INTERNALLY CLUB AND APPROACHABLE FOR LARGER STRUCTURES

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Abstract. We generalize the notion of a fat subset of a regular cardinal \( \kappa \) to a fat subset of \( P_\kappa(X) \), where \( \kappa \subseteq X \). Suppose \( \mu < \kappa \), \( \mu^\kappa = \mu \), and \( \kappa \) is supercompact. Then there is a generic extension in which \( \kappa = \mu^{++} \), and for all regular \( \lambda \geq \mu^{++} \), there are stationarily many \( N \) in \( [H(\lambda)]^{\mu^+} \) which are internally club but not internally approachable.

Suppose \( \mu \) is an infinite cardinal. A set \( N \) is internally approachable with length \( \mu^+ \) if \( N \) is the union of an increasing and continuous sequence \( \langle N_i : i < \mu^+ \rangle \) of sets with size \( \mu \) such that for all \( \alpha < \mu^+ \), \( \langle N_i : i < \alpha \rangle \) is in \( N \). A related idea is that of an internally club set. A set \( N \) with size \( \mu^+ \) is internally club if \( N \cap [N]^\mu \) contains a club subset of \( [N]^\mu \). In other words, \( N \) is the union of an increasing and continuous sequence \( \langle N_i : i < \mu^+ \rangle \) of sets with size \( \mu \) such that each \( N_i \) is in \( N \).

Foreman and Todorčević [3] asked whether the properties of being internally approachable and internally club are equivalent. In [5] we proved that under PFA, for all regular \( \lambda \geq \omega_2 \) there are stationarily many structures \( N \prec H(\lambda) \) with size \( \aleph_1 \) such that \( N \) is internally club but not internally approachable. In this paper we generalize this result to larger structures.

Theorem 1. Suppose \( \mu < \kappa \), \( \mu^\kappa = \mu \), and \( \kappa \) is supercompact. Then there is a \( \mu \)-closed, \( \mu^+ \)-proper forcing poset which collapses \( \kappa \) to become \( \mu^{++} \), and forces that for all regular \( \lambda \geq \mu^{++} \), there are stationarily many \( N \) in \( [H(\lambda)]^{\mu^+} \) which are internally club but not internally approachable.

In the model we construct to prove Theorem 1, we have that \( 2^\mu = \mu^{++} \). In fact, if \( 2^\mu = \mu^+ \), then any elementary substructure \( N \prec H(\lambda) \) with size \( \mu^+ \) and which contains \( \mu^+ \) is internally club iff it is internally approachable; this is shown at the end of the paper.

In Section 1 we review notation and some background material. Section 2 generalizes the idea of a fat subset of a regular cardinal \( \kappa \) to a fat subset of \( P_\kappa(X) \), where \( \kappa \subseteq X \). Section 3 presents the basic forcing poset we use in our consistency result, and in Section 4 we describe how to iterate this poset with a mixed support forcing iteration. In Section 5 we prove Theorem 1.

1. Preliminaries

If \( \kappa \) is regular and \( \kappa \subseteq X \), we say \( C \subseteq P_\kappa(X) \) is club if it is closed under unions of increasing sequences of length less than \( \kappa \), and is cofinal. A set \( S \subseteq P_\kappa(X) \) is stationary if it has non-empty intersection with every club. We will use the fact that if \( C \subseteq P_\kappa(X) \) is club, and \( A \subseteq C \) is a directed set with size less than \( \kappa \), then
\[ \bigcup A \in C \text{ (see Lemma 8.25 of [4] for a proof).} \] By directed we mean that if \( a \) and \( b \) are in \( A \), then there is \( c \) in \( A \) such that \( a \cup b \subseteq c \).

If \( N \) is a set, \( P \) is a forcing poset, and \( G \) is a filter on \( P \), then \( N[G] \) denotes the set \( \{ \dot{a} : a \in N \cap V^P \} \). A filter \( G \) on \( P \) is \( N \)-generic if for every dense set \( D \subseteq P \) in \( N \), \( N \cap D \cap G \) is non-empty. A condition \( q \) in \( P \) is \( N \)-generic if \( q \) forces \( \dot{G} \) is \( N \)-generic, where \( \dot{G} \) is a name for the generic filter. Suppose \( \lambda \) is regular with \( P \in H(\lambda) \), and \( N \prec \langle H(\lambda), \in, P \rangle \). Then for any condition \( q \) in \( P \), the following are equivalent: (1) \( q \) is \( N \)-generic, (2) for every dense set \( D \subseteq P \) in \( N \), \( N \cap D \) is predense below \( \dot{G} \), (3) \( q \) forces \( N[\dot{G}] \cap On = N \cap On \), and (4) \( q \) forces \( N[\dot{G}] \cap V = N \). Note that if \( q \) is \( N \)-generic, then for any set \( X, q \) forces \( N[\dot{G}] \cap X = N \cap X \).

Suppose \( P \) is a forcing poset and \( \lambda \) is regular with \( P \in H(\lambda) \). If \( G \) is generic for \( P \) over \( V \), then \( H(\lambda)^{V[G]} = H(\lambda)^V[\dot{G}] \). Suppose \( N \prec \langle H(\theta), \in, P \rangle \) in \( V \). If \( G \) is generic for \( P \) over \( V \), then \( N[G] \prec H(\theta)^{V[G]} \).

Let \( P \) be a forcing poset and \( \mu \) a regular cardinal with \( \mu^{<\mu} = \mu \). Then \( P \) is \( \mu^+ \)-proper if for any regular cardinal \( \theta > 2^{[\mu]} \) with \( P \in H(\theta) \), if \( N \) is an elementary substructure of \( \langle H(\theta), \in, P \rangle \), \( N \) has size \( \mu \), and \( N^{<\mu} \subseteq N \), then for all \( p \in N \cap P \), there is \( q \leq p \) which is \( N \)-generic. Any \( \mu^+ \)-proper forcing poset preserves \( \mu^+ \). Note that if \( P \) is \( \mu^+ \)-c.c. then any condition in \( P \) is \( N \)-generic, since every maximal antichain of \( P \) in \( N \) is actually a subset of \( N \).

If \( \mu \) is a regular cardinal and \( P \) is a forcing poset, we say \( P \) is \( \mu \)-distributive if for any collection \( D \) of less than or equal to \( \mu \) many dense open subsets of \( P \), \( \bigcap D \) is dense open. This property is equivalent to \( P \) not adding any new sequences of ordinals with order type less than or equal to \( \mu \). If \( \kappa \) is a cardinal we say \( P \) is \( < \kappa \)-distributive if \( P \) is \( \mu \)-distributive for all regular \( \mu < \kappa \).

Let \( P \) be a forcing poset and \( \mu \) a regular cardinal. We say \( P \) is \( \mu \)-closed if whenever \( \langle p_i : i < \xi \rangle \) is a descending sequence of conditions in \( P \) with \( \xi < \mu \), there is \( q \) in \( P \) such that \( q \leq p_i \) for all \( i < \xi \). If \( A \subseteq P \), a greatest lower bound of \( A \), or \( \text{glb} \) of \( A \), is a condition \( q \) such that \( q \leq p \) for all \( p \in A \), and whenever \( r \leq p \) for all \( p \in A \), \( r \leq q \). We say \( P \) is \( \mu \)-glb closed if whenever \( \langle p_i : i < \xi \rangle \) is a descending sequence of conditions in \( P \) with \( \xi < \mu \), there exists a greatest lower bound for the set \( \{ p_i : i < \xi \} \).

2. Generalized Fat Sets

Let \( \kappa \) be a regular uncountable cardinal. Recall that a set \( A \subseteq \kappa \) is fat if for any club set \( C \subseteq \kappa \) and \( \xi < \kappa \), \( A \cap C \) contains a closed subset with order type at least \( \xi \).

**Fact 2.1.** (Abraham and Shelah [1]). Suppose \( \kappa \) is strongly inaccessible or \( \kappa = \mu^+ \) where \( \mu^{<\mu} = \mu \). Then the following are equivalent for a set \( A \subseteq \kappa \):

1. \( A \) is fat.
2. There is a \( < \kappa \)-distributive forcing poset \( P \) which forces that \( A \) contains a club set.

Suppose \( \kappa \) is a regular uncountable cardinal and \( \kappa \subseteq X \). We generalize the idea of fatness to subsets of \( P_\kappa(X) \) with the following definition.

**Definition 2.2.** Suppose \( \kappa \) is a regular uncountable cardinal and \( \kappa \subseteq X \). A set \( A \subseteq P_\kappa(X) \) is fat if for all regular \( \theta \geq \kappa \) with \( X \subseteq H(\theta) \), for any club \( C \subseteq P_\kappa(H(\theta)) \) and \( \xi < \kappa \), there is an increasing and continuous sequence \( \langle N_i : i < \xi \rangle \) such that for all \( i < \xi \), \( N_i \in C \), \( N_i \cap X \in A \), and \( N_i \in N_{i+1} \) when \( i + 1 < \xi \).
Lemma 2.3. Suppose $\kappa = \mu^+$. Then $A \subseteq P_\kappa(X)$ is fat iff for all regular $\theta \geq \kappa$ with $X \subseteq H(\theta)$, for any club $C \subseteq P_\kappa(H(\theta))$, and for any regular cardinal $\lambda \leq \mu$, there is an increasing and continuous sequence $\langle N_i : i \leq \lambda \rangle$ such that for $i \leq \lambda$, $N_i \subseteq C$, $N_i \cap X \in A$, and $N_i \subseteq N_{i+1}$ when $i < \lambda$.

Proof. Suppose $A$ satisfies the second property. Then clearly $A$ is stationary in $P_\kappa(X)$. Fix $\theta \geq \kappa$ regular with $X \subseteq H(\theta)$. We prove by induction on $\xi < \mu^+$ that for any club set $C \subseteq P_\kappa(H(\theta))$, there is an increasing and continuous sequence $\langle N_i : i < \xi \rangle$ such that for all $i < \xi$, $N_i \subseteq C$, $N_i \cap X \in A$, and $N_i \subseteq N_{i+1}$ when $i < \xi$. The successor step of the induction follows from the fact that $A$ is stationary.

Suppose $\delta < \mu^+$ is a limit ordinal and the claim holds for all $\delta' < \delta$. Let $\langle \delta_i : i < \text{cf}(\delta) \rangle$ be increasing and cofinal in $\delta$. Note that $\text{cf}(\delta) \leq \mu$. Let

$A = \langle H(\theta), \in, <, X, A, \delta, \langle \delta_i : i < \text{cf}(\delta) \rangle \rangle,$

where $<$ is a well-ordering of $H(\theta)$. Fix an increasing and continuous sequence $\langle N_i : i < \delta \rangle$ of sets such that for $i \leq \text{cf}(\delta)$, $N_i \subseteq C$, $N_i \subseteq A$, $\mu \subseteq N_i$, $N_i \cap X \in A$, and $N_i \subseteq N_{i+1}$ when $i < \text{cf}(\delta)$.

Fix $i < \text{cf}(\delta)$. By the induction hypothesis, let $\langle M_i^j : j \leq \delta_i \rangle$ be the $<$-least increasing and continuous sequence with length $\delta_i + 1$ such that $\mu \cap \{ N_i \} \subseteq M_i^\delta$, and for $j \leq \delta_i$, $M_i^j \subseteq C$, $N_i \subseteq M_i^j$, $M_i^j \cap X \in A$, and $M_i^j \subseteq M_{i+1}^j$ when $j < \delta_i$. By elementarity, this sequence is in $N_{i+1}$. Then the set

$\{ N_i : i \leq \text{cf}(\delta) \} \cup \{ M_i^j : i < \text{cf}(\delta), j \leq \delta_i \},$

well-ordered by $\in$, is increasing and continuous with order type at least $\delta$, and for all $N$ in this set, $N \subseteq C$ and $N \cap X \in A$. \qed

We will now show that our definition of fatness generalizes the classical notion. Indeed, let $A$ be a fat subset of a regular cardinal $\kappa$. We show $A$ is a fat subset of $P_\kappa(X)$, where $X = \kappa$, according to Definition 2.2. So let $\theta \geq \kappa$ be regular, and let $C \subseteq P_\kappa(H(\theta))$ be club. Fix $\xi < \kappa$. Define by induction an increasing and continuous sequence $\langle M_i : i < \kappa \rangle$ such that for $i < \kappa$, $M_i \cap \kappa \in C$, $M_i \subseteq C$, and $M_i \subseteq M_{i+1}$. Then $\langle M_i \cap \kappa : i < \kappa \rangle$ is a club subset of $\kappa$. Since $A$ is fat, there is a closed set $a \subseteq \kappa$ with order type at least $\xi$ such that $\{ M_i \cap \kappa : i \in a \} \subseteq A$. Then $\langle M_i : i \in a \rangle$ is as required.

Suppose on the other hand that $A \subseteq \kappa$ is fat as a subset of $P_\kappa(\kappa)$ by Definition 2.2, and we show $A$ is fat as a subset of $\kappa$. Let $C \subseteq \kappa$ be club and fix $\xi < \kappa$. Let $\langle N_i : i \leq \xi \rangle$ be an increasing and continuous sequence of sets in $P_\kappa(H(\kappa))$ such that for $i \leq \xi$, $N_i \subseteq \langle H(\kappa), \in, C \rangle$, $N_i \cap \kappa \in C$, $N_i \cap \kappa \subseteq A$, and $N_i \subseteq N_{i+1}$. Then $\{ N_i \cap \kappa : i \leq \xi \}$ is a closed set contained in $A \cap C$.

The next theorem generalizes Fact 2.1.

Theorem 2.4. Suppose $\kappa$ is strongly inaccessible or $\kappa = \mu^+$ where $\mu^\kappa = \mu$. Let $X$ be a set containing $\kappa$. Then the following are equivalent for a set $A \subseteq P_\kappa(X)$:

1. $A$ is fat.

2. There is a $< \kappa$-distributive forcing poset which forces there is an increasing and continuous sequence $\langle a_i : i < \kappa \rangle$ which is cofinal in $P_\kappa(X)$ such that $a_i \subseteq A$ for $i < \kappa$.

Proof. Suppose $A \subseteq P_\kappa(X)$ and $\mathbb{P}$ is a $< \kappa$-distributive forcing poset which forces that $\langle \dot{a}_i : i < \kappa \rangle$ is increasing, continuous, and cofinal in $P_\kappa(X)$ such that $\dot{a}_i \subseteq A$
for $i < \kappa$. We prove that $A$ is fat. So let $\theta \geq \kappa$ be regular with $X \subseteq H(\theta)$. Suppose $C \subseteq P_\kappa(H(\theta))$ is club. Let $G$ be generic for $\mathbb{P}$ over $V$, and let $a_i = a^{C}_{\kappa}$ for $i < \kappa$. Since $\mathbb{P}$ is $<\kappa$-distributive, in $V[G]$ the set $C$ is still a club subset of $P_\kappa(H(\theta))$.

We work in $V[G]$. Since $X = \bigcup \{a_i : i < \kappa\}$ and $|a_i| < \kappa$ for all $i < \kappa$, $X$ has size $\kappa$ in the extension. So let $\langle x_i : i < \kappa\rangle$ enumerate $X$. We define by induction an increasing and continuous sequence $\langle N_i : i < \kappa \rangle$ such that for all $i < \kappa$, $N_i \in N_{i+1}$ and $N_i \in C$. Choose $N_0$ in $C$ arbitrarily. At limits take unions. Suppose $N_i$ is defined. Then $N_i$ is in $H(\theta)^V$, so choose $N_{i+1}$ in $C$ such that $N_i \cup \{N_i\} \cup \{x_i\} \subseteq N_{i+1}$. This completes the definition. Now $\langle a_i : i < \kappa \rangle$ and $\langle N_i \cap X : i < \kappa \rangle$ are both club in $P_\kappa(X)$. So there is a club $D \subseteq \kappa$ such that for all $i \in D$, $a_i = N_i \cap X$. Then $\langle N_i : i \in D \rangle$ is an increasing and continuous sequence such that for all $i \in D$, $N_i \in C$, $N_i \in N_{i+1}$, and $N_i \cap X \in A$. But every initial segment of this sequence is in $V$ since $\mathbb{P}$ is $<\kappa$-distributive. So $A$ is fat.

In the other direction, suppose $A \subseteq P_\kappa(X)$ is fat. Define a forcing poset $\mathbb{P}(A)$ as follows. A condition in $\mathbb{P}(A)$ is an increasing and continuous sequence $\langle a_i : i \leq \gamma \rangle$, where $\gamma < \kappa$, such that $a_i \in A$ for all $i \leq \gamma$. The ordering is by extension of sequences. We claim that $\mathbb{P}(A)$ is $<\kappa$-distributive and $\mathbb{P}(A)$ forces that the union of the generic filter is an increasing and continuous sequence cofinal in $P_\kappa(X)$ with order type $\kappa$ whose elements are in $A$.

Suppose $\langle D_i : i < \xi \rangle$ is a sequence of dense open subsets of $\mathbb{P}(A)$, where $\xi < \kappa$ is a cardinal. Let $p$ be in $\mathbb{P}(A)$, and we find $q \leq p$ which is in $\bigcap \{D_i : i < \xi\}$. Fix a regular cardinal $\theta \gg \kappa$ with $X \in H(\theta)$, and let

$$A = \langle H(\theta), \in, X, A, \mathbb{P}(A), p, \langle D_i : i < \xi \rangle \rangle.$$ 

Since $A$ is fat we can find an increasing and continuous sequence $\langle N_i : i \leq \xi \rangle$ such that for all $i \leq \xi$, $N_i \smallsetminus A$, $N_i \cap \kappa \in \kappa$, $\xi \subseteq N_i$, $N_i \cap X \in A$, and when $i < \xi$, $N_i \in N_{i+1}$.

We define by induction a descending sequence of conditions $\langle p_i : i \leq \xi \rangle$ in $\mathbb{P}(A)$. Our induction hypothesis is that $p_i$ is in $N_{i+1}$ and the maximum element of $p_i$ is $N_i \cap X$. Let $p_0 = p^\times (N_0 \cap X)$. Then $p_0$ is a condition, because $p \in N_0$ and thus all the elements of $p$ are subsets of $N_0 \cap X$. Suppose $i < \xi$, and for all $j \leq i$, $p_j$ is defined, $p_j$ is a member of $N_{j+1}$, and the maximum element of $p_j$ is $N_j \cap X$. Since $\xi \subseteq N_{i+1}$, $D_i$ is in $N_{i+1}$. Fix $p^*_i \leq p_i$ in $D_i \cap N_{i+1}$. Since $p^*_i$ has size less than $\kappa$ and $N_{i+1} \cap \kappa \in \kappa$, $p^*_i \subseteq N_{i+1}$, and so every element of $p^*_i$ is a subset of $N_{i+1}$ as well. Therefore if we let $p_{i+1} = p^*_i \smallsetminus (N_{i+1} \cap X)$, $p_{i+1}$ is a condition in $N_{i+2} \cap D_i$ below $p_i$.

Suppose $\delta \leq \xi$ is a limit ordinal and $p_i \in N_{i+1}$ is defined for all $i < \delta$. Let

$$p_\delta = \bigcup \{p_i : i < \delta\} \smallsetminus (N_\delta \cap X),$$

which is a condition since $N_\delta \cap X \in A$ and $N_\delta = \bigcup \{N_i : i < \delta\}$. We need to show that $p_\delta$ is in $N_{\delta+1}$ when $\delta < \xi$. The sequence $\langle p_i : i < \delta \rangle$ is in $N^{<\xi}_\delta$. Since $\kappa$ is either strongly inaccessible or $\kappa = \mu^+$ where $\mu^{<\mu} = \mu$, $N^{<\xi}_\delta$ has size less than $\kappa$. But $N^{<\xi}_\delta \subseteq N_{\delta+1}$. Since $N_{\delta+1} \cap \kappa \in \kappa$, $N^{<\xi}_\delta \subseteq N_{\delta+1}$. So the sequence $\langle p_i : i < \delta \rangle$ is in $N_{\delta+1}$. Clearly then $p_\delta$ is in $N_{\delta+1}$ as well.

This completes the construction of $\langle p_i : i \leq \xi \rangle$. The condition $p_\xi$ is below $p$ and is in $\bigcap \{D_i : i < \xi\}$. So $\mathbb{P}(A)$ is $<\kappa$-distributive.

For each $\alpha < \kappa$ let $D_\alpha$ be the set of conditions in $\mathbb{P}(A)$ with length at least $\alpha$. Clearly $D_0$ is dense open, and if $D_i$ is dense open, $D_{i+1}$ is dense open as well.
Assume $\delta < \kappa$ is a limit ordinal and $D_i$ is dense open for all $i < \delta$. Since $\mathbb{P}(A)$ is $<\kappa$-distributive, $\bigcap\{D_i : i < \delta\}$ is dense open. But if $p$ is in this intersection, $p$ has length at least $\delta$. So $\mathbb{P}(A)$ forces the union of the generic filter has length $\kappa$. By an easy density argument, $\mathbb{P}(A)$ forces the union of the generic filter is cofinal in $P_\kappa(X)$. \hfill \Box

Since we will use the forcing poset from the last theorem in our consistency proof, we describe it explicitly in the following definition.

**Definition 2.5.** Suppose $\kappa$ is regular, $\kappa \subseteq X$, and $A \subseteq P_\kappa(X)$ is fat. Let $\mathbb{P}(A)$ be the forcing poset consisting of increasing and continuous sequences $\langle a_i : i \leq \gamma \rangle$, where $\gamma < \kappa$ and $a_i \in A$ for $i \leq \gamma$, ordered by extension of sequences.

The forcing poset $\mathbb{P}(A)$ is $<\kappa$-distributive and adds an increasing, continuous, and cofinal sequence $\langle a_i : i < \kappa \rangle$ through $P_\kappa(X)$ such that $a_i \in A$ for $i < \kappa$. In particular, $\mathbb{P}(A)$ collapses the size of $X$ to be $\kappa$.

If $\kappa = \omega_1$ and $\omega_1 \subseteq X$, one can show using Lemma 2.3 that any stationary set $A \subseteq P_{\omega_1}(X)$ is fat. Thus $\mathbb{P}(A)$ is $\omega$-distributive for any stationary set $A \subseteq P_{\omega_1}(X)$.

3. THE BASIC FORCING POSET

We now describe the forcing poset which we will use in our consistency proof.

Suppose $\mu^+ = \mu$ and $\mu^+ \subseteq X$. The basic forcing poset we will use is $\text{Add}(\mu) * \mathbb{P}(\dot{S})$, where $\text{Add}(\mu)$ adds a Cohen subset to $\mu$, $\text{Add}(\mu)$ forces $\dot{S} = [X]^{\mu} \cap V$, and $\mathbb{P}(\dot{S})$ is the forcing poset from Definition 2.5. Thus we need to know that $\text{Add}(\mu)$ forces $\dot{S}$ is fat. If $\mu^\kappa = \mu$ then $\text{Add}(\mu)$ is $\mu^+\text{-c.c.}$, so this follows from the next proposition.

**Proposition 3.1.** Suppose $\kappa$ is regular and $\kappa \subseteq X$. Let $\mathbb{P}$ be a $\kappa\text{-c.c.}$ forcing poset. Then $\mathbb{P}$ forces $P_\kappa(X) \cap V$ is fat.

**Proof.** Let $G$ be generic for $\mathbb{P}$ over $V$. Working in $V[G]$, fix $\theta \geq \kappa$ regular with $X \subseteq H(\theta)$, and let $C \subseteq P_\kappa(H(\theta))$ be club. Fix $\chi \gg \theta$ regular such that $H(\chi)$ contains $C$ and $\mathbb{P}$ as members. Recall that $H(\chi)^{V[G]} = H(\chi)^V$. Let $\dot{C}$ be a name for $C$ in $H(\chi)^V$.

Now back in $V$, define by induction an increasing and continuous sequence $\langle N_i : i < \kappa \rangle$ of elementary substructures of $\langle H(\chi), \in, X, \dot{C}, \mathbb{P} \rangle$ such that for all $i < \kappa$, $|N_i| < \kappa$, $N_i \cap \kappa \in \kappa$, and $N_i \in N_{i+1}$. Then in $V[G]$, for all $i < \kappa$, $N_i[G] \prec \langle H(\chi)^{V[G]}, \in, C \rangle$. By elementarity, $N_i[G] \cap C$ is a directed subset of $C$ with size less than $\kappa$ whose union is equal to $N_i[G] \cap H(\theta)$. So $N_i[G] \cap H(\theta)$ is in $C$. Since $N_i \in N_{i+1}$, $N_i[G] \in N_{i+1}[G]$, and therefore $N_i[G] \cap H(\theta) \in N_{i+1}[G] \cap H(\theta)$. But $\mathbb{P}$ is $\kappa\text{-c.c.}$, so $N_i[G] \cap V = N \cap V$. For $x \in N_i[G] \cap V$, there is a name $\dot{x}$ for $x$ in $N_i$. The maximal antichain of conditions deciding $\dot{x}$ is in $N_i$, and has size less than $\kappa$, so is a subset of $N_i$. But then $x$ is in $N_i$. In particular, $N_i[G] \cap X = N_i \cap X$, which is in $P_\kappa(X) \cap V$. \hfill \Box

The forcing poset $\text{Add}(\mu)$ is $\mu\text{-g.l.b.}$ closed. Indeed, if $\langle p_i : i < \xi \rangle$ is decreasing in $\text{Add}(\mu)$ where $\xi < \mu$, then $\bigcup\{p_i : i < \xi\}$ is the greatest lower bound. Note that any two step forcing iteration of $\mu\text{-g.l.b.}$ closed forcing posets is $\mu\text{-g.l.b.}$ closed.

**Lemma 3.2.** Suppose $\mu^\kappa = \mu$, $\mu^+ \subseteq X$, and $\dot{S}$ is an $\text{Add}(\mu)$-name for $[X]^{\mu} \cap V$. Then $\text{Add}(\mu)$ forces that $\mathbb{P}(\dot{S})$ is $\mu\text{-g.l.b.}$ closed. Hence $\text{Add}(\mu) * \mathbb{P}(\dot{S})$ is $\mu\text{-g.l.b.}$ closed.
**Proof.** Let $G$ be generic for $\text{ADD}(\mu)$. In $V[G]$, suppose $\langle p_i : i < \xi \rangle$ is a descending sequence of conditions in $\mathbb{P}(S)$ where $\xi < \mu$ is a limit ordinal. For each $i$ write $p_i = \langle a_j : j \leq \gamma_i \rangle$. Let $\gamma = \sup(\{\gamma_i : i < \xi\})$ and $a = \bigcup(a_i : i < \gamma)$. Let

$$q = \bigcup\{p_i : i < \xi\} \cup \{\langle \gamma, a \rangle\}.$$

Then $q$ is a condition in $\mathbb{P}(S)$ iff $a$ is in $V$. But since $\text{ADD}(\mu)$ is $\mu$-closed, the sequence $\langle a_{\gamma_i} : i < \xi \rangle$ is in $V$, and hence its union $a$ is in $V$. Clearly any condition which extends each $p_i$ must extend $q$, so $q$ is the greatest lower bound of the sequence. \qed

4. ITERATING THE BASIC FORCING POSET

We now describe a mixed support iteration of the forcing poset introduced in the last section.

Fix a cardinal $\mu$ such that $\mu^{<\mu} = \mu$. We consider a forcing iteration

$$\langle \mathbb{P}_i, \check{Q}_j : i \leq \alpha, j < \alpha \rangle,$$

satisfying the following recursive definition:

1. If $i < \alpha$ is even, $\mathbb{P}_i$ forces $\check{Q}_i = \text{ADD}(\mu)$, and $\mathbb{P}_i$ forces $\check{X}_i$ is a set containing $\mu^+$.  
2. If $i = j + 1 < \alpha$ is odd, $\mathbb{P}_i$ forces $\check{S}_i = [\check{X}_j]^\mu \cap V[\check{G}_j]$, where $\check{G}_j$ is a name for the generic filter for $\mathbb{P}_j$, and $\check{Q}_i = \mathbb{P}(\check{S}_i)$ is the poset from Definition 2.5.  
3. If $i \leq \alpha$ is a limit ordinal, $\mathbb{P}_i$ is the poset consisting of partial functions $p : i \rightarrow V$ such that $p \upharpoonright j \in \mathbb{P}_j$ for $j < i$, $|\text{dom}(p) \cap \{j < i : j \text{ even}\}| < \mu$, and $|\text{dom}(p) \cap \{j < i : j \text{ odd}\}| \leq \mu$.  

We assume the following recursion hypotheses for all $\beta < \alpha$, which guarantee that the definition above makes sense.

4. $\mathbb{P}_\beta$ is $\mu$-glb closed and $\mu^+$-proper, and so preserves cardinals and cofinalities less than or equal to $\mu^+$.  
5. Let $\check{P}_\beta$ be the set of $p$ in $\mathbb{P}_\beta$ such that for all even $j$ in $\text{dom}(p)$, there is $x$ in $\text{ADD}(\mu)$ such that $p(j) = \check{x}$. Then $\check{P}_\beta$ is dense in $\mathbb{P}_\beta$.  
6. If $\langle p_i : i < \xi \rangle$ is a descending sequence of conditions in $\check{P}_\beta$ with $\xi < \mu$, then the greatest lower bound of this sequence is in $\check{P}_\beta$.

We prove that properties (4), (5), and (6) above also hold for $\mathbb{P}_\alpha$.

**Case 1:** $\alpha = \beta + 1$ is a successor ordinal.

We show that $\check{P}_\alpha$ is $\mu$-glb closed. This will follow from the fact that a two-step iteration of $\mu$-glb closed forcing posets is $\mu$-glb closed. If $\beta$ is even, then $\check{P}_\alpha = \mathbb{P}_\beta * \text{ADD}(\mu)$. Since $\mathbb{P}_\beta$ is $\mu$-glb closed by recursion, clearly $\check{P}_\alpha$ is $\mu$-glb closed as well. Suppose $\beta = \gamma + 1$ is odd. Then $\check{P}_\alpha = \check{P}_\gamma * \text{ADD}(\mu) * \mathbb{P}(\check{S}_\beta)$. By recursion, $\check{P}_\gamma$ is $\mu$-glb closed, and by Lemma 3.2, $\mathbb{P}_\gamma$ forces that $\text{ADD}(\mu) * \mathbb{P}(\check{S}_\beta)$ is $\mu$-glb closed. So $\check{P}_\alpha$ is $\mu$-glb closed. We prove in Proposition 4.2 below that $\check{P}_\alpha$ is $\mu^+$-proper.

Now we prove that $\check{P}_\alpha$ is dense in $\check{P}_\beta$. Consider a condition $p$ in $\check{P}_\alpha$. If $\beta$ is not in the domain of $p$ or if $\beta$ is odd, fix $q \leq p \upharpoonright \beta$ in $\check{P}_\beta$. Then $q \leq p$ is in $\check{P}_\alpha$ if $\beta$ is not in $\text{dom}(p)$, and $q^{-1}p(\beta) \leq p$ is in $\check{P}_\alpha$ otherwise. Assume $\beta$ is in $\text{dom}(p)$ and $\beta$ is even. Since $\mathbb{P}_\beta$ is $\mu$-closed, it forces that $p(\beta)$ is an element of $\text{ADD}(\mu)$ in
the ground model. So choose \( r \leq p \upharpoonright \beta \) in \( \mathbb{P}_\beta^* \) and \( x \) in \( \text{Add}(\mu) \) such that \( r \) forces \( p(\beta) = \hat{x} \). Then \( r \upharpoonright \hat{x} \) is as required.

Suppose \( \langle p_i : i < \xi \rangle \) is a descending sequence of conditions in \( \mathbb{P}_\alpha^* \) with \( \xi < \mu \). We show that the greatest lower bound of this sequence is in \( \mathbb{P}_\alpha^* \). Now \( \langle p_i \upharpoonright \beta : i < \xi \rangle \) is a descending sequence in \( \mathbb{P}_\beta^* \). By induction the greatest lower bound \( q \) of this sequence is in \( \mathbb{P}_\beta^* \). If \( \beta \) is not in \( \text{dom}(p_i) \) for all \( i < \xi \), then \( q \) is the greatest lower bound of \( \langle p_i : i < \xi \rangle \) in \( \mathbb{P}_\alpha^* \). Otherwise let \( \gamma < \xi \) be the least ordinal such that \( \beta \) is in \( \text{dom}(p_\gamma) \). If \( \beta \) is odd, let \( \hat{u} \) be a \( \mathbb{P}_\beta \)-name for the greatest lower bound of \( \langle p_i(\beta) : \gamma < i < \xi \rangle \). Then \( q \upharpoonright \hat{u} \) is as required. If \( \beta \) is even, then fix for each \( \gamma < i < \xi \) a condition \( x_i \) in \( \text{Add}(\mu) \) such that \( p_i(\beta) = \hat{x}_i \). Let \( x = \bigcup \{ x_i : \gamma < i < \xi \} \). Then \( q \upharpoonright \hat{x} \) is as required.

**Case 2:** \( \alpha \) is a limit ordinal.

We show that \( \mathbb{P}_\alpha \) is \( \mu \)-glb closed. Suppose \( \langle p_i : i < \xi \rangle \) is a descending sequence of conditions in \( \mathbb{P}_\alpha \), with \( \xi < \mu \). For each \( i < \alpha \), \( p_i \) forces \( \mathcal{Q}_i \) is \( \mu \)-glb closed. Define \( q \) with support equal to \( \bigcup \{ \text{dom}(p_i) : i < \xi \} \), so that for each \( \beta \) in this support, \( q \upharpoonright \beta \) forces \( q(\beta) \) to be the greatest lower bound of \( \langle p_i(\beta) : \gamma_\beta \leq i < \xi \rangle \), where \( \gamma_\beta \) is the least \( i < \xi \) with \( \beta \) in \( \text{dom}(p_i) \). Clearly then \( q \) is the greatest lower bound of \( \langle p_i : i < \xi \rangle \) in \( \mathbb{P}_\alpha \). Suppose moreover that \( p_i \in \mathbb{P}_\alpha^* \) for all \( i < \xi \). Then \( q \) can be chosen to be in \( \mathbb{P}_\alpha^* \) as well. Namely, for each even \( \beta \) in \( \text{dom}(q) \), and for \( \gamma_\beta \leq i < \xi \), choose \( x_i^\beta \) in \( \text{Add}(\mu) \) such that \( p_i(\beta) = \hat{x}_i^\beta \). Then let \( q(\beta) \) be a name for \( \bigcup \{ x_i^\beta : \gamma_\beta \leq i < \xi \} \).

Now we show \( \mathbb{P}_\alpha^* \) is dense in \( \mathbb{P}_\alpha \). First assume \( \text{cf}(\alpha) \geq \mu \), and let \( p \) be in \( \mathbb{P}_\alpha \). Then there is \( \xi < \alpha \) such that \( \text{dom}(p) \cap \{ i < \alpha : \xi \text{ even} \} \subseteq \xi \). By induction we can choose \( q \leq p \upharpoonright \xi \) in \( \mathbb{P}_\xi^* \). Then \( q \upharpoonright \hat{p} \upharpoonright [\xi, \alpha) \) is in \( \mathbb{P}_\alpha^* \) and is below \( p \).

Suppose \( \text{cf}(\alpha) < \mu \) and let \( p \) be in \( \mathbb{P}_\alpha \). Fix an increasing and continuous sequence \( \langle \xi_i : i < \text{cf}(\alpha) \rangle \) cofinal in \( \alpha \) with \( \xi_0 = 0 \), and let \( \xi_{\text{cf}(\alpha)} = \alpha \). We define by induction a descending sequence \( \langle p_i : i \leq \text{cf}(\alpha) \rangle \) so that \( p_i \upharpoonright \xi_i \) is in \( \mathbb{P}_{\xi_i}^* \). Let \( p_0 = p \).

Given \( p_i \), apply the recursion hypotheses to choose \( q \leq p_i \upharpoonright \xi_{i+1} \) in \( \mathbb{P}_{\xi_{i+1}}^* \), and let \( p_{i+1} = q \upharpoonright \xi_{i+1} \upharpoonright [\xi_{i+1}, \alpha) \). Suppose \( \delta \leq \text{cf}(\alpha) \) is a limit ordinal and \( p_i \) is defined for all \( i < \delta \). Let \( q \) be the greatest lower bound of the sequence \( \langle p_i \upharpoonright \xi_i : i < \delta \rangle \). Since each \( p_i \upharpoonright \xi_i \) is in \( \mathbb{P}_{\xi_i}^* \subseteq \mathbb{P}_{\xi_{i+1}}^* \), \( q \) is in \( \mathbb{P}_{\xi_i}^* \). Now define \( p_\delta = q \upharpoonright [\xi_i, \alpha) \). This completes the definition. The condition \( p_{\text{cf}(\alpha)} \) is below \( p \) and is in \( \mathbb{P}_{\alpha}^* \).

Now we prove that \( \mathbb{P}_\alpha \) is \( \mu^+ \)-proper. The proof is the same whether \( \alpha \) is a successor or a limit ordinal.

We will use the following basic observation.

**Lemma 4.1.** Suppose \( p \) and \( q \) are conditions in \( \mathbb{P}_\alpha \) such that for all \( \gamma \) in \( \text{dom}(p) \cap \text{dom}(q) \), either \( p \upharpoonright \gamma \) or \( q \upharpoonright \gamma \) forces \( p(\gamma) \) and \( q(\gamma) \) are compatible in \( \mathcal{Q}_\gamma \). Then \( p \) and \( q \) are compatible.

**Proposition 4.2.** The poset \( \mathbb{P}_\alpha \) is \( \mu^+ \)-proper.

**Proof.** Fix a regular cardinal \( \theta > 2^{\text{cf}(\alpha)} \) such that \( \mathbb{P}_\alpha \) is in \( H(\theta) \). Let \( N < \langle H(\theta), \in, \mathbb{P}_\alpha \rangle \) be a set with size \( \mu \) with \( N^{<\mu} \subseteq N \). We would like to show that for every \( p \) in \( N \cap \mathbb{P}_\alpha \), there is \( q \leq p \) which is \( N \)-generic. In Proposition 4.5 we need \( q \) to satisfy a slightly stronger property, which we describe in the following claim.
Claim 4.3. For all $p$ in $N \cap P_\alpha$, there is $q \leq p$ which satisfies the property that for all $r \leq q$, and for any dense set $D \subseteq P_\alpha$ in $N$, there is $q'$ in $D \cap N$ compatible with $r$ such that for all odd $\gamma$ in dom($q'$), $\gamma \in \text{dom}(r)$ and $r \restriction \gamma$ forces $r(\gamma) \leq q'(\gamma)$.

Let $\langle\langle D_i, f_i \rangle : i < \mu \rangle$ be an enumeration of all pairs $\langle D, f \rangle$ in $N$ such that $D \subseteq P_\alpha$ is dense and $f : \{\beta < \alpha : \beta \text{ even} \} \rightarrow \text{ADD}(\mu)$ is a partial function with $|\text{dom}(f)| < \mu$.

We define by induction a descending sequence $\langle p_i : i < \mu \rangle$ of conditions in $N \cap P_\alpha$ and a sequence $\langle q_i : i < \mu \rangle$ of conditions in $N \cap P_\alpha^*$ such that:

1. for $i < \mu$, $\text{dom}(p_i) \cap \{\beta < \alpha : \beta \text{ even} \} = \text{dom}(p_0) \cap \{\beta < \alpha : \beta \text{ even} \}$,

2. for $i < \mu$, for all even $\beta$ in $\text{dom}(p_i)$, $p_i(\beta) = p_0(\beta)$.

Fix $p_0 \leq p$ in $N \cap P_\alpha^*$. If $\delta < \mu$ is a limit ordinal and $p_\delta$ is defined for all $i < \delta$, let $p_\delta$ be the greatest lower bound of $\{p_i : i < \delta \}$. Since $N^{< \mu} \subseteq N$, $\{p_i : i < \delta \}$ is in $N$, and therefore $p_\delta$ is in $N \cap P_\alpha^*$.

Suppose $p_\delta$ is defined for a fixed $i < \mu$. Consider the pair $\langle D_i, f_i \rangle$. If there is $q$ in $N \cap D_i$ below $p_i$ such that $\text{dom}(f_i) \subseteq \text{dom}(q)$, and for all $\beta$ in $\text{dom}(f_i)$, $q(\beta)$ is a name for $f_i(\beta)$, then choose $q_i$ as such a $q$. Otherwise just pick $q_i \leq p_i$ in $N \cap D_i$. Now define $p_{i+1}$ with support equal to

$$\langle \text{dom}(p_i) \cap \{\beta < \alpha : \beta \text{ even} \} \rangle \cup \langle \text{dom}(q_i) \cap \{\gamma < \alpha : \gamma \text{ odd }\} \rangle$$

so that $p_{i+1}(\beta) = p_i(\beta)$ for even $\beta$, and $p_{i+1}(\gamma) = q_i(\gamma)$ for odd $\gamma$.

We define a lower bound $q$ for $\langle p_i : i < \mu \rangle$, and prove that $q$ satisfies the requirements of Claim 4.3. Clearly then $q$ is $N$-generic. The domain of $q$ is $\bigcup\{\text{dom}(p_i) : i < \mu\}$. In particular, $\text{dom}(q) \cap \{\beta < \alpha : \beta \text{ even} \} = \text{dom}(p_0) \cap \{\beta < \alpha : \beta \text{ even} \}$, which has size less than $\mu$. For even $\beta$ in $\text{dom}(q)$, let $q(\beta) = p_0(\beta)$.

Suppose $\gamma = \beta + 1$ is an odd ordinal in $\text{dom}(q)$. Let $i_\gamma < \mu$ be the least $i$ such that $\gamma$ is in $\text{dom}(p_i)$. For $i_\gamma \leq i < \mu$, fix a name $\hat{\sigma}_i^\gamma$ so that $P_\gamma$ forces $p_i(\gamma)$ has domain $\hat{\sigma}_i^\gamma + 1$. Let $\hat{\sigma}_\gamma$ be a $P_\gamma$-name for $\sup\{\hat{\sigma}_i^\gamma + 1 : i_\gamma \leq i < \mu\}$. Then $P_\gamma$ forces that the union of the conditions in $\langle p_i(\gamma) : i \leq i_\gamma < \mu \rangle$ is a sequence of length $\hat{\sigma}_\gamma$.

Let $\langle \alpha_i^\gamma : i < \hat{\sigma}_\gamma \rangle$ be a sequence of names such that

$$P_\gamma \models \bigcup \{ p_i(\gamma) : i \leq i_\gamma < \mu \} = \langle \alpha_i^\gamma : i < \hat{\sigma}_\gamma \rangle.$$

Let $\check{a}_\gamma^\alpha$ be a name for $\bigcup \{ \check{a}_i^\gamma : i < \hat{\sigma}_\gamma \}$. Finally, let $q(\gamma)$ be a name for the sequence $\langle \check{a}_i^\gamma : i \leq \hat{\sigma}_\gamma \rangle$.

We prove by induction that for all $\gamma \leq \alpha$, $q \restriction \gamma$ is a condition in $P_\gamma$, and is below $p_i \restriction \gamma$ for all $i < \mu$. Limit stages are clear. Suppose $q \restriction \gamma$ satisfies this property. If $\gamma$ is even or if $\gamma$ is not in $\text{dom}(q)$, then clearly $q \restriction \gamma + 1$ is as required. Suppose $\gamma = \beta + 1$ is odd and is in $\text{dom}(q)$. Then $q \restriction \gamma + 1$ is a condition below $p_i \restriction \gamma + 1$ for all $i < \mu$, provided that $q \restriction \gamma$ forces that $\check{a}_\gamma^\alpha$ is in $\check{S}_\gamma = [X]^\mu \cap V[G_\beta]$. Let $G_\beta \ast H$ be generic for $P_\gamma = P_\beta \ast \text{ADD}(\mu)$. Since $\gamma$ is in $\text{dom}(q)$, $\gamma$ is in $\text{dom}(p_i)$ for some $i < \mu$. Since $\mu \subseteq N$, $\text{dom}(p_i) \subseteq N$. Therefore $\gamma$, and hence $\beta$, is in $N$. So $P_\beta$ is in $N$. But $\text{ADD}(\mu)$ is $\mu^{+}$-c.c. in $V[G_\beta]$, so $N[G_\beta \ast H] \cap V[G_\beta] = N[G_\beta]$. In particular, $N[G_\beta \ast H] \cap X_\beta = N[G_\beta] \cap X_\beta$, which is in $[X]^\mu \cap V[G_\beta]$. So it suffices to show that $a_\gamma^\alpha \subseteq N[G_\beta \ast H] \cap X_\beta$.

If $i_\gamma \leq i < \mu$, then the condition $p_i(\gamma) = \langle a_j^\gamma : j \leq \sigma_i^\gamma \rangle$ is a member, and hence a subset, of $N[G_\beta \ast H]$. Therefore each $a_j^\gamma$ is a subset of $N[G_\beta \ast H]$. Hence $\bigcup \{ a_j^\gamma : j < \sigma_i^\gamma \} \subseteq N[G_\beta \ast H] \cap X_\beta$. On the other hand, fix $x$ in $N[G_\beta \ast H] \cap X_\beta$. Fix a $P_\gamma$-name $\dot{x}$ for $x$ in $N$. Then there is a dense subset of $P_\alpha$ in $N$ of conditions $s$ such that $P_\gamma$ forces $\dot{x}$ is in some element of the sequence $s(\gamma)$. Hence for some
$i < \mu$, $q_i$ is in this dense set. Since $q_i(\gamma) = p_{i+1}(\gamma)$, $\mathbb{P}_\gamma$ forces $\dot{x}$ appears in some element of $p_{i+1}(\gamma)$. Therefore $x$ appears in some element of $\{a_{\gamma}^j : j \leq \sigma_{i+1}\}$. So $x$ is in $a_{\gamma}^\omega$. Thus $a_{\gamma}^\omega = \bigcup\{a_{\gamma}^j : i < \sigma_\gamma\} = N[G_\beta * H] \cap X_\beta$.

We prove now that $q$ satisfies the property described in Claim 4.3. Let $r \leq q$ and suppose $D$ is a dense subset of $\mathbb{P}_\alpha$ in $N$. Fix $s \leq r$ in $\mathbb{P}_\alpha \cap D$. Let $f : \alpha \to \text{Add}(\mu)$ be the partial function with $\text{dom}(f) = N \cap \text{dom}(s) \cap \{\beta < \alpha : \beta \text{ even}\}$ such that for all $\beta$ in $\text{dom}(f)$, $s(\beta)$ is a name for $f(\beta)$. Since $N^{<\mu} \subseteq N$, $f$ is in $N$. Fix $i < \mu$ such that $D_i = D$ and $f_i = f$.

Now $H(\theta)$ models that there is $u \leq p_i$ in $D_i$ such that $\text{dom}(f_i) \subseteq \text{dom}(u)$, and for all $\beta$ in $\text{dom}(f_i)$, $u(\beta)$ is a name for $f_i(\beta)$, as witnessed by $u = s$. By elementarity, the same is true in $N$. Hence by construction, $q_i$ also satisfies this property. If $\gamma$ is odd and is in $\text{dom}(q_i)$, then $\gamma$ is in $\text{dom}(p_{i+1})$ and $p_{i+1}(\gamma) = q_i(\gamma)$. Therefore for all odd $\gamma$ in $\text{dom}(q_i)$, $\gamma$ is in $\text{dom}(r)$ and $r \restriction \gamma$ forces $r(\gamma) \leq q_i(\gamma)$.

We show that $q_i$ and $r$ are compatible, which finishes the proof. We apply Lemma 4.1 to show $q_i$ and $s$ are compatible. Suppose $\gamma$ is in $\text{dom}(q_i) \cap \text{dom}(s)$. Since $\text{dom}(q_i) \subseteq N$, $\gamma$ is in $N \cap \text{dom}(s)$. So if $\gamma$ is even, then $\gamma$ is in $\text{dom}(f_i)$. Then $q_i(\gamma)$ and $s(\gamma)$ are both names for $f_i(\gamma)$ and thus are equal. If $\gamma$ is odd, then $q_i(\gamma) = p_{i+1}(\gamma)$, and $s \restriction \gamma$ forces $s(\gamma) \leq p_{i+1}(\gamma)$.

This completes the recursion.

The next proposition describes a special property of $\mathbb{P}_\alpha$ which we will use in the consistency proof of the next section. First we need a technical lemma.

**Lemma 4.4.** Let $p'$ and $q'$ be conditions in $\mathbb{P}_\alpha^*$. Then there are $p \leq p'$ and $q \leq q'$ in $\mathbb{P}_\alpha^*$ such that $\text{dom}(p) \cap \{\gamma < \alpha : \gamma \text{ odd}\} = \text{dom}(q) \cap \{\gamma < \alpha : \gamma \text{ odd}\}$, and for all odd $\gamma$ in this set, $p(\gamma) = q(\gamma)$.

**Proof.** First choose $p(0) \leq p'(0)$ and $q(0) \leq q'(0)$ in $\text{Add}(\mu)$ which are incompatible. Suppose $\beta > 0$ is an even ordinal and $p \not\forces \beta$ and $q \not\forces \beta$ are defined. Let $\beta$ be in $\text{dom}(p)$ iff $\beta$ is in $\text{dom}(p')$, in which case $p(\beta) = p'(\beta)$, and similarly with $q$. Suppose $\gamma$ is odd and $p \not\forces \gamma$ and $q \not\forces \gamma$ are defined. If $\gamma$ is in $\text{dom}(p') \setminus \text{dom}(q')$ then let $p(\gamma) = q(\gamma) = p'(\gamma)$, and similarly if $\gamma$ is in $\text{dom}(q') \setminus \text{dom}(p')$. Suppose $\gamma$ is in $\text{dom}(p') \cap \text{dom}(q')$. Let $\dot{x}_\gamma$ be a $\mathbb{P}_\gamma$-name such that $\mathbb{P}_\gamma$ forces $\dot{x}_\gamma = p'(\gamma)$ if $p \not\forces \gamma$ is in $G_{\gamma}$, and $\dot{x}_\gamma = q'(\gamma)$ otherwise. Then $\dot{x}_\gamma$ is well-defined because $p \not\forces \gamma$ and $q \not\forces \gamma$ are incompatible. Let $p(\gamma) = q(\gamma) = \dot{x}_\gamma$.

**Proposition 4.5.** The poset $\mathbb{P}_\alpha$ forces that whenever $f : \mu^+ \to V$ is a function in the extension such that for all $i < \mu^+$, $f \restriction i$ is in $V$, then $f$ is in $V$.

**Proof.** Suppose for a contradiction that $p$ forces $\dot{f} : \mu^+ \to V$ is a function which is not in $V$, but for all $i < \mu^+$, $\dot{f} \restriction i$ is in $V$.

Fix a regular cardinal $\theta \gg \mu^+$ with $\mathbb{P}_\alpha \in H(\theta)$. Let $N$ be an elementary substructure of $(H(\theta), \in, \mathbb{P}_\alpha, p, \dot{f})$ with size $\mu$ and $N^{<\mu} \subseteq N$. By Claim 4.3, fix $q \leq p$ such that for all $r \leq q$ and for any dense set $D \subseteq \mathbb{P}_\alpha$ in $N$, there is $q' \in D \cap N$ compatible with $r$ such that for all odd $\gamma$ in $\text{dom}(q')$, $\gamma \in \text{dom}(r)$ and $r \restriction \gamma$ forces $r(\gamma) \leq q'(\gamma)$.

Let $r \leq q$ be in $\mathbb{P}_\alpha^*$ such that $r$ decides $\dot{f} \restriction N \cap \mu^+$. Let $g : \alpha \to \text{Add}(\mu)$ be the partial function with domain equal to $N \cap \text{dom}(r) \cap \{\beta < \alpha : \beta \text{ even}\}$ such that for all $\beta$ in $\text{dom}(g)$, $r(\beta)$ is a name for $g(\beta)$. Since $N^{<\mu} \subseteq N$, $g$ is in $N$. 

Define $D$ as the set of $s_0 \leq p$ in $\mathbb{P}_\kappa$ for which there exists $s_1$ in $\mathbb{P}_\kappa$ such that:

1. $\text{dom}(g) \subseteq \text{dom}(s_0)$,
2. there is $i < \mu^+$ and distinct $a_0$ and $a_1$ such that $s_0 \models \hat{f}(i) = a_0$ and $s_1 \models \hat{f}(i) = a_1$,
3. $\text{dom}(s_0) \cap \{ \gamma < \alpha : \gamma \text{ odd} \} = \text{dom}(s_1) \cap \{ \gamma < \alpha : \gamma \text{ odd} \}$,
4. for all odd $\gamma$ in $\text{dom}(s_0)$, $s_0(\gamma) = s_1(\gamma)$,
5. $\text{dom}(g) \subseteq \text{dom}(s_1)$, and for all $\beta$ in $\text{dom}(g)$, if $g(\beta)$ is compatible with the condition named by $s_0(\beta)$, then $s_1(\beta)$ is the name for a condition extending $g(\beta)$.

By elementarity, $D$ is in $N$.

We claim that $D$ is dense below $p$. So let $s \leq p$. Extend $s$ to $s'$ in $\mathbb{P}_\kappa$ so that $\text{dom}(g) \subseteq \text{dom}(s')$. Now define $s'' \leq s'$ with the same domain as $s'$ as follows. For $\beta \in \text{dom}(s') \setminus \text{dom}(g)$, let $s''(\beta) = s'(\beta)$. Suppose $\beta$ is in $\text{dom}(g)$. If $s'(\beta)$ names a condition in $\text{Add}(\mu)$ compatible with $g(\beta)$, let $s''(\beta)$ be a name for a condition which extends $g(\beta)$ and $s'(\beta)$. Otherwise let $s''(\beta) = s'(\beta)$.

Since $\hat{f}$ is not in $V$, there is $i < \mu^+$ such that $s''$ does not decide $\hat{f}(i)$. Fix $s_0', s_1' \leq s''$ in $\mathbb{P}_\kappa$ and distinct $a_0$ and $a_1$ so that $s_0' \models \hat{f}(i) = a_0$ and $s_1' \models \hat{f}(i) = a_1$. Now apply Lemma 4.4 to obtain $s_0 \leq s_0'$ and $s_1 \leq s_1'$ in $\mathbb{P}_\kappa$ satisfying (3) and (4). We check that (5) holds. If $\beta$ is in $\text{dom}(g)$ and $s_0(\beta)$ names a condition compatible with $g(\beta)$, then clearly $s'(\beta)$ names a condition compatible with $g(\beta)$. So $s''(\beta)$ is a name for a condition refining $g(\beta)$. Since $s_1 \leq s'', s_1(\beta)$ is a name for a condition refining $g(\beta)$.

By the genericity property of $q$, we can fix $s_0 \in D \cap N$ which is compatible with $r$, and such that for all odd $\gamma$ in $\text{dom}(s_0)$, $\gamma$ is in $\text{dom}(r)$ and $r \upharpoonright \gamma$ forces that $r(\gamma) \leq s_0(\gamma)$. Fix $s_1$, $i$, $a_0$, and $a_1$ in $N$ as described in the definition of $D$. Since $r$ decides $\hat{f}(i)$ and $r$ and $s_0$ are compatible, $r \upharpoonright \gamma$ forces $r(\gamma) \leq s_0(\gamma)$. So $r$ and $s_1$ are incompatible. We will get a contradiction by showing $r$ and $s_1$ are compatible.

We apply Lemma 4.1. Suppose $\beta$ is in $\text{dom}(r) \cap \text{dom}(s_1)$ and $\beta$ is even. Then $\beta$ is in $N$, so $\beta$ must be in $\text{dom}(g)$. Since $r$ and $s_0$ are compatible, $r(\beta)$ and $s_0(\beta)$ are compatible. By (5), $s_1(\beta)$ is the name for a condition extending $g(\beta)$. Suppose $\gamma$ is in $\text{dom}(r) \cap \text{dom}(s_1)$ and $\gamma$ is odd. Then $\gamma$ is in $\text{dom}(s_0)$. So $\gamma$ is in $\text{dom}(r)$, and $r \upharpoonright \gamma$ forces $r(\gamma) \leq s_0(\gamma)$. But $s_0(\gamma) = s_1(\gamma)$. $\square$

5. The Consistency Result

Suppose $\mu < \kappa$ are cardinals, $\mu < \mu = \mu$, and $\kappa$ is supercompact. We define a forcing iteration of the form given in the last section which collapses $\kappa$ to become $\mu^+$, and forces that for all regular $\lambda \geq \mu^+$, there are stationarily many $N$ in $[H(\lambda)]^{\mu^+}$ such that $N$ is internally club but not internally approachable.

Fix a Laver function $f : \kappa \to V_\kappa$. So for all $x$ and $\lambda$, there is an elementary embedding $j : V \to M$ with critical point $\kappa$ such that $M^\lambda \subseteq M$ and $j(f)(\kappa) = x$.

We define by recursion a forcing iteration

$$\langle \mathbb{P}_i, \dot{Q}_j : i \leq \kappa, j < \kappa \rangle.$$ 

Suppose $\mathbb{P}_i$ is defined for a fixed $i < \kappa$. If $i$ is an even ordinal, let $\dot{Q}_i$ be a $\mathbb{P}_i$-name for $\text{Add}(\mu)$. Suppose $i = j + 1$ is odd. If $f(j)$ is a $\mathbb{P}_j$-name for a set which contains $\mu^+$, let $X_j = f(j)$, and otherwise let $X_j$ be a $\mathbb{P}_j$-name for $\mu^+$. Let $\dot{S}_i$ be a $\mathbb{P}_i$-name for $[X_j]^{\mu^+} \cap V[\dot{G}_j]$, and let $\dot{Q}_i$ be a $\mathbb{P}_i$-name for the poset $\mathbb{P}(\dot{S}_i)$ from Definition 2.5. Suppose $\delta < \kappa$ is a limit ordinal and $\mathbb{P}_i$ is defined for all $i < \delta$. Then let $\mathbb{P}_\delta$ be the
poset consisting of all partial functions $p : \delta \rightarrow V$ such that $p \upharpoonright i \in P_i$ for all $i < \delta$, $|\text{dom}(p) \cap \{i < \delta : i \text{ even}\}| < \mu$, and $|\text{dom}(p) \cap \{i < \delta : i \text{ odd}\}| < \mu$.

Since $f$ is a Laver function, there are stationarily many $\alpha < \kappa$ such that $f(\alpha)$ is a $P_\alpha$-name and $P_\alpha$ forces $f(\alpha) = (\mu^+)\mathbb{V}^{G_{\alpha}}$. Indeed, let $\dot{x}$ be a $P_\alpha$-name for $(\mu^+)\mathbb{V}^{G_{\alpha}}$. Choose $j : V \rightarrow M$ with critical point $\kappa$ such that $j(f)(\kappa) = \dot{x}$ and $M$ is sufficiently closed that it models $P_\alpha = j(P_\alpha) \cap \kappa$ forces $\dot{x} = (\mu^+)\mathbb{V}^{M[G_{\alpha}]}$. If $C$ is club in $\kappa$, then $\kappa \in j(C)$. Hence by elementarity, there is $\alpha < \kappa$ in $C$ such that $f(\alpha)$ is as desired. But then $P_{\alpha+2}$ forces $|(\mu^+)\mathbb{V}^{G_{\alpha}}| = \mu^+$. So $P_\alpha$ collapses all cardinals in the interval $(\mu^+, \kappa)$.

Since $|P_i| < \kappa$ for all $i < \kappa$, there are club many $\delta < \kappa$ such that $|P_i| < \delta$ for all $i < \delta$. Suppose $\mu^+ < \delta \leq \kappa$ is inaccessible and satisfies this property. Then $P_\delta$ is the direct limit of $\{P_i : i < \delta\}$, where each $P_i$ has size less than $\delta$, and there are stationarily many $\alpha < \delta$ such that $P_\alpha$ is the direct limit of $\{P_i : i < \alpha\}$. By a standard $\Delta$-system argument, $P_\delta$ is $\kappa$-c.c. (see Theorem 2.2 of [2]). In particular, $P_\kappa$ is $\kappa$-c.c. and forces that $\kappa = \mu^+$.

Let $G_\kappa$ be generic for $P_\kappa$. In $V[G_\kappa]$ let $\lambda \geq \mu^+$ be regular. In $V$ let $\theta = (2^\kappa)^+$. Let $j : V \rightarrow M$ be an elementary embedding with critical point $\kappa$ such that $M^\theta \subseteq M$ and $j(f)(\kappa)$ is a $P_\kappa$-name for $H(\lambda)^{V[G_\kappa]}$. Then by choice of $j$,

$$j(P_\kappa) = P_\kappa \ast \text{ADD}(\mu) \ast P(\delta) \ast P_{\text{tail}}$$

where

$$P_{\alpha+1} \models \dot{S} = \dot{S}_{\alpha+1} = [H(\lambda)^{V[G_\kappa]}]^{\mu} \cap M[G_{\alpha}]$$

and $P_{\text{tail}}$ is forced to be an iteration of the form given in the previous section.

Let $H \ast K \ast G_{\text{tail}}$ be generic over $V[G_\kappa]$ for $\text{ADD}(\mu) \ast P(\delta) \ast P_{\text{tail}}$. Extend $j$ in $V[G_\kappa \ast H \ast K \ast G_{\text{tail}}]$ to

$$j : V[G_\kappa] \rightarrow M[G_\kappa \ast H \ast K \ast G_{\text{tail}}].$$

Then $j(G_\kappa) = G_\kappa \ast H \ast K \ast G_{\text{tail}}$. Since $P_\kappa$ is $\kappa$-c.c., $M[G_\kappa]^\theta \cap V[G_\kappa] \subseteq M[G_\kappa]$. In particular, $H(\lambda)^{V[G_\kappa]} = H(\lambda)^{M[G_\kappa]}$.

Working in $V[G_\kappa]$, let $C \subseteq [H(\lambda)]^{\mu^+}$ be club. We prove there is a set in $C$ which is internally club but not internally approachable. By elementarity, it suffices to prove the same statement about $j(C)$ in $M[j(G_\kappa)]$. We will prove that in $M[j(G_\kappa)]$, the set $\text{th}^+ H(\lambda)^{V[G_\kappa]}$ is in $j(C)$ and is internally club but not internally approachable.

Let $N^* = \text{th}^+ H(\lambda)^{V[G_\kappa]}$. First we prove that $N^*$ is in $M[j(G_\kappa)]$. The set $\text{th}^+ H(\lambda)^{V[G_\kappa]}$ is in $M[j(G_\kappa)]$ by the closure of $M$. But $H(\lambda)^{V[G_\kappa]} = H(\lambda)^{M[G_\kappa]}$. So every element of $N^*$ is of the form $j(\dot{a}_{G_{\alpha}}) = j(\dot{a})_{j(G_{\alpha})}$, where $\dot{a}$ is in $H(\lambda)^{V[G_\kappa]}$. So $N^* = (\text{th}^+ H(\lambda)^{V[G_\kappa]})^j$. This in $M[j(G_\kappa)]$. Also note that in $M[j(G_\kappa)]$, $|N^*| = |H(\lambda)^{V[G_\kappa]}| = \mu^+$.

We claim that $N^*$ is in $j(C)$. Since $j(C)$ is closed under unions of directed subsets with size less than $\mu^{++}$, it suffices to show that $N^* \cap j(C)$ is directed and $\bigcup(N^* \cap j(C)) = N^*$. Suppose $j(a)$ and $j(b)$ are in $N^* \cap j(C)$. By elementarity, $a$ and $b$ are in $C$. Fix $c$ in $C$ such that $a \cup b \subseteq c$. Then $j(a)$ and $j(b)$ are contained in $j(c)$ and $j(c) \in N^* \cap j(C)$. Hence $N^* \cap j(C)$ is directed.

We show that $\bigcup(N^* \cap j(C)) = N^*$. Let $j(x)$ be in $N^*$. Then $x$ is in $H(\lambda)^{V[G_{\alpha}]}$, so there is $a$ in $C$ such that $x \in a$. Then $j(x) \in j(a) \in N^* \cap j(C)$. So $N^* \subseteq \bigcup(N^* \cap j(C))$. On the other hand, suppose $y$ is in $\bigcup(N^* \cap j(C))$. Fix $j(a) \in N^* \cap j(C)$ so that $y \in j(a)$. Then $a$ is in $C$. In $V[G_\kappa]$, $a$ has size less than $\mu^{++} = \kappa$, and $\kappa$ is the critical point of $j$. So $j(a) = j^\kappa a$, and clearly $j^\kappa a \subseteq N^*$. Thus $y$ is in $N^*$. Therefore $N^* = \bigcup(N^* \cap j(C))$ and $N^*$ is in $j(C)$. 


Now we show that $N^*$ is internally club but not internally approachable. Let $N = H(\lambda)^V[G]$. Since $N$ is transitive and isomorphic to $N^*$ by the map $j \upharpoonright N, N$ is the transitive collapse of $N^*$ and $j^{-1} \upharpoonright N^* = \pi$ is the transitive collapse map.

Recall that $H \ast K$ is generic for $\text{ADD}(\mu, 1) \ast \mathbb{P}\langle \dot{S} \rangle$ over $M[G_{\kappa}]$, and $S = \check{S}^H = [N]^\mu \cap M[G_{\kappa}]$. Write $\bigcup K = \langle a_i : i < \mu^+ \rangle$. Then $N = \bigcup \{a_i : i < \mu^+ \}$. For all $i < \mu^+$, $a_i$ is a subset of $N = H(\lambda)^{M[G]}$ which is in $M[G_{\kappa}]$, and $a_i$ has size $\mu$, which is less than $\lambda$. Therefore $a_i$ is in $H(\lambda)^{M[G]} = N$. Hence $N$ is internally club. But then $\langle j(a_i) : i < \mu^+ \rangle = \langle j^{\alpha a_i} : i < \mu^+ \rangle$ witnesses that $N^*$ is internally club.

Suppose for a contradiction that $N^*$ is internally approachable in $M[j(G_{\kappa})]$, as witnessed by a sequence $\langle N_i^* : i < \mu^+ \rangle$. Note that $N$ is then also internally approachable. Indeed, for all $i < \mu^+$, let $N_i = \pi(N_i^*) = \pi^\alpha(N_i^*)$. Clearly $\langle N_i : i < \mu^+ \rangle$ is increasing and continuous and its union is equal to $N$. For each $\alpha < \mu^+$, choose $f_\alpha$ in $N$ such that $j(f_\alpha) = \langle N_i^* : i < \alpha \rangle$. Then for $i < \alpha$, $j(N_i) = N_i^* = j(f_\alpha)(i) = j(f_\alpha)(j(i)) = j(f_\alpha(i))$. So $N_i = f_\alpha(i)$. Therefore $\langle N_i : i < \alpha \rangle = f_\alpha$, which is in $N$. Hence $\langle N_i : i < \mu^+ \rangle$ witnesses that $N$ is internally approachable.

Let $f = \langle N_i : i < \mu^+ \rangle$. Then for all $i < \mu^+$, $f \upharpoonright i$ is in $N$. Since $N \subseteq M[G_{\kappa}]$, for all $i < \mu^+$, $f \upharpoonright i$ is in $M[G_{\kappa}]$. By Proposition 4.5, $f = \langle N_i : i < \mu^+ \rangle$ is in $M[G_{\kappa}]$. But this implies $N$ has size $\mu^+$ in $M[G_{\kappa}]$, which is false. This completes the proof.

We note that if $\text{GCH}$ holds in $V$, then in $V[G]$, $2^\mu = \mu^{++}$ and $2^\alpha = \alpha^+$ for all infinite cardinals $\alpha$ different from $\mu$. This violation of $\text{GCH}$ is necessary, by the following argument:

Suppose $2^\alpha = \mu^+$ and $\lambda \geq \mu^{++}$ is regular. Let $N < H(\lambda)$ be a model with size $\mu^+$ such that $\mu^+ \subseteq N$, and $N$ satisfies the $\mu^+$-covering property, that is, every subset of $N$ with size less than $\mu^+$ is a subset of a member of $N$ with size less than $\mu^+$. Then $N^\mu \subseteq N$. For if $a \subseteq N$ has size $\mu$, then $a$ is covered by a set $b$ in $N$ with size $\mu$. Since $2^{\mu^+} = \mu^+$, we can enumerate the power set of $b$ by a sequence $\langle x_i : i < \mu^+ \rangle$ in $N$. But $\mu^+ \subseteq N$, so $x_i \in N$ for all $i < \mu^+$. In particular, $a$ is in $N$. Now fix an increasing and continuous sequence $\langle N_i : i < \mu^+ \rangle$ of sets with size $\mu$ whose union is $N$. Since $N^\mu \subseteq N$, each $N_i$ is in $N$, and thus every initial segment of this sequence is in $N$. So $N$ is internally approachable. But if $N$ is internally club, then $N$ satisfies the $\mu^+$-covering property.

Remarks. Mixed support iterations similar to that presented in Section 4 appear in Chapter 8 of [8], where an analogue of Proposition 4.2 is proved for iterations of posets of the form $\mathbb{P} \ast \dot{Q}$, where $\mathbb{P}$ is $\omega_1$-closed, $\mathbb{P}$ satisfies a strengthening of $\omega_2$-c.c., and $\dot{Q}$ is forced to be $\omega_2$-closed. The proof of our consistency result is related to Mitchell’s construction in [6] of a model with no Aronszajn trees on $\omega_2$. See [7] for a recent discussion concerning the special property described in Proposition 4.5.

References


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