INTERNALLY CLUB AND APPROACHABLE

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Abstract. We prove that PFA implies for all regular $\lambda \geq \omega_2$, there are stationarily many $N \subseteq H(\lambda)$ with size $\aleph_1$ which are internally club but not internally approachable.

A set N is *internally approachable with length and cardinality* $\aleph_1$ if there is an increasing and continuous sequence $\langle N_i : i < \omega_1 \rangle$ of countable sets with union N such that for all $\alpha < \omega_1$, $\langle N_i : i < \alpha \rangle$ is in N. N is *internally club* if there is an increasing and continuous sequence $\langle N_i : i < \omega_1 \rangle$ of countable sets with union N such that for all $\alpha < \omega_1$, N_α is in N.

Foreman and Todorčević [2] asked whether these two notions are equivalent for a club of N in $[H(\lambda)]^{\aleph_1}$, where $\lambda \geq \omega_2$ is regular. Clearly any $N \prec H(\lambda)$ which is internally approachable is internally club. We prove the converse fails in general. Namely, if PFA holds, then for all $\lambda \geq \omega_2$ regular, there are stationarily many sets N in $[H(\lambda)]^{\aleph_1}$ which are internally club but not internally approachable.

1. Preliminaries

Suppose κ is regular and X is a set with size at least κ. A set $C \subseteq P_\kappa(X)$ is *club* if it is closed under unions of $\subseteq$-increasing sequences of length less than κ and cofinal. A set is stationary if it has non-empty intersection with every club. If $C \subseteq P_\kappa(X)$ is club, then there is a function $F : [X]^{<\omega} \to P_\kappa(X)$ such that for any a in $P_\kappa(X)$, if for all $x_0, \ldots, x_n$ in a, $F(x_0, \ldots, x_n) \subseteq a$, then a is in C. If κ is regular and X has size κ, a set $S \subseteq P_\kappa(X)$ is stationary iff there is an increasing and continuous sequence $\langle a_i : i < \kappa \rangle$ of sets with size less than κ with union X and a stationary set $B \subseteq \kappa$ such that $\{a_i : i \in B\} \subseteq S$ (or equivalently, for any such sequence there is a stationary set B with $\{a_i : i \in B\} \subseteq S$).

If λ is a regular cardinal, $H(\lambda)$ is the collection of sets whose transitive closure has size less than λ. Any set in $H(\lambda)$ can be coded by a bounded subset of λ, so $|H(\lambda)| = \lambda^{<\lambda}$. Suppose $\lambda \geq \omega_1$ is regular, $\mathbb{P}$ is a forcing poset in $H(\lambda)$, and G is generic for $\mathbb{P}$ over V. Then $H(\lambda)[G] = H(\lambda)^{V[G]}$. If $\mathbb{P}$ is in N and $N \prec \langle H(\lambda), \in \rangle$, then $N[G] \prec \langle H(\lambda)^{V[G]}, \in \rangle$.

If N is a set and G is a filter on $\mathbb{P}$, we say G is *N-generic* if for any dense set $D \subseteq \mathbb{P}$ in N, $N \cap G \cap D$ is non-empty. A condition p is *N-generic* if p forces $\dot{G}$ is N-generic. Suppose $\lambda \geq \omega_2$ is regular, $N \prec H(\lambda)$, and $\mathbb{P}$ is in N. Then the following are equivalent: (1) p is N-generic; (2) $p \Vdash N[\dot{G}] \cap On = N \cap On$; (3) $p \Vdash N[\dot{G}] \cap V = N \cap V$; and (4) $p \Vdash N[\dot{G}] \cap H(\lambda)^V = N$. A sufficient condition for p being N-generic, known as *strong N-genericity*, is that for any dense set $D \subseteq \mathbb{P}$ in N, there is u in $D \cap N$ with $p \leq u$. If $\mathbb{P}$ is ω_1-c.c. then any condition in $\mathbb{P}$

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is N-generic. A condition $p * \dot{q}$ in a two-step iteration $\mathbb{P} * \dot{\mathbb{Q}}$ is N-generic iff p is N-generic for $\mathbb{P}$ and p forces $\dot{q}$ is $N[\dot{G}]$-generic for $\dot{\mathbb{Q}}$.

A forcing poset $\mathbb{P}$ is *proper* if for any set X which contains ω_1, $\mathbb{P}$ preserves the stationarity of stationary subsets of $P_{\omega_1}(X)$. Equivalently, $\mathbb{P}$ is proper iff for all sufficiently large regular $\lambda \geq \omega_2$, there is a club of $N \prec H(\lambda)$ in $P_{\omega_1}(H(\lambda))$ such that for any p in $N \cap \mathbb{P}$, there is $q \leq p$ which is N-generic. Any ω_1-c.c. or ω_1-closed forcing poset is proper.

A forcing poset $\mathbb{P}$ has the *countable covering property* if whenever p forces $\dot{x} \subseteq On$ is countable, there is a countable set a and $q \leq p$ which forces $\dot{x} \subseteq a$. Any proper forcing poset satisfies the countable covering property.

We let $\textsc{Add}(\omega)$ denote the forcing poset for adding one Cohen real. Conditions in $\textsc{Add}(\omega)$ are finite partial functions $p : \omega \to 2$, ordered by extension of functions.

If G is a filter on a poset $\mathbb{P}$ in V and $\dot{a}$ is a $\mathbb{P}$-name for a subset of V, then $\dot{a}^G$ is defined as the collection of x for which there is p in G which forces $x \in \dot{a}$.

Suppose Γ is a class of forcing posets and $\alpha \leq \omega_1$ is a cardinal. Then $\mathsf{MA}_\alpha(\Gamma)$ is the following statement. Suppose $\mathbb{P}$ is in Γ, $\langle \dot{A}_i : i < \alpha \rangle$ is a sequence of names for stationary subsets of ω_1, and $\mathcal{D}$ is a collection of $\aleph_1$ many dense subsets of $\mathbb{P}$. Then there is a filter G on $\mathbb{P}$ which intersects each dense set in $\mathcal{D}$ and for $i < \alpha$, $\dot{A}_i^G$ is stationary in ω_1. If Γ is the class of proper forcing posets, then PFA is the statement $\mathsf{MA}_0(\Gamma)$, and PFA^+ is the statement $\mathsf{MA}_1(\Gamma)$.

Fact 1.1 (H. Woodin [5])**.** *Assume $\alpha \leq \omega_1$ is a cardinal and $MA_\alpha(\Gamma)$ holds for a class of separative forcing posets Γ. Suppose $\mathbb{P}$ is in Γ and $\langle \dot{A}_i : i < \alpha \rangle$ is a sequence of $\mathbb{P}$-names for stationary subsets of ω_1. Then for any regular $\theta \geq \omega_2$ with $\mathbb{P}$ in $H(\theta)$, there is a stationary collection of N in $[H(\theta)]^{\aleph_1}$ for which there exists a filter $G \subseteq \mathbb{P}$ which is N-generic and for all $i < \alpha$, $\dot{A}_i^G$ is stationary in ω_1.*

2. Distinguishing Internally Club and Approachable

We prove the following theorem.

Theorem 2.1. *PFA implies that for all regular $\lambda \geq \omega_2$, there are stationarily many sets N in $[H(\lambda)]^{\aleph_1}$ which are internally club but not internally approachable.*

The structure of the proof will be as follows. We define a proper forcing poset $\mathbb{P}$ and apply Fact 1.1 to obtain stationarily many N in $[H(\theta)]^{\aleph_1}$ for which there exists an N-generic filter G. We then prove that whenever N is such a model, $N \cap H(\lambda)$ is internally club but not internally approachable.

In the proof we will use a forcing poset $\mathbb{P}(S)$ defined as follows. Suppose S is a stationary subset of $P_{\omega_1}(Y)$, where Y is a set containing ω_2 as a subset. A condition in $\mathbb{P}(S)$ is a countable increasing and continuous sequence $\langle a_i : i \leq \gamma \rangle$, with each a_i in S. The ordering is by extension of sequences.

Lemma 2.2. *If $S \subseteq P_{\omega_1}(Y)$ is stationary then $\mathbb{P}(S)$ is ω-distributive.*

Proof. Suppose p forces $\dot{f} : \omega \to \mathbf{On}$. Let $\theta \gg \omega_2$ be a regular cardinal such that $\dot{f}$ and Y are in $H(\theta)$. Since S is stationary, we can fix a countable elementary substructure N of the model

$$\langle H(\theta), \in, S, \mathbb{P}(S), p, \dot{f} \rangle$$

such that $N \cap Y$ is in S. Let $\langle D_n : n < \omega \rangle$ be an enumeration of all the dense subsets of $\mathbb{P}(S)$ in N. Inductively define a decreasing sequence $\langle p_n : n < \omega \rangle$ of elements of $N \cap \mathbb{P}$ such that $p_0 = p$ and p_{n+1} is a refinement of p_n in $D_n \cap N$. Write $\bigcup\{p_n : n < \omega\} = \langle b_i : i < \gamma\rangle$. Clearly $\bigcup\{b_i : i < \gamma\} = N \cap Y$. Since $N \cap Y$ is in S, the sequence $\langle b_i : i < \gamma\rangle \cup \{\langle \gamma, N \cap Y\rangle\}$ is a condition below p which decides $\dot{f}(n)$ for all $n < \omega$. $\square$

The poset $\mathbb{P}(S)$ adds an increasing and continuous sequence of sets in S with order type ω_1 and union Y. In particular, $\mathbb{P}(S)$ collapses the size of Y to $\aleph_1$.

Let $\langle T, <_T\rangle$ denote the tree of all increasing countable sequences of ordinals less than ω_2, ordered by the initial segment relation. Under PFA, $2^\omega = 2^{\omega_1} = \aleph_2$, so T has height ω_1, size $\aleph_2$, and has $\aleph_2$ many cofinal branches.

Fix $\lambda \geq \omega_2$ regular. Let $\dot{X}$ be an $\textsc{Add}(\omega)$-name for $P_{\omega_1}(H(\lambda)^V) \cap V$. Since $\textsc{Add}(\omega)$ is proper, it forces $\dot{X}$ is stationary in $P_{\omega_1}(H(\lambda)^V)$. Let $\mathbb{P}(\dot{X})$ be an $\textsc{Add}(\omega)$-name for the forcing poset discussed above which adds an increasing and continuous sequence through $\dot{X}$ with order type ω_1.

Proposition 2.3. *The forcing poset* $\textsc{Add}(\omega) * \mathbb{P}(\dot{X})$ *is proper.*

Proof. Fix $\theta \gg \lambda$ regular with $\textsc{Add}(\omega) * \mathbb{P}(\dot{X})$ in $H(\theta)$. Let $N \prec H(\theta)$ be a countable set containing $\textsc{Add}(\omega) * \mathbb{P}(\dot{X})$ as a member. Let $p * \dot{q}$ be a condition in N. Since $\textsc{Add}(\omega)$ is ω_1-c.c., p is N-generic for $\textsc{Add}(\omega)$. We will find $\dot{r}$ such that p forces $\dot{r} \leq \dot{q}$ and $\dot{r}$ is $N[\dot{G}]$-generic for $\mathbb{P}(\dot{X})$. Then $p * \dot{r} \leq p * \dot{q}$ and $p * \dot{r}$ is N-generic for $\textsc{Add}(\omega) * \mathbb{P}(\dot{X})$.

Let $\langle \dot{D}_n : n < \omega\rangle$ enumerate all $\textsc{Add}(\omega)$-names in N for dense open subsets of $\mathbb{P}(\dot{X})$. Inductively define a sequence $\langle \dot{q}_n : n < \omega\rangle$ as follows. Let $\dot{q}_0 = \dot{q}$. Given $\dot{q}_n$, let $\dot{q}_{n+1}$ be an $\textsc{Add}(\omega)$-name in N such that p forces $\dot{q}_{n+1} \leq \dot{q}_n$ and $\dot{q}_{n+1}$ is in $\dot{D}_n$. This completes the definition.

For each $\alpha < N \cap \omega_1$, there is n such that p forces $\dot{D}_n$ is the dense open set of conditions in $\mathbb{P}(\dot{X})$ whose domain has length at least α. On the other hand, p forces $N[\dot{G}] \cap \omega_1 = N \cap \omega_1$, so for all n, p forces $\mathrm{dom}(\dot{q}_n) < N \cap \omega_1$. So there is an $\textsc{Add}(\omega)$-name $\langle \dot{x}_i : i < N \cap \omega_1\rangle$ for an increasing and continuous sequence which p forces is equal to $\bigcup\{\dot{q}_n : n < \omega\}$.

We claim that p forces $\bigcup\{\dot{x}_i : i < N \cap \omega_1\} = N \cap H(\lambda)^V$. For if a is in $N \cap H(\lambda)^V$, then there is n such that p forces $\dot{D}_n$ is the dense open set of conditions of the form $\langle y_i : i \leq \gamma\rangle$, where a is in y_i for some $i \leq \gamma$. Since p forces $\dot{q}_{n+1}$ is in $\dot{D}_n$, p forces a is in $\bigcup\{\dot{x}_i : i < N \cap \omega_1\}$. On the other hand, if G is generic containing p and a is in $\bigcup\{x_i : i < N \cap \omega_1\}$, then there is n such that q_n is of the form $\langle x_i : i \leq \zeta\rangle$, where a is in x_i for some $i \leq \zeta$. Since q_n is in $N[G]$, a is in $N[G] \cap H(\lambda)^V = N \cap H(\lambda)^V$.

Let $\dot{r}$ be a name such that p forces $\dot{r} = \langle y_i : i \leq N \cap \omega_1\rangle$, where $y_i = \dot{x}_i$ for $i < N \cap \omega_1$ and $y_{N \cap \omega_1} = N \cap H(\lambda)^V$. Then p forces $\dot{r}$ is a condition in $\mathbb{P}(\dot{X})$ below $\dot{q}$ which is $N[\dot{G}]$-generic. $\square$

After forcing with $\textsc{Add}(\omega) * \mathbb{P}(\dot{X})$, $\aleph_2$ is collapsed, so T becomes a tree with height and size $\aleph_1$. The next proposition is similar to the Key Lemma of [3]. It implies in particular that $\textsc{Add}(\omega) * \mathbb{P}(\dot{X})$ does not add any new cofinal branches in T.

Proposition 2.4. *The poset* $\textsc{Add}(\omega) * \mathbb{P}(\dot{X})$ *forces that whenever* $\dot{b} : \omega_1 \to \mathbf{On}$ *is a function whose initial segments are all in* V*,* $\dot{b}$ *is in* V*.*

Proof. Suppose for a contradiction that $p * \dot{q}$ forces $\dot{b} : \omega_1 \to \mathbf{On}$ is a new sequence whose initial segments are in the ground model. For each $i < \omega_1$ let X_i denote the set of conditions in $\text{ADD}(\omega) * \mathbb{P}(\dot{X})$ below $p * \dot{q}$ which decide for some s in V that $\dot{b} \restriction i = s$. Then X_i is dense open below $p * \dot{q}$. Note that if $i < j < \omega_1$, then $X_j \subseteq X_i$.

Claim 2.5. *There is $p_0 * \dot{q}_0 \leq p * \dot{q}$ and a sequence $\langle \dot{E}_i : i < \omega_1 \rangle$ of $\text{ADD}(\omega)$-names for dense open subsets of $\mathbb{P}(\dot{X})$ such that whenever p_0 forces $\dot{q}_1 \leq \dot{q}_0$ and $\dot{q}_1 \in \dot{E}_i$, then $p_0 * \dot{q}_1$ is in X_i.*

Proof. Let $G * H$ be generic for $\text{ADD}(\omega) * \mathbb{P}(\dot{X})$ containing $p * \dot{q}$. Then for each $i < \omega_1$ there is $p_i * \dot{q}_i$ in $X_i \cap (G * H)$. Since $|\text{ADD}(\omega)| = \aleph_0$, there is p^* such that for cofinally many $i < \omega_1$, $p_i = p^*$. But $X_j \subseteq X_i$ for $i < j < \omega_1$. So for all $i < \omega_1$, there is $\dot{u}_i$ such that $p^* * \dot{u}_i \in X_i \cap (G * H)$.

In V fix $p_0 * \dot{q}_0 \leq p * \dot{q}$, a condition p^* in $\text{ADD}(\omega)$, and a sequence of $\text{ADD}(\omega) * \mathbb{P}(\dot{X})$-names $\langle \dot{u}_i : i < \omega_1 \rangle$ such that $p_0 * \dot{q}_0$ forces $p^* * \dot{u}_i$ is in $X_i \cap (\dot{G} * \dot{H})$ for $i < \omega_1$. By separativity, $p_0 \leq p^*$. For $i < \omega_1$, X_i is open, so $p_0 * \dot{q}_0$ forces $p_0 * \dot{u}_i$ is in $X_i \cap (\dot{G} * \dot{H})$.

Fix $i < \omega_1$. Let $\dot{E}_i$ be an $\text{ADD}(\omega)$-name for the dense open set of conditions in $\mathbb{P}(\dot{X})$ which decide the value of $\dot{u}_i$. Suppose p_0 forces $\dot{q}_1 \leq \dot{q}_0$ and $\dot{q}_1 \in \dot{E}_i$. Then there is an $\text{ADD}(\omega)$-name $\dot{v}_i$ such that $p_0 \Vdash \dot{q}_1 \Vdash \dot{u}_i = \dot{v}_i$ and $\dot{v}_i \in H$. By separativity, $p_0 \Vdash \dot{q}_1 \leq \dot{v}_i$. So $p_0 * \dot{q}_1 \leq p_0 * \dot{v}_i$. Since X_i is open and $p_0 * \dot{q}_1$ forces that $p_0 * \dot{v}_i$ is in X_i, $p_0 * \dot{q}_1$ forces itself to be in X_i. Hence $p_0 * \dot{q}_1$ is in fact in X_i. $\square$

Fix $\theta \gg \lambda$ regular and let N be a countable elementary substructure of $\langle H(\theta), \in \rangle$ containing as members the objects λ, $\text{ADD}(\omega) * \mathbb{P}(\dot{X})$, $p_0 * \dot{q}_0$, $\dot{b}$, $\langle X_i : i < \omega_1 \rangle$, and $\langle \dot{E}_i : i < \omega_1 \rangle$. Let $\langle \dot{D}_n : n < \omega \rangle$ enumerate all $\text{ADD}(\omega)$-names in N for dense open subsets of $\mathbb{P}(\dot{X})$. Let $\langle \gamma_n : n < \omega \rangle$ be increasing and cofinal in $N \cap \omega_1$.

We define an indexed family $\{\langle p_s, \dot{q}_s, \xi_s, t_s, \alpha_s \rangle : s \in {}^{<\omega}2\}$ of elements of N by induction on the length of s in ${}^{<\omega}2$. Let $p_{\langle\rangle} = p_0$, $\dot{q}_{\langle\rangle} = \dot{q}_0$, and $\xi_{\langle\rangle} = 0$. Suppose p_s, $\dot{q}_s$, and ξ_s are defined. Let t_s be the maximal sequence in V such that $p_0 * \dot{q}_s$ forces t_s is an initial segment of $\dot{b}$, and let $\alpha_s = \text{dom}(t_s)$. Then there are distinct ordinals $\xi_{s^\frown 0}$ and $\xi_{s^\frown 1}$ in N and for $i < 2$ a condition $p_{s^\frown i} * \dot{q}'_{s^\frown i} \leq p_0 * \dot{q}_s$ which forces $\dot{b}(\alpha_s) = \xi_{s^\frown i}$. Without loss of generality we can assume p_0 forces $\dot{q}'_{s^\frown i} \leq \dot{q}_s$, otherwise replace it with a name which satisfies this property. Let $\dot{q}_{s^\frown i}$ be a name which p_0 forces is below $\dot{q}'_{s^\frown i}$ and in the dense open set

$$\dot{D}_{|s|} \cap \bigcap \{\dot{E}_i : i \leq \max(\gamma_{|s|}, \alpha_s)\}.$$

Then by Claim 2.5, $p_0 * \dot{q}_{s^\frown i}$ is in

$$\bigcap \{X_i : i \leq \max(\gamma_{|s|}, \alpha_s)\}.$$

Since $p_0 * \dot{q}_{s^\frown i}$ is compatible with $p_{s^\frown i} * \dot{q}'_{s^\frown i}$, $p_0 * \dot{q}_{s^\frown i}$ forces $\dot{b}(\alpha_s) = \xi_{s^\frown i}$. This completes the definition.

If $\dot{x} : \omega \to 2$ is an $\text{ADD}(\omega)$-name for a real, let $q_{\dot{x}}$ be a name such that p_0 forces $q_{\dot{x}} = \langle y_i : i \leq N \cap \omega_1 \rangle$, where $\langle y_i : i < N \cap \omega_1 \rangle$ is the union $\bigcup \{\dot{q}_{\dot{x} \restriction n} : n < \omega\}$ and $y_{N \cap \omega_1} = N \cap H(\lambda)^V$. As in the proof of Proposition 2.3, p_0 forces that $q_{\dot{x}}$ is a condition in $\mathbb{P}(\dot{X})$.

Fix an $\text{ADD}(\omega)$-name for a real $\dot{h} : \omega \to 2$ which is not in V. For each n, p_0 forces $\dot{q}_{\dot{h} \restriction (n+1)}$ is in $\bigcap \{\dot{E}_i : i < \gamma_n\}$, so p_0 forces $q_{\dot{h}}$ is in $\bigcap \{\dot{E}_i : i < N \cap \omega_1\}$. Hence $p_0 * q_{\dot{h}}$ is in $\bigcap \{X_i : i < N \cap \omega_1\} = X_{N \cap \omega_1}$. Fix t such that $p_0 * q_{\dot{h}}$ forces $\dot{b} \restriction N \cap \omega_1 = t$. Define $z : \omega \to 2$ inductively as follows. Given $z \restriction n$, let $z(n) = i$

for $i < 2$ iff $t(\alpha_{z\restriction n}) = \xi_{(z\restriction n)^\frown i}$. Then p_0 forces $z = \dot{h}$ and $\dot{h}$ is in V, which is a contradiction. $\square$

Let $\langle \dot{a}_i : i < \omega_1 \rangle$ be an $\textsc{Add}(\omega) * \mathbb{P}(\dot{X})$-name for an increasing and continuous sequence of countable sets from $\dot{X}$ with union $H(\lambda)^V$ given by a generic for $\mathbb{P}(\dot{X})$. Let $\langle \dot{\xi}_i : i < \omega_1 \rangle$ be a name for an increasing and continuous sequence cofinal in ω_2^V with $\dot{\xi}_0 = 0$.

Let $G * H$ be generic for $\textsc{Add}(\omega) * \mathbb{P}(\dot{X})$ over V. In $V[G * H]$ define a relation R_T on T as follows. Let sR_Tt iff there is $i < \omega_1$ with $t \setminus \xi_i$ non-empty such that $s = t \cap \xi_i$. Clearly $\langle T, R_T \rangle$ is a tree.

Lemma 2.6. *The tree $\langle T, R_T \rangle$ has height and size $\aleph_1$ and has no cofinal branches.*

Proof. To show T has height ω_1, let $\alpha < \omega_1$. Fix $a \subseteq \omega_2^V$ in $V[G*H]$ countable such that $\{i < \omega_1 : a \cap [\xi_i, \xi_{i+1}) \neq \emptyset\}$ has order type at least $\alpha + 1$. Since $\textsc{Add}(\omega) * \mathbb{P}(\dot{X})$ has the countable covering property, there is a countable set b in V such that $a \subseteq b$. Let s be the increasing enumeration of b. Then s is in T and $\{s' \in T : s' R_T s\}$ has order type at least α.

If b is a cofinal branch in $\langle T, R_T \rangle$, then the set $\{i < \omega_1 : \mathrm{ran}(b) \cap [\xi_i, \xi_{i+1}) \neq \emptyset\}$ is cofinal in ω_1. So b is cofinal in ω_2^V and hence is not in V. It is easy to check that every initial segment of b is in T, which contradicts Proposition 2.4. $\square$

In $V[G * H]$ we define $\mathbb{Q}$ as the forcing poset from [1] which adds a specializing function for the tree $\langle T, R_T \rangle$. A condition in $\mathbb{Q}$ is a finite partial function $q : T \to \omega$ such that whenever u and v are in $\mathrm{dom}(q)$ and uR_Tv, then $q(u) \neq q(v)$. The ordering of $\mathbb{Q}$ is by extension of functions. The poset $\mathbb{Q}$ is ω_1-c.c. and forces that there is a total function $h : T \to \omega$ such that whenever uR_Tv, $h(u) \neq h(v)$.

Let $\mathbb{P}$ be the proper forcing poset $\textsc{Add}(\omega) * \mathbb{P}(\dot{X}) * \dot{\mathbb{Q}}$. Let $\dot{R}_T$ be an $\textsc{Add}(\omega) * \mathbb{P}(\dot{X})$-name for the tree relation defined above, and let $\dot{h}$ be a $\mathbb{P}$-name for the specializing function for $\langle T, \dot{R}_T \rangle$ given by a generic for $\mathbb{Q}$.

Now we are ready to prove Theorem 2.1. Assume PFA. Fix a function $F : H(\lambda)^{<\omega} \to [H(\lambda)]^{\aleph_1}$. We will find a set $M \subseteq H(\lambda)$ with size $\aleph_1$ closed under F which is internally club but not internally approachable.

Fix $\theta \gg \lambda$ regular. Let $\mathcal{A}$ denote the structure

$$\langle H(\theta), \in, F, \langle T, <_T \rangle, \mathbb{P}, \langle \dot{a}_i : i < \omega_1 \rangle, \langle \dot{\xi}_i : i < \omega_1 \rangle, \dot{R}_T, \dot{h} \rangle.$$

By Fact 1.1 we can choose a set $N \prec \mathcal{A}$ in $[H(\theta)]^{\aleph_1}$ containing ω_1 and a filter $G * H * I = K \subseteq \mathbb{P}$ which is N-generic for $\mathbb{P}$. It follows that $G * H$ is N-generic for $\textsc{Add}(\omega) * \mathbb{P}(\dot{X})$. Let $M = N \cap H(\lambda)$ and let $\beta = M \cap \omega_2$.

Suppose $x_0, \ldots, x_n$ are in M. Since $N \prec \mathcal{A}$, $F(x_0, \ldots, x_n) \in N$. But $\omega_1 \subseteq N$, so $F(x_0, \ldots, x_n) \subseteq N \cap H(\lambda) = M$. So M is closed under F.

For each $i < \omega_1$, let $a_i = \dot{a}_i^{G*H}$ and $\xi_i = \dot{\xi}_i^{G*H}$. Since $\dot{a}_i$ is forced to be in V, a_i is the unique set such that there is p in $G * H$ which forces $\dot{a}_i = a_i$, and similarly with ξ_i.

Lemma 2.7. *The sequence $\langle \xi_i : i < \omega_1 \rangle$ is an increasing and continuous sequence of ordinals cofinal in β with $\xi_0 = 0$.*

Proof. For each $i < \omega_1$ there is a dense set in N of conditions which decide the value of $\dot{\xi}_i$. So by N-genericity and the elementarity of N, each ξ_i is in $N \cap \omega_2$. For $i < j < \omega_1$, since $\mathbb{P}$ forces $\dot{\xi}_i < \dot{\xi}_j$ and conditions in $G * H$ are compatible,

$\xi_i < \xi_j$. Clearly $\xi_0 = 0$. If $\alpha < N \cap \omega_2$, then there is a dense set in N consisting of conditions which decide for some $i < \omega_1$ that $\dot{\xi}_i > \alpha$. So by N-genericity, the sequence is cofinal in β.

Suppose $\delta < \omega_1$ is limit. Fix p in $G * H$ which forces $\dot{\xi}_\delta = \xi_\delta$. Then for each $\gamma < \xi_\delta$, there is a dense set of conditions below p in N which decide for some $i < \delta$ that $\gamma < \dot{\xi}_i < \xi_\delta$. By N-genericity there is $i < \delta$ with $\gamma < \xi_i < \xi_\delta$. So the sequence is continuous. □

Lemma 2.8. *The sequence $\langle a_i : i < \omega_1 \rangle$ is an increasing and continuous sequence of countable sets with union M.*

Proof. The proof is similar to Lemma 2.7. □

Lemma 2.9. *The set M is internally club.*

Proof. It suffices to show that each a_i is in M. But $\dot{a}_i$ is in N and $\mathbb{P}$ forces $\dot{a}_i$ is in $H(\lambda)^V$. So there is p in $(G * H) \cap N$ and x in $N \cap H(\lambda) = M$ such that p forces $\dot{a}_i = x$. Then $x = a_i$. □

Let $R_T = \dot{R}_T^{G*H}$ and $h = \dot{h}^I$.

Lemma 2.10. *For u and v in $T \cap N$, $u R_T v$ iff there is $i < \omega_1$ with $v \setminus \xi_i$ non-empty such that $u = v \cap \xi_i$. The function h is defined on every u in $T \cap N$, and $u R_T v$ implies $h(u) \neq h(v)$.*

Proof. Use N-genericity as in the previous lemmas. □

Now suppose for a contradiction that M is internally approachable. Fix an increasing and continuous sequence $\langle M_i : i < \omega_1 \rangle$ of countable sets with union M such that for all $i < \omega_1$, $\langle M_j : j < i \rangle$ is in M. Define $b : \omega_1 \to \omega_2$ by letting $b(i) = \sup(M_i \cap \omega_2)$ for $i < \omega_1$. Then b is cofinal in β, and every initial segment of b is in $N \cap T$. For each $i < \omega_1$ let b_i be the maximal initial segment of b such that $\mathrm{ran}(b_i) \subseteq \xi_i$. Since b is cofinal in β, $\{b_i : i < \omega_1\}$ has size $\aleph_1$ and is linearly ordered by R_T. But then h is injective on this set, contradicting the fact that the range of h is countable. This completes the proof.

Acknowledgements. Our proof of Theorem 2.1 bears some resemblance to the earlier proof in [4] that PFA implies the failure of AP_{ω_1}.

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