfies this property. If γ is even or if γ is not in $\mathrm{dom}(q)$, then clearly $q \restriction \gamma + 1$ is as required. Suppose $\gamma = \beta + 1$ is odd and is in $\mathrm{dom}(q)$. Then $q \restriction \gamma + 1$ is a condition below $p_i \restriction \gamma + 1$ for all $i < \mu$, provided that $q \restriction \gamma$ forces that $\dot{a}^\gamma_{\dot{\sigma}_\gamma}$ is in $\dot{S}_\gamma = [\dot{X}_\beta]^\mu \cap V[\dot{G}_\beta]$.

Let $G_\beta * H$ be generic for $\mathbb{P}_\gamma = \mathbb{P}_\beta * \textsc{Add}(\mu)$. Since γ is in $\mathrm{dom}(q)$, γ is in $\mathrm{dom}(p_i)$ for some $i < \mu$. Since $\mu \subseteq N$, $\mathrm{dom}(p_i) \subseteq N$. Therefore γ, and hence β, is in N. So $\mathbb{P}_\beta$ is in N. But $\textsc{Add}(\mu)$ is μ^+-c.c. in $V[G_\beta]$, so $N[G_\beta * H] \cap V[G_\beta] = N[G_\beta]$. In particular, $N[G_\beta * H] \cap X_\beta = N[G_\beta] \cap X_\beta$, which is in $[X_\beta]^\mu \cap V[G_\beta]$. So it suffices to show that $a^\gamma_{\sigma_\gamma} = N[G_\beta * H] \cap X_\beta$.

If $i_\gamma \leq i < \mu$, then the condition $p_i(\gamma) = \langle a^\gamma_j : j \leq \sigma^\gamma_i \rangle$ is a member, and hence a subset, of $N[G_\beta * H]$. Therefore each a^γ_j is a subset of $N[G_\beta * H]$. Hence $\bigcup\{a^\gamma_j : j < \sigma_\gamma\} \subseteq N[G_\beta * H] \cap X_\beta$. On the other hand, fix x in $N[G_\beta * H] \cap X_\beta$. Fix a $\mathbb{P}_\gamma$-name $\dot{x}$ for x in N. Then there is a dense subset of $\mathbb{P}_\alpha$ in N of conditions s such that $\mathbb{P}_\gamma$ forces $\dot{x}$ is in some element of the sequence $s(\gamma)$. Hence for some

$i < \mu$, q_i is in this dense set. Since $q_i(\gamma) = p_{i+1}(\gamma)$, $\mathbb{P}_\gamma$ forces $\dot{x}$ appears in some element of $p_{i+1}(\gamma)$. Therefore x appears in some element of $\langle a^\gamma_j : j \leq \sigma^\gamma_{i+1}\rangle$. So x is in $a^\gamma_{\sigma_\gamma}$. Thus $a^\gamma_{\sigma_\gamma} = \bigcup\{a^\gamma_i : i < \sigma_\gamma\} = N[G_\beta * H] \cap X_\beta$.

We prove now that q satisfies the property described in Claim 4.3. Let $r \leq q$ and suppose D is a dense subset of $\mathbb{P}_\alpha$ in N. Fix $s \leq r$ in $\mathbb{P}^*_\alpha \cap D$. Let $f : \alpha \to \mathrm{A{\scriptsize DD}}(\mu)$ be the partial function with $\mathrm{dom}(f) = N \cap \mathrm{dom}(s) \cap \{\beta < \alpha : \beta \text{ even}\}$ such that for all β in $\mathrm{dom}(f)$, $s(\beta)$ is a name for $f(\beta)$. Since $N^{<\mu} \subseteq N$, f is in N. Fix $i < \mu$ such that $D_i = D$ and $f_i = f$.

Now $H(\theta)$ models that there is $u \leq p_i$ in D_i such that $\mathrm{dom}(f_i) \subseteq \mathrm{dom}(u)$, and for all β in $\mathrm{dom}(f_i)$, $u(\beta)$ is a name for $f_i(\beta)$, as witnessed by $u = s$. By elementarity, the same is true in N. Hence by construction, q_i also satisfies this property. If γ is odd and is in $\mathrm{dom}(q_i)$, then γ is in $\mathrm{dom}(p_{i+1})$ and $p_{i+1}(\gamma) = q_i(\gamma)$. Therefore for all odd γ in $\mathrm{dom}(q_i)$, γ is in $\mathrm{dom}(r)$ and $r \restriction \gamma$ forces $r(\gamma) \leq q_i(\gamma)$.

We show that q_i and r are compatible, which finishes the proof. We apply Lemma 4.1 to show q_i and s are compatible. Suppose γ is in $\mathrm{dom}(q_i) \cap \mathrm{dom}(s)$. Since $\mathrm{dom}(q_i) \subseteq N$, γ is in $N \cap \mathrm{dom}(s)$. So if γ is even, then γ is in $\mathrm{dom}(f_i)$. Then $q_i(\gamma)$ and $s(\gamma)$ are both names for $f_i(\gamma)$ and thus are equal. If γ is odd, then $q_i(\gamma) = p_{i+1}(\gamma)$, and $s \restriction \gamma$ forces $s(\gamma) \leq p_{i+1}(\gamma)$. □

This completes the recursion.

The next proposition describes a special property of $\mathbb{P}_\alpha$ which we will use in the consistency proof of the next section. First we need a technical lemma.

Lemma 4.4. *Let p' and q' be conditions in $\mathbb{P}^*_\alpha$. Then there are $p \leq p'$ and $q \leq q'$ in $\mathbb{P}^*_\alpha$ such that $\mathrm{dom}(p) \cap \{\gamma < \alpha : \gamma \text{ odd}\} = \mathrm{dom}(q) \cap \{\gamma < \alpha : \gamma \text{ odd}\}$, and for all odd γ in this set, $p(\gamma) = q(\gamma)$.*

Proof. First choose $p(0) \leq p'(0)$ and $q(0) \leq q'(0)$ in $\mathrm{A{\scriptsize DD}}(\mu)$ which are incompatible. Suppose $\beta > 0$ is an even ordinal and $p \restriction \beta$ and $q \restriction \beta$ are defined. Let β be in $\mathrm{dom}(p)$ iff β is in $\mathrm{dom}(p')$, in which case $p(\beta) = p'(\beta)$, and similarly with q. Suppose γ is odd and $p \restriction \gamma$ and $q \restriction \gamma$ are defined. If γ is in $\mathrm{dom}(p') \setminus \mathrm{dom}(q')$ then let $p(\gamma) = q(\gamma) = p'(\gamma)$, and similarly if γ is in $\mathrm{dom}(q') \setminus \mathrm{dom}(p')$. Suppose γ is in $\mathrm{dom}(p') \cap \mathrm{dom}(q')$. Let $\dot{x}_\gamma$ be a $\mathbb{P}_\gamma$-name such that $\mathbb{P}_\gamma$ forces $\dot{x}_\gamma = p'(\gamma)$ if $p \restriction \gamma$ is in $\dot{G}_\gamma$, and $\dot{x}_\gamma = q'(\gamma)$ otherwise. Then $\dot{x}_\gamma$ is well-defined because $p \restriction \gamma$ and $q \restriction \gamma$ are incompatible. Let $p(\gamma) = q(\gamma) = \dot{x}_\gamma$. □

Proposition 4.5. *The poset $\mathbb{P}_\alpha$ forces that whenever $f : \mu^+ \to V$ is a function in the extension such that for all $i < \mu^+$, $f \restriction i$ is in V, then f is in V.*

Proof. Suppose for a contradiction that p forces $\dot{f} : \mu^+ \to V$ is a function which is not in V, but for all $i < \mu^+$, $\dot{f} \restriction i$ is in V.

Fix a regular cardinal $\theta \gg \mu^+$ with $\mathbb{P}_\alpha \in H(\theta)$. Let N be an elementary substructure of $\langle H(\theta), \in, \mathbb{P}_\alpha, p, \dot{f}\rangle$ with size μ and $N^{<\mu} \subseteq N$. By Claim 4.3, fix $q \leq p$ such that for all $r \leq q$ and for any dense set $D \subseteq \mathbb{P}_\alpha$ in N, there is q' in $D \cap N$ compatible with r such that for all odd γ in $\mathrm{dom}(q')$, $\gamma \in \mathrm{dom}(r)$ and $r \restriction \gamma$ forces $r(\gamma) \leq q'(\gamma)$.

Let $r \leq q$ be in $\mathbb{P}^*_\alpha$ such that r decides $\dot{f} \restriction N \cap \mu^+$. Let $g : \alpha \to \mathrm{A{\scriptsize DD}}(\mu)$ be the partial function with domain equal to $N \cap \mathrm{dom}(r) \cap \{\beta < \alpha : \beta \text{ even}\}$ such that for all β in $\mathrm{dom}(g)$, $r(\beta)$ is a name for $g(\beta)$. Since $N^{<\mu} \subseteq N$, g is in N.

Define D as the set of $s_0 \leq p$ in $\mathbb{P}^*_\alpha$ for which there exists s_1 in $\mathbb{P}^*_\alpha$ such that:

(1) $\mathrm{dom}(g) \subseteq \mathrm{dom}(s_0)$,

(2) there is $i < \mu^+$ and distinct a_0 and a_1 such that $s_0 \Vdash \dot{f}(i) = a_0$ and $s_1 \Vdash \dot{f}(i) = a_1$,

(3) $\mathrm{dom}(s_0) \cap \{\gamma < \alpha : \gamma \text{ odd}\} = \mathrm{dom}(s_1) \cap \{\gamma < \alpha : \gamma \text{ odd}\}$,

(4) for all odd γ in $\mathrm{dom}(s_0)$, $s_0(\gamma) = s_1(\gamma)$,

(5) $\mathrm{dom}(g) \subseteq \mathrm{dom}(s_1)$, and for all β in $\mathrm{dom}(g)$, if $g(\beta)$ is compatible with the condition named by $s_0(\beta)$, then $s_1(\beta)$ is the name for a condition extending $g(\beta)$.

By elementarity, D is in N.

We claim that D is dense below p. So let $s \leq p$. Extend s to s' in $\mathbb{P}^*_\alpha$ so that $\mathrm{dom}(g) \subseteq \mathrm{dom}(s')$. Now define $s'' \leq s'$ with the same domain as s' as follows. For $\beta \in \mathrm{dom}(s') \setminus \mathrm{dom}(g)$, let $s''(\beta) = s'(\beta)$. Suppose β is in $\mathrm{dom}(g)$. If $s'(\beta)$ names a condition in $\textsc{add}(\mu)$ compatible with $g(\beta)$, let $s''(\beta)$ be a name for a condition which extends $g(\beta)$ and $s'(\beta)$. Otherwise let $s''(\beta) = s'(\beta)$.

Since $\dot{f}$ is not in V, there is $i < \mu^+$ such that s'' does not decide $\dot{f}(i)$. Fix $s'_0, s'_1 \leq s''$ in $\mathbb{P}^*_\alpha$ and distinct a_0 and a_1 so that $s'_0 \Vdash \dot{f}(i) = a_0$ and $s'_1 \Vdash \dot{f}(i) = a_1$. Now apply Lemma 4.4 to obtain $s_0 \leq s'_0$ and $s_1 \leq s'_1$ in $\mathbb{P}^*_\alpha$ satisfying (3) and (4). We check that (5) holds. If β is in $\mathrm{dom}(g)$ and $s_0(\beta)$ names a condition compatible with $g(\beta)$, then clearly $s'(\beta)$ names a condition compatible with $g(\beta)$. So $s''(\beta)$ is a name for a condition refining $g(\beta)$. Since $s_1 \leq s''$, $s_1(\beta)$ is a name for a condition refining $g(\beta)$.

By the genericity property of q, we can fix $s_0 \in D \cap N$ which is compatible with r, and such that for all odd γ in $\mathrm{dom}(s_0)$, γ is in $\mathrm{dom}(r)$ and $r \restriction \gamma$ forces that $r(\gamma) \leq s_0(\gamma)$. Fix s_1, i, a_0, and a_1 in N as described in the definition of D. Since r decides $\dot{f}(i)$ and r and s_0 are compatible, r forces $\dot{f}(i) = a_0$. So r and s_1 are incompatible. We will get a contradiction by showing r and s_1 are compatible.

We apply Lemma 4.1. Suppose β is in $\mathrm{dom}(r) \cap \mathrm{dom}(s_1)$ and β is even. Then β is in N, so β must be in $\mathrm{dom}(g)$. Since r and s_0 are compatible, $r(\beta)$ and $s_0(\beta)$ are compatible. By (5), $s_1(\beta)$ is the name for a condition extending $g(\beta)$. Suppose γ is in $\mathrm{dom}(r) \cap \mathrm{dom}(s_1)$ and γ is odd. Then γ is in $\mathrm{dom}(s_0)$. So γ is in $\mathrm{dom}(r)$, and $r \restriction \gamma$ forces $r(\gamma) \leq s_0(\gamma)$. But $s_0(\gamma) = s_1(\gamma)$. $\square$

5. The Consistency Result

Suppose $\mu < \kappa$ are cardinals, $\mu^{<\mu} = \mu$, and κ is supercompact. We define a forcing iteration of the form given in the last section which collapses κ to become μ^{++}, and forces that for all regular $\lambda \geq \mu^{++}$, there are stationarily many N in $[H(\lambda)]^{\mu^+}$ such that N is internally club but not internally approachable.

Fix a Laver function $f : \kappa \to V_\kappa$. So for all x and λ, there is an elementary embedding $j : V \to M$ with critical point κ such that $M^\lambda \subseteq M$ and $j(f)(\kappa) = x$.

We define by recursion a forcing iteration

$$\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \kappa, j < \kappa \rangle.$$

Suppose $\mathbb{P}_i$ is defined for a fixed $i < \kappa$. If i is an even ordinal, let $\dot{\mathbb{Q}}_i$ be a $\mathbb{P}_i$-name for $\textsc{Add}(\mu)$. Suppose $i = j + 1$ is odd. If $f(j)$ is a $\mathbb{P}_j$-name for a set which contains μ^+, let $\dot{X}_j = f(j)$, and otherwise let $\dot{X}_j$ be a $\mathbb{P}_j$-name for μ^+. Let $\dot{S}_i$ be a $\mathbb{P}_i$-name for $[\dot{X}_j]^\mu \cap V[\dot{G}_j]$, and let $\dot{\mathbb{Q}}_i$ be a $\mathbb{P}_i$-name for the poset $\mathbb{P}(\dot{S}_i)$ from Definition 2.5. Suppose $\delta \leq \kappa$ is a limit ordinal and $\mathbb{P}_i$ is defined for all $i < \delta$. Then let $\mathbb{P}_\delta$ be the poset consisting of all partial functions $p : \delta \to V$ such that $p \restriction i \in \mathbb{P}_i$ for all $i < \delta$, $|\mathrm{dom}(p) \cap \{i < \delta : \ i \text{ even }\}| < \mu$, and $|\mathrm{dom}(p) \cap \{i < \delta : \ i \text{ odd }\}| \leq \mu$.

Since f is a Laver function, there are stationarily many $\alpha < \kappa$ such that $f(\alpha)$ is a $\mathbb{P}_\alpha$-name and $\mathbb{P}_\alpha$ forces $f(\alpha) = (\mu^{++})^{V[\dot{G}_\alpha]}$. Indeed, let $\dot{x}$ be a $\mathbb{P}_\kappa$-name for $(\mu^{++})^{V[\dot{G}_\kappa]}$. Choose $j : V \to M$ with critical point κ such that $j(f)(\kappa) = \dot{x}$ and M is sufficiently closed that it models $\mathbb{P}_\kappa = j(\mathbb{P}_\kappa) \restriction \kappa$ forces $\dot{x} = (\mu^{++})^{M[\dot{G}_\kappa]}$. If C is club in κ, then $\kappa \in j(C)$. Hence by elementarity, there is $\alpha < \kappa$ in C such that $f(\alpha)$ is as desired. But then $\mathbb{P}_{\alpha+2}$ forces $|(\mu^{++})^{V[\dot{G}_\alpha]}| = \mu^+$. So $\mathbb{P}_\kappa$ collapses all cardinals in the interval (μ^+, κ).

Since $|\mathbb{P}_i| < \kappa$ for all $i < \kappa$, there are club many $\delta < \kappa$ such that $|\mathbb{P}_i| < \delta$ for all $i < \delta$. Suppose $\mu^+ < \delta \leq \kappa$ is inaccessible and satisfies this property. Then $\mathbb{P}_\delta$ is the direct limit of $\langle \mathbb{P}_i : i < \delta \rangle$, where each $\mathbb{P}_i$ has size less than δ, and there are stationarily many $\alpha < \delta$ such that $\mathbb{P}_\alpha$ is the direct limit of $\langle \mathbb{P}_i : i < \alpha \rangle$. By a standard Δ-system argument, $\mathbb{P}_\delta$ is δ-c.c. (see Theorem 2.2 of [2]). In particular, $\mathbb{P}_\kappa$ is κ-c.c. and forces that $\kappa = \mu^{++}$.

Let G_κ be generic for $\mathbb{P}_\kappa$. In $V[G_\kappa]$ let $\lambda \geq \mu^{++}$ be regular. In V let $\theta = (2^\lambda)^+$. Let $j : V \to M$ be an elementary embedding with critical point κ such that $M^\theta \subseteq M$ and $j(f)(\kappa)$ is a $\mathbb{P}_\kappa$-name for $H(\lambda)^{V[\dot{G}_\kappa]}$. Then by choice of j,

$$j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \textsc{Add}(\mu) * \mathbb{P}(\dot{S}) * \mathbb{P}_{\text{tail}}$$

where

$$\mathbb{P}_{\kappa+1} \Vdash \dot{S} = \dot{S}_{\kappa+1} = [H(\lambda)^{V[\dot{G}_\kappa]}]^\mu \cap M[\dot{G}_\kappa],$$

and $\mathbb{P}_{\text{tail}}$ is forced to be an iteration of the form given in the previous section. Let $H * K * G_{\text{tail}}$ be generic over $V[G_\kappa]$ for $\textsc{Add}(\mu) * \mathbb{P}(\dot{S}) * \mathbb{P}_{\text{tail}}$. Extend j in $V[G_\kappa * H * K * G_{\text{tail}}]$ to

$$j : V[G_\kappa] \to M[G_\kappa * H * K * G_{\text{tail}}].$$

Then $j(G_\kappa) = G_\kappa * H * K * G_{\text{tail}}$. Since $\mathbb{P}_\kappa$ is κ-c.c., $M[G_\kappa]^\theta \cap V[G_\kappa] \subseteq M[G_\kappa]$. In particular, $H(\lambda)^{V[G_\kappa]} = H(\lambda)^{M[G_\kappa]}$.

Working in $V[G_\kappa]$, let $C \subseteq [H(\lambda)]^{\mu^+}$ be club. We prove there is a set in C which is internally club but not internally approachable. By elementarity, it suffices to prove the same statement about $j(C)$ in $M[j(G_\kappa)]$. We will prove that in $M[j(G_\kappa)]$, the set $j``H(\lambda)^{V[G_\kappa]}$ is in $j(C)$ and is internally club but not internally approachable.

Let $N^* = j``H(\lambda)^{V[G_\kappa]}$. First we prove that $N^*$ is in $M[j(G_\kappa)]$. The set $j``H(\lambda)^V$ is in M by the closure of M. But $H(\lambda)^{V[G_\kappa]} = H(\lambda)^V[G_\kappa]$. So every element of N^* is of the form $j(\dot{a}^{G_\kappa}) = j(\dot{a})^{j(G_\kappa)}$, where $\dot{a}$ is in $H(\lambda)^V$. So $N^* = (j``H(\lambda)^V)[j(G_\kappa)]$, which is in $M[j(G_\kappa)]$. Also note that in $M[j(G_\kappa)]$, $|N^*| = |H(\lambda)^{V[G_\kappa]}| = \mu^+$.

We claim that N^* is in $j(C)$. Since $j(C)$ is closed under unions of directed subsets with size less than $j(\mu^{++})$, it suffices to show that $N^* \cap j(C)$ is directed and $\bigcup(N^* \cap j(C)) = N^*$. Suppose $j(a)$ and $j(b)$ are in $N^* \cap j(C)$. By elementarity, a and b are in C. Fix c in C such that $a \cup b \subseteq c$. Then $j(a)$ and $j(b)$ are contained in $j(c)$ and $j(c) \in N^* \cap j(C)$. Hence $N^* \cap j(C)$ is directed.

We show that $\bigcup(N^* \cap j(C)) = N^*$. Let $j(x)$ be in N^*. Then x is in $H(\lambda)^{V[G_\kappa]}$, so there is a in C such that $x \in a$. Then $j(x) \in j(a) \in N^* \cap j(C)$. So $N^* \subseteq \bigcup(N^* \cap j(C))$. On the other hand, suppose y is in $\bigcup(N^* \cap j(C))$. Fix $j(a) \in N^* \cap j(C)$ so that $y \in j(a)$. Then a is in C. In $V[G_\kappa]$, a has size less than $\mu^{++} = \kappa$, and κ is the critical point of j. So $j(a) = j``a$, and clearly $j``a \subseteq N^*$. Thus y is in N^*. Therefore $N^* = \bigcup(N^* \cap j(C))$ and N^* is in $j(C)$.

Now we show that N^* is internally club but not internally approachable. Let $N = H(\lambda)^{V[G_\kappa]}$. Since N is transitive and isomorphic to N^* by the map $j \restriction N$, N is the transitive collapse of N^* and $j^{-1} \restriction N^* = \pi$ is the transitive collapse map.

Recall that $H * K$ is generic for $\textsc{Add}(\mu) * \mathbb{P}(\dot{S})$ over $M[G_\kappa]$, and $S = \dot{S}^H = [N]^\mu \cap M[G_\kappa]$. Write $\bigcup K = \langle a_i : i < \mu^+ \rangle$. Then $N = \bigcup\{a_i : i < \mu^+\}$. For all $i < \mu^+$, a_i is a subset of $N = H(\lambda)^{M[G_\kappa]}$ which is in $M[G_\kappa]$, and a_i has size μ, which is less than λ. Therefore a_i is in $H(\lambda)^{M[G_\kappa]} = N$. Hence N is internally club. But then $\langle j(a_i) : i < \mu^+ \rangle = \langle j``a_i : i < \mu^+ \rangle$ witnesses that N^* is internally club.

Suppose for a contradiction that N^* is internally approachable in $M[j(G_\kappa)]$, as witnessed by a sequence $\langle N_i^* : i < \mu^+ \rangle$. Note that N is then also internally approachable. Indeed, for all $i < \mu^+$, let $N_i = \pi(N_i^*) = \pi``(N_i^*)$. Clearly $\langle N_i : i < \mu^+ \rangle$ is increasing and continuous and its union is equal to N. For each $\alpha < \mu^+$, choose f_α in N such that $j(f_\alpha) = \langle N_i^* : i < \alpha \rangle$. Then for $i < \alpha$, $j(N_i) = N_i^* = j(f_\alpha)(i) = j(f_\alpha)(j(i)) = j(f_\alpha(i))$. So $N_i = f_\alpha(i)$. Therefore $\langle N_i : i < \alpha \rangle = f_\alpha$, which is in N. Hence $\langle N_i : i < \mu^+ \rangle$ witnesses that N is internally approachable.

Let $f = \langle N_i : i < \mu^+ \rangle$. Then for all $i < \mu^+$, $f \restriction i$ is in N. Since $N \subseteq M[G_\kappa]$, for all $i < \mu^+$, $f \restriction i$ is in $M[G_\kappa]$. By Proposition 4.5, $f = \langle N_i : i < \mu^+ \rangle$ is in $M[G_\kappa]$. But this implies N has size μ^+ in $M[G_\kappa]$, which is false. This completes the proof.

We note that if GCH holds in V, then in $V[G]$, $2^\mu = \mu^{++}$ and $2^\alpha = \alpha^+$ for all infinite cardinals α different from μ. This violation of GCH is necessary, by the following argument:

Suppose $2^\mu = \mu^+$ and $\lambda \geq \mu^{++}$ is regular. Let $N \prec H(\lambda)$ be a model with size μ^+ such that $\mu^+ \subseteq N$, and N satisfies the μ^+-*covering property*, that is, every subset of N with size less than μ^+ is a subset of a member of N with size less than μ^+. Then $N^\mu \subseteq N$. For if $a \subseteq N$ has size μ, then a is covered by a set b in N with size μ. Since $2^\mu = \mu^+$, we can enumerate the power set of b by a sequence $\langle x_i : i < \mu^+ \rangle$ in N. But $\mu^+ \subseteq N$, so $x_i \in N$ for all $i < \mu^+$. In particular, a is in N. Now fix an increasing and continuous sequence $\langle N_i : i < \mu^+ \rangle$ of sets with size μ whose union is N. Since $N^\mu \subseteq N$, each N_i is in N, and thus every initial segment of this sequence is in N. So N is internally approachable. But if N is internally club, then N satisfies the μ^+-covering property.

Remarks. Mixed support iterations similar to that presented in Section 4 appear in Chapter 8 of [8], where an analogue of Proposition 4.2 is proved for iterations of posets of the form $\mathbb{P} * \dot{\mathbb{Q}}$, where $\mathbb{P}$ is ω_1-closed, $\mathbb{P}$ satisfies a strengthening of ω_2-c.c., and $\dot{\mathbb{Q}}$ is forced to be ω_2-closed. The proof of our consistency result is related to Mitchell's construction in [6] of a model with no Aronszajn trees on ω_2. See [7] for a recent discussion concerning the special property described in Proposition 4.5.

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