ADDING CLUBS WITH SQUARE

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Abstract. We present a technique for destroying stationary subsets of $P_\kappa\kappa^+$ using partial square sequences. We combine this method with Gitik's poset for changing the cofinality of a cardinal without adding bounded sets to prove a variety of consistency results concerning saturated ideals and the set $S(\kappa, \kappa^+)$.

In this paper we continue our study of consistency results concerning the set $S(\kappa, \kappa^+) = \{a \in P_\kappa\kappa^+ : \text{o.t.}(a) = (a \cap \kappa)^+\}$ from [5] and [6]. We present a method for destroying the stationarity of certain subsets of $P_\kappa\kappa^+$, where κ is inaccessible, using partial square sequences.

This method of destroying stationary sets has a variety of applications. We prove several results which support the general theme that the structure of $S(\kappa, \kappa^+)$ can vary greatly depending on the particular model considered. For example, if κ is supercompact and $\mu < \kappa$ is regular, then there exists a generic extension in which $S(\kappa, \kappa^+)$ is stationary and for almost all a in $S(\kappa, \kappa^+)$, $a \cap \kappa$ is singular with cofinality μ. On the other hand, it is relatively consistent that for almost all a in $S(\kappa, \kappa^+)$, $a \cap \kappa$ is measurable with Mitchell order $(a \cap \kappa)^{++}$. We also construct a model in which GCH holds and there is a stationary set $S \subseteq P_\kappa\kappa^+$ such that $NS \restriction S$ is saturated and for all regular $\mu < \kappa$, $\{a \in S : \text{cf}(a \cap \kappa) = \mu\}$ is stationary.

This material is related to our previous papers [5] and [6]. However, no prior acquaintance with these papers is required by the reader. Most of our consistency results use Easton support Prikry iterations. The theorems in this paper rely on Gitik's technique from [3] for changing the cofinality of a cardinal without adding bounded sets. We present Gitik's poset in complete detail.

The contents of the paper are as follows. In Section 1 we outline preliminaries and notation. Section 2 provides additional background on forcing posets and large cardinals. Section 3 describes Easton support Prikry iterations. In Section 4 we present our method for destroying stationary sets using partial square sequences. In Section 5 we construct a model in which almost all a in $S(\kappa, \kappa^+)$ satisfy that $a \cap \kappa$ is a measurable cardinal with Mitchell order $(a \cap \kappa)^{++}$. Section 6 describes Gitik's forcing poset for changing a cofinality without adding bounded sets. In Section 7 we show how to construct models in which we can control the cofinality of $a \cap \kappa$ for almost all a in $S(\kappa, \kappa^+)$. In Section 8 we construct a model with a stationary set $S \subseteq P_\kappa\kappa^+$ such that $NS \restriction S$ is saturated, GCH holds, and for every regular μ less than κ, the set $\{a \in S : \text{cf}(a \cap \kappa) = \mu\}$ is stationary.

1. Preliminaries

We assume that the reader is familiar with iterated forcing, supercompact cardinals, and Prikry forcing; see [1] and [4].

If κ is a regular uncountable cardinal and $\kappa \subseteq X$, define $P_\kappa X = \{a \subseteq X : |a| < \kappa,\ a \cap \kappa \in \kappa\}$. A subset of $P_\kappa X$ is *club* if it is closed under unions of $\subseteq$-increasing sequences of length less than κ and is cofinal in $P_\kappa X$. A set is *stationary* if it intersects every club. If A is a subset of a club set C with size less than κ and is *directed*, i.e. for all a and b in C, there is c in C with $a \cup b \subseteq c$, then $\bigcup A$ is in C.

The ideal of non-stationary subsets of $P_\kappa\lambda$ for $\lambda \geq \kappa$ is denoted by $NS_{\kappa,\lambda}$ or NS. If S is stationary then $NS_{\kappa,\lambda} \restriction S$ denotes the ideal generated by the elements of $NS_{\kappa,\lambda}$ along with the complement of S. An ideal on $P_\kappa\lambda$ is *fine* if it contains the set $\{a \in P_\kappa\lambda : \xi \notin a\}$ for every $\xi < \lambda$. A function $f : P_\kappa X \to X$ is *regressive* if $f(a)$ is in a for all a. An ideal I is *normal* if for every set $S \subseteq P_\kappa\lambda$ not in I and for every regressive function $f : S \to \lambda$, there is an $i < \lambda$ such that the set $\{a \in S : f(a) = i\}$ is not in I.

If I is an ideal on $P_\kappa\lambda$, the set $I^* = \{P_\kappa\lambda \setminus A : A \in I\}$ is the *dual filter of* I. The collection of *I-positive sets* $I^+ = \{A \subseteq P_\kappa\lambda : A \notin I\}$ is a forcing poset ordered by $A \leq B$ if $A \setminus B$ is in I.

An ideal I is *μ-saturated* if I^+ is μ-c.c. If $I = NS \restriction S$ for some stationary set S, then I is μ-saturated iff there is no family $\{S_i : i < \mu\}$ of stationary subsets of S such that $S_i \cap S_j$ is non-stationary for $i < j$. We say that an ideal I on $P_\kappa\lambda$ is *saturated* if I is λ^+-saturated.

If I is an ideal on $P_\kappa\lambda$, the forcing poset I^+ adds a generic set U which is an ultrafilter on the Boolean algebra $(\mathcal{P}(P_\kappa\lambda))^V$. If M is the ultrapower of V by U in the generic extension, M is called the *generic ultrapower* and the ultrapower map $j : V \to M$ is the *generic elementary embedding*. If I is saturated then the generic ultrapower M is well-founded and ${}^\lambda M \subseteq M$.

If $M \subseteq N$ are inner models of set theory and λ is a cardinal, we say that M is *λ-closed* in N if ${}^\lambda M \cap N \subseteq M$. The model M is λ-closed in N iff ${}^\lambda On \cap N \subseteq M$.

A cardinal κ is *λ-supercompact* if there is a normal fine ultrafilter on $P_\kappa\lambda$, or equivalently, there is an elementary embedding $j : V \to M$, where M is a transitive inner model, such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \lambda$, and M is λ-closed. If U_0 and U_1 are normal fine ultrafilters on $P_\kappa\lambda$, we write $U_0 \lhd U_1$ if U_0 is in the ultrapower of V by U_1. This ordering, called the *Mitchell ordering*, is transitive and well-founded.

When we mention an ultrafilter on $P_\kappa\lambda$ we always assume that it is non-principal and fine. If U is a ultrafilter on $P_\kappa\lambda$, we write $\mathrm{Ult}(V, U)$ for the transitive collapse of the ultrapower of V by U. Suppose that U is normal and $j : V \to M = \mathrm{Ult}(V, U)$. Then M is $\lambda^{<\kappa}$-closed, and for any function $f : P_\kappa\lambda \to V$, $[f]$ is equal to $j(f)(j``\lambda)$.

If a is a set of ordinals then $\mathrm{o.t.}(a)$ is the order type of a, and if $\mathrm{o.t.}(a)$ is a limit ordinal then $\mathrm{cf}(a)$ is $\mathrm{cf}(\mathrm{o.t.}(a))$. Note that $\mathrm{cf}(a) = \mathrm{cf}(\sup a)$.

Suppose that α is a limit ordinal which is equal to the ordinal exponent ω^γ for some $\gamma \geq 1$. Then any set $X \subseteq \alpha$ which is cobounded in α has order type α.

The expression $\theta \gg \kappa$ means that θ is larger than $2^{2^{|H(\kappa)|}}$. If $\mathcal{A}$ is a structure with underlying set $H(\theta)$, then the collection of elementary substructures of $\mathcal{A}$ in $P_\kappa H(\theta)$ is a club set. If N is such an elementary substructure, then $N \cap \kappa^+$ is closed under suprema of bounded subsets with order type different from $\mathrm{cf}(N \cap \kappa)$.

If $F : \lambda^{<\omega} \to \lambda$ is a partial function and $A \subseteq \lambda$, we say that F is *Jonsson for* A if A is closed under F, and whenever $B \subsetneq A$ is closed under F, it follows that $|B| < |A|$. If κ is a regular cardinal then there exists a Jonsson function for κ^+.

If A is a set of ordinals, then A is an *Easton set* if for any strongly inaccessible cardinal β, $A \cap \beta$ is bounded below β.

We use the phrase *forcing poset* to indicate any ordering $\langle \mathbb{P}, \leq \rangle$ which is reflexive and transitive. We do not require that $\mathbb{P}$ be antisymmetric or separative. We usually assume that $\mathbb{P}$ has a maximum element 1; this should be clear from context.

Suppose that κ is regular and $\mathbb{P}$ is a forcing poset. We say that $\mathbb{P}$ is $< \kappa$-*distributive* if whenever $\{D_i : i < \beta\}$ is a family of dense open subsets of $\mathbb{P}$ and β is less than κ, then $\bigcap D_i$ is dense open. Equivalently, $\mathbb{P}$ is $< \kappa$-distributive if forcing with $\mathbb{P}$ does not add any new sequences of ordinals with order type less than κ.

A forcing poset $\mathbb{P}$ is λ-*strategically closed* for an ordinal λ if there is a strategy for Player II in the following game: Player I starts the game by playing a condition p_1 in $\mathbb{P}$. Player II responds with a condition $p_2 \leq p_1$. The game continues in this manner, each player choosing a condition below the previous one, with Player I playing at odd stages and Player II at even successor stages. At limit stages Player II plays a condition below all the conditions played so far. Player II wins if it is able to play a condition at all stages below λ. If $\mathbb{P}$ is $\lambda+1$-strategically closed, then $\mathbb{P}$ does not add any subsets to λ.

Suppose that $\mathbb{P}$ is a forcing poset and λ is a cardinal. A *canonical name for a subset of* λ is a $\mathbb{P}$-name of the form

$$\{\langle p, \check{\alpha} \rangle : p \in A_\alpha,\ \alpha < \lambda\}$$

where A_α is an antichain for each $\alpha < \lambda$. If p forces that $\dot{A}$ is a subset of λ, then there is a canonical name $\dot{B}$ for a subset of λ such that p forces that $\dot{A} = \dot{B}$. If $\dot{X}$ is a $\mathbb{P}$-name and $\mathbb{P}$ forces that $\dot{f} : \lambda \to \dot{X}$ is a bijection, then a *canonical name for a subset of* $\dot{X}$ is a name of the form

$$\{\langle p, \dot{f}(\alpha) \rangle : p \in A_\alpha,\ \alpha < \lambda\}$$

where each A_α is an antichain. If p forces that $\dot{A}$ is a subset of $\dot{X}$, then there is a canonical name $\dot{B}$ for a subset of $\dot{X}$ such that p forces that $\dot{A} = \dot{B}$.

Suppose that λ is a regular uncountable cardinal and $\mathbb{P}$ is a forcing poset which is λ-c.c. and is a subset of $H(\lambda)$. Let G be generic for $\mathbb{P}$ over V. If a is in $H(\lambda)^{V[G]}$, then there is a $\mathbb{P}$-name $\dot{x}$ in $H(\lambda)^V$ such that $\dot{x}^G = a$. Therefore $H(\lambda)^{V[G]} = H(\lambda)^V[G]$.

If H is a subset of a forcing poset $\mathbb{P}$, we say that H *generates* J if J is the set of q in $\mathbb{P}$ such that there is p in H with $p \leq q$.

Suppose that $\mathbb{P}$ is a forcing poset and $\langle U_i : i < \beta \rangle$ is a $\triangleleft$-increasing sequence of normal ultrafilters on $P_\kappa \lambda$. We say that $\mathbb{P}$ forces that the sequence can be *lifted* if $\mathbb{P}$ forces that there are normal ultrafilters $U_i \subseteq U_i^*$ for $i < \beta$ such that $U_i^* \triangleleft U_j^*$ for $i < j$.

We say that forcing posets $\mathbb{P}$ and $\mathbb{Q}$ are *equivalent* if they have the same generic extensions. This is only an intuitive definition, not a formal one, since generic filters for non-trivial posets do not exist in the universe. To prove that forcing posets $\mathbb{P}$ and $\mathbb{Q}$ are equivalent in this informal sense, we show how to construct a generic filter for one poset, given a generic filter for the other. The following lemma provides a sufficient condition for the equivalence of forcing posets.

Lemma 1. *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are forcing posets, D is a dense subset of $\mathbb{P}$, and E is a dense subset of $\mathbb{Q}$. Assume there is a surjective mapping $i : D \to E$ satisfying:*

(1) $q \leq p$ implies $i(q) \leq i(p)$,

(2) If p and q are incompatible, then $i(p)$ and $i(q)$ are incompatible.
Then $\mathbb{P}$ and $\mathbb{Q}$ are equivalent.

A mapping $\pi : \mathbb{Q} \to \mathbb{P}$ between forcing posets is a *projection mapping* if it satisfies:
(1) $q \leq p$ implies $\pi(q) \leq \pi(p)$,
(2) $\pi(1) = 1$,
(3) if $p \leq \pi(q)$ then there is $r \leq q$ such that $\pi(r) \leq p$.

Conditions (2) and (3) imply that $\pi``\mathbb{Q}$ is dense in $\mathbb{P}$. If $G$ is generic for $\mathbb{Q}$ over $V$, then $\pi``G$ generates a generic filter for $\mathbb{P}$ over V.

Suppose that $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection mapping. Let G be generic for $\mathbb{P}$ over V. In $V[G]$, define a poset $\mathbb{Q}/\mathbb{P}$ as follows. A condition in $\mathbb{Q}/\mathbb{P}$ is any q in $\mathbb{Q}$ such that $\pi(q) \in G$. Let $q \leq p$ in $\mathbb{Q}/\mathbb{P}$ iff $q \leq p$ in $\mathbb{Q}$. Now let $\mathbb{Q}/\mathbb{P}$ be a $\mathbb{P}$-name for this poset. Define $k : \mathbb{Q} \to \mathbb{P} * (\mathbb{Q}/\mathbb{P})$ by letting $k(q) = \pi(q) * \check{q}$. Then k satisfies:
(1) $q \leq p$ implies $k(q) \leq k(p)$,
(2) If p and q are incompatible, then $k(p)$ and $k(q)$ are incompatible,
(3) $k``\mathbb{Q}$ is dense in $\mathbb{P} * (\mathbb{Q}/\mathbb{P})$.
So by Lemma 1, $\mathbb{Q}$ and $\mathbb{P} * (\mathbb{Q}/\mathbb{P})$ are equivalent.

Lemma 2. *Suppose that $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection mapping and μ is a regular uncountable cardinal. Assume that $\mathbb{Q}$ is μ-c.c. Then:*
(1) $\mathbb{P}$ is μ-c.c.,
(2) $\mathbb{P}$ forces that $\mathbb{Q}/\mathbb{P}$ is μ-c.c.

Proof. Since $\mathbb{Q}$ is μ-c.c., so is $\mathbb{P} * (\mathbb{Q}/\mathbb{P})$. If A is an antichain in $\mathbb{P}$, then the family $\{p * 1 : p \in A\}$ is an antichain in $\mathbb{P} * (\mathbb{Q}/\mathbb{P})$. So $\mathbb{P}$ is μ-c.c. Suppose that p forces that $\langle \dot{q}_i : i < \xi \rangle$ is an antichain in $\mathbb{Q}/\mathbb{P}$. Then p forces that $\dot{q}_i$ is incompatible with $\dot{q}_j$ for i less than j. It follows that the family $\{p * \dot{q}_i : i < \xi\}$ is an antichain in $\mathbb{P} * (\mathbb{Q}/\mathbb{P})$, and so ξ is less than μ. Therefore $\mathbb{P}$ forces that $\mathbb{Q}/\mathbb{P}$ is μ-c.c. $\square$

A *Prikry type forcing poset* is a triple $\langle \mathbb{Q}, \leq, \leq^* \rangle$ such that $\langle \mathbb{Q}, \leq \rangle$ and $\langle \mathbb{Q}, \leq^* \rangle$ are forcing posets, $q \leq^* p$ implies $q \leq p$, and $\mathbb{Q}$ satisfies the *Prikry property*: if p is a condition in $\mathbb{Q}$ and φ is a statement in the forcing language for $\langle \mathbb{Q}, \leq \rangle$, then there is $q \leq^* p$ such that q decides φ. If $q \leq^* p$ we say that q is a *direct extension* or a *direct refinement* of p.

We say that $\mathbb{Q}$ is *α-weakly closed* if $\langle \mathbb{Q}, \leq^* \rangle$ is α-closed, and *α-weakly strategically closed* if $\langle \mathbb{Q}, \leq^* \rangle$ is α-strategically closed. The poset $\mathbb{Q}$ has the *direct extension property* if whenever $q, r \leq^* p$, there is $s \leq^* q, r$.

2. Background on Forcing and Large Cardinals

In this section we review some additional material on large cardinals and forcing.

Suppose that $\kappa \leq \lambda_0 \leq \lambda_1$ and U is a normal ultrafilter on $P_\kappa \lambda_1$. Define $U \restriction \lambda_0$ by letting X be in $U \restriction \lambda_0$ iff $X \subseteq P_\kappa \lambda_0$ and the set $\{a \in P_\kappa \lambda_1 : a \cap \lambda_0 \in X\}$ is in U. Equivalently, if $j : V \to M = \mathrm{Ult}(V, U)$, X is in $U \restriction \lambda_0$ iff $j``\lambda_0 \in j(X)$. The set $U \restriction \lambda_0$ is a normal ultrafilter on $P_\kappa \lambda_0$.

Lemma 3. *Suppose that $\kappa \leq \lambda_0 < 2^{\lambda_0^{<\kappa}} \leq \lambda_1$ are cardinals and U is a normal ultrafilter on $P_\kappa \lambda_1$. Let $j : V \to M = Ult(V, U)$ and $i : V \to N = Ult(V, U \restriction \lambda_0)$. Then there is an elementary embedding $k : N \to M$ such that $j = k \circ i$ and* $\mathrm{crit}(k) = (2^{\lambda_0^{<\kappa}})^{+N}$.

Proof. Define k as follows. Let a be in N and let $f : P_\kappa\lambda_0 \to V$ be a function such that $[f] = a$. Define $k(a) = j(f)(j``\lambda_0)$. It is easy to check that k is a well-defined elementary embedding and $j = k \circ i$.

If $\beta \leq \lambda_0$, then $\beta = \text{o.t.}(i``\beta) = \text{o.t.}(i``\lambda_0 \cap i(\beta))$. So β is represented by the function $a \mapsto \text{o.t.}(a \cap \beta)$ in N. By the definition of k, $k(\beta) = \text{o.t.}(j``\lambda_0 \cap j(\beta)) = \text{o.t.}(j``\beta) = \beta$ in M. So $\text{crit}(k) > \lambda_0$.

Since N is $\lambda_0^{<\kappa}$-closed and $\text{crit}(k) > \lambda_0$, $k(P_\kappa\lambda_0) = P_\kappa\lambda_0$ and $k(\mathcal{P}(P_\kappa\lambda_0)) = \mathcal{P}(P_\kappa\lambda_0)$. If a is in $P_\kappa\lambda_0$, then $k(a)$ is in $k(P_\kappa\lambda_0) = P_\kappa\lambda_0$, and $\alpha \in k(a)$ iff $k(\alpha) \in k(a)$ iff $\alpha \in a$. Therefore $k \restriction P_\kappa\lambda_0$ is the identity. The same argument shows that $k \restriction \mathcal{P}(P_\kappa\lambda_0)$ is the identity.

We prove by induction that for all β less than $(2^{\lambda_0^{<\kappa}})^{+N}$, $k(\beta) = \beta$. Fix β and suppose $k(\alpha) = \alpha$ for all α less than β. Since β is less than $(2^{\lambda_0^{<\kappa}})^{+N}$, there is a surjective function $s : \mathcal{P}(P_\kappa\lambda_0)) \to \beta$ in N. By elementarity, $k(s)$ is a surjection of $k(\mathcal{P}(P_\kappa\lambda_0)) = \mathcal{P}(P_\kappa\lambda_0)$ onto $k(\beta)$. If $k(\beta) > \beta$, then there is A in $\mathcal{P}(P_\kappa\lambda)$ such that $\beta = k(s)(A) = k(s)(k(A)) = k(s(A)) = s(A) < \beta$, which is absurd.

Now $(2^{\lambda^{<\kappa}})^{+N} < i(\kappa) < (2^{\lambda^{<\kappa}})^{+} = (2^{\lambda^{<\kappa}})^{+M}$, and $k((2^{\lambda^{<\kappa}})^{+N}) = (2^{\lambda^{<\kappa}})^{+M}$. So $\text{crit}(k) = (2^{\lambda^{<\kappa}})^{+N}$. $\square$

The following lemma is the main tool for extending elementary embeddings.

Lemma 4. *Suppose that $j : M \to N$ is an elementary embedding between transitive models of set theory, $\mathbb{P}$ is a forcing poset in M, G is generic for $\mathbb{P}$ over M, and H is generic for $j(\mathbb{P})$ over N.*

Then j can be lifted to $j : M[G] \to N[H]$ iff $j``G \subseteq H$. In this case, $j(G) = H$. In particular, j can be lifted if there exists a condition s in H such that $s \leq j(p)$ for all p in G.

Proof. The mapping $j(\dot{x}^G) = j(\dot{x})^H$ is well defined and satisfies the required properties. $\square$

We will use Silver's notation and refer to a condition s in $j(\mathbb{P})$ such that $s \leq j(p)$ for all p in G as a *master condition.*

Lemma 5. *Suppose that U is a normal ultrafilter on a cardinal κ and $j : V \to M = Ult(V, U)$. Let $\mathbb{P}$ be a forcing poset and G a generic filter for $\mathbb{P}$ over V. Suppose that in $V[G]$ there a generic filter H for $j(\mathbb{P})$ over M such that $j``G \subseteq H$. Lift j to $j : V[G] \to M[H]$.*

In $V[G]$ define U^* by letting $X \in U^*$ iff $X \subseteq \kappa$ and $\kappa \in j(X)$. Then U^* is a normal ultrafilter extending U and $M[H] = Ult(V[G], U^*)$.

Proof. We omit the standard argument that U^* is a normal ultrafilter extending U. Since any isomorphism between transitive models of set theory is the identity, it suffices to prove that $M[H]$ and $\text{Ult}(V[G], U^*)$ are isomorphic.

Define $k : M[H] \to \text{Ult}(V[G], U^*)$ as follows. Let a be in $M[H]$. Then there is a $j(\mathbb{P})$-name $\dot{x}$ such that $\dot{x}^H = a$. Since $\dot{x}$ is in M, we can choose $f : \kappa \to V$ such that $[f] = \dot{x}$. Moreover, choose f so that $f(\alpha)$ is a $\mathbb{P}$-name for all α. In $V[G]$ define $g : \kappa \to V[G]$ by letting $g(\alpha) = f(\alpha)^G$. Now let $k(a) = [g]$.

First we show that g is well-defined. Suppose that $a = \dot{x}^H = \dot{y}^H$, where $[f_x] = \dot{x}$ and $[f_y] = \dot{y}$. Then there exists a condition q in $H = j(G)$ which forces over M that $\dot{x} = \dot{y}$. In M let $q = [h]$ and write $h(\alpha) = p_\alpha$. Then there exists a set A in U such that for all α in A, p_α forces that $f_x(\alpha) = f_y(\alpha)$. In $M[H]$, q is in $j(G)$. But

$j(h)(\kappa) = q$ in M and hence in $M[H]$. By definition of U^*, there is $A^* \subseteq A$ in U^* such that for all α in A^*, $h(\alpha) = p_\alpha$ is in G. Then for all α in A^*, $f_x(\alpha)^G = f_y(\alpha)^G$. Define g_x and g_y by $g_x(\alpha) = f_x(\alpha)^G$ and $g_y(\alpha) = f_y(\alpha)^G$. Then for all α in A^*, $g_x(\alpha) = g_y(\alpha)$, so $[g_x] = [g_y]$. This proves that k is well-defined.

A similar argument shows that k is injective and that $a \in b$ iff $k(a) \in k(b)$. To show that k is surjective, fix $[h]$ in $\mathrm{Ult}(V[G], U^*)$. For each α let $\dot{b}_\alpha$ be a $\mathbb{P}$-name for $h(\alpha)$. Then $[\alpha \mapsto \dot{b}_\alpha]$ is a $j(\mathbb{P})$-name in M. Let $a = [\alpha \mapsto \dot{b}_\alpha]^H$, which is in $M[H]$. Using the definition of k, it is straightforward to check that $k(a) = [h]$. $\square$

A standard way to extend an elementary embedding is to apply strategic closure to build a generic filter. Suppose that $M \subseteq N$ are models of set theory and λ is an N-cardinal. Let $\mathbb{P}$ be a forcing poset in M and p in $\mathbb{P}$. Suppose that N models that $\mathbb{P}$ is λ-strategically closed and has no more than λ many maximal antichains in M. Enumerate all maximal antichains in M as $\langle A_i : i < \lambda \rangle$. Applying strategic closure we can inductively define a decreasing sequence $\langle p_i : i < \lambda \rangle$ so that $p_0 = p$ and p_{i+1} is below some member of A_i. This sequence of conditions generates a generic filter H for $\mathbb{P}$ over M which contains p.

The following two lemmas show how to obtain closure of generic extensions.

Lemma 6. *Suppose that $M \subseteq N$ are models of set theory and λ is a regular uncountable cardinal in N such that M is $< \lambda$-closed in M. If $\mathbb{P}$ is a forcing poset in M which is λ-c.c. in N and G is generic for $\mathbb{P}$ over N, then $M[G]$ is $< \lambda$-closed in $N[G]$.*

Proof. We prove that ${}^{<\lambda}On \cap N[G] \subseteq M[G]$. Suppose that p forces over N that $\dot{f} : \beta \to On$ for some $\beta < \lambda$. For each $\alpha < \beta$ let A_α be a maximal antichain contained in the dense set of conditions which decide the value of $\dot{f}(\alpha)$. Let X_α be the set of pairs $\langle q, \gamma \rangle$ such that $q \in A_\alpha$ and q forces over N that $\dot{f}(\alpha) = \gamma$. Then $|A_\alpha| = |X_\alpha| < \lambda$, and so $\langle A_\alpha, X_\alpha : \alpha < \beta \rangle$ is in M. Define a name $\dot{g}$ in M by letting $\dot{g}(\alpha)$ be the unique γ so that there is q in $\dot{G} \cap A_\alpha$ such that $\langle q, \gamma \rangle$ is in X_α. Clearly p forces that $\dot{g} = \dot{f}$, and $\dot{g}$ is in $M[\dot{G}]$. $\square$

Lemma 7. *Suppose $M \subseteq N$ are models of set theory, λ is a regular cardinal in N such that M is λ-closed in N. If $\mathbb{P} \in M$ is a forcing poset, $G \in N$ is a generic filter for $\mathbb{P}$ over M, then $M[G]$ is λ-closed in N.*

Proof. In N we have ${}^{\lambda}On \subseteq M \subseteq M[G]$. $\square$

We need some facts about the Mitchell ordering.

Lemma 8. *Let U_0 and U_1 be normal ultrafilters on $P_\kappa\lambda$. For each a in $P_\kappa\lambda$, let $\pi_a : a \to \mathrm{o.t.}(a)$ be the unique order preserving bijection. Then $U_0 \lhd U_1$ iff there exists a function $f : P_\kappa\lambda \to V_\kappa$ such that:*

(1) $\{a \in P_\kappa\lambda : f(a)$ is a normal ultrafilter on $(P_{a\cap\kappa}\ \mathrm{o.t.}(a))\}$ is in U_1,
(2) For every $X \subseteq P_\kappa\lambda$, $X \in U_0$ iff the set of a in $P_\kappa\lambda$ such that

$$X_a = \{b : (\exists c \in (P_{a\cap\kappa}a) \cap X)\ (\pi_a``c = b)\} \in f(a)$$

is in U_1.

Proof. Let $j : V \to M = \mathrm{Ult}(V, U_1)$. Let $[f] = U_0$ in M. In M, $[a \mapsto a] = j``\lambda$, $[a \mapsto a \cap \kappa] = \kappa$, and $[a \mapsto \mathrm{o.t.}(a)] = \lambda$. Clearly (1) holds. For (2), check that $j(a \mapsto X_a)(j``\lambda) = X$, and therefore $[a \mapsto X_a] = X$ in M. So $X \in U_0$ iff $M \models X \in U_0$ iff $\{a \in P_\kappa\lambda : X_a \in f(a)\}$ is in U_1. The converse is similar. $\square$

Corollary 1. *Suppose that U_0 and U_1 are normal ultrafilters on $P_\kappa\lambda$ and $U_0 \lhd U_1$. Let $\mathbb{P}$ be a forcing poset which does not add subsets to $\lambda^{<\kappa}$. Then $\mathbb{P}$ forces that U_0, U_1 are normal ultrafilters on $P_\kappa\lambda$ and $U_0 \lhd U_1$.*

Corollary 2. *Suppose that M is a transitive model of set theory which is $\lambda^{<\kappa}$-closed in V. If U and W are normal ultrafilters on $P_\kappa\lambda$ which are in M, then $U \lhd W$ iff M models that $U \lhd W$.*

Lemma 9. *Suppose that $U_0 \lhd U_1$ are normal ultrafilters on $P_\kappa\lambda$. Let $j_0 : V \to M_0 = Ult(V, U_0)$, $j_1 : V \to M_1 = Ult(V, U_1)$, and $j_{1,0} : M_1 \to N_0 = Ult(M_1, U_0)$. Then the following statements are true:*

(1) If $f \in {}^{P_\kappa\lambda}M_1$ then $[f]_{M_0} = [f]_{N_0}$, so $N_0 \subseteq M_0$,

(2) $j_0 \restriction M_1 = j_{1,0}$.

Suppose that $\mathbb{P}$ is a forcing poset in M_1, G is generic for $\mathbb{P}$ over V, H is generic for $j_0(\mathbb{P}) = j_{1,0}(\mathbb{P})$ over M_0, and j_0“$G = j_{1,0}$“$G \subseteq H$. Extend j_0 and $j_{1,0}$ to $j_0 : V[G] \to M_0[H]$ and $j_{1,0} : M_1[G] \to N_0[H]$. Then $j_0 \restriction M_1[G] = j_{1,0}$.

Proof. Let $[f]_{M_0}$ be given such that f is in ${}^{P_\kappa\lambda}M_1$, and assume that (1) holds for all functions g where $[g]_{M_0}$ has rank less than the rank of $[f]_{M_0}$. We prove that $[f]_{M_0} = [f]_{N_0}$. Note that f is in M_1 by the closure of M_1, so $[f]_{N_0}$ is defined. Suppose that x is in $[f]_{M_0}$. Since f is in ${}^{P_\kappa\lambda}M_1$, there is g is in ${}^{P_\kappa\lambda}M_1$ and A in U_0 such that $[g]_{M_0} = x$ and $g(a) \in f(a)$ for all a in A. By induction, $[g]_{N_0} = x$, and clearly $[g]_{N_0}$ is in $[f]_{N_0}$. So $[f]_{M_0} \subseteq [f]_{N_0}$. The other direction is similar.

To prove (2), let x be in M_1. Then $j_0(x) = [f]_{M_0}$ where $f(a) = x$ for all a in $P_\kappa\lambda$. Since f is in ${}^{P_\kappa\lambda}M_1$, by (1) we have that $[f]_{M_0} = [f]_{N_0}$, and clearly $[f]_{N_0} = j_{1,0}(x)$.

The fact that $j_0 \restriction M_1[G] = j_{1,0}$ follows from (2) and the definition of the extended mappings given in the proof of Lemma 4. □

Lemma 10. *Suppose that κ is a measurable cardinal, $\mathbb{P}$ is a forcing poset with size less than κ, and $\langle U_i : i < \beta\rangle$ is a $\lhd$-increasing sequence of normal ultrafilters on κ. Then $\mathbb{P}$ forces that $\langle U_i : i < \beta\rangle$ can be lifted.*

Proof. Without loss of generality we assume that $\mathbb{P}$ is in V_κ. For $i < \beta$ let $j_i : V \to M_i = \mathrm{Ult}(V, U_i)$ and $k_i : M_{i+1} \to N_i = \mathrm{Ult}(M_{i+1}, U_i)$. Then $j_i(\mathbb{P}) = k_i(\mathbb{P}) = \mathbb{P}$. By Lemma 9, $j_i \restriction M_{i+1} = k_i$.

Let G be generic for $\mathbb{P}$ over V. Then j“$G = G$ and we can lift each j_i and k_i to $j_i : V[G] \to M_i[G]$ and $k_i : M_{i+1}[G] \to N_i[G]$. By Lemma 6 each $M_i[G]$ is κ-closed in $V[G]$. By Lemma 9, $j_i \restriction M_{i+1}[G] = k_i$.

Define U_i^* by letting $X \in U_i^*$ iff $\kappa \in j_i(X)$. Then U_i^* is a normal ultrafilter on κ extending U_i, and by Lemma 5 $\mathrm{Ult}(V[G], U_i^*) = M_i[G]$. Moreover, since $\mathcal{P}(\kappa) \subseteq M_{i+1}[G]$ and $j_i \restriction M_{i+1}[G] = k_i$, $M_{i+1}[G]$ can compute U_i^* using the same definition. So U_i^* is in $M_{i+1}[G]$.

Fix $i < j$. Then U_i^* is in $M_{i+1}[G] \subseteq M_j[G] = \mathrm{Ult}(V[G], U_j^*)$. So $U_i^* \lhd U_j^*$. □

3. Iterations of Prikry type forcing posets

In this section we present the basics of Easton support Prikry iterations which we use in our consistency proofs.

Magidor [7] showed how to iterate Prikry forcing over different cardinals using conditions with full support. To overcome the difficulty in extending an elementary embedding with such an iteration, Gitik [3] developed a method for iterating Prikry type forcing posets using conditions with Easton support.

An *Easton support Prikry iteration* is an iterated forcing

$$\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \kappa \rangle,$$

for some ordinal κ, satisfying the following properties:

(1) There exists a set $A \subseteq \kappa$ consisting of strongly inaccessible cardinals such that if $\mathbb{Q}_\alpha$ is non-trivial, then α is in A,

(2) $\mathbb{P}_\alpha$ forces that $|\mathbb{Q}_\alpha| < \min(A \setminus (\alpha + 1))$,

(3) $\mathbb{P}_\alpha$ forces that $\langle \mathbb{Q}_\alpha, \leq, \leq^* \rangle$ is a Prikry type forcing poset,

(4) $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{Q}_\alpha$,

(5) If α is a limit ordinal, then $\mathbb{P}_\alpha$ consists of functions p with domain an Easton subset of $\alpha \cap A$ such that $p \restriction \beta$ is in $\mathbb{P}_\beta$ for all β less than α,

(6) If α is a limit ordinal and p, q are in $\mathbb{P}_\alpha$, then $q \leq p$ if $q \restriction \beta \leq p \restriction \beta$ for β less than α, and there is a finite set a such that for all β in $\mathrm{dom}(p) \setminus a$, $q \restriction \beta \Vdash q(\beta) \leq^* p(\beta)$.

(7) If α is a limit ordinal, $q \leq^* p$ in $\mathbb{P}_\alpha$ if $q \leq p$ and the finite set a in (6) is empty; i.e. for all β in $\mathrm{supp}(p)$, $q \restriction \beta \Vdash q(\beta) \leq^* p(\beta)$,

(8) $q^\frown \dot{b} \leq^* p^\frown \dot{a}$ in $\mathbb{P}_{\beta+1}$ if $q \leq^* p$ in $\mathbb{P}_\beta$ and $q \Vdash \dot{b} \leq^* \dot{a}$.

In (3) the Prikry poset $\mathbb{Q}_\alpha$ will usually not add bounded subsets to α. To iterate Prikry posets which do add bounded sets, such as Radin forcing, one can use the Magidor iteration; see [6] for an example.

If p is a condition in $\mathbb{P}_\alpha$, the *support of* p, denoted by $\mathrm{supp}(p)$, is the domain of p as a function.

If β is less than α, then $\mathbb{P}_\alpha$ factors into $\mathbb{P}_\beta * \mathbb{Q}_\beta * \mathbb{P}_{\beta,\alpha}$. Suppose that $\mathbb{Q}_\gamma$ is forced to be γ-weakly strategically closed for all γ greater than β; then $\langle \mathbb{P}_{\beta,\alpha}, \leq^* \rangle$ is forced to be $\min(A \setminus (\beta+1))$-weakly strategically closed. It will follow from Proposition 2 that $\mathbb{P}_{\beta,\alpha}$ does not add any bounded subsets to $\min(A \setminus (\beta + 1))$.

The following result follows from the usual proof of the corresponding fact for Easton support iterations.

Proposition 1. *If $|\mathbb{P}_\beta| < \alpha$ for all β less than α and α is a Mahlo cardinal, then $\mathbb{P}_\alpha$ is α-c.c.*

We need the following lemma in order to prove that $\mathbb{P}_\alpha$ satisfies the Prikry property.

Lemma 11. *Suppose that β is less than α, and factor $\mathbb{P}_\alpha = \mathbb{P}_\beta * \mathbb{Q}_\beta * \mathbb{P}_{\beta,\alpha}$. Assume that p is in $\mathbb{P}_\beta$ and p forces that $\dot{a}$ is in $\mathbb{Q}_\beta * \mathbb{P}_{\beta,\alpha}$. Then there is q in the ground model such that $p^\frown q$ is a condition in $\mathbb{P}_\alpha$ and p forces that $q \leq^* \dot{a}$ in $\mathbb{Q}_\beta * \mathbb{P}_{\beta,\alpha}$.*

Proof. First we define the support of q, which we call x. Let γ be in x iff there is $r \leq p$ such that r forces that γ is in the support of $\dot{a}$. We prove that x is an Easton set. Let ξ be a strongly inaccessible cardinal, which we can assume to be larger than $\min(A \setminus (\beta + 1))$. Then $|\mathbb{P}_\beta| < \xi$. Now p forces that $\mathrm{supp}(\dot{a}) \cap \xi$ is bounded below ξ. Let Y be the set of δ such that there is $r \leq p$ which forces that δ is the least ordinal larger than all the elements of $\mathrm{supp}(\dot{a}) \cap \xi$. Since $\mathbb{P}_\beta$ has size less than ξ, $\sup(Y)$ is below ξ, and $x \cap \xi \subseteq \sup(Y)$.

Now define q by induction. Suppose that γ is in x and $q \restriction \gamma$ is already defined so that $p^\frown(q \restriction \gamma)$ is in $\mathbb{P}_\gamma$. Then let $q(\gamma)$ be a $\mathbb{P}_\gamma$-name which $p^\frown(q \restriction \gamma)$ forces is equal to $\dot{a}(\gamma)$, provided that γ is in $\mathrm{supp}(\dot{a})$, and otherwise is some arbitrary element of $\mathbb{Q}_\gamma$.

Suppose that G is generic for $\mathbb{P}_\beta$ and contains p. Let $a = \dot{a}^G$. Then in $V[G]$, $\mathrm{supp}(a) \subseteq x = \mathrm{supp}(q)$, and if γ is in the support of a, then $q(\gamma) = a(\gamma)$. So p forces that $q \leq^* a$ in $\mathbb{Q}_\beta * \mathbb{P}_{\beta,\alpha}$. □

Proposition 2. *The iteration $\langle \mathbb{P}_\alpha, \leq, \leq^* \rangle$ is a Prikry type forcing poset.*

Proof. We prove by induction on α that $\mathbb{P}_\alpha$ satisfies the Prikry property. Suppose that β is less than α and $\mathbb{P}_\beta$ satisfies the Prikry property. We prove the same is true for $\mathbb{P}_{\beta+1} = \mathbb{P}_\beta * \mathbb{Q}_\beta$. Let φ be a statement in the forcing language and let $p^\frown \dot{a}$ be a condition. Fix a $\mathbb{P}_\beta$-name $\dot{b}$ such that p forces $\dot{b} \leq^* \dot{a}$ and $\dot{b}$ decides φ. Now apply the induction hypothesis to obtain $q \leq^* p$ in $\mathbb{P}_\beta$ which decides which way that $\dot{b}$ decides φ. Then $q^\frown \dot{b} \leq^* p^\frown \dot{a}$ and $q^\frown \dot{b}$ decides φ.

Suppose that α is a limit ordinal and for all β less than α, $\mathbb{P}_\beta$ satisfies the Prikry property. Let p be a condition in $\mathbb{P}_\alpha$ and φ a statement in the forcing language.

First assume that the support of p is bounded below α. Then there is β less than α such that p is in $\mathbb{P}_\beta$. Write $\mathbb{P}_\alpha = \mathbb{P}_\beta * \mathbb{Q}_\beta * \mathbb{P}_{\beta,\alpha}$. Let $\dot{a}$ be a $\mathbb{P}_\beta$-name for a condition in $\mathbb{Q}_\beta * \mathbb{P}_{\beta,\alpha}$ which decides φ. Apply the induction hypothesis to obtain $p_0 \leq^* p$ which decides which way that $\dot{a}$ decides φ. By Lemma 11, choose q so that $p_0{}^\frown q$ is a condition in $\mathbb{P}_\alpha$ and p_0 forces that $q \leq^* \dot{a}$. Then $p_0{}^\frown q$ is a direct extension of p which decides φ.

Now assume that the support of p is unbounded in α. Suppose for a contradiction that p does not have a direct extension which decides φ. We construct by induction a condition $q = \langle q(\gamma) : \gamma \in \mathrm{supp}(p) \rangle$ which is a direct refinement of p and has the same support as p.

Suppose that γ is in $\mathrm{supp}(p)$ and $q \restriction \gamma$ is defined. Then $(q \restriction \gamma)^\frown p(\gamma)$ is in $\mathbb{P}_{\gamma+1}$ and forces that $p \setminus (\gamma+1)$ is in $\mathbb{P}_{\gamma,\alpha}$. Choose a $\mathbb{P}_\gamma$-name $q(\gamma)$ such that $q \restriction \gamma$ forces that $q(\gamma) \leq^* p(\gamma)$ and $q(\gamma)$ decides the following statement: there is a direct extension of $p \setminus (\gamma+1)$ in $\mathbb{P}_{\gamma,\alpha}$ which decides φ. This completes the definition of q.

Claim 1. *For all γ less than α, $q \restriction \gamma$ forces that there is no direct extension of $p \setminus \gamma$ in $\mathbb{Q}_\gamma * \mathbb{P}_{\gamma,\alpha}$ which decides φ.*

Proof. We prove the claim by induction. By assumption, there is no direct extension of p which decides φ. So the claim holds for γ less than $\min(A)$.

Suppose that $\gamma \geq \min(A)$ and the claim holds for all γ^* less than γ. First assume that $\gamma = \gamma_0 + 1$. Consider the case when γ_0 is not in $\mathrm{supp}(p)$. If the claim fails then without loss of generality there is $r^\frown \dot{c} \leq q \restriction \gamma$ and $\dot{a}$ such that $r^\frown \dot{c} \Vdash \dot{a} \leq^* p \setminus \gamma$ and $\dot{a}$ forces φ. But then since γ_0 is not in $\mathrm{supp}(p)$, r forces that $\dot{c}^\frown \dot{a}$ is a direct extension of $p \setminus \gamma_0$ which forces φ. This contradicts the induction hypothesis.

Suppose now that γ_0 is in $\mathrm{supp}(p)$. Then $q \restriction \gamma_0$ forces that $q(\gamma_0)$ decides whether there is a direct extension of $p \setminus \gamma$ which decides φ. So if the claim fails, there is $r \leq q \restriction \gamma_0$ and $\dot{b}$ such that

$$r \Vdash q(\gamma_0) \Vdash \dot{b} \leq^* p \setminus \gamma,\ \dot{b} \text{ decides } \varphi.$$

It follows that r forces that $q(\gamma_0)^\frown \dot{b}$ is a direct extension of $p \setminus \gamma_0$ which decides φ, contradicting the induction hypothesis.

Now suppose that γ is a limit ordinal. If the claim fails then without loss of generality there is $r \leq q \restriction \gamma$ and $\dot{b}$ such that r forces that $\dot{b} \leq^* p \setminus \gamma$ and $\dot{b}$ forces φ. Let a be a finite subset of γ such that for all ξ in $\mathrm{supp}(q \restriction \gamma) \setminus a$, $r \Vdash r(\xi) \leq^* q(\xi)$. Since γ is a limit ordinal, there is γ^* less than γ such that a is a subset of γ^*.

Therefore $r \restriction \gamma^*$ forces that $(r \setminus \gamma^*)^\frown \dot{b}$ is a direct extension of $p \setminus \gamma^*$ in $\mathbb{P}_{\gamma^*,\alpha}$ which forces φ. This contradicts the induction hypothesis that $q \restriction \gamma^*$ forces there is no such direct extension. □

Now choose $r \leq q$ in $\mathbb{P}_\alpha$ which decides φ. Let a be the set of ξ in supp(q) such that $r \restriction \xi$ forces that $r(\xi)$ is a non-direct extension of q. Since the support of p is unbounded in α, there is γ in supp(q) such that $a \subseteq \gamma$. Then $r \restriction \gamma$ is a condition below $q \setminus \gamma$ which forces that $r \setminus \gamma$ is a direct extension of $p \setminus \gamma$ which decides φ. This contradicts the fact that $q \restriction \gamma$ forces there is no such direct extension. □

If $\mathbb{P}$ is an α-strategically closed forcing poset, then the triple $\langle \mathbb{P}, \leq, \leq^* \rangle$, where $\leq^* = \leq$, is a Prikry type forcing poset which is α-weakly strategically closed. So we can define Easton support Prikry iterations by combining usual Prikry forcings with strategically closed posets. Gitik showed that in some cases it is possible to use distributive posets as well, by turning them into Prikry type forcing posets using a projection mapping from a strongly compact Prikry forcing; see [5] for details.

4. Adding Clubs with Square

Suppose for the remainder of this section that κ is a weakly inaccessible cardinal. We present a method for destroying certain stationary subsets of $P_\kappa \kappa^+$.

Lemma 12. *There is a club set of a in $P_\kappa \kappa^+$ such that* o.t.$(a) \leq (a \cap \kappa)^+$.

Proof. Fix $\theta \gg \kappa$ regular and let N be an elementary substructure of $\langle H(\theta), \in \rangle$ in $P_\kappa H(\theta)$. Let $a = N \cap \kappa^+$. We claim that o.t.$(a) \leq (a \cap \kappa)^+$.

We show that for all β in a, $|a \cap \beta| \leq |a \cap \kappa|$. This is clear if $\beta < \kappa$. Suppose that $\beta \geq \kappa$ is in a. Fix a bijection $f_\beta : \kappa \to \beta$ in N. By elementarity, for all j in $a \cap \beta$, $f_\beta^{-1}(j)$ is in $a \cap \kappa$. Therefore $f_\beta \restriction a \cap \kappa$ is a bijection of $a \cap \kappa$ onto $a \cap \beta$, and so $|a \cap \beta| = |a \cap \kappa|$. Since each initial segment of a has size at most $|a \cap \kappa|$, clearly o.t.$(a) \leq (a \cap \kappa)^+$. □

Define $S(\kappa, \kappa^+)$ as the set of a in $P_\kappa \kappa^+$ such that o.t.$(a) = (a \cap \kappa)^+$.

Proposition 3. *If κ is κ^+-supercompact then $S(\kappa, \kappa^+)$ is stationary.*

Proof. Fix $j : V \to M$ with crit$(j) = \kappa$, $j(\kappa) > \kappa^+$, and M is κ^+-closed. Then $j``\kappa^+$ is in $M$. Note that $j``\kappa^+$ is in $j(S(\kappa, \kappa^+))$. Suppose that C is a club subset of $P_\kappa \kappa^+$ in V. Then $j``C$ is in $M$ and is a directed subset of $j(C)$ with size less than $j(\kappa)$. So $\bigcup j``C = j``\kappa^+$ is in $j(C)$. Therefore $j(S(\kappa, \kappa^+)) \cap j(C)$ is non-empty. By elementarity, $S(\kappa, \kappa^+) \cap C$ is non-empty. □

If κ is subcompact, which is a weaker assumption than κ^+-supercompactness, then $S(\kappa, \kappa^+)$ is stationary. In [6] we prove that the strong compactness of κ does not imply that $S(\kappa, \kappa^+)$ is stationary.

Let $B \subseteq \kappa^+$ be a set of limit ordinals. We say that $\square_\kappa^B$ holds if there exists a *partial square sequence* $\langle c_\alpha : \alpha \in B \rangle$ satisfying:

(1) c_α is club in α,

(2) if cf$(\alpha) < \kappa$ then o.t.$(c_\alpha) < \kappa$,

(3) if β is in lim$(c_\alpha) \cap B$ then $c_\beta = c_\alpha \cap \beta$.

Define a forcing poset $\mathbb{P}_B$ which adds a $\square_\kappa^B$-sequence as follows. This poset is the obvious generalization of Jensen's poset for adding a $\square_\kappa$-sequence. A condition in $\mathbb{P}_B$ is a sequence $p = \langle p_\alpha : \alpha \in B \cap (\gamma + 1) \rangle$ for some $\gamma < \kappa^+$ satisfying (1), (2), and (3) above for all α in $B \cap (\gamma + 1)$. Let $q \leq p$ if p is an initial segment of q.

Proposition 4. *The forcing poset $\mathbb{P}_B$ is $\kappa+1$-strategically closed.*

Proof. We describe a strategy by considering a run of the game. Suppose $\langle p_i : 0 < i < \beta\rangle$ is the run of the game up to stage β, and it is Player II's turn. For each $0 < i < \beta$ let γ_i be the least ordinal γ such that p_i is of the form $\langle c_\alpha : \alpha \in B\cap(\gamma+1)\rangle$. First assume $\beta = \alpha + 1$. Let ξ be the least element of B larger than γ_α, and define $p_\beta = p_\alpha \cup \{\langle \xi, c_\xi\rangle\}$, where c_ξ is any club subset of ξ with order type $\mathrm{cf}(\xi)$ and $\min c_\xi > \gamma_\alpha$.

Suppose now that β is a limit ordinal. Let $\gamma_\beta = \bigcup \gamma_i$. If γ_β is not in B, then let $p_\beta = \bigcup\{p_i : i < \beta\}$. If γ_β is in B, then define p_β as $\bigcup\{p_i : i < \beta\} \cup \{\langle \gamma_\beta, c_{\gamma_\beta}\rangle\}$, where $c_{\gamma_\beta} = \{\gamma_i : i < \beta\}$. If $\beta < \kappa$ then c_{γ_β} has order type less than κ. Now suppose that γ is a limit point of c_{γ_β} in B. Then $\gamma = \gamma_i$ for some limit ordinal i and $c_\gamma = \{\gamma_j : j < i\} = c_{\gamma_\beta} \cap \gamma$. $\square$

It follows that $\mathbb{P}_B$ does not add subsets to κ. Since $\mathbb{P}_B$ has size 2^κ, if $2^\kappa = \kappa^+$ then $\mathbb{P}_B$ preserves all cardinalities and cofinalities.

We show how to use this forcing poset to destroy stationary sets.

Theorem 1. *Suppose that B is a subset of κ^+ consisting of limit ordinals and there are distinct regular cardinals δ_0 and δ_1 below κ such that B contains all its limit points with cofinality δ_0 or δ_1. Then if $\square^B_\kappa$ holds, the set*

$$S = \{a \in S(\kappa,\kappa^+) : \sup a \in B\}$$

is non-stationary.

Proof. Let $\langle c_\beta : \beta \in B\rangle$ be a partial square sequence witnessing that $\square^B_\kappa$ holds. Suppose for a contradiction that S is stationary.

Fix $\theta \gg \kappa$ regular. Then there exists N in $P_\kappa H(\theta)$ such that $N \cap \kappa^+$ is in S, $N\cap\kappa$ is larger than δ_0 and δ_1, and

$$N \prec \langle H(\theta), \in, \langle c_\beta : \beta \in B\rangle\rangle.$$

Let $a = N \cap \kappa^+$, $\beta = \sup a$, and $\kappa_a = a \cap \kappa$. Since a is in S, β is in B. So c_β is defined. Note that $\mathrm{o.t.}(a) = \kappa_a^+$ so $\mathrm{cf}(\beta) = \kappa_a^+$.

By elementarity, B is unbounded in β. Let δ be one of δ_0 or δ_1 which is different from $\mathrm{cf}(\kappa_a)$. Then B and a are both closed under suprema of subsets with order type δ. Therefore $B \cap a$ is a stationary subset of β.

Since $\mathrm{o.t.}(c_\beta) \geq \mathrm{cf}(\beta) = \kappa_a^+$, there is γ in $a \cap B \cap \lim(c_\beta)$ such that $\mathrm{o.t.}(c_\beta \cap \gamma) \geq \kappa_a$. But $c_\gamma = c_\beta \cap \gamma$, and by elementarity, $\mathrm{o.t.}(c_\gamma)$ is in $N \cap \kappa = \kappa_a$. This is a contradiction since $\mathrm{o.t.}(c_\gamma) \geq \kappa_a$. $\square$

Now let A be any subset of κ^+ consisting of limit ordinals. Let B be the closure of A under suprema of subsets with order type ω or ω_1. We claim that

$$\{a \in S(\kappa,\kappa^+) : \sup a \in A\} = \{a \in S(\kappa,\kappa^+) : \sup a \in B\}$$

modulo clubs.

Clearly if $\sup a$ is in A then $\sup a$ is in B. Suppose that a is in $S(\kappa,\kappa^+)$, $\sup a = \beta$ is in B, and $\kappa_a = a \cap \kappa$ is a limit cardinal. Then $\mathrm{cf}(\beta) = \kappa_a^+$, which is different from ω and ω_1. But B can be written as $A \cup X$ where X consists of ordinals with cofinality ω or ω_1. So β is in A.

Proposition 5. *The forcing poset $\mathbb{P}_B$ preserves cardinals and cofinalities less than or equal to κ^+, does not add subsets to κ, and destroys the stationarity of the set $S = \{a \in S(\kappa,\kappa^+) : \sup a \in A\}$.*

Proof. The poset $\mathbb{P}_B$ adds a partial square sequence $\langle c_\beta : \beta \in B\rangle$. Since $\mathbb{P}_B$ does not add subsets to κ, after forcing with $\mathbb{P}_A$ the set B is still the closure of A under suprema of subsets with order type ω or ω_1. Now apply Theorem 1. $\square$

All of the stationary subsets of $S(\kappa,\kappa^+)$ which we will consider have the form

$$\{a \in S(\kappa,\kappa^+) : \sup a \in A\}$$

for some $A \subseteq \kappa^+$. In [5] we constructed a model in which *all* stationary subsets of $S(\kappa,\kappa^+)$ have this form. Namely, fix a Jonsson function $F : \kappa^{+<\omega} \to \kappa^+$. Using the forcing poset from [5], force to make almost all a in $S(\kappa,\kappa^+)$ satisfy that F is Jonsson for a. Then the function $a \mapsto \sup a$ is injective on a club set intersected with $S(\kappa,\kappa^+)$. So if $S \subseteq S(\kappa,\kappa^+)$ is stationary, then S is equal modulo clubs to the set

$$\{a \in S(\kappa,\kappa^+) : \sup a \in A\}$$

where $A = \{\beta < \kappa^+ : \exists a \in S \ \sup a = \beta\}$. The stationarity of $S(\kappa,\kappa^+)$ can be preserved by preparing the ground model.

Lemma 13. *Suppose that S is a stationary subset of $S(\kappa,\kappa^+)$ such that for some $X \subseteq \kappa$,*

$$S = \{a \in S(\kappa,\kappa^+) : a \cap \kappa \in X\}.$$

Then there is $A \subseteq \kappa^+$ such that

$$S = \{a \in S(\kappa,\kappa^+) : \sup a \in A\}.$$

modulo clubs.

Proof. Let C be the club set of a in $P_\kappa\kappa^+$ such that $a\cap\kappa$ is a limit cardinal. Define A as the set of α less than κ^+ such that $\mathrm{cf}(\alpha) = \mu^+$ for some limit cardinal μ in X. If a is in $S\cap C$, then $\mathrm{cf}(\sup a) = \mathrm{o.t.}(a) = (a\cap\kappa)^+$. But $a\cap\kappa$ is a limit cardinal in X, so $\sup a$ is in A. On the other hand, suppose that a is in $S(\kappa,\kappa^+)\cap C$ and $\sup a$ is in A. Then $\mathrm{cf}(\sup a) = \mu^+$ for some limit cardinal μ in X. But $\mathrm{cf}(\sup a) = (a \cap \kappa)^+$, so $a \cap \kappa = \mu$, and therefore a is in S. $\square$

Note: The method presented here for destroying stationary sets using partial square sequences is simpler and easier to use than previous posets for adding clubs, as found in [2] and [5]. The Radin forcing required by these other posets is eliminated. Also the fact that the partial square poset does not add subsets to κ is very useful, as we see in the following sections.

5. The Structure of $S(\kappa,\kappa^+)$

In this section and in Section 7 we prove several consistency results which contribute to the general theme that the structure of $S(\kappa,\kappa^+)$, unlike its complement, is largely independent of ZFC. In the present section we construct a model in which $S(\kappa,\kappa^+)$ is stationary and for almost all a in $S(\kappa,\kappa^+)$, $a\cap\kappa$ is a measurable cardinal with Mitchell order $(a\cap\kappa)^{++}$.

Let V be a model of set theory in which κ is κ^{++}-supercompact and GCH holds. Let A be the set of α less than κ such that α is α^+-supercompact.

Proposition 6. *If α is in $A \cup \{\kappa\}$, then there exists a $\lhd$-increasing sequence $\langle U(\alpha,i) : i < \alpha^{++}\rangle$ of normal ultrafilters on α such that $A\cap\alpha$ is not in $U(\alpha,i)$ for each i.*

Proof. Let W be a normal ultrafilter on $P_\alpha \alpha^+$ which is minimal in the Mitchell ordering and let $j : V \to M = \mathrm{Ult}(V, W)$. Then α is not α^+-supercompact in M, so α is not in $j(A \cap \alpha)$. Let $W \restriction \alpha$ be the projection of W to α, i.e. X is in $W \restriction \alpha$ iff $X \subseteq \alpha$ and $\alpha \in j(X)$. Note that $A \cap \alpha$ is not in $W \restriction \alpha$. Let $i : V \to N = \mathrm{Ult}(V, W \restriction \alpha)$. By Lemma 3, there exists an elementary embedding $k : N \to M$ such that $j = k \circ i$ and $\mathrm{crit}(k) = \alpha^{++N}$.

If $\vec{U}$ is a $\lhd$-increasing sequence of ultrafilters on α in N with length $(\alpha^{++})^N$, then $k(\vec{U})$ is a $\lhd$-increasing sequence of ultrafilters on $k(\alpha) = \alpha$ with length $k(\alpha^{++N}) = \alpha^{++}$. So it suffices to prove there exists such a sequence in N.

It suffices to prove that in N, whenever β is less than α^{++N} and $\vec{U}$ is a $\lhd$-increasing sequence of ultrafilters on α, then there is an ultrafilter not containing A which is above the sequence. Suppose for a contradiction that this claim fails. Let $\beta < \alpha^{++N}$ be the least ordinal such that there exists a sequence $\vec{U}$ with length β, but there is no ultrafilter above it which does not contain A. Since $\mathrm{crit}(k) = (\alpha^{++})^N$, $k(\vec{U}) = \vec{U}$, and by elementarity there is no ultrafilter above $\vec{U}$ in M which does not contain A. This is a contradiction, since $W \restriction \alpha$ is in M. $\square$

Define an Easton support iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha : \alpha < \kappa \rangle$ as follows. Suppose that $\mathbb{P}_\alpha$ is defined for some α less than κ. If α is not in A then let $\mathbb{Q}_\alpha$ be trivial. Suppose that α is in A.

Let G_α be generic for $\mathbb{P}_\alpha$ over V. In $V[G_\alpha]$, define B_α as the set of β less than α^+ such that $\mathrm{cf}(\beta) \neq \mu^+$ for any μ in $A \cap \alpha$. Clearly B_α is closed under suprema of subsets with order type ω and ω_1. So the forcing poset $\mathbb{P}_{B_\alpha}$ for adding a $\square_\alpha^{B_\alpha}$-sequence forces that almost all a in $S(\alpha, \alpha^+)$ satisfy that $\mathrm{cf}(\sup a) = \mu^+$ for some μ in $A \cap \alpha$. But $\mathrm{cf}(\sup a) = (a \cap \alpha)^+$, so $\mu = a \cap \alpha$ is in A. Let $\mathbb{Q}_\alpha$ be a name for $\mathbb{P}_{B_\alpha}$.

This completes the definition of $\mathbb{P}_\kappa$. Let $B = B_\kappa$. We force with $\mathbb{P}_\kappa * \mathbb{P}_B$. By standard Easton support iteration arguments, this poset preserves all cardinals, cofinalities, and GCH. It also forces that almost all a in $S(\kappa, \kappa^+)$ satisfy that $a \cap \kappa$ is in A. So to complete the proof, we show that $S(\kappa, \kappa^+)$ remains stationary and for each α in A, the sequence of ultrafilters on α can be lifted.

Lemma 14. *For all α in A, the poset $\mathbb{P}_\kappa * \mathbb{P}_B$ forces that the sequence $\langle U(\alpha, i) : i < \alpha^{++} \rangle$ can be lifted.*

Proof. Write $\mathbb{P}_\kappa * \mathbb{P}_B = \mathbb{P}_\alpha * \mathbb{P}_{B_\alpha} * \mathbb{P}_{\mathrm{tail}} * \mathbb{P}_B$. The key point is that $\mathbb{P}_{B_\alpha}$ is forced to be $\alpha + 1$-strategically closed, and therefore $\mathbb{P}_{B_\alpha} * \mathbb{P}_{\mathrm{tail}} * \mathbb{P}_B$ does not add subsets to α. So by Corollary 1, it suffices to prove that $\mathbb{P}_\alpha$ forces that the sequence can be lifted.

If α is not a limit point of $A \cap \alpha$, then $|\mathbb{P}_\alpha| < \alpha$, so the claim holds by Lemma 10.

Suppose that α is a limit point of $A \cap \alpha$. In V fix ultrapower maps $j_i : V \to M_i = \mathrm{Ult}(V, U(\alpha, i))$ and $k_i : M_{i+1} \to N_i = \mathrm{Ult}(M_{i+1}, U(\alpha, i))$. By Lemma 9, $j_i \restriction M_{i+1} = k_i$. We define a normal ultrafilter $U(\alpha, i)^*$ extending $U(\alpha, i)$ in $M_{i+1}[G_\alpha]$.

Write $j_i(\mathbb{P}_\alpha) = \mathbb{P}_\alpha * \mathbb{P}_{\mathrm{tail}}$. We claim that $k_i(\mathbb{P}_\alpha)$ can also be factored as $\mathbb{P}_\alpha * \mathbb{P}_{\mathrm{tail}}$. For each β in $\alpha \setminus A$ let $\mathbb{P}_\alpha = \mathbb{P}_\beta * \mathbb{P}_{\beta,\alpha}$. Since M_{i+1} is α-closed, the sequence $\vec{\mathbb{P}} = \langle \mathbb{P}_{\beta,\alpha} : \beta \in \alpha \setminus A \rangle$ is in M_{i+1}. So $j_i(\vec{\mathbb{P}}) = k_i(\vec{\mathbb{P}})$. Since $\mathbb{P}_{\mathrm{tail}} = j_i(\vec{\mathbb{P}})(\alpha)$, $k_i(\mathbb{P}_\alpha) = \mathbb{P}_\alpha * \mathbb{P}_{\mathrm{tail}}$.

Similarly, for each β in $\alpha \setminus A$ let $\mathcal{A}_\beta$ be a $\mathbb{P}_\beta$-name for the set of maximal antichains in $\mathbb{P}_{\beta,\alpha}$. Since $\mathbb{P}_\alpha$ is α-c.c. and M_{i+1} is α-closed, the sequence $\vec{\mathcal{A}} = \langle \mathcal{A}_\beta : \beta \in \alpha \setminus A \rangle$

is in M_{i+1}, and $j_i(\vec{\mathcal{A}}) = k_i(\vec{\mathcal{A}})$. In particular, $\mathcal{A} = j_i(\vec{\mathcal{A}})(\alpha)$ is a $\mathbb{P}_\alpha$-name for the collection of maximal antichains in $\mathbb{P}_{\text{tail}}$.

In $M_{i+1}[G_\alpha]$ the size of $k_i(\alpha)$ is α^+, so we can enumerate $\mathcal{A}$ as $\langle A_i : i < \alpha^+\rangle$. In $V[G_\alpha]$ this sequence also enumerates all of the maximal antichains of $\mathbb{P}_{\text{tail}}$ in $M_i[G_\alpha]$. Since $N_i[G_\alpha]$ is α-closed, $\mathbb{P}_{\text{tail}}$ is α^+-strategically closed in $M_{i+1}[G_\alpha]$. So we can construct a generic H_i for $\mathbb{P}_{\text{tail}}$ over $N_i[G_\alpha]$. Note that H_i is also a generic for $\mathbb{P}_{\text{tail}}$ over $M_i[G_\alpha]$.

Now lift j_i and k_i to $j_i : V[G_\alpha] \to M_i[G_\alpha * H_i]$ and $k_i : M_{i+1}[G_\alpha] \to N_i[G_\alpha * H_i]$. Note that $j_i \restriction M_{i+1}[G_\alpha] = k_i$. Define $U(\alpha,i)^*$ by letting $X \in U(\alpha,i)^*$ iff $X \subseteq \alpha$ and $\alpha \in j_i(X)$. Since $\mathcal{P}(\alpha) \subseteq M_{i+1}[G_\alpha]$ and $k_i = j_i \restriction M_{i+1}[G_\alpha]$, the ultrafilter $U(\alpha,i)^*$ can be correctly computed in $M_{i+1}[G_\alpha]$.

This defines $U(\alpha,i)^*$ for $i < \alpha^{++}$. Suppose that $i < j$. Then $U(\alpha,i)^*$ is in $M_{i+1}[G_\alpha] \subseteq M_j[G_\alpha] \subseteq M_j[G_\alpha * H_j] = \text{Ult}(V[G], U(\alpha,j)^*)$. So $U(\alpha,i)^* \lhd U(\alpha,j)^*$. □

To prove that $\mathbb{P}_\kappa * \mathbb{P}_B$ preserves the stationarity of $S(\kappa,\kappa^+)$, it suffices to prove that κ remains κ^{++}-supercompact. Let U be a normal ultrafilter on $P_\kappa\kappa^{++}$ and let $j : V \to M = \text{Ult}(V,U)$. Note that κ is in $j(A)$. Let $G_\kappa * G_B$ be generic for $\mathbb{P}_\kappa * \mathbb{P}_B$ over V. Write $j(\mathbb{P}_\kappa * \mathbb{P}_B) = \mathbb{P}_\kappa * \mathbb{P}_B * \mathbb{P}_{\text{tail}} * j(\mathbb{P}_B)$. In $M[G_\kappa * G_B]$, $\mathbb{P}_{\text{tail}}$ is κ^{+3}-strategically closed. By Lemma 6, $M[G_\kappa * G_B]$ is κ^{++}-closed in $V[G_\kappa * G_B]$. Therefore $\mathbb{P}_{\text{tail}}$ is κ^{+3}-strategically closed in $V[G_\kappa * G_B]$. In $M[G_\kappa * G_B]$, $\mathbb{P}_{\text{tail}}$ has $j(\kappa)$-many maximal antichains and $|j(\kappa)| = \kappa^{+3}$. So we can construct a generic filter G_{tail} for $\mathbb{P}_{\text{tail}}$ over $M[G_\kappa * G_B]$. Extend j to $j : V[G_\kappa] \to M[G_\kappa * G_B * G_{\text{tail}}]$.

We construct a master condition. For each p in G_B there is $\beta < \kappa^+$ such that $p = \langle c_\alpha : \alpha \in B \cap (\beta+1)\rangle$. So $j(p) = \langle d_\alpha : \alpha \in j(B) \cap j(\beta+1)\rangle$. Let $s = \bigcup j``G_B$. Then $s = \langle d_\alpha : \alpha \in j(B) \cap \sup j``\kappa^+\rangle$. The sequence s is a condition in $j(\mathbb{P}_B)$ iff the domain of s is equal to $j(B) \cap (\sup j``\kappa^+ + 1)$, that is, iff $\sup j``\kappa^+$ is not in $j(B)$. But $\sup j``\kappa^+$ has cofinality equal to the successor of $\kappa$, which is a member of $j(A)$. So by the definition of $B$, $\sup j``\kappa^+$ is not in $j(B)$.

The poset $j(\mathbb{P}_B)$ has $j(\kappa^{++})$ many maximal antichains. But $j(\kappa^{++})$ has size κ^{+3}. In $M[G_\kappa * G_B * G_{\text{tail}}]$, $j(\mathbb{P}_B)$ is $j(\kappa)$-strategically closed. By Lemma 7, $M[G_\kappa * G_{\mathbb{Q}} * G_{\text{tail}}]$ is κ^{++}-closed and $|j(\kappa)| = \kappa^{+3}$. So in $V[G_\kappa * G]$, $j(\mathbb{P}_B)$ is κ^{+3}-strategically closed. So we can construct a generic H for $j(\mathbb{P}_B)$ which contains the master condition s. Now extend j to $j : V[G_\kappa * G] \to M[G_\kappa * G * G_{\text{tail}} * H]$.

Define U^* by letting $X \in U^*$ iff $X \subseteq P_\kappa\kappa^{++}$ and $j``\kappa^{++} \in j(X)$. Standard arguments show that U^* is a normal ultrafilter on $P_\kappa\kappa^{++}$ in $V[G_\kappa * G_B]$.

6. Changing Cofinalities

In this section we present Gitik's poset for changing a cofinality without adding bounded sets which we will use in the consistency results of the following sections. If V is the core model and $\mathbb{P}$ is a forcing poset which changes the cofinality of a regular cardinal κ to an uncountable value without collapsing cardinals, then $\mathbb{P}$ adds bounded subsets to κ. Therefore we are required to prepare the ground model before defining the poset.

Our exposition differs from that of [3].

Let V be a model of set theory in which κ is a strongly inaccessible cardinal. Assume that there is a function

$$o : A \cup \{\kappa\} \to \kappa,$$

where $A \subseteq \kappa$ is an unbounded set of strongly inaccessible cardinals and $o(\alpha) < \alpha$ for all α in $A \cup \{\kappa\}$. The function o will be a relativized version of the Mitchell order function in a particular inner model.

Assume that for each α in A, there is a fixed set

$$b_\alpha = b^*_\alpha \cup \{\alpha\},$$

which satisfies the following properties:

(1) If $o(\alpha) = 0$ then $b^*_\alpha = \emptyset$ and $b_\alpha = \{\alpha\}$,

(2) If $o(\alpha) > 0$ then b^*_α is closed and unbounded in α, and the order type of b^*_α is equal to the ordinal exponent $\omega^{o(\alpha)}$,

(3) If γ is in b^*_α, then $o(\gamma) < o(\alpha)$,

(4) If γ is in b^*_α and for all β in $b_\alpha \cap \gamma$, $o(\beta) < o(\gamma)$, then $b^*_\gamma = b_\alpha \cap \gamma$,

(5) If γ is in b^*_α and (4) fails for γ, then there is a maximum ordinal γ' in $b_\alpha \cap \gamma$ such that $o(\gamma') \geq o(\gamma)$, and $b^*_\gamma = b_\alpha \cap (\gamma', \gamma)$.

Note that by (5), if γ is in b^*_α and $o(\gamma) = 0$, then γ is not a limit point of b_α.

For each i less than $o(\kappa)$ let A_i denote the set of α in A with $o(\alpha) = i$.

We will define a forcing poset $\mathbb{P}(\kappa, o(\kappa))$ which adds a generic set $b_\kappa = b^*_\kappa \cup \{\kappa\}$ which itself satisfies conditions (1) through (5). We will need to state a few additional properties of the ground model, but first we provide some analysis of the sets b_α.

Lemma 15. *Suppose that α is in A and $o(\alpha) = \gamma + 1$ for some $\gamma \geq 1$. Enumerate b_α in increasing order as $\langle \beta_i : 1 \leq i < \omega^\gamma \cdot \omega \rangle$. Then for any $i < \omega^\gamma \cdot \omega$, $o(\beta_i) = \gamma$ iff there is $1 \leq n < \omega$ such that $i = \omega^\gamma \cdot n$. If $o(\beta_i) = \gamma$, then $b^*_{\beta_i} = \{\beta_j : \omega^\gamma \cdot (n-1) < j < i\}$.*

Proof. By (3), the maximum possible value of $o(\beta_i)$ is γ, and therefore the largest possible order type of $b^*_{\beta_i}$ is ω^γ. By (4) and (5) we can write $b^*_{\beta_i} = \{\beta_j : k \leq j < i\}$ for some $k < i$. Suppose that $i = \omega^\gamma \cdot n$ for some $n \geq 1$. Then any co-bounded subset of $\{\beta_j : 1 \leq j < i\}$ has order type at least ω^γ. Therefore o.t.$(b^*_{\beta_i}) = \omega^\gamma$ and $o(\beta_i) = \gamma$. Conversely, suppose that $\omega^\gamma \cdot (n-1) < i < \omega^\gamma \cdot n$ for some $n \geq 1$. Since $o(\beta_{\omega^\gamma \cdot (n-1)})$ is either undefined (if $n = 1$) or else is equal to γ by what we just proved, (5) implies that $b^*_{\beta_i}$ is contained in the interval $\{\beta_j : \omega^\gamma \cdot (n-1) < j < i\}$, which has order type less than ω^γ. So $o(\beta_i) < \gamma$.

The second statement follows immediately from (4) and (5). □

Lemma 16. *Suppose that α is in A and $o(\alpha)$ is a limit ordinal. Enumerate b_α in increasing order as $\langle \beta_i : 1 \leq i < \omega^{o(\alpha)} \rangle$. Fix $1 \leq i < \omega^{o(\alpha)}$ and suppose that $i = \omega^\gamma$ for some $1 \leq \gamma < o(\alpha)$. Then $o(\beta_i) = \gamma$, and $b_{\beta_i} = \{\beta_j : 1 \leq j \leq i\}$.*

Proof. Since b_α is closed and i is a limit ordinal, β_i is a limit point of b_α and therefore $o(\beta_i) > 0$. By (4) and (5) there exists k such that $b^*_{\beta_i} = \{\beta_j : k \leq j \leq i\}$. But for any choice of k, this set has order type ω^γ. So o.t.$(b^*_{\beta_i}) = \omega^\gamma$ and therefore $o(\beta_i) = \gamma$. If $k > 1$, then by (4) and (5) there is $l < i$ such that $o(\beta_l) \geq \gamma$. This is impossible since any bounded subset of $\{\beta_j : 1 \leq j < \omega^\gamma\}$ has order type less than ω^γ. □

Fix an ordinal $\beta \leq o(\kappa)$. A sequence $\langle \alpha_0, \ldots, \alpha_n \rangle$ is said to be *β-coherent for κ* if it is a finite increasing sequence from A such that $o(\alpha_i) < \beta$ for each i, and it satisfies the following property: For each $1 \leq m \leq n$, let m' be the least index so that for all $m' \leq k < m$, $o(\alpha_k) < o(\alpha_m)$. Then $\bigcup \{b_{\alpha_k} : m' \leq k < m\}$ is an initial segment of b_{α_m}. If m' does not exist, or in other words, $o(\alpha_{m-1}) \geq o(\alpha_m)$, then the minimal element of b_{α_m} is greater than α_{m-1}.

Suppose that $\gamma < \beta \leq o(\kappa)$ and $t = \langle \alpha_0, \ldots, \alpha_n \rangle$ is β-coherent. Define $t(\gamma) = \langle \alpha_i, \ldots, \alpha_n \rangle$, where i is the minimal index such that for each $i \leq k \leq n$, $o(\alpha_k) < \gamma$. If there is no such i, i.e. $o(\alpha_n) \geq \gamma$, then let $t(\gamma)$ be the empty sequence. Note that $t(\gamma)$ is γ-coherent.

Lemma 17. *Suppose that $t = \langle \alpha_0, \ldots, \alpha_n \rangle$ is β-coherent and $i \leq n$. If $j < i$ and $o(\alpha_j) \geq o(\alpha_i)$, then $\min b_{\alpha_i} > \alpha_j$.*

Proof. We prove the statement by induction on i. It is vacuously true for $i = 0$. Suppose that i is greater than 0 and there exists $j < i$ such that $o(\alpha_j) \geq o(\alpha_i)$. Let j be the maximal such index. If $j = i-1$, then $\min b_{\alpha_i} > \alpha_j$ by the definition of β-coherence. Otherwise let $i' = j+1$. Then for all $i' \leq k < i$, $o(\alpha_k) < o(\alpha_i) \leq o(\alpha_j)$. So $\bigcup\{b_{\alpha_k} : i' \leq k < i\}$ is an initial segment of b_{α_i}. By induction, $\min b_{\alpha_k} > \alpha_j$ for all such k, so $\min b_{\alpha_i} > \alpha_j$. $\square$

If t is an $o(\kappa)$-coherent sequence, let b_t denote $\bigcup\{b_\alpha : \alpha \in t\}$.

Lemma 18. *Suppose that $t = \langle \alpha_0, \ldots, \alpha_n \rangle$ is β-coherent and α is an ordinal in $b_t \setminus t$. Let i_0 be the minimal index such that α is less than α_{i_0}, and let i_1 be the minimal index such that α is in $b_{\alpha_{i_1}}$. Then $i_0 = i_1$.*

In particular, for each i, $\bigcup\{b_{\alpha_k} : k \leq i\}$ is an initial segment of b_t.

Proof. Since α is in $b_{\alpha_{i_1}} \setminus t$, α is less than α_{i_1}. So $\alpha_{i_0} \leq \alpha_{i_1}$. Suppose for a contradiction that $\alpha_{i_0} < \alpha_{i_1}$. Let i_1' be the least index such that for all $i_1' \leq k < i_1$, $o(\alpha_k) < o(\alpha_{i_1})$. Since $\alpha < \alpha_{i_0} < \alpha_{i_1}$ and α is in $b_{\alpha_{i_1}}$, by Lemma 17 i_1' exists and $i_1' \leq i_0$. Since $\alpha < \alpha_{i_0}$ and $\bigcup\{b_{\alpha_k} : i_1' \leq k < i_1\}$ is an initial segment of b_{α_1}, there exists $k < i_1$ such that α is in b_{α_k}. This contradicts the minimality of i_1. $\square$

Lemma 19. *Suppose that t is β-coherent and γ is in t. If $o(\xi) < o(\gamma)$ for all ξ in $b_t \cap \gamma$, then $b^*_\gamma = b_t \cap \gamma$. Otherwise there is a maximal γ' in $b_t \cap \gamma$ such that $o(\gamma') \geq o(\gamma)$. In this case, $b^*_\gamma = b_t \cap (\gamma', \gamma)$.*

Proof. Suppose that $o(\xi) < o(\gamma)$ for all ξ in $b_t \cap \gamma$. In particular, $o(\xi) < o(\gamma)$ for all ξ in $t \cap \gamma$. So $\bigcup\{b_\xi : \xi \in t \cap \gamma\}$ is an initial segment of b_γ. This fact along with Lemma 18 imply that $b_t \cap \gamma = b^*_\gamma$. Otherwise let γ' be the maximal ordinal in $t \cap \gamma$ such that $o(\gamma') \geq o(\gamma)$. Then $\bigcup\{b_\xi : \xi \in t \cap (\gamma', \gamma)\}$ is an initial segment of b_γ. It follows from Lemmas 17 and 18 that $b_t \cap (\gamma', \gamma) = b^*_\gamma$. $\square$

Now we complete the description of the ground model. We assume that there is a family

$$\{\ U(\kappa, \beta, t) : \beta < o(\kappa),\ t \textit{ is } \beta\textit{-coherent}\ \}$$

of κ-complete ultrafilters on κ satisfying the following conditions:

(1) $A_\beta = \{\alpha < \kappa : o(\alpha) = \beta\}$ is in $U(\kappa, \beta, t)$,

(2) If s and t are β-coherent sequences and $b_s = b_t$, then $U(\kappa, \beta, s) = U(\kappa, \beta, t)$,

(3) If t is β-coherent and γ is less than β, then the set of α less than κ such that $t^\frown\alpha$ is β-coherent is in the ultrafilter $U(\kappa, \gamma, t(\gamma))$.

Fix $\beta \leq o(\kappa)$. We describe a forcing poset $\mathbb{P}(\kappa, \beta)$.

A set T is said to be a *β-tree* if T is a collection of β-coherent sequences satisfying:

(1) $\langle T, \trianglelefteq \rangle$ is a tree ordering, where $u \trianglelefteq v$ iff u is an initial segment of v. In other words, if v is in T and $u \trianglelefteq v$, then u is in T,

(2) The tree T has a *trunk* t such that t is in T and for all u in T, either $u \trianglelefteq v$ or $v \trianglelefteq u$,

(3) For any u in T, if $t \trianglelefteq u$ where t is the trunk of T, then the set of α less than κ such that $u^\frown\alpha$ is in T can be partitioned into a disjoint family

$$\{ \operatorname{Suc}(T, u, \gamma) : \gamma < \beta \},$$

where $\operatorname{Suc}(T, u, \gamma)$ is a subset of A_γ and is in $U(\kappa, \gamma, u(\gamma))$.

If T is a β-tree with trunk t, u is in T, and $t \trianglelefteq u$, then let T_u denote the tree consisting of those v in T such that either $u \trianglelefteq v$ or $v \trianglelefteq u$. Note that T_u is a β-tree with trunk u.

If $\beta = 0$ then let $\mathbb{P}(\kappa, \beta)$ be the trivial poset. Define $b^*_\kappa = \emptyset$ and $b_\kappa = \{\kappa\}$. Suppose that β is greater than 0.

A condition in $\mathbb{P}(\kappa, \beta)$ is a pair $\langle t, T\rangle$ such that T is a β-tree with trunk t. If $\langle t, T\rangle$ and $\langle s, S\rangle$ are conditions, we let $\langle t, T\rangle \leq \langle s, S\rangle$ if:

(1) there exists t^* in S such that $b_{t^*} = b_t$,

(2) for each u in T, the sequence $t^{*\frown}(u \setminus (\max t^* + 1))$ is in S.

If $\langle t, T\rangle \leq \langle s, S\rangle$ and $s = t$, then let $\langle t, T\rangle \leq^* \langle s, S\rangle$.

The assumptions about the family of ultrafilters imply that β-trees exist, and therefore $\mathbb{P}(\kappa, \beta)$ is a non-trivial forcing poset.

The proof that $\mathbb{P}(\kappa, o(\kappa))$ satisfies the Prikry property will depend on certain details about the preparation forcing; see Proposition 10. For the rest of this section we will just assume that $\langle \mathbb{P}(\kappa, o(\kappa)), \leq, \leq^*\rangle$ is a Prikry type forcing poset.

If T_0 and T_1 are β-trees with the same trunk t, then $T_0 \cap T_1$ is also a β-tree with trunk t. In fact, suppose that $\{\langle t, T_i\rangle : i < \xi\}$ is a family of conditions in $\mathbb{P}(\kappa, \beta)$, where ξ is less than κ. By the κ-completeness of the ultrafilters, $\bigcap T_i$ is a β-tree with trunk t. So $\langle t, \bigcap T_i\rangle$ is a condition which directly extends each $\langle t, T_i\rangle$. It follows that $\mathbb{P}(\kappa, o(\kappa))$ is κ-weakly closed and satisfies the direct extension property. Therefore $\mathbb{P}(\kappa, o(\kappa))$ does not add bounded subsets to κ.

Proposition 7. *The poset $\mathbb{P}(\kappa, \beta)$ is κ^+-c.c.*

Proof. Any two conditions with the same trunk are compatible. There are only κ many possibilities for the trunk. $\square$

So $\mathbb{P}(\kappa, o(\kappa))$ preserves all cardinals.

Lemma 20. *Suppose that $\langle t, T\rangle \leq \langle s, S\rangle$ in $\mathbb{P}(\kappa, \beta)$. Then there is t^* in S and $T^* \subseteq S$ such that $\langle t, T\rangle$ is equivalent to $\langle t^*, T^*\rangle$.*

Proof. Fix t^* in S witnessing that $\langle t, T\rangle \leq \langle s, S\rangle$. Since $b_t = b_{t^*}$, t and t^* have the same maximal element α. Define

$$T^* = \{t^{*\frown}(u \setminus \alpha + 1) : u \in T\}.$$

Note that $T^* \subseteq S$. The relation $\langle t, T\rangle \leq \langle t^*, T^*\rangle$ is witnessed by t^*, and $\langle t^*, T^*\rangle \leq \langle t, T\rangle$ is witnessed by t. $\square$

Lemma 21. *Suppose that $\langle s, S\rangle$ is a condition in $\mathbb{P}(\kappa, \beta)$ and α is in b_s. Then there is a condition $\langle t, T\rangle$ such that $b_t = b_s$, $\langle t, T\rangle$ is equivalent to $\langle s, S\rangle$, and α is in t.*

Proof. Let $\langle s, S\rangle$ be a condition and suppose that α is in b_s. Write $s = \langle \alpha_0, \ldots, \alpha_n\rangle$. If α is in s then we are done. Suppose that α is in $b_s \setminus s$. Fix $i \leq n$ so that α_i is the least ordinal in s larger than α. By Lemma 18, α_i is also the least ordinal in s for which α is in b_{α_i}. We will prove the existence of $\langle t, T\rangle$ by induction on $o(\alpha_i)$.

Since α is in $b^*_{\alpha_i}$, $o(\alpha_i) > 0$. Suppose that $o(\alpha_i) = 1$, so that $b^*_{\alpha_i}$ has order type ω. Let $\beta_0, \ldots, \beta_m$ enumerate in increasing order the ordinals in $b_{\alpha_i} \cap \alpha$, and which are larger than α_{i-1} if $i > 0$. Define

$$t = \langle \alpha_0, \ldots, \alpha_{i-1}, \beta_0, \ldots, \beta_m, \alpha, \alpha_i, \ldots, \alpha_n\rangle.$$

Then t is β-coherent and $b_t = b_s$. Define

$$T = \{t^\frown(u \setminus (\alpha_n + 1)) : u \in S\}.$$

Then $\langle t, T\rangle$ is a condition, $\langle t, T\rangle \leq \langle s, S\rangle$ is witnessed by s, and $\langle s, S\rangle \leq \langle t, T\rangle$ is witnessed by t.

Suppose that $o(\alpha_i) = \gamma$, where $\gamma > 1$, and the claim holds whenever $o(\alpha_i) < \gamma$. Enumerate b_{α_i} in increasing order as $\langle \beta_i : 1 \leq i < \omega^\gamma\rangle$.

Suppose that γ is a successor ordinal, and let $\gamma = \gamma_0 + 1$. Then b_{α_i} has order type $\omega^\gamma = \omega^{\gamma_0} \cdot \omega$. Fix $1 \leq n < \omega$ minimal so that α is in $b_{\alpha_i} \cap (\beta_{\omega^{\gamma_0}\cdot n} + 1)$. Let $\xi_0, \ldots, \xi_m$ enumerate the ordinals in $b_{\alpha_i} \setminus (\alpha_{i-1} + 1)$ of the form $\beta_{\omega^{\gamma_0}\cdot k}$, where $1 \leq k \leq n$. By Lemma 15, these are exactly the ordinals in $b_{\alpha_i} \setminus (\alpha_{i-1} + 1)$ which have order γ_0. Define

$$s^* = \langle \alpha_0, \ldots, \alpha_{i-1}, \xi_0, \ldots, \xi_m, \alpha_i, \ldots, \alpha_n\rangle,$$

which is β-coherent. Also $b_s = b_{s^*}$. Define

$$S^* = \{s^{*\frown}(u \setminus (\alpha_n + 1)) : u \in S\}.$$

Then $\langle s^*, S^*\rangle$ is equivalent to $\langle s, S\rangle$. If α is in s^* we are done. Otherwise, the least ordinal in s^* which is larger than α is now ξ_n, and $o(\xi_n) = \gamma_0$. By induction, there is $\langle t, T\rangle$ equivalent to $\langle s^*, S^*\rangle$ such that α is in t and $b_t = b_{s^*}$.

Now suppose that γ is a limit ordinal. Then $\omega^\gamma = \sup\{\omega^j : j < \gamma\}$. Fix $j < \gamma$ minimal so that α is in $b_{\alpha_i} \cap (\beta_{\omega^j} + 1)$. By Lemma 16, $o(\beta_{\omega^j}) = j$, and $b_{\beta_{\omega^j}} = \{\beta_l : 1 \leq l \leq \omega^i\}$. Define

$$s^* = \langle \alpha_0, \ldots, \alpha_{i-1}, \beta_{\omega^j}, \alpha_i, \ldots, \alpha_n\rangle.$$

Then s^* is β-coherent and $b_{s^*} = b_s$. Define S^* as in the last case. Then $\langle s^*, S^*\rangle$ is equivalent to $\langle s, S\rangle$. If α is in s^* we are done. Otherwise, the least ordinal above α in s^* is β_{ω^j}, which has order less than γ. By induction, there is a condition $\langle t, T\rangle$ which is equivalent to $\langle s^*, S^*\rangle$ such that α is in t and $b_t = b_{s^*}$. $\square$

Lemma 22. *Suppose that $\langle s, S\rangle$ and $\langle t, T\rangle$ are conditions in $\mathbb{P}(\kappa, \beta)$ such that $b_s = b_t$. Then $\langle s, S\rangle$ and $\langle t, T\rangle$ are compatible.*

Proof. Note that s and t have the same maximal element α. Define

$$S_0 = \{v \in S : \text{if } s \trianglelefteq v \text{ then } t^\frown(v \setminus (\alpha + 1)) \in T\}.$$

Then $\langle s, S_0\rangle$ is a condition below $\langle s, S\rangle$. Apply Lemma 21 finitely many times to obtain a condition $\langle u, U\rangle$ which is equivalent to $\langle s, S_0\rangle$ such that $s \cup t \subseteq u$ and $b_u = b_s$. By definition of S_0, $\langle u, U\rangle$ is also below $\langle t, T\rangle$, as witnessed by t. $\square$

Now we analyze the generic object for $\mathbb{P}(\kappa, o(\kappa))$. Let $\dot{b}_\kappa$ and $\dot{b}^*_\kappa$ be $\mathbb{P}(\kappa, o(\kappa))$-names such that

$$\Vdash \dot{b}^*_\kappa = \{\alpha \in \kappa : \exists \langle t, T\rangle \in \dot{G} \ \ \alpha \in t\}.$$

and

$$\Vdash \dot{b}_\kappa = \dot{b}^*_\kappa \cup \{\kappa\},$$

Lemma 23. *The poset $\mathbb{P}(\kappa, o(\kappa))$ forces that b^*_κ is equal to $\bigcup\{b_t : \exists T\ \langle t, T\rangle \in \dot{G}\}$.*

Proof. This is immediate from Lemma 21. □

Lemma 24. *Let $\langle s, S\rangle$ be a condition in $\mathbb{P}(\kappa, o(\kappa))$. Then $\langle s, S\rangle$ forces that b_s is an initial segment of $\dot{b}_\kappa$.*

Proof. By Lemma 23, $\langle s, S\rangle$ forces that b_s is a subset of $\dot{b}_\kappa$. Suppose that $\langle t, T\rangle$ is below $\langle s, S\rangle$ and α is in $t \cap \max s$. Fix t^* in S such that $b_{t^*} = b_t$. Then α is in b_{t^*}. Since $s \trianglelefteq t^*$ and α is less than $\max s$, by Lemma 18 it follows that α is in b_s. Therefore $\langle s, S\rangle$ forces that b_s is an initial segment of $\dot{b}_\kappa$.

□

We show that the generic set b_κ satisfies the properties (1) through (5) which we stated for the b_α's.

Lemma 25. *The poset $\mathbb{P}(\kappa, o(\kappa))$ forces that $\dot{b}^*_\kappa$ is closed and unbounded in κ.*

Proof. By an easy density argument, $\dot{b}^*_\kappa$ is unbounded in κ. To show it is club, suppose that $\langle s, S\rangle$ forces that β is a limit point of $\dot{b}^*_\kappa$. Let $\langle t, T\rangle \leq \langle s, S\rangle$ such that $\max t > \beta$. Then $\langle t, T\rangle$ forces that $\dot{b}_\kappa \cap (\max t + 1) = b_t$, which is closed. So β is in $b_t \subseteq \dot{b}_\kappa$. □

Let G be generic for $\mathbb{P}(\kappa, o(\kappa))$. If α is in b^*_κ, then α is in t for some $\langle t, T\rangle$ in G. But t is $o(\kappa)$-coherent, so $o(\alpha) < o(\kappa)$.

Lemma 26. *Let γ be in b^*_κ. If $o(\beta) < o(\gamma)$ for all β in $b_\kappa \cap \gamma$, then $b^*_\gamma = b_\alpha \cap \gamma$.*

Otherwise, there is a maximal ordinal γ' in b_α below γ such that $o(\gamma') \geq o(\gamma)$. In this case, $b^*_\gamma = b_\alpha \cap (\gamma', \gamma)$.

Proof. If γ is in b^*_κ, then there is a condition $\langle s, S\rangle$ in G such that γ is in s. Since b_s is an initial segment of b_κ, the statement follows from Lemma 19. □

Proposition 8. *The order type of b^*_κ is $\omega^{o(\kappa)}$. In particular, if $o(\kappa)$ is a regular uncountable cardinal, then* o.t.$(b^*_\kappa) = o(\kappa)$.

Proof. Suppose that $o(\kappa) = 1$. Then an $o(\kappa)$-coherent sequence is just a finite increasing sequence from the set $\{\alpha \in A : o(\alpha) = 0\}$, and for any such sequence t, $b_t = t$. It follows that $\mathbb{P}(\kappa, o(\kappa))$ forces that every initial segment of b_κ is finite. So b^*_κ has order type ω.

Suppose that $o(\kappa) = \gamma + 1$. Then the maximum possible order an ordinal in an $o(\kappa)$-coherent sequence is γ. For any finite n, there is a dense set of conditions $\langle t, T\rangle$ such that t has the form $t = \langle \alpha_0, \ldots, \alpha_m, \beta_0, \ldots, \beta_{n-1}\rangle$, where for each $0 \leq i < n$, $o(\beta_i) = \gamma$. Then b_t has order type at least $\omega^\gamma \cdot n$. Therefore b^*_κ is forced to have order type at least $\omega^\gamma \cdot \omega = \omega^{\gamma+1}$. Suppose for a contradiction that there is a condition $\langle s, S\rangle$ in G which forces that the order type of $\dot{b}^*_\kappa$ is larger than $\omega^{\gamma+1}$. Then there is a condition $\langle t, T\rangle$ in G and an ordinal β less than κ such that $\max t$ is larger than β and $\langle t, T\rangle$ forces that the order type of $b_\kappa \cap \beta$ is at least $\omega^{\gamma+1}$. So the order type of b_t is at least $\omega^{\gamma+1}$. But b_t is a finite union of sets with order type at most ω^γ, so we have a contradiction.

Finally, suppose that $o(\kappa)$ is a limit ordinal. Then for each $i < o(\kappa)$, there is a dense set of conditions $\langle t, T\rangle$ such that there is α in t with $o(\alpha) = i$. For any such condition $\langle t, T\rangle$, b_t has order type at least ω^i. It follows that b^*_κ has order type at least $\omega^{o(\kappa)}$. By the same argument as in the last case, b^*_κ cannot have an order type larger than $\omega^{o(\kappa)}$. $\square$

7. The Structure of $S(\kappa, \kappa^+)$, Continued

Let V be a model of set theory in which κ is a κ^+-supercompact cardinal and GCH holds. Let μ be a regular cardinal less than κ. We construct a generic extension of V in which κ is strongly inaccessible, $S(\kappa, \kappa^+)$ is stationary, and for almost all a in $S(\kappa, \kappa^+)$, $a \cap \kappa$ is a singular cardinal with cofinality μ.

For the remainder of the section fix a normal ultrafilter U on $P_\kappa\kappa^+$ and let $j_U : V \to M = \mathrm{Ult}(V, U)$. Let $U \restriction \kappa$ be the projection of U to κ; i.e., $X \in U \restriction \kappa$ iff $X \subseteq \kappa$ and $\kappa \in j_U(X)$. Let $i : V \to N = \mathrm{Ult}(V, U \restriction \kappa)$. Applying Lemma 3, fix an elementary embedding $k : N \to M$ such that $\mathrm{crit}(k) = \kappa^{++N}$ and $j_U = k \circ i$.

Proposition 9. *There exists a $\lhd$-increasing sequence $\langle U(\kappa, i) : i \le \mu\rangle$ of normal ultrafilters on κ such that $U(\kappa, \mu) = U \restriction \kappa$.*

Proof. It suffices to prove that there exists a $\lhd$-increasing sequence of normal ultrafilters on κ with length μ in the model $N = \mathrm{Ult}(V, U \restriction \kappa)$. To prove this claim it is enough to show that any $\lhd$-increasing sequence of ultrafilters in N with length less than μ has an upper bound in N.

If the claim fails then let α be the least ordinal less than μ such that in N, there exists a $\lhd$-increasing sequence $\vec{U}$ with length α with no upper bound. By elementarity, the same statement holds for $k(\vec{U}) = \vec{U}$ in M. This is a contradiction, since $U \restriction \kappa$ is above every ultrafilter in $\vec{U}$. $\square$

Lemma 27. *There is a sequence $\langle X_i : i \le \mu\rangle$ such that each X_i is in $U(\kappa, i)$ and each $X_i \cap X_j$ is empty.*

Proof. For distinct i and j let $X_i(j)$ be a set in $U(\kappa, i) \setminus U(\kappa, j)$. Let $X^*_i = \bigcap\{X_i(j) : j \le \mu, j \ne i\}$. Then for all $j \ne i$, $X^*_i \in U(\kappa, i) \setminus U(\kappa, j)$. Now let $X_i = X^*_i \setminus \bigcup\{X^*_j : j \ne i\}$. $\square$

For each $i \le \mu$ let $j^\kappa_i : V \to N^\kappa_i = \mathrm{Ult}(V, U(\kappa, i))$. For $i < j \le \mu$ let $f^j_i : \kappa \to V_\kappa$ be a function such that $[f^j_i] = U(\kappa, i)$ in N^κ_j. Define a sequence $\langle A_j : j \le \mu\rangle$ by induction, using the following recursion: A_j is the set of α less than κ such that

(1) α is in X_j,
(2) $\alpha > \mu$,
(3) For $i < j$, $f^j_i(\alpha)$ is a normal ultrafilter on α and $A_i \cap \alpha$ is in $f^j_i(\alpha)$,
(4) For $i_0 < i_1 < j$, $f^j_{i_0}(\alpha) \lhd f^j_{i_1}(\alpha)$,
(5) For $i_0 < i_1 < j$, $[f^{i_1}_{i_0} \restriction \alpha]$ in $\mathrm{Ult}(V, f^j_{i_1}(\alpha))$ is equal to $f^j_{i_0}(\alpha)$.

The reader can check by induction that A_j is in $U(\kappa, j)$, using Lemma 9 (1) to prove (5). In particular, κ is in $j_U(A_\mu)$.

Let A be the disjoint union $\bigcup\{A_j : j \le \mu\}$. If α is in A, let $o(\alpha)$ denote the unique β such that α is in A_β. The function o is a relativized version of the Mitchell order function. If α is in A, then for each $i < o(\alpha)$ let $U(\alpha, i) = f^{o(\alpha)}_i(\alpha)$ and let $j^\alpha_i : V \to N^\alpha_i = \mathrm{Ult}(V, U(\alpha, i))$. By (3), for $i < o(\alpha)$, $A_i \cap \alpha$ is in $f^{o(\alpha)}_i(\alpha) = U(\alpha, i)$, and therefore α is in $j_i^\alpha(A_i)$. It follows that $j_i^\alpha(o)(\alpha) = i$ in N_i^α. By (5), for each $i_0 < i_1 < o(\alpha)$, $U(\alpha, i_0)$ is in $N_{i_1}^\alpha$, so $U(\alpha, i_0) \lhd U(\alpha, i_1)$.

Now we are ready to construct our model. Fix a well ordering $<_\kappa$ of V_κ such that for every inaccessible α less than κ, $<_\kappa \cap (V_\alpha \times V_\alpha)$ is a well-ordering of V_α. We define an Easton support Prikry iteration $\langle \mathbb{P}_i, \mathbb{Q}_i : i < \kappa\rangle$ by induction.

Suppose that $\mathbb{P}_\alpha$ is defined. Assume as an induction hypothesis that for all γ less than α, $\mathbb{Q}_\gamma$ is trivial unless γ is in $A \cap \alpha$. If γ is in A then $\mathbb{P}_\gamma$ forces that $\mathbb{Q}_\gamma = \mathbb{P}_{B_\gamma} * \mathbb{P}(\gamma, o(\gamma))$, where $\mathbb{P}_{B_\gamma}$ is a $\gamma + 1$-strategically closed forcing poset and $\mathbb{P}(\gamma, o(\gamma))$ is defined from a family of ultrafilters

$$\{\ U(\gamma, \beta, t) : \beta < o(\gamma),\ t \textit{ is a } \beta\textit{-coherent sequence}\ \}.$$

Let G_α be generic for $\mathbb{P}_\alpha$ over V. If α is not in A then let $\mathbb{Q}_\alpha$ be trivial. Suppose that α is in A. In $V[G_\alpha]$ define B_α as the set of β less than α^+ such that $\mathrm{cf}(\beta)$ is not the successor of a singular cardinal with cofinality μ. Note that B_α is closed under suprema of subsets with order type ω and ω_1. Therefore the forcing poset $\mathbb{P}_{B_\alpha}$ for adding a $\square_\alpha^{B_\alpha}$-sequence forces that for almost all a in $S(\alpha, \alpha^+)$, $a \cap \alpha$ is a singular cardinal with cofinality μ. We consider $\mathbb{P}_{B_\alpha}$ as a Prikry type forcing poset with $\leq^* = \leq$.

Let G^α be generic for $\mathbb{P}_{B_\alpha}$ over $V[G_\alpha]$. If $o(\alpha) = 0$ then $\mathbb{P}(\alpha, o(\alpha))$ is the trivial poset and there is nothing to prove. Suppose that $o(\alpha) > 0$. We define in $V[G_\alpha * G^\alpha]$ a family of α-complete ultrafilters

$$\{\ U(\alpha, \beta, t) : \beta < o(\alpha),\ t \textit{ is a } \beta\textit{-coherent sequence}\ \}$$

which satisfies the requirements (1), (2), and (3) described in the last section. We also assume as an induction hypothesis that the family of ultrafilters defined at any previous stage γ satisfies the same definition as at stage α.

In $V[G_\alpha * G^\alpha]$ fix β less than $o(\alpha)$ and a β-coherent sequence t. We define $U(\alpha, \beta, t)$. Consider $j_\beta^\alpha : V \to N_\beta^\alpha$. Since $j_\beta^\alpha(o)(\alpha) = \beta$, by the induction hypotheses we can write

$$j_\beta^\alpha(\mathbb{P}_\alpha) = \mathbb{P}_\alpha * \mathbb{P}_{B_\alpha} * \mathbb{P}(\alpha, \beta) * \mathbb{P}_{\alpha, j_\beta^\alpha(\alpha)}.$$

By closure, $<_\alpha = <_\kappa \cap (V_\alpha \times V_\alpha)$ is in N_β^α. In order to define $U(\alpha, \beta, t)$ we need to consider all subsets of α in $V[G_\alpha * G^\alpha]$. However, $\mathbb{P}_{B_\alpha}$ does not add subsets to α over $V[G_\alpha]$. So it suffices to consider only $\mathbb{P}_\alpha$-names for subsets of α. Let $\langle \dot{X}_i : i < \alpha^+\rangle$ be the $j_\beta^\alpha(<_\alpha)$-least enumeration of all canonical $\mathbb{P}_\alpha$-names for subsets of α.

We apply the α^+-weak strategic closure of $\mathbb{P}_{\alpha, j_\beta^\alpha(\alpha)}$ to define a $\leq^*$-decreasing sequence of names $\langle \dot{q}_i, \dot{q}_i^* : i < \alpha^+\rangle$ for conditions in $\mathbb{P}_{\alpha, j_\beta^\alpha(\alpha)}$ as follows. Consider a run of the game. Let $\dot{q}_0$ denote 1. Given $\dot{q}_i$, let $\dot{q}_i^*$ be the $j_\beta^\alpha(<_\alpha)$-least name for a direct extension of $\dot{q}_i$ which decides the statement $\alpha \in j_\beta^\alpha(\dot{X}_i)$. Let $\dot{q}_{i+1}$ be a name for Player II's response according to its strategy. Also Player II plays according to its strategy at limit stages.

Now define $U(\alpha, \beta, t)$ by letting X be in $U(\alpha, \beta, t)$ iff there is an index i such that $\dot{X}_i^{G_\alpha} = X$ and there exists a β-tree T such that

$$N_\beta^\alpha[G_\alpha * G^\alpha] \models \langle t, T\rangle \Vdash \dot{q}_{i+1} \Vdash \alpha \in j_\beta^\alpha(\dot{X}_i).$$

Lemma 28. *Suppose that $\beta < \beta^* < o(\alpha)$ and t is β-coherent. Then $U(\alpha, \beta, t)$ is in $N_{\beta^*}^\alpha[G_\alpha * G^\alpha]$ and satisfies the same definition as it does in $V[G_\alpha * G^\alpha]$.*

Proof. Let $k_{\beta,\beta^*}^\alpha : N_{\beta^*}^\alpha \to \mathrm{Ult}(N_{\beta^*}^\alpha, U(\alpha, \beta))$. By Lemma 9 (2), $k_{\beta,\beta^*}^\alpha = j_\beta^\alpha \restriction N_{\beta^*}^\alpha$. So the lemma follows immediately from the definition of $U(\alpha, \beta, t)$. $\square$

Lemma 29. *The set $U(\alpha,\beta,t)$ is an α-complete ultrafilter on α which extends $U(\alpha,\beta)$.*

Proof. We prove that the definition of a set X being in $U(\alpha,\beta,t)$ is independent of the index of its name. Suppose that $X = \dot{X}_i^{G_\alpha} = \dot{X}_j^{G_\alpha}$. Fix p in G_α which forces that $\dot{X}_i = \dot{X}_j$. Then $j^\alpha_\beta(p) = p$ forces over N^α_β that $j^\alpha_\beta(\dot{X}_i) = j^\alpha_\beta(\dot{X}_j)$. Therefore

$$N^\alpha_\beta[G_\alpha * G^\alpha] \models \mathbb{P}(\alpha,\beta) \Vdash \mathbb{P}_{\alpha,j^\alpha_\beta(\alpha)} \Vdash (\alpha \in j^\alpha_\beta(\dot{X}_i) \text{ iff } \alpha \in j^\alpha_\beta(\dot{X}_j)).$$

Suppose that there is T such that

$$N^\alpha_\beta[G_\alpha * G^\alpha] \models \langle t,T\rangle \Vdash \dot{q}_{i+1} \Vdash \alpha \in j^\alpha_\beta(\dot{X}_i).$$

Since $\dot{q}_{j+1}$ is forced to decide the statement $\alpha \in j^\alpha_\beta(\dot{X}_j)$, it follows that

$$N^\alpha_\beta[G_\alpha * G^\alpha] \models \langle t,T\rangle \Vdash \dot{q}_{j+1} \Vdash \alpha \in j^\alpha_\beta(\dot{X}_j).$$

Similar arguments show that $U(\alpha,\beta,t)$ is an ultrafilter which extends $U(\alpha,\beta)$.

Let us prove that $U(\alpha,\beta,t)$ is α-complete. Suppose that p is in G_α and p forces that $\chi : \beta \to \mathcal{P}(\alpha)$ is a partition of α into β many sets, for some β less than α. Then $j^\alpha_\beta(p) = p$ forces over N^α_β that $j^\alpha_\beta(\chi)$ is a partition of $j^\alpha_\beta(\alpha)$ into β many sets. For each i less than β let ξ_i be an index such that p forces that $\chi(i) = \dot{X}_{\xi_i}$. For each i less than β apply the Prikry property of $\mathbb{P}(\alpha,\beta)$ in $N^\alpha_\beta[G_\alpha * G^\alpha]$ to obtain a β-tree T_i such that $\langle t,T_i\rangle$ decides which way that $\dot{q}_{\xi_i+1}$ decides the statement $\alpha \in j^\alpha_\beta(\dot{X}_{\xi_i})$. Then $\langle t,\bigcap T_i\rangle$ is a condition which directly refines each $\langle t,T_i\rangle$. Since $j^\alpha_\beta(\chi)$ is forced to be a partition of $j^\alpha_\beta(\alpha)$, there must be some index j such that $\langle t,\bigcap T_i\rangle$ forces that $\dot{q}_{\xi_j+1}$ forces $\alpha \in j^\alpha_\beta(\dot{X}_{\xi_j})$. $\square$

Lemma 30. *Suppose that s and t are β-coherent sequences and $b_s = b_t$. Then $U(\alpha,\beta,s) = U(\alpha,\beta,t)$.*

Proof. It suffices to prove that $U(\alpha,\beta,s) \subseteq U(\alpha,\beta,t)$, since they are ultrafilters. So let X be in $U(\alpha,\beta,s)$, and fix i such that $X = \dot{X}_i^{G_\alpha}$. Then there is S such that

$$N^\alpha_\beta[G_\alpha * G^\alpha] \models \langle s,S\rangle \Vdash \dot{q}_{i+1} \Vdash \alpha \in j^\alpha_\beta(\dot{X}_i).$$

Apply the Prikry property to find $\langle t,T\rangle$ which decides whether $\dot{q}_{i+1} \Vdash \alpha \in j^\alpha_\beta(\dot{X}_i)$. By Lemma 22, $\langle s,S\rangle$ and $\langle t,T\rangle$ are compatible, so they must decide the statement the same way. So A is in $U(\alpha,\beta,t)$. $\square$

Lemma 31. *Suppose that t is β-coherent and γ is less than β. Then the set of ξ in α such that $t^\frown\xi$ is β-coherent is in $U(\alpha,\gamma,t(\gamma))$.*

Proof. Let X be the set of ξ less than α such that $t^\frown\xi$ is β-coherent. Fix i so that $X = \dot{X}_i^{G_\alpha}$ and let p be a condition in G_α which forces that $\dot{X}_i$ satisfies the definition of X. Since $j^\alpha_\gamma(p) = p$, it follows that

$$N^\alpha_\gamma[G_\alpha * G^\alpha] \models \mathbb{P}(\alpha,\gamma) * \mathbb{P}_{\alpha,j^\alpha_\gamma(\alpha)} \Vdash j^\alpha_\gamma(\dot{X}_i) = \{\xi < j^\alpha_\gamma(\alpha) : t^\frown\xi \text{ is } \beta\text{-coherent}\}.$$

Fix any γ-tree T with trunk $t(\gamma)$. Then by Lemma 24,

$$N^\alpha_\gamma[G_\alpha * G^\alpha] \models \langle t(\gamma),T\rangle \Vdash b_{t(\gamma)} \text{ is an initial segment of } \dot{b}_\alpha.$$

Now $j^\alpha_\gamma(o)(\alpha) = \gamma$ and every ξ in $t(\gamma)$ has order less than γ. It follows that

$$N^\alpha_\beta[G_\alpha * G^\alpha] \Vdash \langle t(\gamma),T\rangle \Vdash \mathbb{P}_{\alpha,j^\alpha_\gamma(\alpha)} \Vdash t(\gamma)^\frown\alpha \text{ is } \gamma\text{-coherent}.$$

So X is in $U(\alpha,\gamma,t(\gamma))$. $\square$

Now define $\mathbb{P}(\alpha, o(\alpha))$ in $V[G_\alpha * G^\alpha]$ using this family of ultrafilters. We show that $\mathbb{P}(\alpha, o(\alpha))$ satisfies the Prikry property.

Proposition 10. *Suppose that φ is a statement in the forcing language for $\mathbb{P}(\alpha, o(\alpha))$ and $\langle t, T\rangle$ is a condition. Then there is a direct extension of $\langle t, T\rangle$ which decides φ.*

Proof. Let φ be a statement in the forcing language and fix a condition $\langle t, T\rangle$. Let φ^0 denote φ and let φ^1 denote $\neg\varphi$.

We define a sequence $\langle T^*_n : n < \omega\rangle$ such that each $T^*_n \subseteq T$ is an $o(\alpha)$-tree with trunk t, and $T^*_{n+1} \subseteq T^*_n$ for each n. Let $T^*_n = T$ for $n \leq |t|$.

Suppose that $n \geq |t|$ and T^*_n is defined. We define T^*_{n+1}. If u is in T^*_n and $|u| \leq n$, then let u be in T^*_{n+1}. Now for each u in T^*_n with $|u| = n$ we define $(T^*_{n+1})_u$. Fix γ less than $o(\alpha)$. For each ξ in $\mathrm{Suc}(T^*_n, u, \gamma)$ define $n_\xi = l$ for $l < 2$ if there exists a tree $T_\xi \subseteq T^*_n$ with trunk $u^\frown\xi$ such that $\langle u^\frown\xi, T_\xi\rangle$ forces φ^l. If there is no such tree T_ξ, then let $n_\xi = 2$. Now fix $l \leq 2$ and $X_{u,\gamma}$ in $U(\alpha, \gamma, u(\gamma))$ such that $n_\xi = l$ for all ξ in $X_{u,\gamma}$. If $l = 2$ then define

$$(T^*_{n+1})_u = \{u^\frown\xi^\frown v \in T^*_n : \xi \in X_{u,\gamma}, \gamma < o(\alpha)\}$$

If $l < 2$ then define

$$(T^*_{n+1})_u = \{u^\frown\xi^\frown v \in T_\xi : \xi \in X_{u,\gamma}, \gamma < o(\alpha)\}.$$

Clearly T^*_{n+1} is an $o(\alpha)$-tree with trunk t.

Now let $T^* = \bigcap T^*_n$. Then $\langle t, T^*\rangle$ is a direct refinement of $\langle t, T\rangle$. The choice of T^* guarantees that the following property holds: Suppose that u is in T^* such that $t \trianglelefteq u$, γ is less than $o(\alpha)$, and there is an $o(\alpha)$-tree $S \subseteq T^*$ such that for some ξ in $\mathrm{Suc}(T^*, u, \gamma)$, $\langle u^\frown\xi, S\rangle$ forces φ^l. Then for all ξ in $\mathrm{Suc}(T^*, u, \gamma)$, $\langle u^\frown\xi, T^*_{u^\frown\xi}\rangle$ forces φ^l.

Claim 2. *Suppose that $\langle t^\frown u^\frown\xi, T^*_{t^\frown u^\frown\xi}\rangle$ forces φ^l for some $l < 2$. Then for all $\gamma \geq o(\xi)$ and any β in $\mathrm{Suc}(T^*, t^\frown u, \gamma)$, $\langle t^\frown u^\frown\beta, T^*_{t^\frown u^\frown\beta}\rangle$ forces φ^l.*

Proof. It suffices to prove the statement is true for some β in $\mathrm{Suc}(T^*, t^\frown u, \gamma)$. We already know that this is true when $\gamma = o(\xi)$. Suppose that γ is larger than $o(\xi)$.

We claim that there is an ordinal β^* in the set $\mathrm{Suc}(T^*, t^\frown u, o(\xi))$ such that

$$\mathrm{Suc}(T^*, t^\frown u, \gamma) \cap \mathrm{Suc}(T^*, t^\frown u^\frown\beta^*, \gamma) \neq \emptyset.$$

It suffices to prove that there is an ordinal β^* in $\mathrm{Suc}(T^*, t^\frown u, o(\xi))$ such that $\mathrm{Suc}(T^*, t^\frown u, \gamma)$ is in $U(\alpha, \gamma, (t^\frown u^\frown\beta^*)(\gamma))$.

Fix i so that $\mathrm{Suc}(T^*, t^\frown u, \gamma) = \dot{X}_i^{G_\alpha}$. Since $\mathrm{Suc}(T^*, t^\frown u, \gamma)$ is in $U(\alpha, \gamma, (t^\frown u)(\gamma))$, by the definition of this ultrafilter in $V[G_\alpha * G^\alpha]$ there is S such that

$$N^\alpha_\gamma[G_\alpha * G^\alpha] \models \langle (t^\frown u)(\gamma), S\rangle \Vdash \dot{q}_{i+1} \Vdash \alpha \in j^\alpha_\gamma(\dot{X}_i).$$

Since γ is greater than $o(\xi)$, $(t^\frown u)(o(\xi)) = (t^\frown u)(\gamma)(o(\xi))$. It follows that the set $\mathrm{Suc}(S, (t^\frown u)(\gamma), o(\xi))$ is in $U(\alpha, o(\xi), (t^\frown u)(o(\xi)))$. So fix an ordinal β^* which is in the set

$$\mathrm{Suc}(S, (t^\frown u)(\gamma), o(\xi)) \cap \mathrm{Suc}(T^*, t^\frown u, o(\xi)).$$

Now $\langle (t^\frown u)(\gamma)^\frown\beta^*, S_{(t^\frown u)(\gamma)^\frown\beta^*}\rangle$ is below $\langle (t^\frown u)(\gamma), S\rangle$, therefore

$$N^\alpha_\gamma[G_\alpha * G^\alpha] \models \langle (t^\frown u)(\gamma)^\frown\beta^*, S_{(t^\frown u)(\gamma)^\frown\beta^*}\rangle \Vdash \dot{q}_{i+1} \Vdash \alpha \in j^\alpha_\gamma(\dot{X}_i).$$

Therefore $\mathrm{Suc}(T^*, t^\frown u, \gamma)$ is in $U(\alpha, \gamma, (t^\frown u)(\gamma)^\frown\beta^*)) = U(\alpha, \gamma, (t^\frown u^\frown\beta^*)(\gamma))$, as desired.

Now choose β in $\mathrm{Suc}(T^*, t^\frown u, \gamma) \cap \mathrm{Suc}(T^*, t^\frown u^\frown \beta^*, \gamma)$. Since $o(\beta^*) = o(\xi)$, $\langle t^\frown u^\frown \beta^*, T^*_{t^\frown u^\frown \beta^*}\rangle$ forces φ^l. Moreover, $\langle t^\frown u^\frown \beta^{*\frown} \beta, T^*_{t^\frown u^\frown \beta^{*\frown}\beta}\rangle$ is below this condition, so it also forces φ^l. Since $o(\beta) > o(\beta^*)$, β^* is in b_β. So $b_{t^\frown u^\frown \beta} = b_{t^\frown u^\frown \beta^{*\frown}\beta}$. Therefore for all δ less than $o(\alpha)$, $U(\alpha, \delta, (t^\frown u^\frown \beta)(\delta)) = U(\alpha, \delta, (t^\frown u^\frown \beta^{*\frown}\beta)(\delta))$. It follows that $S = \{v \setminus \{\beta^*\} : v \in T^*_{t^\frown u^\frown \beta^{*\frown}\beta}\}$ is an $o(\alpha)$-tree. Also the conditions $\langle t^\frown u^\frown \beta^{*\frown}\beta, T^*_{t^\frown u^\frown \beta^{*\frown}\beta}\rangle$ and $\langle t^\frown u^\frown \beta, S\rangle$ are equivalent. Therefore $\langle t^\frown u^\frown \beta, S \cap T^*_{t^\frown u^\frown \beta}\rangle$ forces φ^l. By the choice of T^*, $\langle t^\frown u^\frown \beta, T^*_{t^\frown u^\frown \beta}\rangle$ forces φ^l.

□

Suppose for a contradiction that $\langle t, T^*\rangle$ does not have a direct extension which decides φ. Using Lemma 20, choose a condition $\langle t^\frown u^\frown \xi, T_0\rangle$ below $\langle t, T^*\rangle$ which decides φ such that $t^\frown u^\frown \xi$ is in T^*, $T_0 \subseteq T^*$, and $|u|$ is minimal. Fix $l < 2$ such that $\langle t^\frown u^\frown \xi, T_0\rangle$ forces φ^l. By Claim 2 and the choice of T^*, for all $\gamma \geq o(\xi)$ and β in $\mathrm{Suc}(T^*, t^\frown u, \gamma)$, $\langle t^\frown u^\frown \beta, T^*_{t^\frown u^\frown \beta}\rangle$ forces φ^l.

We will prove that the condition $\langle t^\frown u, T^*_{t^\frown u}\rangle$ forces φ^l, which contradicts the minimality of u. By Lemma 20 it suffices to prove that for all γ less than $o(\alpha)$ and for all β in $\mathrm{Suc}(T^*, t^\frown u, \gamma)$, $\langle t^\frown u^\frown \beta, T^*_{t^\frown u^\frown \beta}\rangle$ forces φ^l. We already know by Claim 2 that this is true for $\gamma \geq o(\xi)$. So by the choice of T^* it is enough to show that for all γ less than $o(\xi)$ there is β in $\mathrm{Suc}(T^*, t^\frown u, \gamma)$ such that $\langle t^\frown u^\frown \beta, T^*_{t^\frown u^\frown \beta}\rangle$ forces φ^l.

Fix i so that $\mathrm{Suc}(T^*, t^\frown u, o(\xi)) = \dot{X}_i^{G_\alpha}$. Since the set $\mathrm{Suc}(T^*, t^\frown u, o(\xi))$ is in $U(\alpha, o(\xi), (t^\frown u)(o(\xi)))$, there exists an $o(\xi)$-tree S such that

$$N^\alpha_{o(\xi)}[G_\alpha * G^\alpha] \models \langle (t^\frown u)(o(\xi)), S\rangle \Vdash \dot{q}_{i+1} \Vdash \alpha \in j^\alpha_{o(\xi)}(\dot{X}_i).$$

Define T_1 as the set of sequences of the form $t^\frown u^\frown \langle \alpha_0, \ldots, \alpha_n\rangle$ which are in T^* and satisfy the following property: Let $i \leq n$ be the maximal index such that for all $j \leq i$, $o(\alpha_j)$ is less than $o(\xi)$; then $(t^\frown u)(o(\xi))^\frown \langle \alpha_0, \ldots, \alpha_i\rangle$ is in S.

Fix some ordinal β in the set $\mathrm{Suc}(T_1, t^\frown u, \gamma)$. We claim that $\langle t^\frown u^\frown \beta, (T_1)_{t^\frown u^\frown \beta}\rangle$ forces φ^l. If this is true, then by the choice of T^* the condition $\langle t^\frown u^\frown \beta, T^*_{t^\frown u^\frown \beta}\rangle$ forces φ^l and we are done. If the claim fails, then by the choice of T^* there exists a condition

$$\langle t^\frown u^\frown \beta^\frown \langle \alpha_0, \ldots, \alpha_n\rangle, T^*_{t^\frown u^\frown \beta^\frown \langle \alpha_0, \ldots, \alpha_n\rangle}\rangle$$

which forces the negation of φ^l, where $t^\frown u^\frown \beta^\frown \langle \alpha_0, \ldots, \alpha_n\rangle$ is in T_1. Let $i \leq n$ be maximal so that $o(\alpha_j) < o(\xi)$ for all $j \leq i$. Then by the definition of T_1, the sequence $(t^\frown u)(o(\xi))^\frown \beta^\frown \langle \alpha_0, \ldots, \alpha_i\rangle$ is in S. Therefore $N^\alpha_{o(\xi)}[G_\alpha * G^\alpha]$ models that

$$\langle (t^\frown u)(o(\xi))^\frown \beta^\frown \langle \alpha_0, \ldots, \alpha_i\rangle, S_{(t^\frown u)(o(\xi))^\frown \beta^\frown \langle \alpha_0, \ldots, \alpha_i\rangle}\rangle \Vdash \dot{q}_{i+1} \Vdash \alpha \in j^\alpha_{o(\xi)}(\dot{X}_i).$$

But $(t^\frown u)(o(\xi))^\frown \beta^\frown \langle \alpha_0, \ldots, \alpha_i\rangle = (t^\frown u^\frown \beta^\frown \langle \alpha_0, \ldots, \alpha_i\rangle)(o(\xi))$. It follows that $\mathrm{Suc}(T^*, t^\frown u, o(\xi))$ is in the ultrafilter

$$U(\alpha, o(\xi), (t^\frown u^\frown \beta^\frown \langle \alpha_0, \ldots, \alpha_i\rangle)(o(\xi))).$$

So we can choose an ordinal β^* in the set

$$\mathrm{Suc}(T^*, t^\frown u, o(\xi)) \cap \mathrm{Suc}(T^*, t^\frown u^\frown \beta^\frown \langle \alpha_0, \ldots, \alpha_i\rangle, o(\xi)).$$

Since $o(\beta^*) = o(\xi)$, it follows that $b_{t^\frown u^\frown \beta^\frown \langle \alpha_0, \ldots, \alpha_i\rangle^\frown \beta^*} = b_{t^\frown u^\frown \beta^*}$. Define T_2 as the set of sequences of the form

$$t^\frown u^\frown \beta^\frown \langle \alpha_0, \ldots, \alpha_i\rangle^\frown \beta^{*\frown} v$$

such that $t^\frown u^\frown \beta^{*\frown} v$ is in T^*. Then the condition

$$\langle t^\frown u^\frown \beta^\frown \langle \alpha_0, \ldots, \alpha_i\rangle^\frown \beta^*, T_2\rangle$$

is equivalent to the condition

$$\langle t^\frown u^\frown \beta^*, T^*_{t^\frown u^\frown \beta^*}\rangle.$$

Since $o(\xi) = o(\beta^*)$, these conditions force φ^l. Therefore the condition

$$\langle t^\frown u^\frown \beta^\frown \langle \alpha_0, \ldots, \alpha_i\rangle^\frown \beta^*, T_2 \cap T^*_{t^\frown u^\frown \beta^\frown \langle \alpha_0,\ldots,\alpha_i\rangle^\frown \beta^*}\rangle$$

also forces φ^l. By choice of T^*, the condition

$$\langle t^\frown u^\frown \beta^\frown \langle \alpha_0, \ldots, \alpha_i\rangle^\frown \beta^*, T^*_{t^\frown u^\frown \beta^\frown \langle \alpha_0,\ldots,\alpha_i\rangle^\frown \beta^*}\rangle$$

forces ϕ^l. By Claim 2, the same holds if we replace β^* by any δ in the set $\mathrm{Suc}(T^*, t^\frown u^\frown \beta^\frown \langle \alpha_0, \ldots, \alpha_i\rangle, \gamma^*)$, where $\gamma^* \geq o(\xi)$. But this is a contradiction since we cannot replace it with α_{i+1}, since this condition has an extension which forces the negation of φ^l. □

This completes the definition of $\mathbb{P}_\kappa$. The poset $\mathbb{P}_\kappa$ preserves all cardinals and preserves GCH. Now we force with the poset $\mathbb{P}_\kappa * \mathbb{P}_B$, where B is a name for the set of β less than κ^+ such that the cofinality of β is not equal to the successor of a singular cardinal with cofinality μ. After forcing with $\mathbb{P}_\kappa * \mathbb{P}_B$, κ remains strongly inaccessible and for almost all a in $S(\kappa, \kappa^+)$, $a \cap \kappa$ is a singular cardinal with cofinality μ. We show that $S(\kappa, \kappa^+)$ remains stationary.

Recall that U is a normal ultrafilter on $P_\kappa \kappa^+$ in V and $j_U : V \to M = \mathrm{Ult}(V, U)$. The sequence $\langle U(\kappa, i) : i \leq \mu\rangle$ is a $\lhd$-increasing sequence of ultrafilters on κ and $U(\kappa, \mu)$ is equal to $U \restriction \kappa$, i.e. X is in $U(\kappa, \mu)$ iff $X \subseteq \kappa$ and $\kappa \in j_U(X)$. In particular, $\kappa \in j_U(A_\mu)$. So $j_U(o)(\kappa) = \mu$. Therefore

$$j_U(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{P}_B * \mathbb{P}(\kappa, \mu) * \mathbb{P}_{\mathrm{tail}}.$$

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Let $G_\kappa * G_B$ be generic for $\mathbb{P}_\kappa * \mathbb{P}_B$ over V. Let $G(\kappa, \mu) * G_{\mathrm{tail}}$ be generic for $\mathbb{P}(\kappa, \mu) * \mathbb{P}_{\mathrm{tail}}$ over $V[G_\kappa * G_B]$. Let $H = G_\kappa * G_B * G(\kappa, \mu) * G_{\mathrm{tail}}$. Since $j_U``G_\kappa = G_\kappa$, in $V[H]$ we can lift j_U to

$$j_U : V[G_\kappa] \to M[H].$$

Now $j_U \restriction H(\kappa^+)^V$ is in M. For any x in $H(\kappa^+)^{V[G_\kappa]}$, define $j^*(x) = ((j_U \restriction H(\kappa^+)^V)(\dot{x}))^H$, where $\dot{x}$ is a name in $H(\kappa^+)^V$ and $\dot{x}^{G_\kappa} = x$. Then in $V[H]$, $j^* = j_U \restriction H(\kappa^+)^{V[G_\kappa]}$ and j^* is in $M[H]$. It follows that $j_U \restriction \mathbb{P}_B$ is in $M[H]$, and therefore also $\bigcup j_U``G_B$ is in $M[H]$.

Consider p in G_B, and write $p = \langle c_\alpha : \alpha \in B \cap (\beta + 1)\rangle$ for some β less than κ^+. Then $j_U(p) = \langle d_\alpha : \alpha \in j(B) \cap (\beta + 1)\rangle$. It follows that $\bigcup j``G_B = \langle d_\alpha : \alpha \in j(B) \cap \sup j``\kappa^+\rangle$. So $\bigcup j``G_B$ is a master condition in $j_U(\mathbb{P}_B)$ iff $\sup j``\kappa^+$ is not in $j(B)$, which is true iff $\sup j``\kappa^+$ has cofinality equal to the successor of a singular cardinal with cofinality $\mu$. But $\mathrm{cf}(\sup j``\kappa^+) = \kappa^+$ and κ has cofinality μ in $M[H]$. So $\bigcup j_U``G_B$ is a master condition.

Let g be a generic filter for $j_U(\mathbb{P}_B)$ over $V[H]$ which contains the master condition $\bigcup j_U``G_B$. Then in $V[H * g]$ we can lift j_U to

$$j_U : V[G_\kappa * G_B] \to M[H * g].$$

Now we prove that $S(\kappa, \kappa^+)$ is stationary in $V[G_\kappa * G_B]$. Let C be a club subset of $P_\kappa \kappa^+$ in $V[G_\kappa * G_B]$. Since $\mathbb{P}_B$ is $\kappa + 1$-strategically closed, C is a subset of

$H(\kappa^+)^{V[G_\kappa]}$. By Lemma 6, C is in $M[G_\kappa * G_B]$. Therefore $j_U \restriction C$ is in $M[H]$ and so j_U "C is in $M[H]$. Now in $M[H * g]$, j_U "C is a directed subset of the club set $j_U(C)$ with size less than $j_U(\kappa)$. Therefore $\bigcup j_U$ "$C = j$ "κ^+ is in $j_U(C)$. But j "κ^+ is in $j(S(\kappa,\kappa^+))$, so $j_U(C) \cap j(S(\kappa,\kappa^+))$ is non-empty. By elementarity, $S(\kappa,\kappa^+) \cap C$ is non-empty. So $S(\kappa,\kappa^+)$ is stationary.

Note: There are variations of this consistency proof to arrange a set of possible cofinalities for $a \cap \kappa$, instead of a single cofinality. To obtain such models, modify the definition of the B_α's and replace the ultrafilter U above with a coherent sequence of supercompact ultrafilters (see next section).

8. Saturated Ideals and GCH

In this section we construct a model in which κ is strongly inaccessible, GCH holds, and there is a stationary set $S \subseteq P_\kappa \kappa^+$ such that $NS \restriction S$ is saturated. Moreover, we will arrange that for all regular cardinals μ less than κ, the set

$$\{a \in S : \mathrm{cf}(a \cap \kappa) = \mu\}$$

is stationary.

Previously Gitik [2] constructed a model in which $NS \restriction S$ is κ^+-saturated for a set $S \subseteq P_\kappa \kappa^+$, where κ is inaccessible. Unlike the partial square forcing poset, Gitik's poset for destroying stationary sets adds subsets to κ. Therefore $2^\kappa = \kappa^{++}$ in his model. It is an open question whether the GCH is consistent with the existence of a stationary set S such that $NS \restriction S$ is κ^+-saturated.

Before constructing our model we introduce one of the basic forcing posets, which is an iteration of the partial square poset over the same cardinal. Let V be a model of set theory in which κ is strongly inaccessible and GCH holds. We define a κ-support forcing iteration $\mathbb{P}^\kappa$ with length κ^{++}, which we denote by

$$\langle \mathbb{P}^\kappa_i, \mathbb{Q}^\kappa_i : i < \kappa^{++} \rangle.$$

The following induction hypotheses will be maintained:

(1) $\mathbb{P}^\kappa_\alpha$ is $\kappa + 1$-strategically closed,

(2) Let D^κ_α be the set of conditions p in $\mathbb{P}^\kappa_\alpha$ such that for all β in the support of p, there is x in V such that $p(\beta) = \check{x}$. Then D^κ_α is dense in $\mathbb{P}^\kappa_\alpha$, and if $\alpha < \kappa^{++}$ then $|D^\kappa_\alpha| \leq \kappa^+$.

Fix a surjective function $f : \kappa^{++} \to \kappa^{++} \times \kappa^{++}$ such that

$$\forall \alpha, i, j < \kappa^{++} \ (f(\alpha) = \langle i, j \rangle \longrightarrow i \leq \alpha),$$

and each $f^{-1}(\langle i, j \rangle)$ is unbounded in κ^{++}. Suppose that $\mathbb{P}^\kappa_\alpha$ is defined for a fixed α less than κ^{++} and satisfies the induction hypotheses. Let $\langle \dot{B}^\alpha_i : i < \kappa^{++} \rangle$ enumerate all canonical D^κ_α-names $\dot{B}$ for a subset of κ^+ such that $\mathbb{P}^\kappa_\alpha$ forces that $\dot{B}$ is closed under suprema of subsets with order type ω or ω_1. Write $f(\alpha) = \langle i, j \rangle$. Then $i \leq \alpha$, so consider the name $\dot{B}^i_j$ which was defined at stage i. Let $\mathbb{Q}_\alpha$ be a $\mathbb{P}^\kappa_\alpha$-name for the $\kappa + 1$-strategically closed forcing poset $\mathbb{P}_{\dot{B}^i_j}$ which adds a $\square_\kappa^{\dot{B}^i_j}$-sequence. Clearly $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{Q}_\alpha$ satisfies the induction hypotheses.

Now suppose that $\beta \leq \kappa^{++}$ is a limit ordinal and $\mathbb{P}^\kappa_\alpha$ is defined for all α less than β. Define $\mathbb{P}^\kappa_\beta$ as the set of conditions p with support of size no larger than κ such that $p \restriction \alpha$ is in $\mathbb{P}^\kappa_\alpha$ for all α less than β.

Proposition 11. *The iteration $\mathbb{P}^\kappa_\beta$ is $\kappa + 1$-strategically closed.*

Proof. For each α less than β, let σ_α be a $\mathbb{P}^\kappa_\alpha$-name for a strategy for Player II in the game for the poset $\mathbb{Q}^\kappa_\alpha$. We describe a strategy for Player II in $\mathbb{P}^\kappa_\beta$ by considering a run of the game $\langle p_i : 0 < i \leq \kappa\rangle$.

Suppose that $i < \kappa$ and Player I has just played condition p_i. Define p_{i+1} as the condition with the same support as p_i such that for all α in the support, $p_{i+1}(\alpha)$ is a $\mathbb{P}^\kappa_\alpha$-name for the condition obtained by applying σ_α to the sequence of conditions played so far on coordinate α. An easy induction shows that for all α less than β, $p_{i+1} \restriction \alpha$ is a condition in $\mathbb{P}^\kappa_\alpha$ below $p_i \restriction \alpha$. So p_{i+1} is a condition in $\mathbb{P}^\kappa_\beta$ below p_i.

Suppose that $i \leq \kappa$ is a limit ordinal and $\langle p_j : 0 < j < i\rangle$ is defined. Let a be equal to $\bigcup\{\mathrm{supp}(p_j) : 0 < j < i\}$. Since $i \leq \kappa$, $|a| \leq \kappa$. Define p_i as the condition with support a such that for all α in a, $p_i(\alpha)$ is a name for the condition obtained by applying σ_α to the sequence of conditions played thus far on coordinate α. As in the last case, p_i is a condition and extends each p_j for $0 < j < i$. □

Proposition 12. *The set D^κ_β is dense in $\mathbb{P}^\kappa_\beta$, and if $\beta < \kappa^{++}$ then $|D^\kappa_\beta| \leq \kappa^+$.*

Proof. Suppose first that $\mathrm{cf}(\beta) \geq \kappa^+$. Then $\mathbb{P}^\kappa_\beta = \bigcup\{\mathbb{P}^\kappa_i : i < \beta\}$ and $D^\kappa_\beta = \bigcup\{D^\kappa_i : i < \beta\}$. So in this case the statement follows from the induction hypotheses.

Suppose that $\mathrm{cf}(\beta) \leq \kappa$. Let p be a condition in $\mathbb{P}^\kappa_\beta$. We construct a decreasing sequence $\langle p_n : n < \omega\rangle$. Let $p_0 = p$.

Assume that $n < \omega$ and p_n is defined. Since $\mathbb{P}^\kappa_\beta$ is $\kappa + 1$-strategically closed, it forces that the sequence

$$\langle p_n(\alpha)^{\dot{G}} : \alpha \in \mathrm{supp}(p_n)\rangle$$

is in the ground model. So there is a condition $p_{n+1} \leq p_n$ and a sequence

$$\langle s^\alpha_n : \alpha \in \mathrm{supp}(p_n)\rangle$$

in the ground model such that p_{n+1} forces that $\check{s}^\alpha_n = p_n(\alpha)$ for all α in $\mathrm{supp}(p_n)$.

Now define q as follows. The support of q is $\bigcup\{\mathrm{supp}(p_n) : n < \omega\}$. Fix α in $\mathrm{supp}(q)$ and let n_α be the least $n < \omega$ such that α is in $\mathrm{supp}(p_n)$. For $m \geq n_\alpha$, s^α_m is defined and is a sequence of clubs. If s^α_m is eventually constant, define s_α as this constant value. Otherwise let ξ_α be the suprema of the set $\{\sup \mathrm{dom}(s^\alpha_m) : m \geq n_\alpha\}$, and choose c_α as some cofinal subset of ξ_α with order type ω.

Let B_α be a $\mathbb{P}^\kappa_\alpha$-name for the set such that $\mathbb{Q}^\kappa_\alpha = \mathbb{P}_{B_\alpha}$. Since $\mathrm{cf}(\xi_\alpha) = \omega$ and B_α is ω-closed, ξ_α is forced to be in B_α. Define

$$s_\alpha = \bigcup\{s^\alpha_m : m \geq n_\alpha\} \cup \langle \xi_\alpha, c_\alpha\rangle.$$

Now define q by letting $q(\alpha) = \check{s}_\alpha$ for all α in $\mathrm{supp}(q)$. Then q is in D^κ_β and $q \leq p$.

Any condition in D^κ_β is coded by its support and a set in ${}^{\leq\kappa}H(\kappa^+)$. So $|D^\kappa_\beta| \leq \kappa^+$. □

This completes the definition of $\mathbb{P}^\kappa$. Let D^κ be the set $D^\kappa_{\kappa^{++}}$, which is dense in $\mathbb{P}^\kappa$. Note that $D^\kappa = \bigcup\{D^\kappa_i : i < \kappa^{++}\}$.

Proposition 13. *The iteration $\mathbb{P}^\kappa$ is κ^{++}-c.c.*

Proof. It suffices to prove that D^κ is κ^{++}-c.c. Suppose that $\langle p_i : i < \kappa^{++}\rangle$ is a sequence of conditions in D^κ. By the Δ-system lemma, we can assume that there is a set d such that $\mathrm{supp}(p_i) \cap \mathrm{supp}(p_j) = d$ for $i < j$. Since there are at most κ^+ many possibilities for $p_i \restriction d$, we assume $p_i \restriction d = p_j \restriction d$ for $i < j$.

Fix $i < j$ and let $q = p_i \cup p_j$. It is easy to check by induction that q is a condition below p_i and p_j. □

Now we begin to construct our model. Let V be a model of set theory in which κ is κ^{++}-supercompact and GCH holds.

First we construct a coherent sequence of supercompact ultrafilters.

Proposition 14. *There exists a $\triangleleft$-increasing sequence $\langle U(\kappa, i) : i < \kappa\rangle$ of normal ultrafilters on $P_\kappa\kappa^+$ satisfying the following properties:*

(1) for $i < j$, $U(\kappa, i) \restriction \kappa \triangleleft U(\kappa, j) \restriction \kappa$,

(2) For each i let $j_i : V \to Ult(V, U(\kappa, i))$. Then for $\alpha < \beta$ there exists a function $f^\beta_\alpha : \kappa \to V_\kappa$ such that $j_\beta(f^\beta_\alpha)(\kappa) = U(\kappa, \alpha)$.

Proof. It suffices to prove that for any $\triangleleft$-increasing sequence $\vec{U}$ of normal ultrafilters on $P_\kappa\kappa^+$ with length less than κ, there exists a normal ultrafilter U on $P_\kappa\kappa^+$ satisfying the following properties:

(a) for each U_i in $\vec{U}$, $U_i \triangleleft U$ and $U_i \restriction \kappa \triangleleft U \restriction \kappa$,

(b) Let $j_U : V \to \mathrm{Ult}(V, U)$. Then for each U_i in $\vec{U}$, there is a function $f_i : \kappa \to V_\kappa$ such that $j_U(f_i)(\kappa) = U_i$.

If the claim holds then we can construct the desired sequence by induction.

Suppose for a contradiction that the claim fails. Let α be the least ordinal less than κ such that there exists a sequence $\vec{U}$ with length α for which no such U exists. Let W be a normal ultrafilter on $P_\kappa\kappa^{++}$ and let $j : V \to M = \mathrm{Ult}(V, W)$. By closure, the above statement about α holds in M.

Let $U = W \restriction \kappa^+$. Let $j_0 : V \to M_0 = \mathrm{Ult}(V, U \restriction \kappa)$ and $j_1 : V \to M_1 = \mathrm{Ult}(V, U)$. Apply Lemma 3 to obtain $k_0 : M_0 \to M_1$ with $\mathrm{crit}(k_0) = \kappa^{++M_0}$ and $k_1 : M_1 \to M$ with $\mathrm{crit}(k_1) = \kappa^{+3M_1}$.

By elementarity, there exists a sequence $\langle U^*_i : i < \alpha\rangle$ which is a witness for the statement about α in M_0. For each i let $U_i = k_0(U^*_i)$. Then $k_0(\langle U^*_i : i < \alpha\rangle) = \langle U_i : i < \alpha\rangle$ is a witness for the statement about α in M_1. By the closure of M_1, this sequence really is a $\triangleleft$-increasing sequence of ultrafilters on $P_\kappa\kappa^+$. Moreover, U is above each U_i. By the closure of M_0, $U^*_i \restriction \kappa$ as computed in M_0 is exactly $U_i \restriction \kappa$. Therefore $U_i \restriction \kappa \triangleleft U \restriction \kappa$ for each i. For each i fix $f_i : \kappa \to V_\kappa$ such that $[f_i] = U^*_i$ in M_0. By the definition of k_0 as given in the proof of Lemma 3, $k_0([f_i]) = j_1(f_i)(\kappa)$. So $j_1(f_i)(\kappa) = k_0([f_i]) = k_0(U^*_i) = U_i$.

By elementarity $k_1(\langle U_i : i < \alpha\rangle) = \langle U_i : i < \alpha\rangle$ satisfies the statement about α in M. But this sequence and U satisfy (a) and (b) in M, so we have a contradiction. □

Fix $\langle U(\kappa, i) : i < \kappa\rangle$ as in Proposition 14.

Lemma 32. *There exists a sequence $\langle X_i : i < \kappa\rangle$ of subsets of κ such that X_i is in $U(\kappa, i) \restriction \kappa$, and for $i \neq j$, $X_i \cap X_j$ is empty.*

Proof. For distinct i and j, let $X_i(j)$ be a set in $(U(\kappa, i) \restriction \kappa) \setminus (U(\kappa, j) \restriction \kappa)$. Let $X_i(i) = \kappa$. Define $X^*_i = \triangle\{X_i(j) : j < \kappa\}$, which is in $U(\kappa, i) \restriction \kappa$. If i and j are distinct, then X^*_i is not in $U(\kappa, j) \restriction \kappa$, since $X^*_i \setminus (j + 1) \subseteq X_i(j)$. Now define $X_i = X^*_i \setminus \bigcup\{X^*_j : j < i\}$. □

For each $i < \kappa$, let $j^\kappa_i : V \to N^\kappa_i = \mathrm{Ult}(V, U(\kappa, i))$. For $\alpha < \beta < \kappa$, fix $f^\beta_\alpha : \kappa \to V_\kappa$ such that $j^\kappa_\beta(f^\beta_\alpha)(\kappa) = U(\kappa, \alpha)$. Define a sequence $\langle A_j : j < \kappa\rangle$ by induction, using the following recursion: A_j is the set of α less than κ satisfying

(1) α is in X_j and $\alpha > j$,

(2) For $i < j$, $f^j_i(\alpha)$ is a normal ultrafilter on $P_\alpha\alpha^+$ such that $A_i \cap \alpha$ is in

$f_i^j(\alpha) \restriction \alpha$,

(3) For $i_0 < i_1 < j$, $f_{i_0}^j(\alpha) \lhd f_{i_1}^j(\alpha)$,

(4) Let $i_0 < i_1 < j$ and $k : V \to Ult(V, f_{i_1}^j(\alpha))$. Then $k(f_{i_0}^{i_1} \restriction \alpha)(\alpha) = f_{i_0}^j(\alpha)$.

For each j, A_j is in $U(\kappa, j) \restriction \kappa$, as can be proved by induction, using Lemma 9 (2) to prove (4).

Let A be equal to $\bigcup\{A_i : i < \kappa\}$. For each α in A, define $o(\alpha)$ as the unique i such that α is in A_i. If α is in A, then for each $i < o(\alpha)$ let $U(\alpha, i) = f_i^{o(\alpha)}(\alpha)$ and let $j_i^\alpha : V \to N_i^\alpha = \mathrm{Ult}(V, U(\alpha, i))$. By (2), for $i < o(\alpha)$, $A_i \cap \alpha$ is in $f_i^{o(\alpha)}(\alpha) \restriction \alpha = U(\alpha, i) \restriction \alpha$, and therefore α is in $j_i^\alpha(A_i)$. It follows that $j_i^\alpha(o)(\alpha) = i$ in N_i^α. By (4), for each $i_0 < i_1 < o(\alpha)$, $U(\alpha, i_0)$ is in $N_{i_1}^\alpha$, so $U(\alpha, i_0) \lhd U(\alpha, i_1)$.

Fix $<_\kappa$ a well-ordering of V_κ such that for any inaccessible α less than κ, $<_\alpha = <_\kappa \cap (V_\alpha \times V_\alpha)$ is a well-ordering of V_α.

We define an Easton support Prikry iteration $\langle \mathbb{P}_i, \mathbb{Q}_i : i < \kappa \rangle$.

Suppose that $\mathbb{P}_\alpha$ is defined. Assume as an induction hypothesis that for all γ less than α, $\mathbb{Q}_\gamma$ is trivial unless γ is in $A \cap \alpha$. If γ is in A then $\mathbb{P}_\gamma$ forces that $\mathbb{Q}_\gamma = \mathbb{P}^\gamma * \mathbb{P}(\gamma, o(\gamma))$, where $\mathbb{P}^\gamma$ is a $\gamma + 1$-strategically closed forcing poset and $\mathbb{P}(\gamma, o(\gamma))$ is defined from a family of ultrafilters

$$\{\ U(\gamma, \beta, t) : \beta < o(\gamma),\ t \textit{ is a } \beta\textit{-coherent sequence}\ \}.$$

Let G_α be generic for $\mathbb{P}_\alpha$ over V. If α is not in A then let $\mathbb{Q}_\alpha$ be trivial. Suppose that α is in A. In $V[G_\alpha]$ let $\mathbb{P}^\alpha$ be the iteration of partial square posets which was defined earlier in this section. We consider $\mathbb{P}^\alpha$ as a Prikry type forcing poset by letting $\leq^* = \leq$. Let G^α be generic for $\mathbb{P}^\alpha$ over $V[G_\alpha]$.

If $o(\alpha) = 0$ then $\mathbb{P}(\alpha, o(\alpha))$ is the trivial poset and there is nothing to prove. Suppose that $o(\alpha) > 0$. We define in $V[G_\alpha * G^\alpha]$ a family of α-complete ultrafilters

$$\{\ U(\alpha, \beta, t) : \beta < o(\alpha),\ t \textit{ is a } \beta\textit{-coherent sequence}\ \}$$

which satisfies the requirements (1), (2), and (3) described in the last section. We also assume as an induction hypothesis that the family of ultrafilters defined at any previous stage γ satisfies the same definition as at stage α.

In $V[G_\alpha * G^\alpha]$ fix β less than $o(\alpha)$ and a β-coherent sequence t. We define $U(\alpha, \beta, t)$. Consider $j_\beta^\alpha : V \to N_\beta^\alpha$. Since $j_\beta^\alpha(o)(\alpha) = \beta$, by the induction hypotheses we can write

$$j_\beta^\alpha(\mathbb{P}_\alpha) = \mathbb{P}_\alpha * \mathbb{P}^\alpha * \mathbb{P}(\alpha, \beta) * \mathbb{P}_{\alpha, j_\beta^\alpha(\alpha)}.$$

By closure, $<_\alpha = <_\kappa \cap (V_\alpha \times V_\alpha)$ is in N_β^α. In order to define $U(\alpha, \beta, t)$ we need to consider all subsets of α in $V[G_\alpha * G^\alpha]$. However, $\mathbb{P}^\alpha$ does not add subsets to α over $V[G_\alpha]$. So it suffices to consider only $\mathbb{P}_\alpha$-names for subsets of α. Let $\langle \dot{X}_i : i < \alpha^+ \rangle$ be the $j_\beta^\alpha(<_\alpha)$-least enumeration of all canonical $\mathbb{P}_\alpha$-names for subsets of α.

By the α^{++}-weak strategic closure of $\mathbb{P}_{\alpha, j_\beta^\alpha(\alpha)}$, there exists a name $\dot{q}$ for a condition in $\mathbb{P}_{\alpha, j_\beta^\alpha(\alpha)}$ which decides the statement $\alpha \in j_\beta^\alpha(\dot{X}_i)$ for each $i < \alpha^+$. Let $\dot{q}$ be the $j_\beta^\alpha(<_\alpha)$-least such name.

Define $U(\alpha, \beta, t)$ by letting X be in $U(\alpha, \beta, t)$ if there exists an index i such that $X = \dot{X}_i^{G_\alpha}$ and a β-tree T such that

$$N_\beta^\alpha[G_\alpha * G^\alpha] \models \langle t, T \rangle \Vdash \dot{q} \Vdash \alpha \in j_\beta^\alpha(\dot{X}_i).$$

Clearly Lemmas 28 through 31 hold for these ultrafilters. Let $\mathbb{P}(\alpha, o(\alpha))$ be the forcing poset defined from this family in $V[G_\alpha * G^\alpha]$. By Proposition 10, $\mathbb{P}(\alpha, o(\alpha))$ satisfies the Prikry property.

This completes the definition of $\mathbb{P}_\kappa$. By standard Easton support arguments, $\mathbb{P}_\kappa$ preserves GCH.

In V let $F : \kappa^{+<\omega} \to \kappa^+$ be a Jonsson function for κ^+. So if $X \subseteq \kappa^+$ has size κ^+ and is closed under F, then $X = \kappa^+$. Define S^* as the set of a in $S(\kappa, \kappa^+)$ such that $a \cap \kappa$ is strongly inaccessible, F is Jonsson for a, and a is closed under suprema of bounded subsets with order type less than $a \cap \kappa$.

Lemma 33. *If U is any normal ultrafilter on $P_\kappa \kappa^+$, then S^* is in U.*

Proof. Let $j : V \to M = \mathrm{Ult}(V, U)$. Then $j``\kappa^+$ is in $j(S(\kappa, \kappa^+))$ and $j``\kappa^+ \cap j(\kappa) = \kappa$ is strongly inaccessible in M. Since the critical point of j is κ, $j``\kappa^+$ is closed under suprema of subsets with order type less than $\kappa = j``\kappa^+ \cap j(\kappa)$.

To show that $j(F)$ is Jonsson for $j``\kappa^+$, suppose that $X$ is a subset of $j``\kappa^+$ which is closed under $j(F)$ and has size equal to $|j``\kappa^+| = \kappa^+$. Then there is $Y \subseteq \kappa^+$ with size $\kappa^+$ such that $j``Y = X$. Since X is closed under $j(F)$, Y is closed under F. The fact that F is a Jonsson function implies that $Y = \kappa^+$, and therefore $X = j``\kappa^+$. It follows that $j``\kappa^+$ is in $j(S^*)$, so S^* is in U. □

For each i less than κ, let

$$S_i = \{a \in S^* : a \cap \kappa \in A_i\}.$$

Then S_i is in $U(\kappa, i)$. Define

$$S = \bigcup \{S_i : i < \kappa\}.$$

Lemma 34. *The function $a \mapsto \sup a$ is injective on S. So if T is a subset of S, there is $X \subseteq \kappa^+$ such that $T = \{a \in S : \sup a \in X\}$.*

Proof. Suppose that a and b are distinct elements in S with the same supremum. Since a and b are closed under suprema of subsets with order type ω, the set $a \cap b$ is unbounded in $\sup a$. Also $\mathrm{cf}(\sup a) = \mathrm{o.t.}(a) = (a \cap \kappa)^+$. It follows that $|a \cap b| = |a| = |b|$. But $a \cap b$ is closed under F and is a subset of a and b. Since F is Jonsson for a and b, $a = a \cap b = b$.

For the second statement, let $X = \{\sup a : a \in T\}$. Since $a \mapsto \sup a$ is injective, $T = \{a \in S : \sup a \in X\}$. □

Let G_κ be a generic filter for $\mathbb{P}_\kappa$ over V. In $V[G_\kappa]$ we construct a subiteration $\mathbb{P}^\kappa_B$ of $\mathbb{P}^\kappa$ such that $\mathbb{P}^\kappa_B$ forces that $NS \restriction S$ is saturated, and for all regular μ less than κ, the set $\{a \in S : \mathrm{cf}(a \cap \kappa) = \mu\}$ is stationary.

In $V[G_\kappa]$ let $\langle \dot{Y}_i : i < \kappa^{++} \rangle$ be a sequence of D^κ-names such that for all $i < \kappa^{++}$, $\mathbb{P}^\kappa_i$ forces that $\dot{\mathbb{Q}}^\kappa_i$ is the forcing poset $\mathbb{P}_{\dot{Y}_i}$ for adding a $\Box^{\dot{Y}_i}_\kappa$-sequence.

For each $i < \kappa$ write

$$j^\kappa_i(\mathbb{P}_\kappa * \mathbb{P}^\kappa) = \mathbb{P}_\kappa * \mathbb{P}^\kappa * \mathbb{P}(\kappa, i) * \mathbb{P}^i_{\mathrm{tail}} * j^\kappa_i(\mathbb{P}^\kappa).$$

We introduce some notation for generic filters for these posets. We use G^κ, $G(\kappa, i)$, and G^i_{tail} to denote generic filters of D^κ, $\mathbb{P}(\kappa, i)$, and $\mathbb{P}^i_{\mathrm{tail}}$ respectively (or sometimes for the names of such filters). For each α less than κ^{++}, let $G^\kappa_\alpha = G^\kappa \restriction \alpha$, i.e. the generic for D^κ_α given by G^κ. Let H^κ_α be the generic for $\mathbb{P}_{Y_\alpha}$ given by G^κ over $V[G_\kappa * G^\kappa_\alpha]$.

We construct by induction an iteration

$$\langle \mathbb{P}^\kappa_{B_\alpha}, \mathbb{Q}^\kappa_{B_\alpha} : \alpha < \kappa^{++} \rangle$$

and functions

$$\langle \pi_\alpha : \alpha < \kappa^{++} \rangle$$

such that for each α, $\pi_\alpha : D^\kappa_\alpha \to \mathbb{P}^\kappa_{B_\alpha}$ is a projection mapping satisfying:

(1) For $\alpha < \beta$ and p in D^κ_β, $\pi_\beta(p) \restriction \alpha = \pi_\alpha(p \restriction \alpha)$,

(2) $D^\kappa_{B_\alpha} = D^\kappa_\alpha \cap \mathbb{P}^\kappa_{B_\alpha}$ is dense in $\mathbb{P}^\kappa_{B_\alpha}$ and $\pi_\alpha \restriction D^\kappa_{B_\alpha}$ is the identity,

(3) For each α, $B_{\alpha+1}$ is a $\mathbb{P}^\kappa_{B_\alpha}$-name for a subset of $\alpha + 1$,

(4) $\mathbb{P}^\kappa_{B_\alpha}$ is $\kappa + 1$-strategically closed.

In addition we construct for each i less than κ a $\leq^*$-descending sequence of $\mathbb{P}^\kappa * \mathbb{P}(\kappa, i)$-names

$$\langle q^i_\alpha(0), q^i_\alpha(1) : \alpha < \kappa^{++} \rangle$$

for conditions in $\mathbb{P}^i_{\text{tail}}$, and a descending sequence of $\mathbb{P}^\kappa * \mathbb{P}(\kappa, i) * \mathbb{P}^i_{\text{tail}}$-names

$$\langle s^i_\alpha(0), s^i_\alpha(1) : \alpha < \kappa^{++} \rangle,$$

where $s^i_\alpha(0)$ and $s^i_\alpha(1)$ are forced to be conditions in $j^\kappa_i(\mathbb{P}^\kappa_{B_\alpha})$ such that for all p in G^κ_α, $s^i_\alpha(0) \leq j^\kappa_i(\pi_\alpha(p))$ (i.e., $s^i_\alpha(0)$ is a master condition). Both of the sequences will constitute a run of the game in their respective posets, with the 0 condition interpreted as Player II's play and the 1 condition as Player I's play.

Let $B_0 = \emptyset$ and for each i less than κ let $q^i_0(0) = 1$ and $s^i_0(1) = 1$. Fix $\alpha < \kappa^{++}$ and suppose that $\mathbb{P}^\kappa_{B_\alpha}$ and π_α are defined, and for all i less than κ, $q^i_\alpha(0)$ and $s^i_\alpha(0)$ are defined.

We consider several cases. First suppose that $\dot{Y}_\alpha$ is not a $D^\kappa_{B_\alpha}$-name. In this case let $B_{\alpha+1}$ be a $\mathbb{P}^\kappa_{B_\alpha}$-name for B_α. So $\mathbb{P}^\kappa_{B_\alpha}$ forces that α is not in $B_{\alpha+1}$. Let $\mathbb{P}^\kappa_{B_{\alpha+1}}$ be equal to $\mathbb{P}^\kappa_{B_\alpha} * \langle 0 \rangle$. Clearly properties (1) through (4) hold.

Fix i less than κ. Let $s^i_\alpha(1) = s^i_\alpha(0)$ and $q^i_\alpha(1) = q^i_\alpha(0)$. Let $q^i_{\alpha+1}(0)$ be a name for Player II's response to the run of the game so far in $\mathbb{P}^i_{\text{tail}}$. Define $s^i_{\alpha+1}(0)$ as follows. The support of $s^i_{\alpha+1}(0)$ is $\operatorname{supp}(s^i_\alpha(1))$. Fix γ in $\operatorname{supp}(s^i_\alpha(1))$. Consider the run of the game in coordinate γ which begins when Player I plays $s^i_\beta(1)(\gamma)$ for the first value of β for which this term is defined. Define $s^i_{\alpha+1}(0)(\gamma)$ as Player II's response to this run of the game according to its strategy.

Now consider the case when $\dot{Y}_\alpha$ is a $D^\kappa_{B_\alpha}$-name. Fix i less than κ. Since $s^i_\alpha(0)$ is a master condition, $s^i_\alpha(0)$ forces that j^κ_i can be lifted to

$$j^\kappa_i : V[G_\kappa * G^\kappa_{B_\alpha}] \to N^\kappa_i[G_\kappa * G^\kappa * G(\kappa, i) * G^i_{\text{tail}} * g^i_\alpha]$$

where $G^\kappa_{B_\alpha}$ is the filter generated by $\pi_\alpha``(G^\kappa_\alpha)$ and g^i_α is a name for the generic for $j^\kappa_i(\mathbb{P}^\kappa_{B_\alpha})$. Choose $s^i_\alpha(1) \leq s^i_\alpha$ in $j^\kappa_i(\mathbb{P}^\kappa_{B_\alpha})$ which decides the statement

$$\sup j^\kappa_i``\kappa^+ \in j^\kappa_i(\dot{Y}_\alpha).$$

Choose $q^i_\alpha(1)$ as a name for a condition directly extending $q^i_\alpha(0)$ which decides which way that $s^i_\alpha(1)$ decides the given statement. Let $q^i_{\alpha+1}(0)$ be a name for Player II's response to the run of the game in $\mathbb{P}^i_{\text{tail}}$.

Let $B_{\alpha+1}$ be a $\mathbb{P}^\kappa_{B_\alpha}$-name which is forced to be equal to B_α unless for all i less than κ,

$$N^\kappa_i[G_\kappa * G^\kappa_{B_\alpha}] \models (\mathbb{P}^\kappa/\mathbb{P}^\kappa_{B_\alpha}) * \mathbb{P}(\kappa, i) \Vdash q^i_{\alpha+1}(0) \Vdash s^i_\alpha(1) \Vdash \sup j^\kappa_i``\kappa^+ \notin j^\kappa_i(\dot{Y}_\alpha),$$

in which case $B_{\alpha+1} = B_\alpha \cup \{\alpha\}$. Define $\mathbb{P}^\kappa_{B_{\alpha+1}}$ as $\mathbb{P}^\kappa_{B_\alpha} * \mathbb{Q}^\kappa_{B_\alpha}$, where $\mathbb{Q}^\kappa_{B_\alpha}$ is forced to be trivial if α is not in $B_{\alpha+1}$ and is equal to $\mathbb{P}_{\dot{Y}_\alpha}$ if α is in $B_{\alpha+1}$.

Note that if α is in $B_{\alpha+1}$, then for all i less than κ the condition $s^i_\alpha(1)$ forces that $\sup j^\kappa_i``\kappa^+ \notin j^\kappa_i(\dot{Y}_\alpha)$. Therefore $\bigcup j^\kappa_i``H^\kappa_\alpha$ is a master condition in $j^\kappa_i(\mathbb{P}_{\dot{Y}_\alpha})$. Define $s^i_{\alpha+1}(0) \restriction j^\kappa_i(\alpha)$ exactly as $s^i_{\alpha+1}(0)$ was defined in the last case by using the strategy at each coordinate in the support. If α is not in $B_{\alpha+1}$ then let $s^i_{\alpha+1}(0) = s^i_{\alpha+1}(0) \restriction j^\kappa_i(\alpha)$. If α is in $B_{\alpha+1}$ then let

$$s^i_{\alpha+1}(0) = (s^i_{\alpha+1}(0) \restriction j^\kappa_i(\alpha)) \cup \langle j^\kappa_i(\alpha), \bigcup j^\kappa_i``H^\kappa_\alpha\rangle.$$

Note that by the induction hypotheses and the comments above, $s^i_{\alpha+1}(0)$ is a master condition in $k_i(\mathbb{P}^\kappa_{B_{\alpha+1}})$.

Lemma 35. *The poset $\mathbb{P}^\kappa_{B_{\alpha+1}}$ is $\kappa+1$-strategically closed and $D^\kappa_{B_{\alpha+1}}$ is dense in $\mathbb{P}^\kappa_{B_{\alpha+1}}$.*

Proof. Straightforward. □

Lemma 36. *There is a projection mapping $\pi_{\alpha+1} : D^\kappa_{\alpha+1} \to \mathbb{P}^\kappa_{B_{\alpha+1}}$ such that:*
(a) $\pi_{\alpha+1} \restriction D^\kappa_{B_{\alpha+1}}$ is the identity,
(b) For all q in $D^\kappa_{\alpha+1}$ and $\beta < \alpha+1$, $\pi_{\alpha+1}(q) \restriction \beta = \pi_\beta(q \restriction \beta)$.

Proof. For each set x in $V[G_\kappa]$ let $\dot{a}_x$ be a $\mathbb{P}^\kappa_{B_\alpha}$-name which is forced to be equal to $\check{x}$ if α is in $B_{\alpha+1}$ and is 1 otherwise. Define $\pi_{\alpha+1}(p^\frown\check{x}) = \pi_\alpha(q)^\frown\dot{a}_x$. Statements (a) and (b) follow from induction hypotheses (1) and (2). □

This completes the definition for successor stages. Suppose that $\delta \leq \kappa^{++}$ is a limit ordinal and $\mathbb{P}^\kappa_{B_\alpha}$ is defined for $\alpha < \delta$. Define $\mathbb{P}^\kappa_{B_\delta}$ as the set of functions p with domain a subset of δ with size no larger than κ such that $p \restriction \alpha$ is in $\mathbb{P}^\kappa_{B_\alpha}$ for $\alpha < \delta$. Let B_δ be a $\mathbb{P}^\kappa_{B_\delta}$-name for $\bigcup\{B_\alpha : \alpha < \delta\}$.

Lemma 37. *The poset $\mathbb{P}^\kappa_{B_\delta}$ is $\kappa+1$-strategically closed and $D^\kappa_{B_\delta} = D^\kappa_\delta \cap \mathbb{P}^\kappa_{B_\delta}$ is dense in $\mathbb{P}^\kappa_{B_\delta}$.*

Proof. The proof is identical to Propositions 11 and 12. □

Fix i less than κ. Define $q^i_\delta(0)$ as a name for Player II's response to the run of the game defined thus far in $\mathbb{P}^i_{\mathrm{tail}}$. Define $s^i_\delta(0)$ as follows. The support of $s^i_\delta(0)$ is equal to $\bigcup\{\mathrm{supp}(s^i_\alpha(0)) : \alpha < \delta\}$. If γ is in $\mathrm{supp}(s^i_\delta(0))$, define $s^i_\delta(\gamma)$ by applying the strategy to the run of the game determined on coordinate α.

Lemma 38. *There is a projection mapping $\pi_\delta : D^\kappa_\delta \to \mathbb{P}^\kappa_{B_\delta}$ such that:*
(a) $\pi_\delta \restriction D^\kappa_{B_\delta}$ is the identity,
(b) for all p in D^κ_δ and $\alpha < \delta$, $\pi_\delta(p) \restriction \alpha = \pi_\alpha(p \restriction \alpha)$.

Proof. Define $\pi_\delta(p) = \bigcup\{\pi_\alpha(p \restriction \alpha) : \alpha < \delta\}$. The induction hypotheses (1) and (2) imply that the definition of π_δ makes sense and (a) and (b) hold. We show that π_δ is a projection mapping.

Obviously $\pi_\delta(1) = 1$ and $q \leq p$ implies $\pi_\delta(q) \leq \pi_\delta(p)$. Suppose that $q \leq \pi_\delta(p)$ in $\mathbb{P}^\kappa_{B_\delta}$. We find $r \leq p$ in D^κ_δ such that $\pi_\delta(r) \leq q$. If $\mathrm{cf}(\delta) \geq \kappa^+$ then the supports of p and q are bounded below δ, so r exists by the induction hypotheses.

Suppose that $\mathrm{cf}(\delta) \leq \kappa$. Let $\langle \alpha_i : i < \mathrm{cf}(\delta)\rangle$ be increasing and unbounded in δ, with $\alpha_0 = 0$. We define $\langle r_i, r^*_i : i < \mathrm{cf}(\delta)\rangle$ so that r_i is in $D^\kappa_{\alpha_i}$ and is below $p \restriction \alpha_i$.

Let $r_0 = \emptyset$. Suppose that $i < \mathrm{cf}(\delta)$ and r_i is defined such that $\pi_{\alpha_i}(r_i) \leq q \restriction \alpha_i$. Since $q \leq \pi_\delta(p)$,

$$\pi_{\alpha_i}(r_i)^\frown(q \restriction [\alpha_i, \alpha_{i+1})) \leq \pi_{\alpha_{i+1}}(r_i{}^\frown(p \restriction [\alpha_i, \alpha_{i+1})))$$

in $\mathbb{P}^\kappa_{B_{\alpha_{i+1}}}$. Apply the fact that $\pi_{\alpha_{i+1}}$ is a projection mapping to obtain $r_i^* \leq r_i{}^\frown(p \restriction [\alpha_i, \alpha_{i+1}))$ in $D^\kappa_{\alpha_{i+1}}$ such that

$$\pi_{\alpha_{i+1}}(r_i^*) \leq \pi_{\alpha_i}(r_i)^\frown(q \restriction [\alpha_i, \alpha_{i+1})).$$

Now define r_{i+1} as follows. The support of r_{i+1} is the same as the support of r_i^*. For α in the support of r_{i+1}, we consider the following run of the game. Player I begins by playing condition $r_j^*(\alpha)$ for the first value of j for which this term is defined. Player II responds according to its strategy with $r_{j+1}(\alpha)$ and at limit stages. The last play is $r_i^*(\alpha)$ defined above. Let $r_{i+1}(\alpha)$ be the condition obtained by applying the strategy to this run of the game.

Suppose $\delta_0 \leq \delta$ is a limit ordinal. Define r_{δ_0} with support $\bigcup\{\mathrm{supp}(r_i) : i < \delta_0\}$, such that for each α in this support, $r_{\delta_0}(\alpha)$ is obtained by applying the strategy to the run of the game at coordinate α.

Let r be an extension of r_δ in $D^\kappa_{B_\delta}$. Then $r \leq p$ and $\pi_\delta(r) \leq q$. $\square$

This completes the definition of $\mathbb{P}^\kappa_B$. By Lemma 2, $\mathbb{P}^\kappa_B$ and $\mathbb{P}^\kappa/\mathbb{P}^\kappa_B$ are κ^{++}-c.c. Suppose that G^κ_B is generic for $\mathbb{P}^\kappa_B$ over $V[G_\kappa]$.

Proposition 15. *In $V[G_\kappa * G^\kappa_B]$, $NS \restriction S$ is saturated.*

Proof. For each set $X \subseteq \kappa^+$, $i < \kappa$, and $\alpha < \kappa^{++}$, let $\Phi(X, i, \alpha)$ be the statement

$$N^\kappa_i[G_\kappa * G^\kappa_B] \models (\mathbb{P}^\kappa/\mathbb{P}^\kappa_B) * \mathbb{P}(\kappa, i) \Vdash q^i_{\alpha+1}(0) \Vdash s^i_\alpha(1) \Vdash \sup j^\kappa_i``\kappa^+ \notin j^\kappa_i(X).$$

Let T be a stationary subset of S. If a is in T, then $a \cap \kappa$ is in A and $o(a \cap \kappa)$ is less than $a \cap \kappa$. Therefore the map $a \mapsto o(a \cap \kappa)$ is regressive. By Fodor's Lemma, there exists an index i less than κ such that $T \cap S_i$ is stationary. So in order to prove that $NS \restriction S$ is saturated, it suffices to prove that $NS \restriction S_i$ is saturated for all i less than κ.

Fix i less than κ and let T be a stationary subset of S_i. Fix $X \subseteq \kappa^+$ such that

$$T = \{a \in S : \sup a \in X\}.$$

We claim that T is non-stationary iff there is α less than κ^{++} such that $\Phi(X, i, \alpha)$ holds.

Suppose that T is disjoint from a club set C. Since $\mathbb{P}^\kappa_B$ is κ^{++}, there is $\alpha < \kappa^{++}$ such that C and T are in $V[G_\kappa * G^\kappa_{B_\alpha}]$. Fix $p^\frown s$ in $G_\kappa * G^\kappa_{B_\alpha}$ which forces that $\dot{T} = \{a \in S : \sup a \in \dot{X}\}$ and $\dot{C} \cap \dot{T}$ is empty, where $\dot{T}$, $\dot{X}$, and $\dot{C}$ are names for T, X, and C. Since $s^i_\alpha(0)$ is a master condition,

$$N^i_\kappa[G_\kappa * G^\kappa_{B_\alpha}] \models (\mathbb{P}^\kappa/\mathbb{P}^\kappa_{B_\alpha}) * \mathbb{P}(\kappa, i) * \mathbb{P}^i_{\mathrm{tail}} \Vdash s^i_\alpha(0) \Vdash j^\kappa_i(\dot{C}) \cap j^\kappa_i(\dot{T}) = \emptyset.$$

But it is also forced that $j^\kappa_i``\kappa^+ = \bigcup j^\kappa_i``\dot{C} \in j^\kappa_i(\dot{C})$, so $j^\kappa_i``\kappa^+$ is not in $j^\kappa_i(T)$, i.e. $\sup j^\kappa_i``\kappa^+$ is not in $j^\kappa_i(X)$. Therefore $\Phi(X, i, \alpha)$ holds.

Suppose on the other hand that there exists α such that $\Phi(X, i, \alpha)$ holds. Let Y be the closure of X under suprema of subsets with order type ω or ω_1. Since $\sup j^\kappa_i``\kappa^+$ does not have cofinality ω or ω_1, $\Phi(X, i, \alpha)$ implies $\Phi(Y, i, \alpha)$. The poset $\mathbb{P}^\kappa_B$ is κ^{++}-c.c, so there exists γ less than κ^{++} such that Y is in $V[G_\kappa * G^\kappa_{B_\gamma}]$. Moreover, we can choose γ so that there is a canonical $D^\kappa_{B_\gamma}$-name $\dot{Y}$ for Y and p

in $G^\kappa_{B_\gamma}$ which forces $\Phi(\dot{Y}, i, \alpha)$ and the other properties of Y stated above. Then there is $\beta < \kappa^{++}$ greater than α and γ such that $\dot{Y} = \dot{Y}_\beta$ and for all ξ less than κ,

$$N^\kappa_\xi[G_\kappa * G^\kappa_{B_\beta}] \models (\mathbb{P}^\kappa/\mathbb{P}^\kappa_{B_\beta}) * \mathbb{P}(\kappa, \xi) \Vdash q^\xi_{\beta+1}(0) \Vdash s^\xi_\beta(1) \Vdash \sup j^\kappa_\xi \text{``}\kappa^+ \notin j^\kappa_\xi(\dot{Y}_\beta).$$

So β is in B, and therefore G^κ_B adds a $\square^Y_\kappa$-sequence and destroys the stationarity of the set $\{a \in S(\kappa, \kappa^+) : \sup a \in Y\}$. But T is contained in this set, so it is non-stationary in $V[G_\kappa * G^\kappa_B]$.

Suppose for a contradiction that $NS \restriction S_i$ is not saturated. Let $\langle T_{X_\alpha} : \alpha < \kappa^{++}\rangle$ be a sequence of stationary subsets of S_i such that $T_{X_\alpha} \cap T_{X_\beta}$ is non-stationary for $\alpha < \beta$. Since T_{X_α} is stationary, there is p_α in $(\mathbb{P}^\kappa/\mathbb{P}^\kappa_B) * \mathbb{P}(\kappa, i)$ and γ_α such that

$$N^\kappa_i[G_\kappa * G^\kappa_B] \models p_\alpha \Vdash q^i_{\gamma_\alpha+1}(0) \Vdash s^i_{\gamma_\alpha}(1) \Vdash \sup j^\kappa_i \text{``}\kappa^+ \in j^\kappa_i(X_\alpha).$$

Then p_α and p_β are incompatible for $\alpha < \beta$. For if $q \leq p_\alpha, p_\beta$, then letting $\gamma = \max\{\gamma_\alpha, \gamma_\beta\}$,

$$N^\kappa_i[G_\kappa * G^\kappa_B] \models q \Vdash q^i_{\gamma+1}(0) \Vdash s^i_\gamma(1) \Vdash \sup j^\kappa_i \text{``}\kappa^+ \in j^\kappa_i(X_\alpha \cap X_\beta),$$

and therefore $T_{X_\alpha} \cap T_{X_\beta}$ is stationary. We have a contradiction since the poset $(\mathbb{P}^\kappa/\mathbb{P}^\kappa_B) * \mathbb{P}(\kappa, i)$ is κ^{++}-c.c. $\square$

Proposition 16. *In $V[G_\kappa * G^\kappa_B]$ let μ be a regular cardinal less than κ. Then the set*

$$\{a \in S : \mathrm{cf}(a \cap \kappa) = \mu\}$$

is stationary.

Proof. If a is in $S_\mu = \{a \in S : o(a \cap \kappa) = \mu\}$, then $\mathrm{cf}(a \cap \kappa) = \mu$. Let C be a club subset of $P_\kappa \kappa^+$ in $V[G_\kappa * G^\kappa_B]$. Since $\mathbb{P}^\kappa_B$ is κ^{++}-c.c., there is γ less than κ^{++} such that C is in $V[G_\kappa * G^\kappa_{B_\gamma}]$.

Let H be generic for $j^\kappa_\mu(\mathbb{P}_\kappa)/(\mathbb{P}_\kappa * \mathbb{P}^\kappa_B)$ over $V[G_\kappa * G^\kappa_B]$ and let h be generic for $j^\kappa_\mu(\mathbb{P}^\kappa_{B_\gamma})$ over $V[G_\kappa * G^\kappa_B * H]$ which contains the condition $s^\mu_\gamma(0)$. Since $s^\mu_\gamma(0)$ is a master condition, in $V[G_\kappa * G^\kappa_B * H * h]$ we can lift j^κ_μ to

$$j^\kappa_\mu : V[G_\kappa * G^\kappa_{B_\gamma}] \to N^\kappa_\mu[G_\kappa * G^\kappa_B * H * h].$$

Recall that $j^\kappa_\mu \text{``}\kappa^+$ is in $j^\kappa_\mu(S_\mu)$. Also $j^\kappa_\mu \text{``}\kappa^+ = \bigcup j^\kappa_\mu \text{``}C$ is in $j^\kappa_\mu(C)$. By elementarity, $S_\mu \cap C$ is non-empty. $\square$

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