ADDI NG CLUBS WITH SQUARE

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Abstract. We present a technique for destroying stationary subsets of $P_\kappa^\kappa$ using partial square sequences. We combine this method with Gitik’s poset for changing the cofinality of a cardinal without adding bounded sets to prove a variety of consistency results concerning saturated ideals and the set $S(\kappa, \kappa^+)$. In this paper we continue our study of consistency results concerning the set $S(\kappa, \kappa^+)$ from [5] and [6]. We present a method for destroying the stationarity of certain subsets of $P_\kappa^\kappa$, where $\kappa$ is inaccessible, using partial square sequences.

This method of destroying stationary sets has a variety of applications. We prove several results which support the general theme that the structure of $S(\kappa, \kappa^+)$ can vary greatly depending on the particular model considered. For example, if $\kappa$ is supercompact and $\mu < \kappa$ is regular, then there exists a generic extension in which $S(\kappa, \kappa^+)$ is stationary and for almost all $a$ in $S(\kappa, \kappa^+)$, $a\cap \kappa$ is singular with cofinality $\mu$. On the other hand, it is relatively consistent that for almost all $a$ in $S(\kappa, \kappa^+)$, $a\cap \kappa$ is measurable with Mitchell order $(a\cap \kappa)^{++}$. We also construct a model in which GCH holds and there is a stationary set $S \subseteq P_\kappa^\kappa$ such that $NS\upharpoonright S$ is saturated and for all regular $\mu < \kappa$, $\{a \in S : \text{cf}(a\cap \kappa) = \mu\}$ is stationary.

This material is related to our previous papers [5] and [6]. However, no prior acquaintance with these papers is required by the reader. Most of our consistency results use Easton support Prikry iterations. The theorems in this paper rely on Gitik’s technique from [3] for changing the cofinality of a cardinal without adding bounded sets. We present Gitik’s poset in complete detail.

The contents of the paper are as follows. In Section 1 we outline preliminaries and notation. Section 2 provides additional background on forcing posets and large cardinals. Section 3 describes Easton support Prikry iterations. In Section 4 we present our method for destroying stationary sets using partial square sequences. In Section 5 we construct a model in which almost all $a$ in $S(\kappa, \kappa^+)$ satisfy that $a\cap \kappa$ is a measurable cardinal with Mitchell order $(a\cap \kappa)^{++}$. We also construct a model in which GCH holds and there is a stationary set $S \subseteq P_\kappa^\kappa$ such that $NS\upharpoonright S$ is saturated and for all regular $\mu < \kappa$, $\{a \in S : \text{cf}(a\cap \kappa) = \mu\}$ is stationary.

Section 6 describes Gitik’s forcing poset for changing a cofinality without adding bounded sets. In Section 7 we show how to construct models in which we can control the cofinality of $a\cap \kappa$ for almost all $a$ in $S(\kappa, \kappa^+)$. In Section 8 we construct a model with a stationary set $S \subseteq P_\kappa^\kappa$ such that $NS\upharpoonright S$ is saturated, GCH holds, and for every regular $\mu$ less than $\kappa$, the set $\{a \in S : \text{cf}(a\cap \kappa) = \mu\}$ is stationary.

1. Preliminaries

We assume that the reader is familiar with iterated forcing, supercompact cardinals, and Prikry forcing; see [1] and [4].

If $\kappa$ is a regular uncountable cardinal and $\kappa \subseteq X$, define $P_\kappa X = \{a \subseteq X : |a| < \kappa, a \cap \kappa \in \kappa\}$. A subset of $P_\kappa X$ is club if it is closed under unions of $\subseteq$-increasing
sequences of length less than $\kappa$ and is cofinal in $P_\kappa X$. A set is stationary if it intersects every club. If $A$ is a subset of a club set $C$ with size less than $\kappa$ and is directed, i.e. for all $a$ and $b$ in $C$, there is $c$ in $C$ with $a \cup b \subseteq c$, then $\bigcup A$ is in $C$.

The ideal of non-stationary subsets of $P_\kappa \lambda$ for $\lambda \geq \kappa$ is denoted by $NS_{P_\kappa \lambda}$ or $NS$. If $S$ is stationary then $NS_{P_\kappa \lambda} \upharpoonright S$ denotes the ideal generated by the elements of $NS_{P_\kappa \lambda}$ along with the complement of $S$. An ideal on $P_\kappa \lambda$ is fine if it contains the set $\{ a \in P_\kappa \lambda : \xi \notin a \}$ for every $\xi < \lambda$. A function $f : P_\kappa X \to X$ is regressive if $f(a)$ is in $a$ for all $a$. An ideal $I$ is normal if for every set $S \subseteq P_\kappa \lambda$ not in $I$ and for every regressive function $f : S \to \lambda$, there is an $i < \lambda$ such that the set $\{ a \in S : f(a) = i \}$ is not in $I$.

If $I$ is an ideal on $P_\kappa \lambda$, the set $I^* = \{ P_\kappa \lambda \setminus A : A \in I \}$ is the dual filter of $I$. The collection of $I$-positive sets $I^+ = \{ A \subseteq P_\kappa \lambda : A \notin I \}$ is a forcing poset ordered by $A \leq B$ if $A \setminus B$ is in $I$.

An ideal $I$ is $\mu$-saturated if $I^+$ is $\mu$-c.c. If $I = NS \upharpoonright S$ for some stationary set $S$, then $I$ is $\mu$-saturated iff there is no family $\{ S_i : i < \mu \}$ of stationary subsets of $S$ such that $S_i \cap S_j$ is non-stationary for $i < j$. We say that an ideal $I$ on $P_\kappa \lambda$ is saturated if $I$ is $\lambda^+$-saturated.

If $I$ is an ideal on $P_\kappa \lambda$, the forcing poset $I^+$ adds a generic set $U$ which is an ultrafilter on the Boolean algebra $(\mathcal{P}(P_\kappa \lambda))^X$. If $M$ is the ultrapower of $V$ by $U$ in the generic extension, $M$ is called the generic ultrapower and the ultrapower map $j : V \to M$ is the generic elementary embedding. If $I$ is saturated then the generic ultrapower $M$ is well-founded and $^\lambda M \subseteq M$.

If $M \subseteq N$ are inner models of set theory and $\lambda$ is a cardinal, we say that $M$ is $\lambda$-closed in $N$ if $^\lambda M \cap N \subseteq M$. The model $M$ is $\lambda$-closed in $N$ iff $^\lambda On \cap N \subseteq M$.

A cardinal $\kappa$ is $\lambda$-supercompact if there is a normal fine ultrafilter on $P_\kappa \lambda$, or equivalently, there is an elementary embedding $j : V \to M$, where $M$ is a transitive inner model, such that crit$(j) = \kappa$, $j(\kappa) > \lambda$, and $M$ is $\lambda$-closed. If $U_0$ and $U_1$ are normal fine ultrafilters on $P_\kappa \lambda$, we write $U_0 \leq U_1$ if $U_0$ is in the ultrapower of $V$ by $U_1$. This ordering, called the Mitchell ordering, is transitive and well-founded.

When we mention an ultrafilter on $P_\kappa \lambda$ we always assume that it is non-principal and fine. If $U$ is an ultrafilter on $P_\kappa \lambda$, we write $\text{Ult}(V, U)$ for the transitive collapse of the ultrapower of $V$ by $U$. Suppose that $U$ is normal and $j : V \to M = \text{Ult}(V, U)$. Then $M$ is $\lambda^+$-closed, and for any function $f : P_\kappa \lambda \to V$, $[f]$ is equal to $j(f)(j^+ \lambda)$.

If $a$ is a set of ordinals then $\text{o.t.}(a)$ is the order type of $a$, and if $\text{o.t.}(a)$ is a limit ordinal then $\text{cf}(a)$ is $\text{cf}(\text{o.t.}(a))$. Note that $\text{cf}(a) = \text{cf}(\sup a)$.

Suppose that $\alpha$ is a limit ordinal which is equal to the ordinal exponent $\omega^\gamma$ for some $\gamma \geq 1$. Then any set $X \subseteq \alpha$ which is clubed in $\alpha$ has order type $\alpha$.

The expression $\theta \gg \kappa$ means that $\theta$ is larger than $2^{2^{|\kappa|}}$. If $A$ is a structure with underlying set $H(\theta)$, then the collection of elementary substructures of $A$ in $P_\kappa H(\theta)$ is a club set. If $N$ is such an elementary substructure, then $N \cap \kappa^+$ is closed under suprema of bounded subsets with order type different from $\text{cf}(N \cap \kappa)$.

If $F : \lambda^{<\omega} \to \lambda$ is a partial function and $A \subseteq \lambda$, we say that $F$ is Jonsson for $A$ if $A$ is closed under $F$, and whenever $B \subseteq A$ is closed under $F$, it follows that $|B| < |A|$. If $\kappa$ is a regular cardinal then there exists a Jonsson function for $\kappa^+$.

If $A$ is a set of ordinals, then $A$ is an Easton set if for any strongly inaccessible cardinal $\beta$, $A \cap \beta$ is bounded below $\beta$. 
We use the phrase forcing poset to indicate any ordering \( \langle P, \leq \rangle \) which is reflexive and transitive. We do not require that \( P \) be antisymmetric or separative. We usually assume that \( P \) has a maximum element 1; this should be clear from context.

Suppose that \( \kappa \) is regular and \( P \) is a forcing poset. We say that \( P \) is \( \kappa \)-distributive if whenever \( \{ D_i : i < \beta \} \) is a family of dense open subsets of \( P \) and \( \beta \) is less than \( \kappa \), then \( \bigcap D_i \) is dense open. Equivalently, \( P \) is \( \kappa \)-distributive if forcing with \( P \) does not add any new sequences of ordinals with order type less than \( \kappa \).

A forcing poset \( P \) is \( \lambda \)-strategically closed for an ordinal \( \lambda \) if there is a strategy for Player II in the following game: Player I starts the game by playing a condition \( p_1 \) in \( P \). Player II responds with a condition \( p_2 \leq p_1 \). The game continues in this manner, each player choosing a condition below the previous one, with Player I playing at odd stages and Player II at even successor stages. At limit stages Player II plays a condition below all the conditions played so far. Player II wins if it is able to play a condition at all stages below \( \lambda \). If \( P \) is \( \lambda + 1 \)-strategically closed, then \( P \) does not add any subsets to \( \lambda \).

Suppose that \( P \) is a forcing poset and \( \lambda \) is a cardinal. A canonical name for a subset of \( \lambda \) is a \( P \)-name of the form
\[
\{ \langle p, \hat{a} \rangle : p \in A_\alpha, \alpha < \lambda \}
\]
where \( A_\alpha \) is an antichain for each \( \alpha < \lambda \). If \( p \) forces that \( \hat{A} \) is a subset of \( \lambda \), then there is a canonical name \( \hat{B} \) for a subset of \( \lambda \) such that \( \hat{p} \) forces that \( \hat{A} = \hat{B} \). If \( \hat{X} \) is a \( P \)-name and \( P \) forces that \( \hat{f} : \lambda \rightarrow \hat{X} \) is a bijection, then a canonical name for a subset of \( \hat{X} \) is a name of the form
\[
\{ \langle p, \hat{f}(\alpha) \rangle : p \in A_\alpha, \alpha < \lambda \}
\]
where each \( A_\alpha \) is an antichain. If \( p \) forces that \( \hat{A} \) is a subset of \( \hat{X} \), then there is a canonical name \( \hat{B} \) for a subset of \( \hat{X} \) such that \( \hat{p} \) forces that \( \hat{A} = \hat{B} \).

Suppose that \( \lambda \) is a regular uncountable cardinal and \( P \) is a forcing poset which is \( \lambda \)-c.c. and is a subset of \( H(\lambda) \). Let \( G \) be generic for \( P \) over \( V \). If \( a \) is in \( H(\lambda)^V[G] \), then there is a \( P \)-name \( \hat{x} \) in \( H(\lambda)^V \) such that \( \hat{x}^G = a \). Therefore \( H(\lambda)^V[G] = H(\lambda)^V[G] \).

If \( H \) is a subset of a forcing poset \( \mathbb{P} \), we say that \( H \) generates \( J \) if \( J \) is the set of \( q \) in \( \mathbb{P} \) such that there is \( p \) in \( H \) with \( p \leq q \).

Suppose that \( \mathbb{P} \) is a forcing poset and \( \{ U_i : i < \beta \} \) is a \( \mathbb{P} \)-increasing sequence of normal ultrafilters on \( \mathbb{P}, \lambda \). We say that \( \mathbb{P} \) forces that the sequence can be lifted if \( \mathbb{P} \) forces that there are normal ultrafilters \( U_i \subseteq U_i^\ast \) for \( i < \beta \) such that \( U_i^\ast < U_j^\ast \) for \( i < j \).

We say that forcing posets \( \mathbb{P} \) and \( \mathbb{Q} \) are equivalent if they have the same generic extensions. This is only an intuitive definition, not a formal one, since generic filters for non-trivial posets do not exist in the universe. To prove that forcing posets \( \mathbb{P} \) and \( \mathbb{Q} \) are equivalent in this informal sense, we show how to construct a generic filter for one poset, given a generic filter for the other. The following lemma provides a sufficient condition for the equivalence of forcing posets.

**Lemma 1.** Suppose that \( \mathbb{P} \) and \( \mathbb{Q} \) are forcing posets, \( D \) is a dense subset of \( \mathbb{P} \), and \( E \) is a dense subset of \( \mathbb{Q} \). Assume there is a surjective mapping \( i : D \rightarrow E \) satisfying:

1. \( q \leq p \) implies \( i(q) \leq i(p) \),
(2) If $p$ and $q$ are incompatible, then $i(p)$ and $i(q)$ are incompatible.

Then $\mathbb{P}$ and $\mathbb{Q}$ are equivalent.

A mapping $\pi : \mathbb{Q} \to \mathbb{P}$ between forcing posets is a projection mapping if it satisfies:

1. $q \leq p$ implies $\pi(q) \leq \pi(p)$,
2. $\pi(1) = 1$,
3. if $p \leq \pi(q)$ then there is $r \leq q$ such that $\pi(r) \leq p$.

Conditions (2) and (3) imply that $\pi -\text{c.c.}$ is dense in $\mathbb{P}$. If $G$ is generic for $\mathbb{Q}$ over $V$, then $\pi^*G$ generates a generic filter for $\mathbb{P}$ over $V$.

Suppose that $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection mapping. Let $G$ be generic for $\mathbb{P}$ over $V$. In $V[G]$, define a poset $\mathbb{Q}/\mathbb{P}$ as follows. A condition in $\mathbb{Q}/\mathbb{P}$ is any $q$ in $\mathbb{Q}$ such that $\pi(q) \in G$. Let $q \leq p$ in $\mathbb{Q}/\mathbb{P}$ iff $q \leq p$ in $\mathbb{Q}$. Now let $\mathbb{Q}/\mathbb{P}$ be a $\mathbb{P}$-name for this poset. Define $k : \mathbb{Q} \to \mathbb{P} \ast (\mathbb{Q}/\mathbb{P})$ by letting $k(q) = \pi(q) \ast \dot{q}$. Then $k$ satisfies:

1. $q \leq p$ implies $k(q) \leq k(p)$,
2. if $p$ and $q$ are incompatible, then $k(p)$ and $k(q)$ are incompatible,
3. $k^*\mathbb{Q}$ is dense in $\mathbb{P} \ast (\mathbb{Q}/\mathbb{P})$.

So by Lemma 1, $\mathbb{Q}$ and $\mathbb{P} \ast (\mathbb{Q}/\mathbb{P})$ are equivalent.

Lemma 2. Suppose that $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection mapping and $\mu$ is a regular uncountable cardinal. Assume that $\mathbb{Q}$ is $\mu$-c.c. Then:

1. $\mathbb{P}$ is $\mu$-c.c.,
2. $\mathbb{P}$ forces that $\mathbb{Q}/\mathbb{P}$ is $\mu$-c.c.

Proof. Since $\mathbb{Q}$ is $\mu$-c.c., so is $\mathbb{P} \ast (\mathbb{Q}/\mathbb{P})$. If $A$ is an antichain in $\mathbb{P}$, then the family $\{p \ast 1 : p \in A\}$ is an antichain in $\mathbb{P} \ast (\mathbb{Q}/\mathbb{P})$. So $\mathbb{P}$ is $\mu$-c.c. Suppose that $p$ forces that $\{\dot{q}_i : i < \xi\}$ is an antichain in $\mathbb{Q}/\mathbb{P}$. Then $p$ forces that $\dot{q}_i$ is incompatible with $\dot{q}_j$ for $i$ less than $j$. It follows that the family $\{p \ast \dot{q}_i : i < \xi\}$ is an antichain in $\mathbb{P} \ast (\mathbb{Q}/\mathbb{P})$, and so $\xi$ is less than $\mu$. Therefore $\mathbb{P}$ forces that $\mathbb{Q}/\mathbb{P}$ is $\mu$-c.c.

A Prikry type forcing poset is a triple $(\mathbb{Q}, \leq, \leq^*)$ such that $(\mathbb{Q}, \leq)$ and $(\mathbb{Q}, \leq^*)$ are forcing posets, $q \leq^* p$ implies $q \leq p$, and $\mathbb{Q}$ satisfies the Prikry property: if $p$ is a condition in $\mathbb{Q}$ and $\varphi$ is a statement in the forcing language for $(\mathbb{Q}, \leq)$, then there is $q \leq^* p$ such that $q$ decides $\varphi$. If $q \leq^* p$ we say that $q$ is a direct extension or a direct refinement of $p$.

We say that $\mathbb{Q}$ is $\alpha$-weakly closed if $(\mathbb{Q}, \leq^*)$ is $\alpha$-closed, and $\alpha$-weakly strategically closed if $(\mathbb{Q}, \leq^*)$ is $\alpha$-strategically closed. The poset $\mathbb{Q}$ has the direct extension property if whenever $q, r \leq^* p$, there is $s \leq^* q, r$.

2. Background on Forcing and Large Cardinals

In this section we review some additional material on large cardinals and forcing.

Suppose that $\kappa \leq \lambda_0 \leq \lambda_1$ and $U$ is a normal ultrafilter on $P_{\kappa}\lambda_1$. Define $U \restriction \lambda_0$ by letting $X$ be in $U \restriction \lambda_0$ iff $X \subseteq P_{\kappa}\lambda_0$ and the set $\{a \in P_{\kappa}\lambda_1 : a \cap \lambda_0 \in X\}$ is in $U$. Equivalently, if $j : V \to M = \text{Ult}(V, U)$, $X$ is in $U \restriction \lambda_0$ iff $j^*\lambda_0 \in j(X)$. The set $U \restriction \lambda_0$ is a normal ultrafilter on $P_{\kappa}\lambda_0$.

Lemma 3. Suppose that $\kappa \leq \lambda_0 < 2^{\lambda_0^+} \leq \lambda_1$ are cardinals and $U$ is a normal ultrafilter on $P_{\kappa}\lambda_1$. Let $j : V \to M = \text{Ult}(V, U)$ and $i : V \to N = \text{Ult}(V, U \restriction \lambda_0)$. Then there is an elementary embedding $k : N \to M$ such that $j = k \circ i$ and $\text{crit}(k) = (2^{\lambda_0^+})^+ N$. 

Proof. Define $k$ as follows. Let $a$ be in $N$ and let $f : P_\kappa \lambda_0 \to V$ be a function such that $[f] = a$. Define $k(a) = j(f)(j^* \lambda_0)$. It is easy to check that $k$ is a well-defined elementary embedding and $j = k \circ i$.

If $\beta \leq \lambda_0$, then $\beta = \text{o.t.}(i^* \beta) = \text{o.t.}(i^* \lambda_0 \cap i(\beta))$. So $\beta$ is represented by the function $a \mapsto \text{o.t.}(a \cap \beta)$ in $N$. By the definition of $k$, $k(\beta) = \text{o.t.}(j^* \lambda_0 \cap j(\beta)) = \text{o.t.}(j^* \beta) = \beta$ in $M$. So $\text{crit}(k) > \lambda_0$.

Since $N$ is $\lambda_0^{< \kappa}$-closed and $\text{crit}(k) > \lambda_0$, $k(P_\kappa \lambda_0) = P_\kappa \lambda_0$ and $k(P(P_\kappa \lambda_0)) = P(P_\kappa \lambda_0)$. If $a$ is in $P_\kappa \lambda_0$, then $k(a)$ is in $k(P_\kappa \lambda_0) = P_\kappa \lambda_0$, and $a \in k(a)$ iff $k(a) \in k(a)$ iff $a \in a$. Therefore $k \upharpoonright P_\kappa \lambda_0$ is the identity. The same argument shows that $k \upharpoonright P(P_\kappa \lambda_0)$ is the identity.

We prove by induction that for all $\beta$ less than $(2^{< \kappa})^+ + N$, $k(\beta) = \beta$. Fix $\beta$ and suppose $k(\alpha) = \alpha$ for all $\alpha$ less than $\beta$. Since $\beta$ is less than $(2^{< \kappa})^+ + N$, there is a surjective function $s : P(\kappa) \to \beta$ in $N$. By elementarity, $k(s)$ is a surjection of $k(P(\kappa)) = P(\kappa)$ onto $k(\beta)$. If $k(\beta) > \beta$, then there is $A$ in $P(\kappa)$ such that $\beta = k(s)(A) = k(s)(k(A)) = k(s(A)) = s(A) < \beta$, which is absurd.

Now $(2^{< \kappa})^+ + N < i(\kappa) < (2^{< \kappa})^+ = (2^{< \kappa})^+ + M$, and $k((2^{< \kappa})^+ + N) = (2^{< \kappa})^+ + M$. So $\text{crit}(k) = (2^{< \kappa})^+ + N$.

The following lemma is the main tool for extending elementary embeddings.

**Lemma 4.** Suppose that $j : M \to N$ is an elementary embedding between transitive models of set theory, $\mathbb{P}$ is a forcing poset in $M$, $G$ is generic for $\mathbb{P}$ over $M$, and $H$ is generic for $j(\mathbb{P})$ over $N$.

Then $j$ can be lifted to $j : M[G] \to N[H]$ iff $j^* G \subseteq H$. In this case, $j(G) = H$. In particular, $j$ can be lifted if there exists a condition $s$ in $H$ such that $s \leq j(p)$ for all $p$ in $G$.

**Proof.** The mapping $j(\dot{x}) = j(\dot{x})^H$ is well defined and satisfies the required properties.

We will use Silver’s notation and refer to a condition $s$ in $j(\mathbb{P})$ such that $s \leq j(p)$ for all $p$ in $G$ as a master condition.

**Lemma 5.** Suppose that $U$ is a normal ultrafilter on a cardinal $\kappa$ and $j : V \to M = \text{Ult}(V, U)$. Let $\mathbb{P}$ be a forcing poset and $G$ a generic filter for $\mathbb{P}$ over $V$. Suppose that in $V[G]$ there a generic filter $H$ for $j(\mathbb{P})$ over $M$ such that $j^* G \subseteq H$. Lift $j$ to $j : V[G] \to M[H]$.

In $V[G]$ define $U^*$ by letting $X \in U^*$ iff $X \subseteq \kappa$ and $\kappa \in j(X)$. Then $U^*$ is a normal ultrafilter extending $U$ and $M[H] = \text{Ult}(V[G], U^*)$.

**Proof.** We omit the standard argument that $U^*$ is a normal ultrafilter extending $U$. Since any isomorphism between transitive models of set theory is the identity, it suffices to prove that $M[H]$ and $\text{Ult}(V[G], U^*)$ are isomorphic.

Define $k : M[H] \to \text{Ult}(V[G], U^*)$ as follows. Let $a$ be in $M[H]$. Then there is a $j(\mathbb{P})$-name $\dot{x}$ such that $\dot{x}^H = a$. Since $\dot{x}$ is in $M$, we can choose $\dot{f} : \kappa \to V$ such that $[\dot{f}] = \dot{x}$. Moreover, choose $\dot{f}$ so that $\dot{f}(\alpha)$ is a $\mathbb{P}$-name for all $\alpha$. In $V[G]$ define $g : \kappa \to V[G]$ by letting $g(\alpha) = f(\alpha)^G$. Now let $k(a) = [g]$.

First we show that $g$ is well-defined. Suppose that $\alpha = \dot{x}^H = \dot{y}^H$, where $[f_\alpha] = \dot{x}$ and $[f_\beta] = \dot{y}$. Then there exists a condition $q$ in $H = j(G)$ which forces over $M$ that $\dot{x} = \dot{y}$. In $M$ let $q = [h]$ and write $h(\alpha) = p_\alpha$. Then there exists a set $A$ in $U$ such that for all $\alpha$ in $A$, $p_\alpha$ forces that $f_\alpha(\alpha) = f_\beta(\alpha)$. In $M[H]$, $q$ is in $j(G)$. But
such that for all $h$, $p_h$ is in $G$. Then for all $\alpha$ in $A^*$, $f_\alpha(p_h) = f_\alpha(g_h)$. Define $g_x$ and $g_y$ by $g_x(\alpha) = f_x(\alpha)G$ and $g_y(\alpha) = f_y(\alpha)G$. Then for all $\alpha$ in $A^*$, $g_x(\alpha) = g_y(\alpha)$, so $\{g_x\} = \{g_y\}$. This proves that $k$ is well-defined.

A similar argument shows that $k$ is injective and that $a \in b$ iff $k(a) \in k(b)$. To show that $k$ is surjective, fix $[h]$ in $\text{Ult}(V[G], U^*)$. For each $\alpha$ let $\dot{b}_\alpha$ be a $\mathbb{P}$-name for $h(\alpha)$. Then $[\alpha \mapsto \dot{b}_\alpha]$ is a $j(\mathbb{P})$-name in $M$. Let $a = [\alpha \mapsto \dot{b}_\alpha]^H$, which is in $M[H]$. Using the definition of $k$, it is straightforward to check that $k(a) = [h]$. □

A standard way to extend an elementary embedding is to apply strategic closure to build a generic filter. Suppose that $M \subseteq N$ are models of set theory and $\lambda$ is an $N$-cardinal. Let $\mathbb{P}$ be a forcing poset in $M$ and $p$ in $\mathbb{P}$. Suppose that $N$ models that $\mathbb{P}$ is $\lambda$-strategically closed and has no more than $\lambda$ many maximal antichains in $M$. Enumerate all maximal antichains in $M$ as $\langle A_i : i < \lambda \rangle$. Applying strategic closure we can inductively define a decreasing sequence $\langle p_i : i < \lambda \rangle$ so that $p_0 = p$ and $p_{i+1}$ is below some member of $A_i$. This sequence of conditions generates a generic filter $H$ for $\mathbb{P}$ over $M$ which contains $p$.

The following two lemmas show how to obtain closure of generic extensions.

**Lemma 6.** Suppose that $M \subseteq N$ are models of set theory and $\lambda$ is a regular uncountable cardinal in $N$ such that $M$ is $<\lambda$-closed in $M$. If $\mathbb{P}$ is a forcing poset in $M$ which is $\lambda$-c.c. in $N$ and $G$ is generic for $\mathbb{P}$ over $N$, then $M[G]$ is $<\lambda$-closed in $N[G]$.

Proof. We prove that $\langle \lambda \rangle^N \cap N[G] \subseteq M[G]$. Suppose that $p$ forces over $N$ that $\dot{f} : \beta \rightarrow \lambda$ for some $\beta < \lambda$. For each $\alpha < \beta$ let $A_\alpha$ be a maximal antichain contained in the dense set of conditions which decide the value of $\dot{f}(\alpha)$. Let $X_\alpha$ be the set of pairs $(q, \gamma)$ such that $q \in A_\alpha$ and $q$ forces over $N$ that $\dot{f}(\alpha) = \gamma$. Then $|A_\alpha| = |X_\alpha| < \lambda$, and so $(A_\alpha, X_\alpha : \alpha < \beta)$ is in $M$. Define a name $\dot{g}$ in $M$ by letting $\dot{g}(\alpha)$ be the unique $\gamma$ so that there is $q$ in $\dot{G} \cap A_\alpha$ such that $(q, \gamma)$ is in $X_\alpha$. Clearly $p$ forces that $\dot{g} = \dot{f}$, and $\dot{g}$ is in $M[G]$. □

**Lemma 7.** Suppose $M \subseteq N$ are models of set theory, $\lambda$ is a regular cardinal in $N$ such that $M$ is $\lambda$-closed in $N$. If $\mathbb{P} \in M$ is a forcing poset, $G \in N$ is a generic filter for $\mathbb{P}$ over $M$, then $M[G]$ is $\lambda$-closed in $N$.

Proof. In $N$ we have $\langle \lambda \rangle^N \subseteq M \subseteq M[G]$. □

We need some facts about the Mitchell ordering.

**Lemma 8.** Let $U_0$ and $U_1$ be normal ultrafilters on $P_\alpha \lambda$. For each $a$ in $P_\alpha \lambda$, let $\pi_a : a \rightarrow \text{o.t.}(a)$ be the unique order preserving bijection. Then $U_0 \preceq U_1$ iff there exists a function $f : P_\alpha \lambda \rightarrow V_\kappa$ such that:

1. $\{a \in P_\alpha \lambda : f(a) \text{ is a normal ultrafilter on } (P_\alpha \cap \kappa \text{ o.t.}(a)) \}$ is in $U_1$,

2. For every $X \subseteq P_\alpha \lambda$, $X \in U_0$ iff the set of $a$ in $P_\alpha \lambda$ such that $X_a = \{b : (\exists c \in (P_\alpha \cap \kappa) \cap X)(\pi_a \upharpoonright c = b)\} \in f(a)$ is in $U_1$.

Proof. Let $j : V \rightarrow M = \text{Ult}(V, U_1)$. Let $[f] = U_0$ in $M$. In $M$, $[a \mapsto a] = j^*\lambda$, $[a \mapsto \cap \kappa] = \kappa$, and $[a \mapsto \text{o.t.}(a)] = \lambda$. Clearly (1) holds. For (2), check that $j(a \mapsto X_a)[j^\lambda] = X$, and therefore $[a \mapsto X_a] = X$ in $M$. So $X \in U_0$ iff $M \models X \in U_0$ iff $\{a \in P_\alpha \lambda : X_a \in f(a)\}$ is in $U_1$. The converse is similar. □
Corollary 1. Suppose that $U_0$ and $U_1$ are normal ultrafilters on $P_\kappa \lambda$ and $U_0 \prec U_1$. Let $\mathbb{P}$ be a forcing poset which does not add subsets to $\lambda^{<\kappa}$. Then $\mathbb{P}$ forces that $U_0$, $U_1$ are normal ultrafilters on $P_\kappa \lambda$ and $U_0 \prec U_1$.

Corollary 2. Suppose that $M$ is a transitive model of set theory which is $\lambda^{<\kappa}$-closed in $V$. If $U$ and $W$ are normal ultrafilters on $P_\kappa \lambda$ which are in $M$, then $U \prec W$ iff $M$ models that $U \prec W$.

Lemma 9. Suppose that $U_0 \prec U_1$ are normal ultrafilters on $P_\kappa \lambda$. Let $j_0 : V \rightarrow M_0 = \text{Ult}(V, U_0)$, $j_1 : V \rightarrow M_1 = \text{Ult}(V, U_1)$, and $j_{1,0} : M_1 \rightarrow N_0 = \text{Ult}(M_1, U_0)$. Then the following statements are true:

(1) If $f \in P_\kappa \lambda M_1$ then $[f]_{M_0} = [f]_{N_0}$, so $N_0 \subseteq M_0$.

(2) $j_0 \models M_1 = j_{1,0}$.

Suppose that $\mathbb{P}$ is a forcing poset in $M_1$, $G$ is generic for $\mathbb{P}$ over $V$, $H$ is generic for $j_0(\mathbb{P}) = j_{1,0}(\mathbb{P})$ over $M_0$, and $j_0 G = j_{1,0} G \subseteq H$. Extend $j_0$ and $j_{1,0}$ to $j_0 : V[G] \rightarrow M_0[H]$ and $j_{1,0} : M_1[G] \rightarrow N_0[H]$. Then $j_0 \models M_1[G] = j_{1,0}$.

Proof. Let $[f]_{M_0}$ be given such that $f$ is in $P_\kappa \lambda M_1$, and assume that (1) holds for all functions $g$ where $[g]_{M_0}$ has rank less than the rank of $[f]_{M_0}$. We prove that $[f]_{M_0} = [f]_{N_0}$. Note that $f$ is in $M_1$ by the closure of $M_1$, so $[f]_{N_0}$ is defined. Suppose that $x$ is in $[f]_{M_0}$. Since $f$ is in $P_\kappa \lambda M_1$, there is $g$ in $P_\kappa \lambda M_1$ and $A$ in $U_0$ such that $[g]_{M_0} = x$ and $g(a) \in f(a)$ for all $a$ in $A$. By induction, $[g]_{N_0} = x$, and clearly $[g]_{N_0}$ is in $[f]_{N_0}$. So $[f]_{M_0} \subseteq [f]_{N_0}$. The other direction is similar.

To prove (2), let $x$ be in $M_1$. Then $j_0(x) = [f]_{M_0}$ where $f(a) = x$ for all $a$ in $P_\kappa \lambda$. Since $f$ is in $P_\kappa \lambda M_1$, by (1) we have that $[f]_{M_0} = [f]_{N_0}$, and clearly $[f]_{N_0} = j_{1,0}(x)$.

The fact that $j_0 \models M_1[G] = j_{1,0}$ follows from (2) and the definition of the extended mappings given in the proof of Lemma 4. □

Lemma 10. Suppose that $\kappa$ is a measurable cardinal, $\mathbb{P}$ is a forcing poset with size less than $\kappa$, and $\langle U_i : i < \beta \rangle$ is a $\kappa$-increasing sequence of normal ultrafilters on $\kappa$. Then $\mathbb{P}$ forces that $\langle U_i : i < \beta \rangle$ can be lifted.

Proof. Without loss of generality we assume that $\mathbb{P}$ is in $V_\kappa$. For $i < \beta$ let $j_i : V \rightarrow M_i = \text{Ult}(V, U_i)$ and $k_i : M_{i+1} \rightarrow N_i = \text{Ult}(M_{i+1}, U_i)$. Then $j_i(\mathbb{P}) = k_i(\mathbb{P}) = \mathbb{P}$. By Lemma 9, $j_i \models M_{i+1} = k_i$.

Let $G$ be generic for $\mathbb{P}$ over $V$. Then $j_\beta G = G$ and we can lift each $j_i$ and $k_i$ to $j_i : V[G] \rightarrow M_i[G]$ and $k_i : M_{i+1}[G] \rightarrow N_i[G]$. By Lemma 6 each $M_i[G]$ is $\kappa$-closed in $V[G]$. By Lemma 9, $j_i \models M_{i+1}[G] = k_i$.

Define $U^*_i$ by letting $X \in U^*_i$ iff $\kappa \models j_i(X)$. Then $U^*_i$ is a normal ultrafilter on $\kappa$ extending $U_i$, and by Lemma 5 $\text{Ult}(V[G], U^*_i) = M_i[G]$. Moreover, since $\mathbb{P}(\kappa) \subseteq M_{i+1}[G]$ and $j_i \models M_{i+1}[G] = k_i$, $M_{i+1}[G]$ can compute $U^*_i$ using the same definition. So $U^*_i$ is in $M_{i+1}[G]$.

Fix $i < j$. Then $U^*_i$ is in $M_{i+1}[G] \subseteq N_j[G] = \text{Ult}(V[G], U^*_j)$. So $U^*_i \prec U^*_j$. □

3. Iterations of Prikry type forcing posets

In this section we present the basics of Easton support Prikry iterations which we use in our consistency proofs.

Magidor [7] showed how to iterate Prikry forcing over different cardinals using conditions with full support. To overcome the difficulty in extending an elementary embedding with such an iteration, Gitik [3] developed a method for iterating Prikry type forcing posets using conditions with Easton support.
An Easton support Prikry iteration is an iterated forcing
\[ \langle P_\alpha, Q_\alpha : \alpha < \kappa \rangle, \]
for some ordinal \( \kappa \), satisfying the following properties:

1. There exists a set \( A \subseteq \kappa \) consisting of strongly inaccessible cardinals such that if \( Q_\alpha \) is non-trivial, then \( \alpha \) is in \( A \),
2. \( P_\alpha \) forces that \( |Q_\alpha| < \min(A \setminus (\alpha + 1)) \),
3. \( P_\alpha \) forces that \( \langle Q_\alpha, \leq, <^* \rangle \) is a Prikry type forcing poset,
4. \( P_{\alpha+1} = P_\alpha * Q_\alpha \),
5. If \( \alpha \) is a limit ordinal, then \( P_\alpha \) consists of functions \( p \) with domain an Easton subset of \( \alpha \cap A \) such that \( p \restriction \beta \) is in \( P_\beta \) for all \( \beta \) less than \( \alpha \),
6. If \( \alpha \) is a limit ordinal and \( p, q \) are in \( P_\alpha \), then \( q \leq p \) if \( q \restriction \beta \leq p \restriction \beta \) for \( \beta \) less than \( \alpha \), and there is a finite set \( a \) such that for all \( \beta \) in \( \text{dom}(p) \setminus a \), \( q \restriction \beta \Vdash q(\beta) \leq^* p(\beta) \),
7. If \( \alpha \) is a limit ordinal, \( q \leq^* p \) in \( P_\alpha \) if \( q \leq p \) and the finite set \( a \) in (6) is empty; i.e. for all \( \beta \) in \( \text{supp}(p) \), \( q \restriction \beta \Vdash q(\beta) \leq^* p(\beta) \).
8. \( q \Vdash q^* \leq^* p^* \) in \( P_{\beta+1} \) if \( q \leq^* p \) in \( P_\beta \) and \( q \Vdash \hat{b} \leq^* \hat{a} \).

In (3) the Prikry poset \( Q_\alpha \) will usually not add bounded subsets to \( \alpha \). To iterate Prikry posets which do add bounded sets, such as Radin forcing, one can use the Magidor iteration; see [6] for an example.

If \( p \) is a condition in \( P_\alpha \), the support of \( p \), denoted by \( \text{supp}(p) \), is the domain of \( p \) as a function.

If \( \beta \) is less than \( \alpha \), then \( P_\alpha \) factors into \( P_\beta * Q_\beta * P_{\beta, \alpha} \). Suppose that \( Q_\gamma \), is forced to be \( \gamma \)-weakly strategically closed for all \( \gamma \) greater than \( \beta \); then \( \langle P_{\beta, \alpha}, \leq^* \rangle \) is forced to be \( \min(A \setminus (\beta + 1)) \)-weakly strategically closed. It will follow from Proposition 2 that \( P_{\beta, \alpha} \) does not add any bounded subsets to \( \min(A \setminus (\beta + 1)) \).

The following result follows from the usual proof of the corresponding fact for Easton support iterations.

**Proposition 1.** If \(|P_\beta| < \alpha\) for all \( \beta \) less than \( \alpha \) and \( \alpha \) is a Mahlo cardinal, then \( P_\alpha \) is \( \alpha \)-c.c.

We need the following lemma in order to prove that \( P_\alpha \) satisfies the Prikry property.

**Lemma 11.** Suppose that \( \beta \) is less than \( \alpha \), and factor \( P_\alpha = P_\beta * Q_\beta * P_{\beta, \alpha} \). Assume that \( p \in P_\beta \) and \( p \) forces that \( \check{a} \) is in \( Q_\beta * P_{\beta, \alpha} \). Then there is \( q \) in the ground model such that \( p^* q \) is a condition in \( P_\alpha \) and \( p \) forces that \( q \leq^* \check{a} \) in \( Q_\beta * P_{\beta, \alpha} \).

**Proof.** First we define the support of \( q \), which we call \( x \). Let \( \gamma \) be in \( x \) iff there is \( r \leq p \) such that \( r \) forces that \( \gamma \) is in the support of \( \check{a} \). We prove that \( x \) is an Easton set. Let \( \xi \) be a strongly inaccessible cardinal, which we can assume to be larger than \( \min(A \setminus (\beta + 1)) \). Then \( |P_\beta| < \xi \). Now \( p \) forces that \( \sup(\check{a}) \cap \xi \) is bounded below \( \xi \). Let \( Y \) be the set of \( \delta \) such that there is \( r \leq p \) which forces that \( \delta \) is the least ordinal larger than all the elements of \( \sup(\check{a}) \cap \xi \). Since \( P_\beta \) has size less than \( \xi \), \( \sup(Y) \) is below \( \xi \), and \( x \cap \xi \subseteq \sup(Y) \).

Now define \( q \) by induction. Suppose that \( \gamma \) is in \( x \) and \( q \restriction \gamma \) is already defined so that \( p^* (q \restriction \gamma) \) is in \( P_\gamma \). Then let \( q(\gamma) \) be a \( P_\gamma \)-name which \( p^* (q \restriction \gamma) \) forces is equal to \( \check{a}(\gamma) \), provided that \( \gamma \) is in \( \sup(\check{a}) \), and otherwise is some arbitrary element of \( Q_\gamma \).
Suppose that \( G \) is generic for \( P_\beta \) and contains \( p \). Let \( a = 0^G \). Then in \( V[G] \), \( \text{supp}(a) \subseteq x = \text{supp}(q) \), and if \( \gamma \) is in the support of \( a \), then \( q(\gamma) = a(\gamma) \). So \( p \) forces that \( q \leq^* a \) in \( Q_\beta \star P_\beta,\alpha \). \( \square \)

**Proposition 2.** The iteration \( (P_\alpha, \leq, \leq^*) \) is a Prikry type forcing poset.

**Proof.** We prove by induction on \( \alpha \) that \( P_\alpha \) satisfies the Prikry property. Suppose that \( \beta \) is less than \( \alpha \) and \( P_\beta \) satisfies the Prikry property. We prove the same is true for \( P_{\beta+1} = P_\beta \star Q_\beta \). Let \( \varphi \) be a name in the forcing language and let \( p^* \) be a condition. Fix a \( P_\beta \)-name \( b \) such that \( p \forces b \leq^* \dot{a} \) and \( \dot{b} \) decides \( \varphi \). Now apply the induction hypothesis to obtain \( q \leq^* p \) in \( P_\beta \) which decides which way that \( \dot{b} \) decides \( \varphi \). Then \( q \forces \dot{b} \leq^* \dot{a} \) and \( q \forces \dot{b} \) decides \( \varphi \).

Suppose that \( \alpha \) is a limit ordinal and for all \( \beta \) less than \( \alpha \), \( P_\beta \) satisfies the Prikry property. Let \( p \) be a condition in \( P_\alpha \) and \( \varphi \) a statement in the forcing language.

First assume that the support of \( p \) is bounded below \( \alpha \). Then there is \( \beta \leq \alpha \) such that \( p \) is in \( P_\beta \). Write \( P_\alpha = P_\beta \star Q_\beta \star P_{\beta,\alpha} \). Let \( \dot{a} \) be a \( P_\beta \)-name for a condition in \( Q_\beta \star P_{\beta,\alpha} \) which decides \( \varphi \). Apply the induction hypothesis to obtain \( p_0 \leq^* p \) which decides which way \( \dot{a} \) decides \( \varphi \). By Lemma 11, choose \( q \) so that \( p_0 \forces q \) is a condition in \( P_\alpha \) and \( p_0 \forces q \leq^* \dot{a} \). Then \( p_0 \forces q \) is a direct extension of \( p \) from which \( \varphi \) is decided.

Now assume that the support of \( p \) is unbounded in \( \alpha \). Suppose for a contradiction that \( p \) does not have a direct extension from which \( \varphi \) is decided. By construction, \( p \) has the same support as \( p \). Suppose that \( \gamma \) is in \( \text{supp}(p) \) and \( q \upharpoonright \gamma \) is defined. Then \( q \upharpoonright \gamma \copies p(\gamma) \) is in \( P_{\gamma+1} \) and \( Q_{\gamma+1} \) forces that \( p \upharpoonright (\gamma + 1) \) is in \( P_{\gamma,\alpha} \). Choose a \( P_\gamma \)-name \( q(\gamma) \) such that \( q \upharpoonright \gamma \forces q(\gamma) \leq^* p(\gamma) \) and \( q(\gamma) \) decides the following statement: there is a direct extension of \( p \upharpoonright (\gamma + 1) \) in \( P_{\gamma,\alpha} \) which decides \( \varphi \). This completes the definition of \( q \).

**Claim 1.** For all \( \gamma \) less than \( \alpha \), \( q \upharpoonright \gamma \forces q \) is no direct extension of \( p \upharpoonright \gamma \) in \( P_{\gamma,\alpha} \) which decides \( \varphi \).

**Proof.** We prove the claim by induction. By assumption, there is no direct extension of \( p \) which decides \( \varphi \). So the claim holds for \( \gamma \) less than \( \text{min}(A) \).

Suppose that \( \gamma \geq \text{min}(A) \) and the claim holds for all \( \gamma^+ \) less than \( \gamma \). First assume that \( \gamma = \gamma_0 + 1 \). Consider the case when \( \gamma_0 \) is not in \( \text{supp}(p) \). If the claim fails then without loss of generality there is \( r \forces \dot{c} \leq q \upharpoonright \gamma \) and \( \dot{a} \) such that \( r \forces \dot{c} \forces p \upharpoonright \gamma \) and \( a \) forces \( \varphi \). But then since \( \gamma_0 \) is not in \( \text{supp}(p) \), \( r \) forces that \( \dot{c} \forces p \upharpoonright \gamma \) which decides \( \varphi \). This contradicts the induction hypothesis.

Suppose now that \( \gamma_0 \) is in \( \text{supp}(p) \). Then \( q \upharpoonright \gamma_0 \forces q(\gamma_0) \) decides whether there is a direct extension of \( p \upharpoontright \gamma \) which decides \( \varphi \). So if the claim fails, there is \( r \leq q \upharpoonright \gamma_0 \) and \( \dot{b} \) such that

\[
r \forces q(\gamma_0) \forces \dot{b} \leq^* p \upharpoontright \gamma, \dot{b} \forces \varphi.
\]

It follows that \( r \) forces that \( q(\gamma_0) \forces \dot{b} \) is a direct extension of \( p \upharpoontright \gamma \) which decides \( \varphi \), contradicting the induction hypothesis.

Now suppose that \( \gamma \) is a limit ordinal. If the claim fails then without loss of generality there is \( r \leq q \upharpoonright \gamma \) and \( \dot{b} \) such that \( r \forces \dot{b} \leq^* p \upharpoontright \gamma \) and \( \dot{b} \) forces \( \varphi \). Let \( a \) be a finite subset of \( \gamma \) such that for all \( \xi \) in \( \text{supp}(q \upharpoontright \gamma) \setminus a \), \( r \forces r(\xi) \leq^* q(\xi) \). Since \( \gamma \) is a limit ordinal, there is \( \gamma^+ \) less than \( \gamma \) such that \( a \) is a subset of \( \gamma^+ \).
Therefore $r \upharpoonright \gamma^*$ forces that $(r \setminus \gamma^*) \upharpoonright b$ is a direct extension of $p \setminus \gamma^*$ in $P_{\gamma, \alpha}$ which forces $\varphi$. This contradicts the induction hypothesis that $q \upharpoonright \gamma^*$ forces there is no such direct extension.

Now choose $r \leq q$ in $P_{\alpha}$ which decides $\varphi$. Let $a$ be the set of $\xi$ in $\text{supp}(q)$ such that $r \upharpoonright \xi$ forces that $r(\xi)$ is a non-direct extension of $q$. Since the support of $p$ is unbounded in $\alpha$, there is $\gamma$ in $\text{supp}(q)$ such that $a \subseteq \gamma$. Then $r \upharpoonright \gamma$ is a condition below $q \setminus \gamma$ which forces that $r \setminus \gamma$ is a direct extension of $p \setminus \gamma$ which decides $\varphi$. This contradicts the fact that $q \upharpoonright \gamma$ forces there is no such direct extension.

If $P$ is an $\alpha$-strategically closed forcing poset, then the triple $(\langle P, \leq, \leq^* \rangle)$, where $\leq^* \leq$, is a Prikry type forcing poset which is $\alpha$-weakly strategically closed. So we can define Easton support Prikry iterations by combining usual Prikry forcings with strategically closed posets. Gitik showed that in some cases it is possible to use distributive posets as well, by turning them into Prikry type forcing posets using a projection mapping from a strongly compact Prikry forcing; see [5] for details.

4. ADDING CLUBS WITH SQUARE

Suppose for the remainder of this section that $\kappa$ is a weakly inaccessible cardinal. We present a method for destroying certain stationary subsets of $P_{\kappa} \kappa^+$.

**Lemma 12.** There is a club set of $a$ in $P_{\kappa} \kappa^+$ such that $o.t.(a) \leq (a \cap \kappa)^+$.  

**Proof.** Fix $\theta \gg \kappa$ regular and let $N$ be an elementary substructure of $(H(\theta), \in)$ in $P_{\kappa} H(\theta)$. Let $a = N \cap \kappa^+$. We claim that $o.t.(a) \leq (a \cap \kappa)^+$.

We show that for all $\beta$ in $a$, $|a \cap \beta| \leq |a \cap \kappa|$. This is clear if $\beta < \kappa$. Suppose that $\beta \geq \kappa$ is in $a$. Fix a bijection $f_\beta : \kappa \rightarrow \beta$ in $N$. By elementarity, for all $j$ in $a \cap \beta$, $f_\beta^{-1}(j)$ is in $a \cap \kappa$. Therefore $f_\beta \upharpoonright a \cap \kappa$ is a bijection of $a \cap \kappa$ onto $a \cap \beta$, and so $|a \cap \beta| = |a \cap \kappa|$. Since each initial segment of $a$ has size at most $|a \cap \kappa|$, clearly $o.t.(a) \leq (a \cap \kappa)^+$. 

Define $S(\kappa, \kappa^+)$ as the set of $a$ in $P_{\kappa} \kappa^+$ such that $o.t.(a) = (a \cap \kappa)^+$.

**Proposition 3.** If $\kappa$ is $\kappa^+$-supercompact then $S(\kappa, \kappa^+)$ is stationary.

**Proof.** Fix $j : V \rightarrow M$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \kappa^+$, and $M$ is $\kappa^+$-closed. Then $j^* \kappa^+$ is in $M$. Note that $j^* \kappa^+$ is in $j(S(\kappa, \kappa^+))$. Suppose that $C$ is a club subset of $P_{\kappa} \kappa^+$ in $V$. Then $j^* C$ is in $M$ and is a directed subset of $j(C)$ with size less than $j(\kappa)$. So $\bigcup j^* C = j^* \kappa^+$ is in $j(C)$. Therefore $j(S(\kappa, \kappa^+)) \cap j(C)$ is non-empty. By elementarity, $S(\kappa, \kappa^+) \cap C$ is non-empty. 

If $\kappa$ is subcompact, which is a weaker assumption than $\kappa^+$-supercompactness, then $S(\kappa, \kappa^+)$ is stationary. In [6] we prove that the strong compactness of $\kappa$ does not imply that $S(\kappa, \kappa^+)$ is stationary.

Let $B \subseteq \kappa^+$ be a set of limit ordinals. We say that $\square^B_{\kappa}$ holds if there exists a partial square sequence $(c_{\alpha} : \alpha \in B)$ satisfying:

1. $c_{\alpha}$ is club in $\alpha$,
2. if $\text{cf}(\alpha) < \kappa$ then $o.t.(c_{\alpha}) < \kappa$,
3. if $\beta$ is in $\text{lim}(c_{\alpha}) \cap B$ then $c_{\beta} = c_{\alpha} \cap \beta$.

Define a forcing poset $P_B$ which adds a $\square^B_{\kappa}$-sequence as follows. This poset is the obvious generalization of Jensen’s poset for adding a $\square_{\kappa}$-sequence. A condition in $P_B$ is a sequence $p = \langle p_\alpha : \alpha \in B \cap (\gamma + 1) \rangle$ for some $\gamma < \kappa^+$ satisfying (1), (2), and (3) above for all $\alpha$ in $B \cap (\gamma + 1)$. Let $q \leq p$ if $p$ is an initial segment of $q$.  


Proposition 4. The forcing poset $\mathbb{P}_B$ is $\kappa + 1$-strategically closed.

Proof. We describe a strategy by considering a run of the game. Suppose $(p_i : 0 < i < \beta)$ is the run of the game up to stage $\beta$, and it is Player II’s turn. For each $0 < i < \beta$ let $\gamma_i$ be the least ordinal $\gamma$ such that $p_i$ is of the form $\langle c_\alpha : \alpha \in B \cap (\gamma + 1) \rangle$. First assume $\beta = \alpha + 1$. Let $\xi$ be the least element of $B$ larger than $\gamma_\alpha$, and define $p_\beta = p_\alpha \cup \{\langle \xi, c_\xi \rangle\}$, where $c_\xi$ is any club subset of $\xi$ with order type $\text{cf}(\xi)$ and $\min c_\xi > \gamma_\alpha$.

Suppose now that $\beta$ is a limit ordinal. Let $\gamma_\beta = \bigcup \gamma_i$. If $\gamma_\beta$ is not in $B$, then let $p_\beta = \bigcup \{p_i : i < \beta\}$. If $\gamma_\beta$ is in $B$, then define $p_\beta$ as $\bigcup \{p_i : i < \beta\} \cup \{\langle \gamma_\beta, c_{\gamma_\beta} \rangle\}$, where $c_{\gamma_\beta} = \{\gamma_i : i < \beta\}$. If $\beta < \kappa$ then $c_{\gamma_\beta}$ has order type less than $\kappa$. Now suppose that $\gamma$ is a limit point of $c_{\gamma_\beta}$ in $B$. Then $\gamma = \gamma_i$ for some limit ordinal $i$ and $c_i = \{\gamma_j : j < i\} = c_{\gamma_\beta} \cap \gamma$. □

It follows that $\mathbb{P}_B$ does not add subsets to $\kappa$. Since $\mathbb{P}_B$ has size $2^\kappa$, if $2^\kappa = \kappa^+$ then $\mathbb{P}_B$ preserves all cardinalities and cofinalities.

We show how to use this forcing poset to destroy stationary sets.

Theorem 1. Suppose that $B$ is a subset of $\kappa^+$ consisting of limit ordinals and there are distinct regular cardinals $\delta_0$ and $\delta_1$ below $\kappa$ such that $B$ contains all its limit points with cofinality $\delta_0$ or $\delta_1$. Then if $\square^B_\kappa$ holds, the set

$$S = \{a \in S(\kappa, \kappa^+) : \sup a \in B\}$$

is non-stationary.

Proof. Let $\langle c_\beta : \beta \in B \rangle$ be a partial square sequence witnessing that $\square^B_\kappa$ holds. Suppose for a contradiction that $S$ is stationary.

Fix $\theta \gg \kappa$ regular. Then there exists $N$ in $P_\kappa H(\theta)$ such that $N \cap \kappa^+$ is in $S$, $N \cap \kappa$ is larger than $\delta_0$ and $\delta_1$, and

$$N \prec H(\theta), \in, \{c_\beta : \beta \in B\}.$$ 

Let $a = N \cap \kappa^+$, $\beta = \sup a$, and $\kappa_a = a \cap \kappa$. Since $a$ is in $S$, $\beta$ is in $B$. So $c_\beta$ is defined. Note that o.t.$(a) = \kappa_a^+$ so $\text{cf}(\beta) = \kappa_a^+$.

By elementarity, $B$ is unbounded in $\beta$. Let $\delta$ be one of $\delta_0$ or $\delta_1$ which is different from $\text{cf}(\kappa_a)$. Then $B \cap a$ and $a$ are both closed under suprema of subsets with order type $\delta$. Therefore $B \cap a$ is a stationary subset of $\beta$. Since o.t.$(c_\beta) \geq \text{cf}(\beta) = \kappa_a^+$, there is $\gamma$ in $a \cap B \cap \text{lim}(c_\beta)$ such that o.t.$(c_\beta \cap \gamma) \geq \kappa_a$. But $c_\gamma = c_\beta \cap \gamma$, and by elementarity, o.t.$(c_\gamma)$ is in $N \cap \kappa = \kappa_a$. This is a contradiction since o.t.$(c_\gamma) \geq \kappa_a$. □

Now let $A$ be any subset of $\kappa^+$ consisting of limit ordinals. Let $B$ be the closure of $A$ under suprema of subsets with order type $\omega$ or $\omega_1$. We claim that

$$\{a \in S(\kappa, \kappa^+) : \sup a \in A\} = \{a \in S(\kappa, \kappa^+) : \sup a \in B\}$$

modulo clubs.

Clearly if $\sup a$ is in $A$ then $\sup a$ is in $B$. Suppose that $a$ is in $S(\kappa, \kappa^+)$, $\sup a = \beta$ is in $B$, and $\kappa_a = a \cap \kappa$ is a limit cardinal. Then $\text{cf}(\beta) = \kappa_a^+$, which is different from $\omega$ and $\omega_1$. But $B$ can be written as $A \cup X$ where $X$ consists of ordinals with cofinality $\omega$ or $\omega_1$. So $\beta$ is in $A$.

Proposition 5. The forcing poset $\mathbb{P}_B$ preserves cardinals and cofinalities less than or equal to $\kappa^+$, does not add subsets to $\kappa$, and destroys the stationarity of the set $S = \{a \in S(\kappa, \kappa^+) : \sup a \in A\}$. 
Proof. The poset $\mathbb{P}_B$ adds a partial square sequence $\langle c_\beta : \beta \in B \rangle$. Since $\mathbb{P}_B$ does not add subsets to $\kappa$, after forcing with $\mathbb{P}_A$ the set $B$ is still the closure of $A$ under suprema of subsets with order type $\omega$ or $\omega_1$. Now apply Theorem 1. \qed

All of the stationary subsets of $S(\kappa, \kappa^+)$ which we will consider have the form
\[ \{ a \in S(\kappa, \kappa^+) : \sup a \in A \} \]
for some $A \subseteq \kappa^+$. In [5] we constructed a model in which all stationary subsets of $S(\kappa, \kappa^+)$ have this form. Namely, fix a Jonsson function $F : \kappa^+ \rightarrow \kappa^+$. Using the forcing poset from [5], force to make almost all $a$ in $S(\kappa, \kappa^+)$ satisfy that $F$ is Jonsson for $a$. Then the function $a \mapsto \sup a$ is injective on a club set intersected with $S(\kappa, \kappa^+)$. So if $S \subseteq S(\kappa, \kappa^+)$ is stationary, then $S$ is equal modulo clubs to the set
\[ \{ a \in S(\kappa, \kappa^+) : \sup a \in A \} \]
where $A = \{ \beta < \kappa^+ : \exists a \in S \sup a = \beta \}$. The stationarity of $S(\kappa, \kappa^+)$ can be preserved by preparing the ground model.

Lemma 13. Suppose that $S$ is a stationary subset of $S(\kappa, \kappa^+)$ such that for some $X \subseteq \kappa$,
\[ S = \{ a \in S(\kappa, \kappa^+) : a \cap \kappa \in X \} . \]
Then there is $A \subseteq \kappa^+$ such that
\[ S = \{ a \in S(\kappa, \kappa^+) : \sup a \in A \} . \]
modulo clubs.

Proof. Let $C$ be the club set of $a$ in $P_\kappa \kappa^+$ such that $a \cap \kappa$ is a limit cardinal. Define $A$ as the set of $a$ less than $\kappa^+$ such that $\text{cf}(a) = \mu^+$ for some limit cardinal $\mu$ in $X$. If $a$ is in $S \cap C$, then $\text{cf}(\sup a) = \text{ot}(a) = (a \cap \kappa)^+$. But $a \cap \kappa$ is a limit cardinal in $X$, so $\sup a$ is in $A$. On the other hand, suppose that $a$ is in $S(\kappa, \kappa^+) \cap C$ and $\sup a$ is in $A$. Then $\text{cf}(\sup a) = \mu^+$ for some limit cardinal $\mu$ in $X$. But $\text{cf}(\sup a) = (a \cap \kappa)^+$, so $a \cap \kappa = \mu$, and therefore $a$ is in $S$. \qed

Note: The method presented here for destroying stationary sets using partial square sequences is simpler and easier to use than previous posets for adding clubs, as found in [2] and [5]. The Radin forcing required by these other posets is eliminated. Also the fact that the partial square poset does not add subsets to $\kappa$ is very useful, as we see in the following sections.

5. The Structure of $S(\kappa, \kappa^+)$

In this section and in Section 7 we prove several consistency results which contribute to the general theme that the structure of $S(\kappa, \kappa^+)$, unlike its complement, is largely independent of ZFC. In the present section we construct a model in which $S(\kappa, \kappa^+)$ is stationary and for almost all $a$ in $S(\kappa, \kappa^+)$, $a \cap \kappa$ is a measurable cardinal with Mitchell order $(a \cap \kappa)^{++}$.

Let $V$ be a model of set theory in which $\kappa$ is $\kappa^{++}$-supercompact and GCH holds. Let $A$ be the set of $\alpha$ less than $\kappa$ such that $\alpha$ is $\kappa^{++}$-supercompact.

Proposition 6. If $\alpha$ is in $A \cup \{ \kappa \}$, then there exists a $\triangleleft$-increasing sequence $\langle U(\alpha, i) : i < \alpha^{++} \rangle$ of normal ultrafilters on $\alpha$ such that $A \cap \alpha$ is not in $U(\alpha, i)$ for each $i$. 
\textit{Proof.} Let $W$ be a normal ultrafilter on $P_\alpha \alpha^+$ which is minimal in the Mitchell ordering and let $j : V \to M = \text{Ult}(V,W)$. Then $\alpha$ is not $\alpha^+$-supercompact in $M$, so $\alpha$ is not in $j(A \cap \alpha)$. Let $W \upharpoonright \alpha$ be the projection of $W$ to $\alpha$, i.e. $X$ is in $W \upharpoonright \alpha$ iff $X \subseteq \alpha$ and $\alpha \in j(X)$. Note that $A \cap \alpha$ is not in $W \upharpoonright \alpha$. Let $i : V \to N = \text{Ult}(V,W \upharpoonright \alpha)$. By Lemma 3, there exists an elementary embedding $k : N \to M$ such that $j = k \circ i$ and $\text{crit}(k) = \alpha^{++N}$.

If $\bar{U}$ is a $\lt$-$\text{increasing}$ sequence of ultrafilters on $\alpha$ in $N$ with length $(\alpha^{++N})^N$, then $\bar{k}(\bar{U})$ is a $\lt$-$\text{increasing}$ sequence of ultrafilters on $k(\alpha) = \alpha$ with length $k(\alpha^{++N}) = \alpha^{++N}$. So it suffices to prove there exists such a sequence in $N$.

It suffices to prove that in $N$, whenever $\beta < \alpha^{+\beta}$ the least ordinal such that there exists a sequence $\bar{U}$ with length $\beta$, but there is no ultrafilter above $\bar{U}$ which does not contain $A$. Since $\text{crit}(k) = (\alpha^{+\beta})^N$.

Define an Easton support iteration $\langle \bar{P}_\alpha, \bar{Q}_\alpha : \alpha < \kappa \rangle$ as follows. Suppose that $\bar{P}_\alpha$ is defined for some $\alpha$ less than $\kappa$. If $\alpha$ is not in $A$ then let $\bar{Q}_\alpha$ be trivial. Suppose that $\alpha$ is in $A$.

Let $G_\alpha$ be generic for $\bar{P}_\alpha$ over $V$. In $V[G_\alpha]$, define $B_\alpha$ as the set of $\beta$ less than $\alpha^+$ such that $\text{cf}(\beta) \neq \mu^+$ for any $\mu$ in $A \cap \alpha$. Clearly $B_\alpha$ is closed under suprema of subsets with order type $\omega$ and $\omega_1$. So the forcing poset $\bar{P}_{B_\alpha}$ for adding a $\square_{\alpha^+}$-sequence forces that almost all $a$ in $S(\alpha, \alpha^+)$ satisfy that $\text{cf}(\sup a) = \mu^+$ for some $\mu$ in $A \cap \alpha$. But $\text{cf}(\sup a) = (\alpha \cap \alpha)^+$, so $\mu = a \cap \alpha$ is in $A$. Let $\bar{Q}_\alpha$ be a name for $\bar{P}_{B_\alpha}$.

This completes the definition of $\bar{P}_\kappa$. Let $B = B_\kappa$. We force with $\bar{P}_\kappa \ast \bar{P}_B$. By standard Easton support iteration arguments, this poset preserves all cardinals, cofinalities, and GCH. It also forces that almost all $a$ in $S(\kappa, \kappa^+)$ satisfy that $a \cap \kappa$ is in $A$. So to complete the proof, we show that $S(\kappa, \kappa^+)$ remains stationary and for each $\alpha$ in $A$, the sequence of ultrafilters on $\alpha$ can be lifted.

\textbf{Lemma 14.} For all $\alpha$ in $A$, the poset $\bar{P}_\alpha \ast \bar{P}_B$ forces that the sequence $\langle U(\alpha, i) : i \in \alpha^{++N} \rangle$ can be lifted.

\textit{Proof.} Write $\bar{P}_\alpha \ast \bar{P}_B = \bar{P}_\alpha \ast \bar{P}_{B, \alpha} \ast \bar{P}_{\text{tail}} \ast \bar{P}_B$. The key point is that $\bar{P}_{B, \alpha}$ is forced to be $\alpha + 1$-strategically closed, and therefore $\bar{P}_{B, \alpha} \ast \bar{P}_{\text{tail}} \ast \bar{P}_B$ does not add subsets to $\alpha$. So by Corollary 1, it suffices to prove that $\bar{P}_\alpha$ forces that the sequence can be lifted.

If $\alpha$ is not a limit point of $A \cap \alpha$, then $|\bar{P}_\alpha| < \alpha$, so the claim holds by Lemma 10.

Suppose that $\alpha$ is a limit point of $A \cap \alpha$. In $V$ fix ultrapower maps $j_i : V \to M_i = \text{Ult}(V, U(\alpha, i))$ and $k_i : M_{i+1} \to N_i = \text{Ult}(M_{i+1}, U(\alpha, i))$. By Lemma 9, $j_i \upharpoonright M_{i+1} = k_i$. We define a normal ultrafilter $U(\alpha, i)^*$ extending $U(\alpha, i)$ in $M_{i+1}[G_\alpha]$.

Write $j_i(\bar{P}_\alpha) = \bar{P}_\alpha \ast \bar{P}_{\text{tail}}$. We claim that $k_i(\bar{P}_\alpha)$ can also be factored as $\bar{P}_\alpha \ast \bar{P}_{\text{tail}}$. For each $\beta$ in $\alpha \setminus A$ let $\bar{P}_\beta = \bar{P}_\beta \ast \bar{P}_{\beta, \alpha}$. Since $M_{i+1}$ is $\alpha$-closed, the sequence $\bar{P} = \langle \bar{P}_{\beta, \alpha} : \beta \in \alpha \setminus A \rangle$ is in $M_{i+1}$. So $j_i(\bar{P}) = k_i(\bar{P})$. Since $\bar{P}_{\text{tail}} = j_i(\bar{P})(\alpha)$, $k_i(\bar{P}_\alpha) = \bar{P}_\alpha \ast \bar{P}_{\text{tail}}$.

Similarly, for each $\beta$ in $\alpha \setminus A$ let $A_{\beta}$ be a $\bar{P}_\beta$-name for the set of maximal antichains in $\bar{P}_{\beta, \alpha}$. Since $\bar{P}_\alpha$ is $\alpha$-c.c. and $M_{i+1}$ is $\alpha$-closed, the sequence $\bar{A} = \langle A_{\beta} : \beta \in \alpha \setminus A \rangle$
is in $M_{i+1}$ and $j_i(\vec{A}) = k_i(\vec{A})$. In particular, $A = j_i(\vec{A})(\alpha)$ is a $\mathbb{P}_\alpha$-name for the collection of maximal antichains in $\mathbb{P}_{\text{tail}}$.

In $M_{i+1}[\mathcal{G}_\alpha]$ the size of $k_i(\alpha)$ is $\alpha^+$, so we can enumerate $A$ as $\langle A_i : i < \alpha^+ \rangle$. In $V[\mathcal{G}_\alpha]$ this sequence also enumerates all of the maximal antichains of $\mathbb{P}_{\text{tail}}$ in $M_i[\mathcal{G}_\alpha]$. Since $N_i[\mathcal{G}_\alpha]$ is $\alpha$-closed, $\mathbb{P}_{\text{tail}}$ is $\alpha^+$-strategically closed in $M_{i+1}[\mathcal{G}_\alpha]$. So we can construct a generic $H_i$ for $\mathbb{P}_{\text{tail}}$ over $N_i[\mathcal{G}_\alpha]$. Note that $H_i$ is also a generic for $\mathbb{P}_{\text{tail}}$ over $M_i[\mathcal{G}_\alpha]$.

Now lift $j_i$ and $k_i$ to $j_i : V[\mathcal{G}_\alpha] \rightarrow M_i[\mathcal{G}_\alpha * H_i]$ and $k_i : M_{i+1}[\mathcal{G}_\alpha] \rightarrow N_i[\mathcal{G}_\alpha * H_i]$. Note that $j_i | M_{i+1}[\mathcal{G}_\alpha] = k_i$. Define $U(\alpha, i)^*$ by letting $X \in U(\alpha, i)^*$ iff $X \subseteq \alpha$ and $\alpha \in j_i(X)$. Since $\mathcal{P}(\alpha) \subseteq M_{i+1}[\mathcal{G}_\alpha]$ and $k_i = j_i | M_{i+1}[\mathcal{G}_\alpha]$, the ultrafilter $U(\alpha, i)^*$ can be correctly computed in $M_{i+1}[\mathcal{G}_\alpha]$.

This defines $U(\alpha, i)^*$ for $i < \alpha^+$. Suppose that $i < j$. Then $U(\alpha, i)^*$ is in $M_{i+1}[\mathcal{G}_\alpha] \subseteq M_j[\mathcal{G}_\alpha] \subseteq M_j[\mathcal{G}_\alpha * H_j] = \text{Ult}(V[G], U(\alpha, j)^*)$. So $U(\alpha, i)^* < U(\alpha, j)^*$. \hfill $\square$

To prove that $\mathbb{P}_\kappa * \mathbb{P}_B$ preserves the stationarity of $S(\kappa, \kappa^+)$, it suffices to prove that $\kappa$ remains $\kappa^+$-supercompact. Let $U$ be a normal ultrafilter on $\mathbb{P}_\kappa * \mathbb{P}_B$ and let $j : V \rightarrow M = \text{Ult}(V, U)$. Note that $\kappa$ is in $j(A)$. Let $G_\kappa * G_B$ be generic for $\mathbb{P}_\kappa * \mathbb{P}_B$ over $V$. Write $j(\mathbb{P}_\kappa * \mathbb{P}_B) = \mathbb{P}_\kappa * \mathbb{P}_B * \mathbb{P}_{\text{tail}} * j(\mathbb{P}_B)$. In $M[\mathcal{G}_\kappa * \mathcal{G}_B]$, $\mathbb{P}_{\text{tail}}$ is $\kappa^+$-strategically closed. By Lemma 6, $M[\mathcal{G}_\kappa * \mathcal{G}_B]$ is $\kappa^+$-closed in $V[\mathcal{G}_\kappa * \mathcal{G}_B]$. Therefore $\mathbb{P}_{\text{tail}}$ is $\kappa^+$-strategically closed in $V[\mathcal{G}_\kappa * \mathcal{G}_B]$. In $M[\mathcal{G}_\kappa * \mathcal{G}_B]$, $\mathbb{P}_{\text{tail}}$ has $j(\kappa)$-many maximal antichains and $j(\kappa) = \kappa^+$. So we can construct a generic filter $G_{\text{tail}}$ for $\mathbb{P}_{\text{tail}}$ over $M[\mathcal{G}_\kappa * \mathcal{G}_B]$. Extend $j$ to $j : V[\mathcal{G}_\alpha] \rightarrow M[\mathcal{G}_\kappa * \mathcal{G}_B * G_{\text{tail}}]$.

We construct a master condition. For each $p$ in $G_B$ there is $\beta < \kappa$ such that $p = \langle c_\alpha : \alpha \in B \cap (\beta + 1) \rangle$. So $j(p) = \langle d_\alpha : \alpha \in j(B) \cap j(\beta + 1) \rangle$. Let $s = \bigcup j^*G_\alpha$. Then $s = \langle d_\alpha : \alpha \in j(B) \cap \text{sup} \ j^*\kappa^+ \rangle$. The sequence $s$ is a condition in $j(\mathbb{P}_B)$ iff the domain of $s$ is equal to $j(B) \cap (\text{sup} \ j^*\kappa^+ + 1)$, that is, iff $j^*\kappa^+$ is not in $j(B)$. But sup $j^*\kappa^+$ has cofinality equal to the successor of $\kappa$, which is a member of $j(A)$. So by the definition of $B$, sup $j^*\kappa^+$ is not in $j(B)$.

The poset $j(\mathbb{P}_B)$ has $j(\kappa^+)$ many maximal antichains. But $j(\kappa^+)$ has size $\kappa^+$. In $M[\mathcal{G}_\kappa * \mathcal{G}_B * G_{\text{tail}}]$, $j(\mathbb{P}_B)$ is $j(\kappa)$-strategically closed. By Lemma 7, $M[\mathcal{G}_\kappa * \mathcal{G}_B * G_{\text{tail}}]$ is $\kappa^+$-closed and $j(\kappa) = \kappa^+$. So in $V[\mathcal{G}_\kappa * \mathcal{G}_B]$, $j(\mathbb{P}_B)$ is $\kappa^+$-strategically closed. So we can construct a generic $H$ for $j(\mathbb{P}_B)$ which contains the master condition $s$. Now extend $j$ to $j : V[\mathcal{G}_\kappa * \mathcal{G}_B] \rightarrow M[\mathcal{G}_\kappa * \mathcal{G}_B * G_{\text{tail}} * H]$.

Define $U^*$ by letting $X \in U^*$ iff $X \subseteq P_\kappa * \kappa^+$ and $j^*\kappa^+ \in j(X)$. Standard arguments show that $U^*$ is a normal ultrafilter on $P_\kappa * \kappa^+$ in $V[\mathcal{G}_\kappa * \mathcal{G}_B]$.

6. Changing Cofinalities

In this section we present Gitik’s poset for changing a cofinality without adding bounded sets which we will use in the consistency results of the following sections. If $V$ is the core model and $\mathbb{P}$ is a forcing poset which changes the cofinality of a regular cardinal $\kappa$ to an uncountable value without collapsing cardinals, then $\mathbb{P}$ adds bounded subsets to $\kappa$. Therefore we are required to prepare the ground model before defining the poset.

Our exposition differs from that of [3].

Let $V$ be a model of set theory in which $\kappa$ is a strongly inaccessible cardinal. Assume that there is a function $\sigma : A \cup \{ \kappa \} \rightarrow \kappa$. 

where $A \subseteq \kappa$ is an unbounded set of strongly inaccessible cardinals and $o(\alpha) < \alpha$ for all $\alpha$ in $A \cup \{\kappa\}$. The function $o$ will be a relativized version of the Mitchell order function in a particular inner model.

Assume that for each $\alpha$ in $A$, there is a fixed set
\[ b_\alpha = b_\alpha^* \cup \{\alpha\}, \]
which satisfies the following properties:

1. If $o(\alpha) = 0$ then $b_\alpha^* = \emptyset$ and $b_\alpha = \{\alpha\}$.
2. If $o(\alpha) > 0$ then $b_\alpha^*$ is closed and unbounded in $\alpha$, and the order type of $b_\alpha^*$ is equal to the ordinal exponent $\omega^{o(\alpha)}$.
3. If $\gamma$ is in $b_\alpha^*$, then $o(\gamma) < o(\alpha)$.
4. If $\gamma$ is in $b_\alpha^*$ and for all $\beta$ in $b_\alpha \cap \gamma$, $o(\beta) < o(\gamma)$, then $b_\alpha^* = b_\alpha \cap \gamma$.
5. If $\gamma$ is in $b_\alpha^*$ and (4) fails for $\gamma$, then there is a maximum ordinal $\gamma'$ in $b_\alpha \cap \gamma$ such that $o(\gamma') \geq o(\gamma)$, and $b_\alpha^* = b_\alpha \cap (\gamma', \gamma)$.

Note that by (5), if $\gamma$ is in $b_\alpha^*$ and $o(\gamma) = 0$, then $\gamma$ is not a limit point of $b_\alpha$.

For each $i$ less than $o(\kappa)$ let $A_i$ denote the set of $\alpha$ in $A$ with $o(\alpha) = i$.

We will define a forcing poset $P(\kappa,o(\kappa))$ which adds a generic set $b_\kappa = b_\kappa^* \cup \{\kappa\}$ which itself satisfies conditions (1) through (5). We will need to state a few additional properties of the ground model, but first we provide some analysis of the sets $b_\alpha$.

**Lemma 15.** Suppose that $\alpha$ is in $A$ and $o(\alpha) = \gamma + 1$ for some $\gamma \geq 1$. Enumerate $b_\alpha$ in increasing order as $\langle \beta_i : 1 \leq i < \omega^{\gamma} \cdot \omega \rangle$. Then for any $i < \omega^{\gamma} \cdot \omega$, $o(\beta_i) = \gamma$ iff there is $1 \leq n < \omega$ such that $i = \omega^{\gamma} \cdot n$. If $o(\beta_i) = \gamma$, then $b_\alpha^* = \{\beta_j : \omega^{\gamma} \cdot (n-1) < j < i\}$.

**Proof.** By (3), the maximum possible value of $o(\beta_i)$ is $\gamma$, and therefore the largest possible order type of $b_\alpha^*$ is $\omega^{\gamma}$. By (4) and (5) we can write $b_\alpha^* = \{\beta_j : k \leq j < i\}$ for some $k < i$. Suppose that $i = \omega^{\gamma} \cdot n$ for some $n \geq 1$. Then any co-bounded subset of $\{\beta_j : 1 \leq j < i\}$ has order type at least $\omega^{\gamma}$. Therefore, $o(\beta_i) = \omega^{\gamma}$ and $o(\beta_i) = \gamma$. Conversely, suppose that $\omega^{\gamma} \cdot (n-1) < i < \omega^{\gamma} \cdot n$ for some $n \geq 1$. Since $o(\beta_{\omega^{\gamma} \cdot (n-1)})$ is either undefined (if $n = 1$) or else is equal to $\gamma$ by what we just proved, (5) implies that $b_\alpha^*$ is contained in the interval $\{\beta_j : \omega^{\gamma} \cdot (n-1) < j < i\}$, which has order type less than $\omega^{\gamma}$. So $o(\beta_i) < \gamma$.

The second statement follows immediately from (4) and (5).

**Lemma 16.** Suppose that $\alpha$ is in $A$ and $o(\alpha)$ is a limit ordinal. Enumerate $b_\alpha$ in increasing order as $\langle \beta_i : 1 \leq i < \omega^{o(\alpha)} \rangle$. Fix $1 \leq i < \omega^{o(\alpha)}$ and suppose that $i = \omega^{\gamma}$ for some $1 \leq \gamma < o(\alpha)$. Then $o(\beta_i) = \gamma$, and $b_\alpha^* = \{\beta_j : 1 \leq j \leq i\}$.

**Proof.** Since $b_\alpha$ is closed and $i$ is a limit ordinal, $\beta_i$ is a limit point of $b_\alpha$ and therefore $o(\beta_i) > 0$. By (4) and (5) there exists $k$ such that $b_{\beta_k}^* = \{\beta_j : k \leq j \leq i\}$. But for any choice of $k$, this set has order type $\omega^{\gamma}$. So $o(\beta_i) = \omega^{\gamma}$ and therefore $o(\beta_i) = \gamma$. If $k > 1$, then by (4) and (5) there is $l < i$ such that $o(\beta_l) \geq \gamma$. This is impossible since any bounded subset of $\{\beta_j : 1 \leq j < \omega^{\gamma}\}$ has order type less than $\omega^{\gamma}$.

Fix an ordinal $\beta \leq o(\kappa)$. A sequence $\langle \alpha_0, \ldots, \alpha_n \rangle$ is said to be $\beta$-coherent for $\kappa$ if it is a finite increasing sequence from $A$ such that $o(\alpha_i) < \beta$ for each $i$, and it satisfies the following property: For each $1 \leq m \leq n$, let $m'$ be the least index so that for all $m' \leq k < m$, $o(\alpha_k) < o(\alpha_m)$. Then $\bigcup\{b_{\alpha_k} : m' \leq k < m\}$ is an initial
Lemma 17. Suppose that \( b \) segment of \( b_{\alpha_n} \). If \( m' \) does not exist, or in other words, \( o(\alpha_{m-1}) \geq o(\alpha_m) \), then the minimal element of \( b_{\alpha_m} \) is greater than \( \alpha_{m-1} \).

Suppose that \( \gamma < \beta \leq o(\kappa) \) and \( t = \langle \alpha_0, \ldots, \alpha_n \rangle \) is \( \beta \)-coherent. Define \( t(\gamma) = \langle \alpha_i, \ldots, \alpha_n \rangle \), where \( i \) is the minimal index such that for each \( i \leq k \leq n \), \( o(\alpha_k) < \gamma \).

If there is no such \( i \), i.e. \( o(\alpha_n) \geq \gamma \), then let \( t(\gamma) \) be the empty sequence. Note that \( t(\gamma) \) is \( \gamma \)-coherent.

Lemma 18. Suppose that \( t = \langle \alpha_0, \ldots, \alpha_n \rangle \) is \( \beta \)-coherent and \( i \leq n \). If \( j < i \) and \( o(\alpha_j) \geq o(\alpha_i) \), then \( \min b_{\alpha_j} > \alpha_j \).

Proof. We prove the statement by induction on \( i \). It is vacuously true for \( i = 0 \).

Suppose that \( i \) is greater than 0 and there exists \( j < i \) such that \( o(\alpha_j) \geq o(\alpha_i) \). Let \( j \) be the maximal such index. If \( j = i - 1 \), then \( \min b_{\alpha_j} > \alpha_j \) by the definition of \( \beta \)-coherence.

Otherwise let \( i' = j + 1 \). Then for all \( i' \leq k < i \), \( o(\alpha_k) < o(\alpha_i) \leq o(\alpha_j) \). So \( \bigcup \{ b_{\alpha_k} : i' \leq k < i \} \) is an initial segment of \( b_{\alpha_i} \). By induction, \( \min b_{\alpha_j} > \alpha_j \) for all such \( k \), so \( \min b_{\alpha_i} > \alpha_j \).

If \( t \) is an \( (\kappa) \)-coherent sequence, let \( b_t \) denote \( \bigcup \{ b_{\alpha} : \alpha \in t \} \).

Lemma 19. Suppose that \( t = \langle \alpha_0, \ldots, \alpha_n \rangle \) is \( \beta \)-coherent and \( \alpha \) is an ordinal in \( b_{\alpha_0} \setminus t \). Let \( i_0 \) be the minimal index such that \( \alpha \) is less than \( \alpha_{i_0} \), and let \( i_1 \) be the minimal index such that \( \alpha \) is in \( b_{\alpha_{i_1}} \). Then \( i_0 = i_1 \).

In particular, for each \( i \), \( \bigcup \{ b_{\alpha_k} : k \leq i \} \) is an initial segment of \( b_t \).

Proof. Since \( \alpha \) is in \( b_{\alpha_{i_0}} \setminus t \), \( \alpha \) is less than \( \alpha_{i_0} \). So \( \alpha_{i_0} \leq \alpha_{i_1} \). Suppose for a contradiction that \( \alpha_{i_0} < \alpha_{i_1} \). Let \( i' \) be the least index such that for all \( i' \leq k < i_1 \), \( o(\alpha_k) < o(\alpha_{i_1}) \). Since \( \alpha < \alpha_{i_0} < \alpha_{i_1} \), and \( \alpha \) is in \( b_{\alpha_{i_0}} \), by Lemma 17 \( i' \) exists and \( i' \leq i_0 \). Since \( \alpha < \alpha_{i_0} \) and \( \bigcup \{ b_{\alpha_k} : i' \leq k < i_1 \} \) is an initial segment of \( b_{\alpha_{i_1}} \), there exists \( k < i_1 \) such that \( \alpha \) is in \( b_{\alpha_k} \). This contradicts the minimality of \( i_1 \).

Now we complete the description of the ground model. We assume that there is a family
\[
\{ U(\kappa, \beta, t) : \beta < o(\kappa), \ t \text{ is } \beta \text{-coherent} \}
\]
of \( \kappa \)-complete ultrafilters on \( \kappa \) satisfying the following conditions:

(1) \( A_\beta = \{ \alpha < \kappa : o(\alpha) = \beta \} \) is in \( U(\kappa, \beta, t) \),

(2) If \( s \) and \( t \) are \( \beta \)-coherent sequences and \( b_s = b_t \), then \( U(\kappa, \beta, s) = U(\kappa, \beta, t) \),

(3) If \( t \) is \( \beta \)-coherent and \( \gamma \) is less than \( \beta \), then the set of \( \alpha \) less than \( \kappa \) such that \( t^\cdot \alpha \) is \( \beta \)-coherent is in the ultrafilter \( U(\kappa, \gamma, t(\gamma)) \).

Fix \( \beta \leq o(\kappa) \). We describe a forcing poset \( \mathbb{P}(\kappa, \beta) \).

A set \( T \) is said to be a \( \beta \)-tree if \( T \) is a collection of \( \beta \)-coherent sequences satisfying:

(1) \( \langle T, \triangleright \rangle \) is a tree ordering, where \( u \triangleright v \) if \( u \) is an initial segment of \( v \). In other words, if \( v \) is in \( T \) and \( u \triangleright v \), then \( u \) is in \( T \).
(2) The tree $T$ has a trunk $t$ such that $t$ is in $T$ and for all $u$ in $T$, either $u \leq v$ or $v \leq u$.

(3) For any $u$ in $T$, if $t \leq u$ where $t$ is the trunk of $T$, then the set of $\alpha$ less than $\kappa$ such that $u \sim \alpha$ is in $T$ can be partitioned into a disjoint family

$$\{ \text{Suc}(T, u, \gamma) : \gamma < \beta \},$$

where $\text{Suc}(T, u, \gamma)$ is a subset of $A_\gamma$ and is in $U(\kappa, \gamma, u(\gamma))$.

If $T$ is a $\beta$-tree with trunk $t$, $u$ is in $T$, and $t \leq u$, then let $T_u$ denote the tree consisting of those $v$ in $T$ such that either $u \leq v$ or $v \leq u$. Note that $T_u$ is a $\beta$-tree with trunk $u$.

If $\beta = 0$ then let $\mathcal{P}(\kappa, \beta)$ be the trivial poset. Define $b_\kappa^* = 0$ and $b_\kappa = \{ \kappa \}$.

Suppose that $\beta$ is greater than 0.

A condition in $\mathcal{P}(\kappa, \beta)$ is a pair $\langle t, T \rangle$ such that $T$ is a $\beta$-tree with trunk $t$. If $\langle t, T \rangle$ and $\langle s, S \rangle$ are conditions, we let $\langle t, T \rangle \leq \langle s, S \rangle$ if:

1. there exists $t^*$ in $S$ such that $b_{t^*} = b_t$,
2. for each $u$ in $T$, the sequence $t^*(u \setminus (\max t^* + 1))$ is in $S$.

If $\langle t, T \rangle \leq \langle s, S \rangle$ and $s = t$, then let $\langle t, T \rangle \leq^* \langle s, S \rangle$.

The assumptions about the family of ultrafilters imply that $\beta$-trees exist, and therefore $\mathcal{P}(\kappa, \beta)$ is a non-trivial forcing poset.

The proof that $\mathcal{P}(\kappa, o(\kappa))$ satisfies the Prikry property will depend on certain details about the preparation forcing; see Proposition 10. For the rest of this section we will just assume that $\langle \mathcal{P}(\kappa, o(\kappa)), \leq, \leq^* \rangle$ is a Prikry type forcing poset.

If $T_0$ and $T_1$ are $\beta$-trees with the same trunk $t$, then $T_0 \cap T_1$ is also a $\beta$-tree with trunk $t$. In fact, suppose that $\{ \langle t_i, T_i \rangle : i < \xi \}$ is a family of conditions in $\mathcal{P}(\kappa, \beta)$, where $\xi$ is less than $\kappa$. By the $\kappa$-completeness of the ultrafilters, $\bigcap T_i$ is a $\beta$-tree with trunk $t$. So $\langle t, \bigcap T_i \rangle$ is a condition which directly extends each $\langle t, T_i \rangle$. It follows that $\mathcal{P}(\kappa, o(\kappa))$ is $\kappa$-weakly closed and satisfies the direct extension property. Therefore $\mathcal{P}(\kappa, o(\kappa))$ does not add bounded subsets to $\kappa$.

**Proposition 7.** The poset $\mathcal{P}(\kappa, \beta)$ is $\kappa^+\text{-c.c.}$

**Proof.** Any two conditions with the same trunk are compatible. There are only $\kappa$ many possibilities for the trunk. □

So $\mathcal{P}(\kappa, o(\kappa))$ preserves all cardinals.

**Lemma 20.** Suppose that $\langle t, T \rangle \leq \langle s, S \rangle$ in $\mathcal{P}(\kappa, \beta)$. Then there is $t^*$ in $S$ and $T^* \subseteq S$ such that $\langle t, T \rangle$ is equivalent to $\langle t^*, T^* \rangle$.

**Proof.** Fix $t^*$ in $S$ witnessing that $\langle t, T \rangle \leq \langle s, S \rangle$. Since $b_t = b_{t^*}$, $t$ and $t^*$ have the same maximal element $\alpha$. Define

$$T^* = \{ t^*(u \setminus (\alpha + 1)) : u \in T \}.$$

Note that $T^* \subseteq S$. The relation $\langle t, T \rangle \leq \langle t^*, T^* \rangle$ is witnessed by $t^*$, and $\langle t^*, T^* \rangle \leq \langle t, T \rangle$ is witnessed by $t$. □

**Lemma 21.** Suppose that $\langle s, S \rangle$ is a condition in $\mathcal{P}(\kappa, \beta)$ and $\alpha$ is in $b_s$. Then there is a condition $\langle t, T \rangle$ such that $b_t = b_s$, $\langle t, T \rangle$ is equivalent to $\langle s, S \rangle$, and $\alpha$ is in $t$.

**Proof.** Let $\langle s, S \rangle$ be a condition and suppose that $\alpha$ is in $b_s$. Write $s = \langle \alpha_0, \ldots, \alpha_n \rangle$. If $\alpha$ is in $s$ then we are done. Suppose that $\alpha$ is in $b_s \setminus s$. Fix $i \leq n$ so that $\alpha_i$ is
the least ordinal in $s$ larger than $\alpha$. By Lemma 18, $\alpha_i$ is also the least ordinal in $s$
for which $\alpha$ is in $b_{\alpha_i}$. We will prove the existence of $(t, T)$ by induction on $o(\alpha_i)$.

Since $\alpha$ is in $b_{\alpha_i}^*$, $o(\alpha_i) > 0$. Suppose that $o(\alpha_i) = 1$, so that $b_{\alpha_i}^*$
has order type $\omega$. Let $\beta_0, \ldots, \beta_m$ enumerate in increasing order the ordinals in
$b_{\alpha_i} \cap \alpha$, and which are larger than $\alpha_{i-1}$ if $i > 0$. Define

$$t = (\alpha_0, \ldots, \alpha_{i-1}, \beta_0, \ldots, \beta_m, \alpha, \alpha_i, \ldots, \alpha_n).$$

Then $t$ is $\beta$-coherent and $b_t = b_s$. Define

$$T = \{t : (u \setminus (\alpha_n + 1)) : u \in S\}.$$

Then $(t, T)$ is a condition, $(t, T) \leq (s, S)$ is witnessed by $s$, and $(s, S) \leq (t, T)$ is
witnessed by $t$.

Suppose that $o(\alpha_i) = \gamma$, where $\gamma > 1$, and the claim holds whenever $o(\alpha_i) < \gamma$.
Enumerate $b_{\alpha_i}$ in increasing order as $(\beta_i : 1 \leq i < \omega^\gamma)$.

Suppose that $\gamma$ is a successor ordinal, and let $\gamma = \gamma_0 + 1$. Then $b_{\alpha_i}$ has order
type $\omega^\gamma = \omega^{\gamma_0} \cdot \omega$. Fix $1 \leq n < \omega$ minimal so that $\alpha$ is in $b_{\alpha_i} \cap (\beta_{\gamma_0 + 1})$.
Let $\xi_0, \ldots, \xi_n$ enumerate the ordinals in $b_{\alpha_i} \setminus (\alpha_{i-1} + 1)$ of the form $\beta_{\gamma_0 k}$, where
$1 \leq k \leq n$. By Lemma 15, these are exactly the ordinals in $b_{\alpha_i} \setminus (\alpha_{i-1} + 1)$ which
have order $\gamma_0$. Define

$$s^* = \langle \alpha_0, \ldots, \alpha_{i-1}, \xi_0, \ldots, \xi_m, \alpha_i, \ldots, \alpha_n \rangle,$$

which is $\beta$-coherent. Also $b_s = b_{s^*}$. Define

$$S^* = \{s^* : (u \setminus (\alpha_n + 1)) : u \in S\}.$$

Then $(s^*, S^*)$ is equivalent to $(s, S)$. If $\alpha$ is in $s^*$ we are done. Otherwise, the least
ordinal in $s^*$ which is larger than $\alpha$ is now $\xi_n$, and $o(\xi_n) = \gamma_0$. By induction, there is
$(t, T)$ equivalent to $(s^*, S^*)$ such that $\alpha$ is in $t$ and $b_t = b_{s^*}$.

Now suppose that $\gamma$ is a limit ordinal. Then $\omega^\gamma = \sup \{\omega^j : j < \gamma\}$. Fix
$j < \gamma$ minimal so that $\alpha$ is in $b_{\alpha_i} \cap (\beta_{\omega^j} + 1)$. By Lemma 16, $o(\beta_{\omega^j}) = j$, and
$b_{\beta_{\omega^j}} = \{\beta_l : 1 \leq l \leq \omega^j\}$. Define

$$s^* = \langle \alpha_0, \ldots, \alpha_{i-1}, \beta_{\omega^j}, \alpha_i, \ldots, \alpha_n \rangle.$$

Then $s^*$ is $\beta$-coherent and $b_{s^*} = b_s$. Define $S^*$ as in the last case. Then $(s^*, S^*)$ is
equivalent to $(s, S)$. If $\alpha$ is in $s^*$ we are done. Otherwise, the least ordinal above $\alpha$
in $s^*$ is $\beta_{\omega^j}$, which has order less than $\gamma$. By induction, there is a condition $(t, T)$
which is equivalent to $(s^*, S^*)$ such that $\alpha$ is in $t$ and $b_t = b_{s^*}$.

Lemma 22. Suppose that $(s, S)$ and $(t, T)$ are conditions in $P(\kappa, \beta)$ such that
$b_s = b_t$. Then $(s, S)$ and $(t, T)$ are compatible.

Proof. Note that $s$ and $t$ have the same maximal element $\alpha$. Define

$$S_0 = \{v \in S : s \subseteq v \text{ then } t \setminus (v \setminus (\alpha + 1)) \in T\}.$$

Then $(s, S_0)$ is a condition below $(s, S)$. Apply Lemma 21 finitely many times to
obtain a condition $(u, U)$ which is equivalent to $(s, S_0)$ such that $s \cup t \subseteq u$ and
$b_u = b_s$. By definition of $S_0$, $(u, U)$ is also below $(t, T)$, as witnessed by $t$.

Now we analyze the generic object for $P(\kappa, o(\kappa))$. Let $\hat{b}_{\kappa}$ and $\hat{b}^*_{\kappa}$ be $P(\kappa, o(\kappa))$-
names such that

$$\models \hat{b}^*_{\kappa} = \{\alpha \in \kappa : \exists (t, T) \in \hat{G} \quad \alpha \in t\}.$$
Lemma 23. The poset $\mathbb{P}(\kappa, o(\kappa))$ forces that $b^*_\kappa$ is equal to $\bigcup\{b_t : \exists T \langle t, T \rangle \in \dot{G}\}$.

Proof. This is immediate from Lemma 21. □

Lemma 24. Let $\langle s, S \rangle$ be a condition in $\mathbb{P}(\kappa, o(\kappa))$. Then $\langle s, S \rangle$ forces that $b_s$ is an initial segment of $b_\kappa$.

Proof. By Lemma 23, $\langle s, S \rangle$ forces that $b_s$ is a subset of $b_\kappa$. Suppose that $\langle t, T \rangle$ is below $\langle s, S \rangle$ and $\alpha$ is in $t \cap \text{max} s$. Fix $t^*$ in $S$ such that $b_{t^*} = b_t$. Then $\alpha$ is in $b_{t^*}$. Since $s \supseteq t^*$ and $\alpha$ is less than $\text{max} s$, by Lemma 18 it follows that $\alpha$ is in $b_s$. Therefore $\langle s, S \rangle$ forces that $b_s$ is an initial segment of $b_\kappa$. □

We show that the generic set $b_\kappa$ satisfies the properties (1) through (5) which we stated for the $b_n$’s.

Lemma 25. The poset $\mathbb{P}(\kappa, o(\kappa))$ forces that $b^*_\kappa$ is closed and unbounded in $\kappa$.

Proof. By an easy density argument, $b^*_\kappa$ is unbounded in $\kappa$. To show it is club, suppose that $\langle s, S \rangle$ forces that $\beta$ is a limit point of $b^*_\kappa$. Let $\langle t, T \rangle \leq \langle s, S \rangle$ such that $\text{max} t > \beta$. Then $\langle t, T \rangle$ forces that $b_\kappa \cap (\text{max} t + 1) = b_t$, which is closed. So $\beta$ is in $b_t \subseteq b_\kappa$. □

Let $G$ be generic for $\mathbb{P}(\kappa, o(\kappa))$. If $\alpha$ is in $b^*_\kappa$, then $\alpha$ is in $t$ for some $\langle t, T \rangle$ in $G$. But $t$ is $o(\kappa)$-coherent, so $o(\alpha) < o(\kappa)$.

Lemma 26. Let $\gamma$ be in $b^*_\kappa$. If $o(\beta) < o(\gamma)$ for all $\beta$ in $b_\kappa \cap \gamma$, then $b^*_\kappa = b_\kappa \cap \gamma$.

Otherwise, there is a maximal ordinal $\gamma'$ in $b_\kappa$ below $\gamma$ such that $o(\gamma') \geq o(\gamma)$. In this case, $b^*_\gamma = b_\kappa \cap (\gamma', \gamma)$.

Proof. If $\gamma$ is in $b^*_\kappa$, then there is a condition $\langle s, S \rangle$ in $G$ such that $\gamma$ is in $s$. Since $b_s$ is an initial segment of $b_\kappa$, the statement follows from Lemma 19. □

Proposition 8. The order type of $b^*_\kappa$ is $\omega^{o(\kappa)}$. In particular, if $o(\kappa)$ is a regular uncountable cardinal, then $\text{o.t.}(b^*_\kappa) = o(\kappa)$.

Proof. Suppose that $o(\kappa) = 1$. Then an $o(\kappa)$-coherent sequence is just a finite increasing sequence from the set $\{\alpha \in A : o(\alpha) = 0\}$, and for any such sequence $t$, $b_t = t$. It follows that $\mathbb{P}(\kappa, o(\kappa))$ forces that every initial segment of $b_\kappa$ is finite. So $b^*_\kappa$ has order type $\omega$.

Suppose that $o(\kappa) = \gamma + 1$. Then the maximum possible order an ordinal in an $o(\kappa)$-coherent sequence is $\gamma$. For any finite $n$, there is a dense set of conditions $\langle t, T \rangle$ such that $t$ has the form $t = (\alpha_0, \ldots, \alpha_m, \beta_0, \ldots, \beta_{n-1})$, where for each $0 \leq i < n$, $o(\beta_i) = \gamma$. Then $b_t$ has order type at least $\omega^\gamma \cdot n$. Therefore $b^*_\kappa$ is forced to have order type at least $\omega^\gamma \cdot \omega = \omega^{\gamma + 1}$. Suppose for a contradiction that there is a condition $\langle s, S \rangle$ in $G$ which forces that the order type of $b^*_s$ is larger than $\omega^{\gamma + 1}$. Then there is a condition $\langle t, T \rangle$ in $G$ and an ordinal $\beta$ less than $\kappa$ such that $\text{max} t$ is larger than $\beta$ and $\langle t, T \rangle$ forces that the order type of $b_\kappa \cap \beta$ is at least $\omega^{\gamma + 1}$. So the order type of $b_t$ is at least $\omega^{\gamma + 1}$ but $b_t$ is a finite union of sets with order type at most $\omega^\gamma$, so we have a contradiction.
Finally, suppose that \( o(\kappa) \) is a limit ordinal. Then for each \( i < o(\kappa) \), there is a dense set of conditions \( (t,T) \) such that there is \( \alpha \) in \( t \) with \( o(\alpha) = i \). For any such condition \( (t,T) \), \( b_i \) has order type at least \( \omega^i \). It follows that \( b_k^\ast \) has order type at least \( \omega^{o(\kappa)} \). By the same argument as in the last case, \( b_k^\ast \) cannot have an order type larger than \( \omega^{o(\kappa)} \).

\[ \square \]

7. The Structure of \( S(\kappa, \kappa^+) \), Continued

Let \( V \) be a model of set theory in which \( \kappa \) is a \( \kappa^+ \)-supercompact cardinal and GCH holds. Let \( \mu \) be a regular cardinal less than \( \kappa \). We construct a generic extension of \( V \) in which \( \kappa \) is strongly inaccessible, \( S(\kappa, \kappa^+) \) is stationary, and for almost all \( a \) in \( S(\kappa, \kappa^+) \), \( a \cap \kappa \) is a singular cardinal with cofinality \( \mu \).

For the remainder of the section fix a normal ultrafilter \( U \) on \( P_\kappa \kappa^+ \) and let \( j_U : V \rightarrow M = \text{Ult}(V,U) \). Let \( U \upharpoonright \kappa \) be the projection of \( U \) to \( \kappa \); i.e., \( X \in U \upharpoonright \kappa \) iff \( X \subseteq \kappa \) and \( \kappa \in j_U(X) \). Let \( i : V \rightarrow N = \text{Ult}(V,U \upharpoonright \kappa) \). Applying Lemma 3, fix an elementary embedding \( k : N \rightarrow M \) such that \( \text{crit}(k) = \kappa^{\kappa+1} \) and \( j_U = k \circ i \).

**Proposition 9.** There exists a \( < \)-increasing sequence \( \langle U(\kappa,i) : i \leq \mu \rangle \) of normal ultrafilters on \( \kappa \) such that \( U(\kappa, \mu) = U \upharpoonright \kappa \).

**Proof.** It suffices to prove that there exists a \( < \)-increasing sequence of normal ultrafilters on \( \kappa \) with length \( \mu \) in the model \( N = \text{Ult}(V,U \upharpoonright \kappa) \). To prove this claim it is enough to show that any \( < \)-increasing sequence of ultrafilters in \( N \) with length less than \( \mu \) has an upper bound in \( N \).

If the claim fails then let \( \alpha \) be the least ordinal less than \( \mu \) such that in \( N \), there exists a \( < \)-increasing sequence \( \tilde{U} \) with length \( \alpha \) with no upper bound. By elementarity, the same statement holds for \( k(\tilde{U}) = \tilde{U} \) in \( M \). This is a contradiction, since \( U \upharpoonright \kappa \) is above every ultrafilter in \( \tilde{U} \).

**Lemma 27.** There is a sequence \( \langle X_i : i \leq \mu \rangle \) such that each \( X_i \) is in \( U(\kappa,i) \) and each \( X_i \cap X_j \) is empty.

**Proof.** For distinct \( i \) and \( j \) let \( X_i(j) \) be a set in \( U(\kappa,i) \setminus U(\kappa,j) \). Let \( X_i^* = \{X_i(j) : j \leq \mu, j \neq i \} \). Then for all \( j \neq i \), \( X_i^* \in U(\kappa,i) \setminus U(\kappa,j) \). Now let \( X_i = X_i^* \setminus \bigcup\{X_j^* : j \neq i \} \).

For each \( i \leq \mu \) let \( j_i^\ast : V \rightarrow N_i^\ast = \text{Ult}(V,U(\kappa,i)) \). For \( i < j \leq \mu \) let \( f_i^j : \kappa \rightarrow V_\kappa \) be a function such that \( [f_i^j] = U(\kappa,i) \) in \( N_i^\ast \). Define a sequence \( \langle A_j : j \leq \mu \rangle \) by induction, using the following recursion: \( A_j \) is the set of \( \alpha \) less than \( \kappa \) such that

1. \( \alpha \) is in \( X_j \),
2. \( \alpha > \mu \),
3. For \( i < j \), \( f_i^j(\alpha) \) is a normal ultrafilter on \( \alpha \) and \( A_i \cap \alpha \) is in \( f_i^j(\alpha) \),
4. For \( i_0 < i_1 < j \), \( f_i^j(\alpha) \) in \( f_i^j(\alpha) \),
5. For \( i_0 < i_1 < j \), \( [f_i^j \upharpoonright \alpha] \) in \( \text{Ult}(V,f_i^j(\alpha)) \) is equal to \( f_i^j(\alpha) \).

The reader can check by induction that \( A_j \) is in \( U(\kappa,j) \), using Lemma 9 (1) to prove (5). In particular, \( \kappa \) is in \( j_U(A_\mu) \).

Let \( A \) be the disjoint union \( \bigcup\{A_j : j \leq \mu \} \). If \( \alpha \) is in \( A \), let \( o(\alpha) \) denote the unique \( \beta \) such that \( \alpha \) is in \( A_\beta \). The function \( o \) is a relativized version of the Mitchell order function. If \( \alpha \) is in \( A \), then for each \( i < o(\alpha) \) let \( U(\alpha,i) = f_i^{o(\alpha)}(\alpha) \) and let \( j_i^\ast : V \rightarrow N_i^\ast = \text{Ult}(V,U(\alpha,i)) \). By (3), for \( i < o(\alpha) \), \( A_i \cap \alpha \) is in \( f_i^{o(\alpha)}(\alpha) = U(\alpha,i) \),
and therefore \( \alpha \) is in \( j_0^\kappa(A_\lambda) \). It follows that \( j_0^\kappa(\alpha)(\alpha) = i \) in \( N_0^\alpha \). By (5), for each \( i_0 < i_1 < o(\alpha) \), \( U(\alpha, i_0) \) is in \( N_0^\alpha \), so \( U(\alpha, i_0) \not< U(\alpha, i_1) \).

Now we are ready to construct our model. Fix a well ordering \(<_\kappa \) of \( V_\kappa \) such that for every inaccessible \( \alpha < \kappa \), \( <_\kappa \cap (V_\alpha \times V_\alpha) \) is a well-ordering of \( V_\alpha \).

We define an Easton support Prikry iteration \( (\mathbb{P}_i, Q_i : i < \kappa) \) by induction.

Suppose that \( \mathbb{P}_\alpha \) is defined. Assume as an induction hypothesis that for all \( \gamma < \kappa \), \( Q_\alpha = \mathbb{P}_B \ast \mathbb{P}(\gamma, o(\gamma)) \), where \( \mathbb{P}_B \) is a \( \gamma + 1 \)-strategically closed forcing poset and \( \mathbb{P}(\gamma, o(\gamma)) \) is defined from a family of ultrafilters

\[
\{ U(\gamma, \beta, t) : \beta < o(\gamma) \text{, } t \text{ is a } \beta \text{-coherent sequence} \}.
\]

Let \( G_\alpha \) be generic for \( \mathbb{P}_\alpha \) over \( V \). If \( \alpha \) is not in \( A \) then let \( Q_\alpha \) be trivial. Suppose that \( \alpha \) is in \( A \). In \( V[G_\alpha] \) define \( B_\alpha \) as the set of \( \beta < \alpha^+ \) such that \( cf(\beta) \) is not the successor of a singular cardinal with cofinality \( \mu \). Note that \( B_\alpha \) is closed under suprema of subsets with order type \( \omega \) and \( \omega_1 \). Therefore the forcing poset \( \mathbb{P}_{B_\alpha} \) for adding a \( \Pi^0_\alpha \)-sequence forces that for almost all \( \alpha \in S(\alpha, \alpha^+) \), \( A \cap \alpha \) is a singular cardinal with cofinality \( \mu \). We consider \( \mathbb{P}_{B_\alpha} \) as a Prikry type forcing poset with \( \leq^* = \leq \).

Let \( G^\alpha \) be generic for \( \mathbb{P}_{B_\alpha} \) over \( V[G_\alpha] \). If \( o(\alpha) = 0 \) then \( \mathbb{P}(\alpha, o(\alpha)) \) is the trivial poset and there is nothing to prove. Suppose that \( o(\alpha) > 0 \). We define in \( V[G_\alpha \ast G^\alpha] \) a family of \( \alpha \)-complete ultrafilters

\[
\{ U(\alpha, \beta, t) : \beta < o(\alpha) \text{, } t \text{ is a } \beta \text{-coherent sequence} \}
\]

which satisfies the requirements (1), (2), and (3) described in the last section. We also assume as an induction hypothesis that the family of ultrafilters defined at any previous stage \( \gamma \) satisfies the same definition as at stage \( \alpha \).

In \( V[G_\alpha \ast G^\alpha] \) fix \( \beta < o(\alpha) \) and a \( \beta \)-coherent sequence \( t \). We define \( U(\alpha, \beta, t) \). Consider \( j_\beta^\alpha : V \rightarrow N_\beta^\alpha \). Since \( j_\beta^\alpha(\alpha) = \beta \), by the induction hypotheses we can write

\[
j_\beta^\alpha(\mathbb{P}_\alpha) = \mathbb{P}_\alpha \ast \mathbb{P}_{B_\alpha} \ast \mathbb{P}(\alpha, \beta) \ast \mathbb{P}(\alpha, j_\beta^\alpha(\alpha)).
\]

By closure, \( <_\alpha = <_\kappa \cap (V_\alpha \times V_\alpha) \) is in \( N_\beta^\alpha \). In order to define \( U(\alpha, \beta, t) \) we need to consider all subsets of \( \alpha \) in \( V[G_\alpha \ast G^\alpha] \). However, \( \mathbb{P}_{B_\alpha} \) does not add subsets to \( \alpha \) over \( V[G_\alpha] \). So it suffices to consider only \( \mathbb{P}_\alpha \)-names for subsets of \( \alpha \). Let \( \langle X_i : i < \alpha^+ \rangle \) be the \( j_\beta^\alpha(\langle \alpha \rangle) \)-least enumeration of all canonical \( \mathbb{P}_\alpha \)-names for subsets of \( \alpha \).

We apply the \( \alpha^+ \)-weak strategic closure of \( \mathbb{P}_\alpha \ast j_\beta^\alpha(\alpha) \) to define a \( \leq^* \)-decreasing sequence of names \( \langle \dot{q}_i : i < \alpha^+ \rangle \) for conditions in \( \mathbb{P}_\alpha \ast j_\beta^\alpha(\alpha) \) as follows. Consider a run of the game. Let \( \dot{q}_0 \) denote 1. Given \( \dot{q}_i \), let \( \dot{q}_i^* \) be the \( j_\beta^\alpha(\langle \alpha \rangle) \)-least name for a direct extension of \( \dot{q}_i \) which decides the statement \( \alpha \in j_\beta^\alpha(X_i) \). Let \( \dot{q}_{i+1} \) be a name for Player II’s response according to its strategy. Also Player II plays according to its strategy at limit stages.

Now define \( U(\alpha, \beta, t) \) by letting \( X \) be in \( U(\alpha, \beta, t) \) iff there is an index \( i \) such that \( X_i^{\dot{q}_i} = X \) and there exists a \( \beta \)-tree \( T \) such that

\[
N_\beta^\alpha[G_\alpha \ast G^\alpha] \models (t, T) \not\models \dot{q}_{i+1} \models \alpha \in j_\beta^\alpha(X_i).
\]

**Lemma 28.** Suppose that \( \beta < \beta^* < o(\alpha) \) and \( t \) is \( \beta \)-coherent. Then \( U(\alpha, \beta, t) \) is in \( N_\beta^\alpha[G_\alpha \ast G^\alpha] \) and satisfies the same definition as it does in \( V[G_\alpha \ast G^\alpha] \).

**Proof.** Let \( k_{\beta, \beta^*} : N_\beta^\alpha \rightarrow \text{Ult}(N_\beta^\alpha, U(\alpha, \beta^*)) \). By Lemma 9 (2), \( k_{\beta, \beta^*} = j_\beta^\alpha \restriction N_\beta^\alpha \).

So the lemma follows immediately from the definition of \( U(\alpha, \beta, t) \). \( \square \)
Lemma 29. The set \( U(\alpha, \beta, t) \) is an \( \alpha \)-complete ultrafilter on \( \alpha \) which extends \( U(\alpha, \beta) \).

Proof. We prove that the definition of a set \( X \) being in \( U(\alpha, \beta, t) \) is independent of the index of its name. Suppose that \( X = \hat{X}_i \). Fix \( p \) in \( G_\alpha \) which forces that \( \hat{X}_i = \hat{X}_j \). Then \( j_\beta^\alpha(p) = p \) forces over \( N_\beta^\alpha \) that \( j_\beta^\alpha(\hat{X}_i) = j_\beta^\alpha(\hat{X}_j) \). Therefore

\[
N_\beta^\alpha[G_\alpha \ast G^\alpha] \models \mathbb{P}(\alpha, \beta) \models \mathbb{P}_{\alpha,j_\beta^\alpha(\alpha)}(\alpha \in j_\beta^\alpha(\hat{X}_i) \iff \alpha \in j_\beta^\alpha(\hat{X}_j)).
\]

Suppose that there is \( T \) such that

\[
N_\beta^\alpha[G_\alpha \ast G^\alpha] \models \langle t, T \rangle \Vdash \hat{q}_{t+1} \Vdash \alpha \in j_\beta^\alpha(\hat{X}_i).
\]

Since \( \hat{q}_{t+1} \) is forced to decide the statement \( \alpha \in j_\beta^\alpha(\hat{X}_j) \), it follows that

\[
N_\beta^\alpha[G_\alpha \ast G^\alpha] \models \langle t, T \rangle \Vdash \hat{q}_{t+1} \Vdash \alpha \in j_\beta^\alpha(\hat{X}_j).
\]

Similar arguments show that \( U(\alpha, \beta, t) \) is an ultrafilter which extends \( U(\alpha, \beta) \).

Let us prove that \( U(\alpha, \beta, t) \) is \( \alpha \)-complete. Suppose that \( p \) is in \( G_\alpha \) and \( p \) forces that \( \chi : \beta \rightarrow \mathbb{P}(\alpha) \) is a partition of \( \alpha \) into \( \beta \) many sets, for some \( \beta \) less than \( \alpha \). Then \( j_\beta^\alpha(p) = p \) forces over \( N_\beta^\alpha \) that \( j_\beta^\alpha(\chi) \) is a partition of \( j_\beta^\alpha(\alpha) \) into \( \beta \) many sets. For each \( i \) less than \( \beta \) let \( \xi_i \) be an index such that \( p \) forces that \( \chi(i) = \hat{X}_{\xi_i} \). For each \( i \) less than \( \beta \) apply the Prikry property of \( \mathbb{P}(\alpha) \) in \( N_\beta^\alpha[G_\alpha \ast G^\alpha] \) to obtain a \( \beta \)-tree \( T_i \) such that \( \langle t, T_i \rangle \) decides which way that \( \hat{q}_{t+1} \) decides the statement \( \alpha \in j_\beta^\alpha(\hat{X}_{\xi_i}) \). Then \( \langle t, \bigcap T_i \rangle \) is a condition which directly refines each \( \langle t, T_i \rangle \). Since \( j_\beta^\alpha(\chi) \) is forced to be a partition of \( j_\beta^\alpha(\alpha) \), there must be some index \( j \) such that \( \langle t, \bigcap T_i \rangle \) forces that \( \hat{q}_{t+1} \) forces \( \alpha \in j_\beta^\alpha(\hat{X}_{\xi_j}) \). \( \square \)

Lemma 30. Suppose that \( s \) and \( t \) are \( \beta \)-coherent sequences and \( b_s = b_t \). Then \( U(\alpha, \beta, s) = U(\alpha, \beta, t) \).

Proof. It suffices to prove that \( U(\alpha, \beta, s) \subseteq U(\alpha, \beta, t) \), since they are ultrafilters. So let \( X \) be in \( U(\alpha, \beta, s) \), and fix \( i \) such that \( X = \hat{X}_i^G_\alpha \). Then there is \( S \) such that

\[
N_\beta^\alpha[G_\alpha \ast G^\alpha] \models \langle s, S \rangle \Vdash \hat{q}_{t+1} \Vdash \alpha \in j_\beta^\alpha(\hat{X}_i).
\]

Apply the Prikry property to \( \langle t, T \rangle \) which decides whether \( \hat{q}_{t+1} \Vdash \alpha \in j_\beta^\alpha(\hat{X}_i) \). By Lemma 22, \( \langle s, S \rangle \) and \( \langle t, T \rangle \) are compatible, so they must decide the statement the same way. So \( A \) is in \( U(\alpha, \beta, t) \). \( \square \)

Lemma 31. Suppose that \( t \) is \( \beta \)-coherent and \( \gamma \) is less than \( \beta \). Then the set of \( \xi \) in \( \alpha \) such that \( t \prec \xi \) is \( \beta \)-coherent is in \( U(\alpha, \gamma, t(\gamma)) \).

Proof. Let \( X \) be the set of \( \xi \) less than \( \alpha \) such that \( t \prec \xi \) is \( \beta \)-coherent. Fix \( i \) so that \( X = \hat{X}_i^G_\alpha \) and let \( p \) be a condition in \( G_\alpha \) which forces that \( \hat{X}_i \) satisfies the definition of \( X \). Since \( j_\beta^\alpha(p) = p \), it follows that

\[
N_\gamma^\alpha[G_\alpha \ast G^\alpha] \models \mathbb{P}(\alpha, \gamma) \ast \mathbb{P}_{\alpha,j_\beta^\alpha(\alpha)}(\xi < j_\beta^\alpha(\alpha) : t \prec \xi \text{ is } \beta\text{-coherent}).
\]

Fix any \( \gamma \)-tree \( T \) with trunk \( t(\gamma) \). Then by Lemma 24,

\[
N_\gamma^\alpha[G_\alpha \ast G^\alpha] \models \langle t(\gamma), T \rangle \Vdash b_{t(\gamma)} \text{ is an initial segment of } b_\alpha.
\]

Now \( j_\beta^\alpha(\alpha) = \gamma \) and every \( \xi \) in \( t(\gamma) \) has order less than \( \gamma \). It follows that

\[
N_\beta^\alpha[G_\alpha \ast G^\alpha] \models \langle t(\gamma), T \rangle \Vdash \mathbb{P}_{\alpha,j_\beta^\alpha(\alpha)}(\alpha \text{ is } \gamma\text{-coherent}).
\]

So \( X \) is in \( U(\alpha, \gamma, t(\gamma)) \). \( \square \)
Now define $\mathbb{P}(\alpha, o(\alpha))$ in $V[G_\alpha * G^\alpha]$ using this family of ultrafilters. We show that $\mathbb{P}(\alpha, o(\alpha))$ satisfies the Prikry property.

**Proposition 10.** Suppose that $\varphi$ is a statement in the forcing language for $\mathbb{P}(\alpha, o(\alpha))$ and $(t, T)$ is a condition. Then there is a direct extension of $(t, T)$ which decides $\varphi$.

**Proof.** Let $\varphi$ be a statement in the forcing language and fix a condition $(t, T)$. Let $\varphi^0$ denote $\varphi$ and let $\varphi^1$ denote $\neg \varphi$.

We define a sequence $\langle T^*_n : n < \omega \rangle$ such that each $T^*_n \subseteq T$ is an $o(\alpha)$-tree with trunk $t$, and $T^*_{n+1} \subseteq T^*_n$ for each $n$. Let $T^*_0 = T$ for $n \leq |t|$.

Suppose that $n \geq |t|$ and $T^*_n$ is defined. We define $T^*_{n+1}$. If $u$ is in $T^*_n$ and $|u| \leq n$, then let $u$ be in $T^*_{n+1}$. Now for each $u$ in $T^*_n$ with $|u| = n$ we define $(T^*_{n+1})_u$. Fix $\gamma$ less than $o(\alpha)$. For each $\xi$ in $\text{Suc}(T^*_n, u, \gamma)$ define $n_\xi = l$ for $l < 2$ if there exists a tree $T^*_\xi \subseteq T^*_n$ with trunk $u^\prec \xi$ such that $\langle u^\prec \xi, T^*_\xi \rangle$ forces $\varphi^1$. If there is no such tree $T^*_\xi$, then let $n_\xi = 2$. Now fix $l \leq 2$ and $X_{u, \gamma}$ in $U(\alpha, \gamma, u(\gamma))$ such that $n_\xi = l$ for all $\xi$ in $X_{u, \gamma}$. If $l = 2$ then define

$$\langle T^*_{n+1} \rangle_u = \{ u^\prec \xi \in T^*_n : \xi \in X_{u, \gamma}, \gamma < o(\alpha) \}$$

If $l < 2$ then define

$$\langle T^*_{n+1} \rangle_u = \{ u^\prec \xi \in T^*_n : \xi \in X_{u, \gamma}, \gamma < o(\alpha) \}.$$}

Clearly $T^*_{n+1}$ is an $o(\alpha)$-tree with trunk $t$.

Now let $T^* = \bigcap T^*_n$. Then $(t, T^*)$ is a direct refinement of $(t, T)$. The choice of $T^*$ guarantees that the following property holds: Suppose that $u$ is in $T^*$ such that $t \subseteq u$. $\gamma$ is less than $o(\alpha)$, and there is an $o(\alpha)$-tree $S \subseteq T^*$ such that for some $\xi$ in $\text{Suc}(T^*, u, \gamma)$, $\langle u^\prec \xi, S \rangle$ forces $\varphi^1$. Then for all $\xi$ in $\text{Suc}(T^*, u, \gamma)$, $\langle u^\prec \xi, T^*_\xi - \xi \rangle$ forces $\varphi^1$.

**Claim 2.** Suppose that $(t^\prec u, \xi, T^*_\xi - \xi)$ forces $\varphi^1$ for some $l < 2$. Then for all $\alpha \geq o(\xi)$ and any $\beta$ in $\text{Suc}(T^*, t^\prec u, \gamma)$, $(t^\prec u, \beta, T^*_\xi - \xi)$ forces $\varphi^1$.

**Proof.** It suffices to prove the statement is true for some $\beta$ in $\text{Suc}(T^*, t^\prec u, \gamma)$. We already know that this is true when $\gamma = o(\xi)$. Suppose that $\gamma$ is larger than $o(\xi)$.

We claim that there is an ordinal $\beta^*$ in the set $\text{Suc}(T^*, t^\prec u, \alpha(\xi))$ such that

$$\text{Suc}(T^*, t^\prec u, \gamma) \cap \text{Suc}(T^*, t^\prec u, \beta^*) \neq \emptyset.$$  

It suffices to prove that there is an ordinal $\beta^*$ in $\text{Suc}(T^*, t^\prec u, o(\xi))$ such that $\text{Suc}(T^*, t^\prec u, \gamma)$ is in $U(\alpha, \gamma, (t^\prec u^\beta)(\gamma))$.

Fix $i$ so that $\text{Suc}(T^*, t^\prec u, \gamma) = X_i$. Since $\text{Suc}(T^*, t^\prec u, \gamma)$ is in $U(\alpha, \gamma, (t^\prec u)(\gamma))$, by the definition of this ultrafilter in $V[G_\alpha * G^\alpha]$ there is $S$ such that

$$N^\alpha_\gamma[G_\alpha * G^\alpha] \models (t^\prec u)(\gamma), S \models \dot{q}_{i+1} \models \alpha \in j_\gamma^\alpha(\dot{X}_i).$$

Since $\gamma$ is greater than $o(\xi)$, $(t^\prec u)(o(\xi)) = (t^\prec u)(\gamma)(o(\xi))$. It follows that the set $\text{Suc}(S, (t^\prec u)(\gamma), o(\xi))$ is in $U(\alpha, o(\xi), (t^\prec u)(o(\xi)))$. So fix an ordinal $\beta^*$ which is in the set

$$\text{Suc}(S, (t^\prec u)(\gamma), o(\xi)) \cap \text{Suc}(T^*, t^\prec u, o(\xi)).$$

Now $(t^\prec u)(\gamma)^\beta)$ is below $(t^\prec u)(\gamma)$, therefore

$$N^\alpha_\gamma[G_\alpha * G^\alpha] \models (t^\prec u)(\gamma)^\beta, S(t^\prec u)(\gamma)^\beta \models \dot{q}_{i+1} \models \alpha \in j_\gamma^\alpha(\dot{X}_i).$$

Therefore $\text{Suc}(T^*, t^\prec u, \gamma)$ is in $U(\alpha, \gamma, (t^\prec u)(\gamma)^\beta)(\gamma)) = U(\alpha, \gamma, (t^\prec u^\beta)(\gamma))$, as desired.
Now choose $\beta$ in $\text{Suc}(T^*, t^{-u}, \gamma) \cap \text{Suc}(T^*, t^{-u} \beta^* \gamma)$. Since $o(\beta^*) = o(\xi)$, $(t^{-u}\beta^* \gamma, T_{t^{-u} \beta^* \gamma})$ forces $\phi^f$. Moreover, $(t^{-u}\beta^* \gamma, T_{t^{-u} \beta^* \gamma})$ is below this condition, so it also forces $\phi^f$. Since $o(\beta^*) > o(\beta^*)$, $\beta^*$ is in $b_\beta$. Therefore for all $\delta$ less than $o(\alpha)$, $U(\alpha, \delta, (t^{-u}\beta^* \gamma)(\delta)) = U(\alpha, \delta, (t^{-u}\beta^* \gamma)(\delta))$. It follows that $S = \{ v \setminus \{ \beta^* \} : v \in T_{t^{-u}\beta^* \gamma} \}$ is an $o(\alpha)$-tree. Also the conditions $(t^{-u}\beta^* \gamma, T_{t^{-u} \beta^* \gamma})$ and $(t^{-u}\beta^*, S)$ are equivalent. Therefore $(t^{-u}\beta^*, S \cap T_{t^{-u} \beta^* \gamma})$ forces $\phi^f$. By the choice of $T^*$, $(t^{-u}\beta^*, T_{t^{-u} \beta^* \gamma})$ forces $\phi^f$.

Suppose for a contradiction that $(t, T^*)$ does not have a direct extension which decides $\phi$. Using Lemma 20, choose a condition $(t^{-u}\beta^* \xi, T_0)$ below $(t, T^*)$ which decides $\phi$ such that $t^{-u}\beta^* \xi$ is in $T^*$, $T_0 \subseteq T^*$, and $|u|$ is minimal. Fix $l < 2$ such that $(t^{-u}\beta^* \xi, T_0)$ forces $\phi^f$. By Claim 2 and the choice of $T^*$, for all $\gamma \geq o(\xi)$ and $\beta$ in $\text{Suc}(T^*, t^{-u}, \gamma)$, $(t^{-u}\beta^* \gamma, T_{t^{-u} \beta^* \gamma})$ forces $\phi^f$.

We will prove that the condition $(t^{-u}, T_{t^{-u} \beta^* \gamma})$ forces $\phi^f$, which contradicts the minimality of $u$. By Lemma 20 it suffices to prove that for all $\gamma$ less than $o(\alpha)$ and for all $\beta$ in $\text{Suc}(T^*, t^{-u}, \gamma)$, $(t^{-u}\beta^* \gamma, T_{t^{-u} \beta^* \gamma})$ forces $\phi^f$. We already know by Claim 2 that this is true for $\gamma \geq o(\xi)$. So by the choice of $T^*$ it is enough to show that for all $\gamma$ less than $o(\xi)$ there is $\beta$ in $\text{Suc}(T^*, t^{-u}, \gamma)$ such that $(t^{-u}\beta^* \gamma, T_{t^{-u} \beta^* \gamma})$ forces $\phi^f$.

Fix $i$ so that $\text{Suc}(T^*, t^{-u}, o(\xi)) = X_{\text{Suc}}^G$. Since the set $\text{Suc}(T^*, t^{-u}, o(\xi))$ is in $U(\alpha, o(\xi), (t^{-u})(o(\xi)))$, there exists an $o(\xi)$-tree $S$ such that

$$N^{o(\xi)}_{\alpha}[G \ast G^\alpha] \models ((t^{-u})(o(\xi)), S) \models \hat{\alpha}_{i+1} \models \alpha \in j_{o(\xi)}(X_{\alpha})$$

Define $T_1$ as the set of sequences of the form $t^{-u} \beta^*(\alpha_0, \ldots, \alpha_n)$ which are in $T^*$ and satisfy the following property: Let $i \leq n$ be the maximal index such that for all $j < i$, $o(\alpha_j)$ is less than $o(\xi)$; then $(t^{-u})(o(\xi)) \neg \beta^*(\alpha_0, \ldots, \alpha_i)$ is in $S$.

Fix some ordinal $\beta$ in the set $\text{Suc}(T_1, t^{-u}, \gamma)$. We claim that $(t^{-u}\beta^* \gamma, T_{t^{-u} \beta^* \gamma})$ forces $\phi^f$. If this is true, then by the choice of $T^*$ the condition $(t^{-u}\beta^* \gamma, T_{t^{-u} \beta^* \gamma})$ forces $\phi^f$ and we are done. If the claim fails, then by the choice of $T^*$ there exists a condition

$$(t^{-u}\beta^* \gamma, (\alpha_0, \ldots, \alpha_n), T_{t^{-u} \beta^* \gamma} - (\alpha_0, \ldots, \alpha_n))$$

which forces the negation of $\phi^f$, where $t^{-u}\beta^*(\alpha_0, \ldots, \alpha_n)$ is in $T_1$. Let $i \leq n$ be maximal so that $o(\alpha_j) < o(\xi)$ for all $j \leq i$. Then by the definition of $T_1$, the sequence $(t^{-u})(o(\xi)) \neg \beta^*(\alpha_0, \ldots, \alpha_i)$ is in $S$. Therefore $N^{o(\xi)}_{\alpha}[G \ast G^\alpha]$ models that

$$(t^{-u})(o(\xi)) \neg \beta^*(\alpha_0, \ldots, \alpha_i), S(\alpha_n)(o(\xi)) \neg \beta^*(\alpha_0, \ldots, \alpha_i) \models \hat{\alpha}_{i+1} \models \alpha \in j_{o(\xi)}(X_{\alpha})$$

But $(t^{-u})(o(\xi)) \neg \beta^*(\alpha_0, \ldots, \alpha_i) = (t^{-u}\beta^*(\alpha_0, \ldots, \alpha_i))(o(\xi))$. It follows that $\text{Suc}(T^*, t^{-u}, o(\xi))$ is in the ultrafilter

$$U(\alpha, o(\xi), (t^{-u}\beta^*(\alpha_0, \ldots, \alpha_i))(o(\xi)))$$

So we can choose an ordinal $\beta^*$ in the set $\text{Suc}(T^*, t^{-u}, o(\xi)) \cap \text{Suc}(T^*, t^{-u}\beta^*(\alpha_0, \ldots, \alpha_i), o(\xi))$.

Since $o(\beta^*) = o(\xi)$, it follows that $b_{t^{-u}\beta^*(\alpha_0, \ldots, \alpha_i)} = b_{t^{-u}\beta^* \gamma}$. Define $T_2$ as the set of sequences of the form

$$(t^{-u}\beta^*(\alpha_0, \ldots, \alpha_i) \neg \beta^* \gamma$$
such that $t^{-u \neg \beta - v}$ is in $T^*$. Then the condition
\[(t^{-u \neg \beta} \langle \alpha_0, \ldots, \alpha_i \rangle \neg \beta^*, T_2)\]
is equivalent to the condition
\[(t^{-u \neg \beta}^*, T_{t^{-u \neg \beta^*}}).\]
Since $o(\xi) = o(\beta^*)$, these conditions force $\varphi^1$. Therefore the condition
\[(t^{-u \neg \beta} \langle \alpha_0, \ldots, \alpha_i \rangle \neg \beta^*, T_2 \cap T_{t^{-u \neg \beta} \langle \alpha_0, \ldots, \alpha_i \rangle \neg \beta^*})\]
also forces $\varphi^1$. By choice of $T^*$, the condition
\[(t^{-u \neg \beta} \langle \alpha_0, \ldots, \alpha_i \rangle \neg \beta^*, T_{t^{-u \neg \beta} \langle \alpha_0, \ldots, \alpha_i \rangle \neg \beta^*})\]
forces $\varphi^1$. By Claim 2, the same holds if we replace $\beta^*$ by any $\delta$ in the set $\text{Suc}(T^*, t^{-u \neg \beta} \langle \alpha_0, \ldots, \alpha_i \rangle, \gamma^*)$, where $\gamma^* \geq o(\xi)$. But this is a contradiction since we cannot replace it with $\alpha_{i+1}$, since this condition has an extension which forces the negation of $\varphi^1$. □

This completes the definition of $\mathbb{P}_\kappa$. The poset $\mathbb{P}_\kappa$ preserves all cardinals and preserves GCH. Now we force with the poset $\mathbb{P}_\kappa \ast \mathbb{P}_B$, where $B$ is a name for the set of $\beta$ less than $\kappa^+$ such that the cofinality of $\beta$ is not equal to the successor of a singular cardinal with cofinality $\mu$. After forcing with $\mathbb{P}_\kappa \ast \mathbb{P}_B$, $\kappa$ remains strongly inaccessible and for almost all $a$ in $S(\kappa, \kappa^+)$, $a \cap \kappa$ is a singular cardinal with cofinality $\mu$. We show that $S(\kappa, \kappa^+)$ remains stationary.

Recall that $U$ is a normal ultrafilter on $\mathbb{P}_\kappa \ast \mathbb{P}_B$ and $j_U : V \rightarrow M = \text{Ult}(V, U)$. The sequence $\langle U(\kappa, i) : i \leq \mu \rangle$ is a $<\mu$-increasing sequence of ultrafilters on $\kappa$ and $U(\kappa, \mu)$ is equal to $U \upharpoonright \kappa$, i.e. $X$ is in $U(\kappa, \mu)$ if and only if $X \subseteq \kappa$ and $\kappa \in j_U(X)$. In particular, $\kappa \in j_U(A_\mu)$. So $j_U(\kappa)(\kappa) = \mu$. Therefore
\[j_U(\mathbb{P}_\kappa) = \mathbb{P}_\kappa \ast \mathbb{P}_B \ast \mathbb{P}(\kappa, \mu) \ast \mathbb{P}_{\text{tail}}.\]

Let $G_\kappa \ast G_B$ be generic for $\mathbb{P}_\kappa \ast \mathbb{P}_B$ over $V$. Let $G(\kappa, \mu) \ast G_{\text{tail}}$ be generic for $\mathbb{P}(\kappa, \mu) \ast \mathbb{P}_{\text{tail}}$ over $V[G_\kappa \ast G_B]$. Let $H = G_\kappa \ast G_B \ast G(\kappa, \mu) \ast G_{\text{tail}}$. Since $j_U(\kappa) = \kappa$, in $V[H]$ we can lift $j_U$ to
\[j_U : V[G_\kappa] \rightarrow M[H].\]
Now $j_U \upharpoonright H(\kappa^+)^V$ is in $M$. For any $x$ in $H(\kappa^+)^V[G_\kappa]$, define $j^*(x) = ((j_U \upharpoonright H(\kappa^+)^V)(x))^H$, where $\check{x}$ is a name in $H(\kappa^+)^V$ and $\check{x}^{G_\kappa} = x$. Then in $V[H]$, $j^* = j_U \upharpoonright H(\kappa^+)^V[G_\kappa]$ and $j^*$ is in $M[H]$. It follows that $j_U \upharpoonright \mathbb{P}_B$ is in $M[H]$, and therefore also $j_U(\mathbb{P}_B)$ is in $M[H]$.

Consider $p$ in $G_B$, and write $p = \langle c_\alpha : \alpha \in B \cap (\beta + 1) \rangle$ for some $\beta$ less than $\kappa^+$. Then $j_U(p) = \langle d_\alpha : \alpha \in j(B) \cap (\beta + 1) \rangle$. It follows that $\bigcup j^*G_B = \langle d_\alpha : \alpha \in j(B) \cap \kappa^* \rangle$. So $\bigcup j^*G_B$ is a master condition in $j_U(\mathbb{P}_B)$ if and only if $j^*G_B$ is not in $j(B)$, which is true if and only if $\text{sup}(j^* \kappa^*)$ has cofinality equal to the successor of a singular cardinal with cofinality $\mu$. But $\text{cf}(\text{sup}(j^* \kappa^*)) = \kappa^+$ and $\kappa$ has cofinality $\mu$ in $M[H]$. So $\bigcup j^*G_B$ is a master condition.

Let $g$ be a generic filter for $j_U(\mathbb{P}_B)$ over $V[H]$ which contains the master condition $\bigcup j^*G_B$. Then in $V[H * g]$ we can lift $j_U$ to
\[j_U : V[G_\kappa \ast G_B] \rightarrow M[H * g].\]

Now we prove that $S(\kappa, \kappa^+)$ is stationary in $V[G_\kappa \ast G_B]$. Let $C$ be a club subset of $\mathbb{P}_\kappa \ast \mathbb{P}_B$ in $V[G_\kappa \ast G_B]$. Since $\mathbb{P}_B$ is $\kappa + 1$-strategically closed, $C$ is a subset of...
By Lemma 6, $C$ is in $M[G_\kappa \ast GB]$. Therefore $j_{U \upharpoonright} C$ is in $M[H]$ and so $j_{U \upharpoonright} C$ is in $M[H]$. Now in $M[H \ast g]$, $j_{U \upharpoonright} C$ is a directed subset of the club set $j_{U \upharpoonright}(C)$ with size less than $j_{U \upharpoonright}(\kappa)$. Therefore $\bigcup j_{U \upharpoonright} C = j^{\kappa^+}$ is in $j_{U \upharpoonright}(C)$. But $j^{\kappa^+}$ is in $j(S(\kappa, \kappa^+))$, so $j_{U \upharpoonright}(C) \cap j(S(\kappa, \kappa^+))$ is non-empty. By elementarity, $S(\kappa, \kappa^+) \cap C$ is non-empty. So $S(\kappa, \kappa^+)$ is stationary.

Note: There are variations of this consistency proof to arrange a set of possible cofinalities for $a \cap \kappa$, instead of a single cofinality. To obtain such models, modify the definition of the $B_\alpha$’s and replace the ultrafilter $U$ above with a coherent sequence of supercompact ultrafilters (see next section).

8. Saturated Ideals and GCH

In this section we construct a model in which $\kappa$ is strongly inaccessible, GCH holds, and there is a stationary set $S \subseteq P_\kappa \kappa^+$ such that $\text{NS} \upharpoonright S$ is saturated. Moreover, we will arrange that for all regular cardinals $\mu$ less than $\kappa$, the set

$$\{a \in S : \text{cf}(a \cap \kappa) = \mu\}$$

is stationary.

Previously Gitik [2] constructed a model in which $\text{NS} \upharpoonright S$ is $\kappa^+$-saturated for a set $S \subseteq P_\kappa \kappa^+$, where $\kappa$ is inaccessible. Unlike the partial square forcing poset, Gitik’s poset for destroying stationary sets adds subsets to $\kappa$. Therefore $2^{\kappa^+} = \kappa^{++}$ in his model. It is an open question whether the GCH is consistent with the existence of a stationary set $S$ such that $\text{NS} \upharpoonright S$ is $\kappa^+$-saturated.

Before constructing our model we introduce one of the basic forcing posets, which is an iteration of the partial square poset over the same cardinal. Let $V$ be a model of set theory in which $\kappa$ is stationary. In his model. It is an open question whether the GCH is consistent with the $\kappa^+$-satisfied for

$$\kappa^+ \times \kappa^+$$

such that

$$\forall \alpha, i, j < \kappa^+ \quad (f(\alpha) = \langle i, j \rangle \quad \rightarrow i \leq \alpha),$$

and each $f^{-1}(\langle i, j \rangle)$ is unbounded in $\kappa^+$. Suppose that $P_\alpha^\kappa$ is defined for a fixed $\alpha$ less than $\kappa^{++}$ and satisfies the induction hypotheses. Let $\langle B_\alpha^i : i < \kappa^{++} \rangle$ enumerate all canonical $D_\kappa^{\kappa^+}$-names $\dot{B}$ for a subset of $\kappa^+$ such that $P_\alpha^\kappa$ forces that $\dot{B}$ is closed under suprema of subsets with order type $\omega$ or $\omega_1$. Write $f(\alpha) = \langle i, j \rangle$. Then $i \leq \alpha$, so consider the name $\dot{B}_j^i$ which was defined at stage $i$. Let $Q_\alpha$ be a $P_\alpha^\kappa$-name for the $\kappa+1$-strategically closed forcing poset $P_{B_j^i}$ which adds a $\text{NS}^\kappa$-sequence. Clearly $P_{\alpha+1} = P_\alpha \ast Q_\alpha$ satisfies the induction hypotheses.

Now suppose that $\beta < \kappa^{++}$ is a limit ordinal and $P_\alpha^\kappa$ is defined for all $\alpha$ less than $\beta$. Define $P_\beta$ as the set of conditions $p$ with support of size no larger than $\kappa$ such that $p \upharpoonright \alpha$ is in $P_\alpha^\kappa$ for all $\alpha < \beta$.

**Proposition 11.** The iteration $P_\beta \upharpoonright \kappa^+$ is $\kappa+1$-strategically closed.
Proof. For each $\alpha$ less than $\beta$, let $\sigma_\alpha$ be a $\mathbb{P}_\alpha^\kappa$-name for a strategy for Player II in the game for the poset $Q_\alpha^\kappa$. We describe a strategy for Player II in $\mathbb{P}_\beta^\kappa$ by considering a run of the game $(p_i : 0 < i \leq \kappa)$.

Suppose that $i < \kappa$ and Player I has just played condition $p_i$. Define $p_{i+1}$ as the condition with the same support as $p_i$ such that for all $\alpha$ in the support, $p_{i+1}(\alpha)$ is a $\mathbb{P}_\alpha^\kappa$-name for the condition obtained by applying $\sigma_\alpha$ to the sequence of conditions played so far on coordinate $\alpha$. An easy induction shows that for all $\alpha$ less than $\beta$, $p_{i+1} | \alpha$ is a condition in $\mathbb{P}_\alpha^\kappa$ below $p_i | \alpha$. So $p_{i+1}$ is a condition in $\mathbb{P}_\beta^\kappa$ below $p_i$.

Suppose that $i \leq \kappa$ is a limit ordinal and $(p_j : 0 < j < i)$ is defined. Let $a$ be equal to $\bigcup\{\text{supp}(p_j) : 0 < j < i\}$. Since $i \leq \kappa$, $|a| \leq \kappa$. Define $p_i$ as the condition with support $a$ such that for all $\alpha$ in $a$, $p_i(\alpha)$ is a name for the condition obtained by applying $\sigma_\alpha$ to the sequence of conditions played thus far on coordinate $\alpha$. As in the last case, $p_i$ is a condition and extends each $p_j$ for $0 < j < i$.

**Proposition 12.** The set $D_\beta^\kappa$ is dense in $\mathbb{P}_\beta^\kappa$, and if $\beta < \kappa^{++}$ then $|D_\beta^\kappa| \leq \kappa^+$. 

**Proof.** Suppose first that $\text{cf}(\beta) \geq \kappa^+$. Then $\mathbb{P}_\beta^\kappa = \bigcup\{\mathbb{P}_i^\kappa : i < \beta\}$ and $D_\beta^\kappa = \bigcup\{D_i^\kappa : i < \beta\}$. So in this case the statement follows from the induction hypotheses.

Suppose that $\text{cf}(\beta) \leq \kappa$. Let $p$ be a condition in $\mathbb{P}_\beta^\kappa$. We construct a decreasing sequence $\langle p_n : n < \omega \rangle$. Let $p_0 = p$.

Assume that $n < \omega$ and $p_n$ is defined. Since $\mathbb{P}_\beta^\kappa$ is $\kappa + 1$-strategically closed, it forces that the sequence 

$$\langle p_n(\alpha)^G : \alpha \in \text{supp}(p_n) \rangle$$

is in the ground model. So there is a condition $p_{n+1} \leq p_n$ and a sequence 

$$\langle s_n^\kappa : \alpha \in \text{supp}(p_n) \rangle$$

in the ground model such that $p_{n+1}$ forces that $s_n^\kappa = p_n(\alpha)$ for all $\alpha$ in $\text{supp}(p_n)$.

Now define $q$ as follows. The support of $q$ is $\bigcup\{\text{supp}(p_n) : n < \omega\}$. Fix $\alpha$ in $\text{supp}(q)$ and let $n_\alpha$ be the least $n < \omega$ such that $\alpha$ is in $\text{supp}(p_n)$. For $m \geq n_\alpha$, $s_m^\kappa$ is defined and is a sequence of clubs. If $s_m^\kappa$ is eventually constant, define $\alpha$ as this constant value. Otherwise let $\xi_\alpha$ be the suprema of the set $\{\sup \text{dom}(s_m^\kappa) : m \geq n_\alpha\}$, and choose $c_\alpha$ as some cofinal subset of $\xi_\alpha$ with order type $\omega$.

Let $B_\alpha$ be a $\mathbb{P}_\alpha^\kappa$-name for the set such that $Q_\alpha^\kappa = \mathbb{P}_{B_\alpha}$. Since $\text{cf}(\xi_\alpha) = \omega$ and $B_\alpha$ is $\omega$-closed, $\xi_\alpha$ is forced to be in $B_\alpha$. Define

$$s_\alpha = \bigcup \{s_m^\kappa : m \geq n_\alpha\} \cup \langle \xi_\alpha, c_\alpha \rangle.$$ 

Now define $q$ by letting $q(\alpha) = s_\alpha$ for all $\alpha$ in $\text{supp}(q)$. Then $q$ is in $D_\beta^\kappa$ and $q \leq p$.

Any condition in $D_\beta^\kappa$ is coded by its support and a set in $\dot{\kappa}^\kappa H(\kappa^+)$. So $|D_\beta^\kappa| \leq \kappa^+$. 

This completes the definition of $\mathbb{P}^\kappa$. Let $D^\kappa$ be the set $D_{\kappa^{++}}^\kappa$, which is dense in $\mathbb{P}^\kappa$. Note that $D^\kappa = \bigcup\{D_i^\kappa : i < \kappa^{++}\}$.

**Proposition 13.** The iteration $\mathbb{P}^\kappa$ is $\kappa^{++}$-c.c.

**Proof.** It suffices to prove that $D^\kappa$ is $\kappa^{++}$-c.c. Suppose that $\langle p_i : i < \kappa^{++} \rangle$ is a sequence of conditions in $D^\kappa$. By the $\Delta$-system lemma, we can assume that there is a set $d$ such that $\text{supp}(p_i) \cap \text{supp}(p_j) = d$ for $i < j$. Since there are at most $\kappa^+$ many possibilities for $p_i | d$, we assume $p_i | d = p_j | d$ for $i < j$.

Fix $i < j$ and let $q = p_i \cup p_j$. It is easy to check by induction that $q$ is a condition below $p_i$ and $p_j$. 

□
Now we begin to construct our model. Let $V$ be a model of set theory in which $\kappa$ is $\kappa^+\text{-supercompact}$ and GCH holds.

First we construct a coherent sequence of supercompact ultrafilters.

**Proposition 14.** There exists a $\prec$-increasing sequence $(U(\kappa, i) : i < \kappa)$ of normal ultrafilters on $P_\kappa^\kappa$ satisfying the following properties:

1. for $i < j$, $U(\kappa, i) \preceq U(\kappa, j)$.

2. For each $i$ let $j_i : V \rightarrow \text{Ult}(V, U(\kappa, i))$. Then for $\alpha < \beta$ there exists a function $f^\alpha_\beta : \kappa \rightarrow V_\kappa$ such that $j_\beta(f^\alpha_\beta)(\kappa) = U(\kappa, \alpha)$.

**Proof.** It suffices to prove that for any $\prec$-increasing sequence $\vec{U}$ of normal ultrafilters on $P_\kappa^\kappa$ with length less than $\kappa$, there exists a normal ultrafilter $U$ on $P_\kappa^\kappa$ satisfying the following properties:

(a) for each $U_i$ in $\vec{U}$, $U_i \preceq U$ and $U_0 \preceq U \upharpoonright \kappa$.

(b) Let $j : V \rightarrow \text{Ult}(V, U)$. Then for each $U_i$ in $\vec{U}$, there is a function $f_i : \kappa \rightarrow V_\kappa$ such that $j_U(f_i)(\kappa) = U_i$.

If the claim holds then we can construct the desired sequence by induction.

Suppose for a contradiction that the claim fails. Let $\alpha$ be the least ordinal less than $\kappa$ such that there exists a sequence $\vec{U}$ with length $\alpha$ for which no such $U$ exists. Let $W$ be a normal ultrafilter on $P_\kappa^\kappa$ and let $j : V \rightarrow M = \text{Ult}(V, W)$. By closure, the above statement about $\alpha$ holds in $M$.

Let $U = W \upharpoonright \kappa^+$. Let $j_0 : V \rightarrow M_0 = \text{Ult}(V, U \upharpoonright \kappa)$ and $j_1 : V \rightarrow M_1 = \text{Ult}(V, U)$. Apply Lemma 3 to obtain $k_0 : M_0 \rightarrow M_1$ with $\text{crit}(k_0) = \kappa^+ + M_0$ and $k_1 : M_1 \rightarrow M$ with $\text{crit}(k_1) = \kappa^+ + M_1$.

By elementarity, there exists a sequence $(U^*_i : i < \alpha)$ which is a witness for the statement about $\alpha$ in $M_0$. For each $i$ let $U_i = k_0(U^*_i)$. Then $k_0((U^*_i : i < \alpha)) = (U_i : i < \alpha)$ is a witness for the statement about $\alpha$ in $M_1$. By the closure of $M_1$, this sequence really is a $\prec$-increasing sequence of ultrafilters on $P_\kappa^\kappa$. Moreover, $U$ is above each $U_i$. By the closure of $M_0$, $U^*_i \preceq U \upharpoonright \kappa$ as computed in $M_0$ is exactly $U_i \preceq \kappa$.

Therefore $U_i \preceq U \upharpoonright \kappa$ for each $i$. For each $i$ fix $f_i : \kappa \rightarrow V_\kappa$ such that $[f_i] = U^*_i \upharpoonright M_0$. By the definition of $k_0$ as given in the proof of Lemma 3, $k_0([f_i]) = j_1(f_i)(\kappa)$.

So $j_1(f_i)(\kappa) = k_0([f_i]) = k_0(U^*_i) = U_i$.

By elementarity $k_1((U_i : i < \alpha)) = (U_i : i < \alpha)$ satisfies the statement about $\alpha$ in $M$. But this sequence and $U$ satisfy (a) and (b) in $M$, so we have a contradiction. \hfill \Box

Fix $(U(\kappa, i) : i < \kappa)$ as in Proposition 14.

**Lemma 32.** There exists a sequence $(X_i : i < \kappa)$ of subsets of $\kappa$ such that $X_i$ is in $U(\kappa, i) \upharpoonright \kappa$, and for $i \neq j$, $X_i \cap X_j$ is empty.

**Proof.** For distinct $i$ and $j$, let $X_i(j)$ be a set in $(U(\kappa, i) \upharpoonright \kappa) \setminus (U(\kappa, j) \upharpoonright \kappa)$. Let $X_i(i) = \kappa$. Define $X_i^* = \Delta \{X_i(j) : j < \kappa\}$, which is in $U(\kappa, i) \upharpoonright \kappa$. If $i$ and $j$ are distinct, then $X_i^*$ is not in $U(\kappa, j) \upharpoonright \kappa$, since $X_i^* \setminus \{j + 1\} \subseteq X_i(j)$. Now define $X_i = X_i^* \cup \{X_j^* : j < i\}$. \hfill \Box

For each $i < \kappa$, let $j^\alpha_i : V \rightarrow N^\alpha_i = \text{Ult}(V, U(\kappa, i))$. For $\alpha < \beta < \kappa$, fix $f^\alpha_\beta : \kappa \rightarrow V_\kappa$ such that $j^\alpha_\beta(f^\alpha_\beta)(\kappa) = U(\kappa, \alpha)$. Define a sequence $(A_j : j < \kappa)$ by induction, using the following recursion: $A_j$ is the set of $\alpha$ less than $\kappa$ satisfying

1. $\alpha$ is in $X_j$ and $\alpha > j$.

2. For $i < j$, $f^i_j(\alpha)$ is a normal ultrafilter on $P_\alpha^\alpha$ such that $A_i \cap \alpha$ is in...
$f^j_i(\alpha) \upharpoonright \alpha$.

(3) For $i_0 < i < j$, $f^j_i(\alpha) < f^j_{i_0}(\alpha)$.

(4) Let $i_0 < i < j$ and $k : V \rightarrow \text{Ult}(V, f^1_i(\alpha))$. Then $k(f^1_{i_0} \upharpoonright \alpha)(\alpha) = f^1_{i_0}(\alpha)$.

For each $j$, $A_j$ is in $U(\kappa, j) \upharpoonright \kappa$, as can be proved by induction, using Lemma 9 (2) to prove (4).

Let $A$ be equal to $\bigcup \{ A_i : i < \kappa \}$. For each $\alpha$ in $A$, define $o(\alpha)$ as the unique $i$ such that $\alpha$ is in $A_i$. If $\alpha$ is in $A$, then for each $i < o(\alpha)$ let $U(\alpha, i) = f^i_{i+}(\alpha)$ and let $j^i_{\alpha} : V \rightarrow N^\alpha_i = \text{Ult}(V, U(\alpha, i))$. By (2), for $i < o(\alpha)$, $A_i \cap \alpha$ is in $f^i_{\alpha}(\alpha) \upharpoonright \alpha = U(\alpha, i) \upharpoonright \alpha$, and therefore $\alpha$ is in $j^i_{\alpha}(A_i)$. It follows that $j^i_{\alpha}(o(\alpha)) = i$ in $N^\alpha_i$. By (4), for each $i_0 < i_1 < o(\alpha)$, $U(\alpha, i_0)$ is in $N^\alpha_{i_1}$, so $U(\alpha, i_0) < U(\alpha, i_1)$.

Fix $<_\kappa$ a well-ordering of $V_\kappa$ such that for any inaccessible $\alpha$ less than $\kappa$, $<_\alpha = <_{\kappa} \cap (V_\alpha \times V_\alpha)$ is a well-ordering of $V_\alpha$.

We define an Easton support Prikry iteration $(\mathbb{P}_i, Q_i : i < \kappa)$.

Suppose that $\mathbb{P}_\alpha$ is defined. Assume as an induction hypothesis that for all $\gamma$ less than $\alpha$, $Q_\gamma$ is trivial unless $\gamma$ is in $A \cap \alpha$. If $\gamma$ is in $A$ then $\mathbb{P}_\gamma$ forces that $Q_\gamma = \mathbb{P}^\gamma \ast \mathbb{P}(\gamma, o(\gamma))$, where $\mathbb{P}^\gamma$ is a $\gamma + 1$-strategically closed forcing poset and $\mathbb{P}(\gamma, o(\gamma))$ is defined from a family of ultrafilters

\[
\{ U(\gamma, \beta, t) : \beta < o(\gamma), t \text{ is a } \beta\text{-coherent sequence } \}.
\]

Let $G_\alpha$ be generic for $\mathbb{P}_\alpha$ over $V$. If $\alpha$ is not in $A$ then let $Q_\alpha$ be trivial. Suppose that $\alpha$ is in $A$. In $V[G_\alpha]$ let $P^\alpha$ be the iteration of partial square posets which was defined earlier in this section. We consider $P^\alpha$ as a Prikry type forcing poset by letting $\leq^\infty = \leq$. Let $G^\alpha$ be generic for $P^\alpha$ over $V[G_\alpha]$.

If $o(\alpha) = 0$ then $\mathbb{P}(\alpha, o(\alpha))$ is the trivial poset and there is nothing to prove. Suppose that $o(\alpha) > 0$. We define in $V[G_\alpha \ast G^\alpha]$ a family of $\alpha$-complete ultrafilters

\[
\{ U(\alpha, \beta, t) : \beta < o(\alpha), t \text{ is a } \beta\text{-coherent sequence } \}
\]

which satisfies the requirements (1), (2), and (3) described in the last section. We also assume as an induction hypothesis that the family of ultrafilters defined at any previous stage $\gamma$ satisfies the same definition as at stage $\alpha$.

In $V[G_\alpha \ast G^\alpha]$ fix $\beta$ less than $o(\alpha)$ and a $\beta$-coherent sequence $t$. We define $U(\alpha, \beta, t)$. Consider $j^0_\beta : V \rightarrow N^\beta_\alpha$. Since $j^0_\beta(\alpha)(\alpha) = \beta$, by the induction hypotheses we can write

\[
j^0_\beta(P_\alpha) = P_\alpha \ast P^\alpha \ast P(\alpha, \beta) \ast P_{\alpha, j^0_\beta}(\alpha).
\]

By closure, $<_\alpha = <_{\kappa} \cap (V_\alpha \times V_\alpha)$ is in $N^\beta_\alpha$. In order to define $U(\alpha, \beta, t)$ we need to consider all subsets of $\alpha$ in $V[G_\alpha \ast G^\alpha]$. However, $P^\alpha$ does not add subsets to $\alpha$ over $V[G_\alpha]$. So it suffices to consider only $P_\alpha$-names for subsets of $\alpha$. Let $\langle \dot{X}_i : i < \alpha^+ \rangle$ be the $j^0_\beta(\dot{\alpha})$-least enumeration of all canonical $P_\alpha$-names for subsets of $\alpha$.

By the $\alpha^+\hat{\tau}$-weak strategic closure of $P_{\alpha, j^0_\beta}(\alpha)$, there exists a name $\dot{q}$ for a condition in $P_{\alpha, j^0_\beta}(\alpha)$ which decides the statement $\alpha \in j^0_\beta(\dot{X}_i)$ for each $i < \alpha^+$. Let $\dot{q}$ be the $j^0_\beta(\dot{\alpha})$-least such name.

Define $U(\alpha, \beta, t)$ by letting $X$ be in $U(\alpha, \beta, t)$ if there exists an index $i$ such that $X = \dot{X}_i^{G_\alpha}$ and a $\beta$-tree $T$ such that

\[
N^\alpha_\beta[G_\alpha \ast G^\alpha] \models \langle t, T \rangle \Vdash \dot{q} \Vdash \alpha \in j^0_\beta(\dot{X}_i).
\]
Clearly Lemmas 28 through 31 hold for these ultrafilters. Let $\mathbb{P}(\alpha, o(\alpha))$ be the forcing poset defined from this family in $V[G_\alpha * G^\alpha]$. By Proposition 10, $\mathbb{P}(\alpha, o(\alpha))$ satisfies the Prikry property.

This completes the definition of $\mathbb{P}_\kappa$. By standard Easton support arguments, $\mathbb{P}_\kappa$ preserves GCH.

In $V$ let $F : \kappa^+ < \omega \to \kappa^+$ be a Jonsson function for $\kappa^+$. So if $X \subseteq \kappa^+$ has size $\kappa^+$ and is closed under $F$, then $X = \kappa^+$. Define $S^*$ as the set of $a$ in $S(\kappa, \kappa^+)$ such that $a \cap \kappa$ is strongly inaccessible, $F$ is Jonsson for $a$, and $a$ is closed under suprema of bounded subsets with order type less than $a \cap \kappa$.

**Lemma 33.** If $U$ is any normal ultrafilter on $P_\kappa \kappa^+$, then $S^*$ is in $U$.

**Proof.** Let $j : V \to M = \text{Ult}(V, U)$. Then $j^* \kappa^+$ is in $j(\kappa(\kappa^+))$ and $j^* \kappa^+ \cap j(\kappa) = \kappa$ is strongly inaccessible in $M$. Since the critical point of $j$ is $\kappa$, $j^* \kappa^+$ is closed under suprema of subsets with order type less than $\kappa = j^* \kappa^+ \cap j(\kappa)$.

To show that $j(F)$ is Jonsson for $j^* \kappa^+$, suppose that $X$ is a subset of $j^* \kappa^+$ which is closed under $j(F)$ and has size equal to $|j^* \kappa^+| = \kappa^+$. Then there is $Y \subseteq \kappa^+$ with size $\kappa^+$ such that $j^* Y = X$. Since $X$ is closed under $j(F)$, $Y$ is closed under $F$. The fact that $F$ is a Jonsson function implies that $Y = \kappa^+$, and therefore $X = j^* \kappa^+$. It follows that $j^* \kappa^+$ is in $j(S^*)$, so $S^*$ is in $U$.

For each $i$ less than $\kappa$, let

$$S_i = \{a \in S^* : a \cap \kappa \in A_i\}.$$ 

Then $S_i$ is in $U(\kappa, i)$. Define

$$S = \bigcup \{S_i : i < \kappa\}.$$ 

**Lemma 34.** The function $a \mapsto \sup a$ is injective on $S$. So if $T$ is a subset of $S$, there is $X \subseteq \kappa^+$ such that $T = \{a \in S : \sup a \in X\}$.

**Proof.** Suppose that $a$ and $b$ are distinct elements in $S$ with the same supremum. Since $a$ and $b$ are closed under suprema of subsets with order type $\omega$, the set $a \cap b$ is unbounded in $\sup a$. Also $\text{cf}(\sup a) = \text{o.t.}(a) = (a \cap \kappa)^+$. It follows that $|a \cap b| = |a| = |b|$. But $a \cap b$ is closed under $F$ and is a subset of $a$ and $b$. Since $F$ is Jonsson for $a$ and $b$, $a = a \cap b = b$.

For the second statement, let $X = \{\sup a : a \in T\}$. Since $a \mapsto \sup a$ is injective, $T = \{a \in S : \sup a \in X\}$.

Let $G_\kappa$ be a generic filter for $\mathbb{P}_\kappa$ over $V$. In $V[G_\kappa]$ we construct a subiteration $\mathbb{P}_B^i$ of $\mathbb{P}_B^\kappa$ such that $\mathbb{P}_B^i$ forces that $\text{NS} \upharpoonright S$ is saturated, and for all regular $\mu$ less than $\kappa$, the set $\{a \in S : \text{cf}(a \cap \kappa) = \mu\}$ is stationary.

In $V[G_\kappa]$ let $\langle \dot{Y}_i : i < \kappa^+ \rangle$ be a sequence of $D^\kappa$-names such that for all $i < \kappa^+$, $\mathbb{P}_i^\kappa$ forces that $\dot{Q}_i^\kappa$ is the forcing poset $\mathbb{P}_i^\kappa$ for adding a $\square^\kappa_Y$-sequence.

For each $i < \kappa$ write

$$j_i^* (\mathbb{P}_\kappa * \mathbb{P}_B^\kappa) = \mathbb{P}_\kappa * \mathbb{P}_i^\kappa * \mathbb{P}_i^\text{tail} * j_i^* (\mathbb{P}_B^\kappa).$$

We introduce some notation for generic filters for these posets. We use $G^\kappa$, $G(\kappa, i)$, and $G^\text{tail}$ to denote generic filters of $D^\kappa$, $\mathbb{P}(\kappa, i)$, and $\mathbb{P}_i^\text{tail}$ respectively (or sometimes for the names of such filters). For each $\alpha$ less than $\kappa^+$, let $G_\alpha^\kappa = G^\kappa \upharpoonright \alpha$, i.e. the generic for $D_\alpha^\kappa$ given by $G^\kappa$. Let $H_\alpha^\kappa$ be the generic for $\mathbb{P}_\alpha^\kappa$ given by $G^\kappa$ over $V[G_\kappa * G_\alpha^\kappa]$. 


We construct by induction an iteration
\[ \langle \mathcal{P}_{B_n}, g_{B_n} : \alpha < \kappa^{++} \rangle \]
and functions
\[ \langle \pi_\alpha : \alpha < \kappa^{++} \rangle \]
such that for each \( \alpha, \pi_\alpha : D_\alpha^P \to \mathcal{P}_{B_n} \) is a projection mapping satisfying:
1. For \( \alpha < \beta \) and \( p \) in \( D_\beta^P \), \( \pi_\beta(p) \upharpoonright \alpha = \pi_\alpha(p \upharpoonright \alpha) \),
2. \( D_{B_n}^P = D_\alpha^P \cap \mathcal{P}_{B_n} \) is dense in \( \mathcal{P}_{B_n} \) and \( \pi_\alpha \upharpoonright D_{B_n}^P \) is the identity,
3. For each \( \alpha \), \( B_{\alpha+1} \) is a \( \mathcal{P}_{B_n} \)-name for a subset of \( \alpha + 1 \),
4. \( \mathcal{P}_{B_n} \) is \( \kappa + 1 \)-strategically closed.

In addition we construct for each \( i \) less than \( \kappa \) a \( \leq^* \)-descending sequence of \( \mathcal{P}^* \)-names
\[ \langle q_i^\alpha(0), q_i^\alpha(1) : \alpha < \kappa^{++} \rangle \]
for conditions in \( \mathcal{P}^i_{\text{tail}} \), and a descending sequence of \( \mathcal{P}^\kappa \)-names
\[ \langle s_i^\alpha(0), s_i^\alpha(1) : \alpha < \kappa^{++} \rangle \]
where \( s_i^\alpha(0) \) and \( s_i^\alpha(1) \) are forced to be conditions in \( j_i^\alpha(\mathcal{P}_{B_n}) \) such that for all \( p \) in \( G^\alpha_{\mathcal{B}_n} \), \( s_i^\alpha(0) \leq j_i^\alpha(p_\alpha(p)) \) (i.e., \( s_i^\alpha(0) \) is a master condition). Both of the sequences will constitute a run of the game in their respective posets, with the 0 condition interpreted as Player II’s play and the 1 condition as Player I’s play.

Let \( B_0 = \emptyset \) and for each \( i \) less than \( \kappa \) let \( q_i^0(0) = 1 \) and \( s_i^0(1) = 1 \). Fix \( \alpha < \kappa^{++} \) and suppose that \( \mathcal{P}_{B_n} \) and \( \pi_\alpha \) are defined, and for all \( i \) less than \( \kappa \), \( g_i^\alpha(0) \) and \( s_i^\alpha(0) \) are defined.

We consider several cases. First suppose that \( \tilde{Y}_\alpha \) is not a \( \mathcal{D}_{B_{\alpha+1}} \)-name. In this case let \( B_{\alpha+1} \) be a \( \mathcal{P}_{\tilde{B}_\alpha} \)-name for \( B_\alpha \). So \( \mathcal{P}_{\tilde{B}_\alpha} \) forces that \( \alpha \) is not in \( B_{\alpha+1} \). Let \( \mathcal{P}_{B_{\alpha+1}} \) be equal to \( \mathcal{P}_{\tilde{B}_\alpha} \ast (0) \). Clearly properties (1) through (4) hold.

Fix \( i \) less than \( \kappa \). Let \( s_i^\alpha(1) = s_i^\alpha(0) \) and \( q_i^\alpha(1) = q_i^\alpha(0) \). Let \( q_i^{\alpha+1}(0) \) be a name for Player II’s response to the run of the game so far in \( \mathcal{P}^i_{\text{tail}} \). Define \( s_i^{\alpha+1}(0) \) as follows. The support of \( s_i^{\alpha+1}(0) \) is \( \text{supp}(s_i^\alpha(1)) \). Fix \( \gamma \) in \( \text{supp}(s_i^\alpha(1)) \). Consider the run of the game in coordinate \( \gamma \) which begins when Player I plays \( s_i^\beta(1)(\gamma) \) for the first value of \( \beta \) for which this term is defined. Define \( s_i^{\alpha+1}(0)(\gamma) \) as Player II’s response to this run of the game according to its strategy.

Now consider the case when \( \tilde{Y}_\alpha \) is a \( \mathcal{D}_{B_{\alpha+1}} \)-name. Fix \( i \) less than \( \kappa \). Since \( s_i^\alpha(0) \) is a master condition, \( s_i^\alpha(0) \) forces that \( j_i^\alpha \) can be lifted to
\[ j_i^\alpha : V[G_\kappa \ast G_{B_n}^\alpha] \to N_i^\kappa[G_\kappa \ast G_\kappa^\alpha \ast G(\kappa, i) \ast G_{\text{tail}}^i \ast g_i^\alpha] \]
where \( G_{B_n}^\alpha \) is the filter generated by \( \pi_\alpha^{-1}(G_{\mathcal{B}_n}^\alpha) \) and \( g_i^\alpha \) is a name for the generic for \( j_i^\alpha(\mathcal{P}_{B_n}) \). Choose \( s_i^\alpha(1) \leq s_i^\alpha \) in \( j_i^\alpha(\mathcal{P}_{\tilde{B}_\alpha}) \) which decides the statement
\[ \sup j_i^\kappa \downarrow^\kappa^+ \in j_i^\kappa(\tilde{Y}_\alpha) \]
Choose \( q_i^\alpha(1) \) as a name for a condition directly extending \( q_i^\alpha(0) \) which decides which way that \( s_i^\alpha(1) \) decides the given statement. Let \( q_i^{\alpha+1}(0) \) be a name for Player II’s response to the run of the game in \( \mathcal{P}^i_{\text{tail}} \).

Let \( B_{\alpha+1} \) be a \( \mathcal{P}_{B_n} \)-name which is forced to be equal to \( B_\alpha \) unless for all \( i \) less than \( \kappa \),
\[ N_i^\kappa[G_\kappa \ast G_{B_n}^\alpha] = (\mathcal{P}^\kappa / \mathcal{P}_{B_n}) \ast \mathcal{P}(\kappa, i) \ast q_i^{\alpha+1}(0) \ast s_i^\alpha(1) \ast \sup j_i^\kappa \downarrow^\kappa^+ \notin j_i^\kappa(\tilde{Y}_\alpha) \]
in which case \( B_{\alpha+1} = B_\alpha \cup \{ \alpha \} \). Define \( \mathbb{P}^\kappa_{B_{\alpha+1}} \) as \( \mathbb{P}^\kappa_{B_\alpha} + \mathbb{Q}^\kappa_{B_\alpha} \), where \( \mathbb{Q}^\kappa_{B_\alpha} \) is forced to be trivial if \( \alpha \) is not in \( B_{\alpha+1} \) and is equal to \( \mathbb{P}_\gamma \) if \( \alpha \) is in \( B_{\alpha+1} \).

Note that if \( \alpha \) is in \( B_{\alpha+1} \), then for all \( i \) less than \( k \) the condition \( s^i_\alpha(1) \) forces that \( \sup j^*_i \) is \( \kappa \)-Borel in \( \gamma \), and \( j^*_i(Y_\alpha) \). Therefore \( \bigcup j^*_i(\mathbb{P}_\gamma) \) is a master condition in \( j^*_i(\mathbb{P}_\gamma) \).

Define \( s^i_{\alpha+1}(0) \) exactly as \( s^i_{\alpha+1}(0) \) was defined in the last case by using the strategy at each coordinate in the support. If \( \alpha \) is not in \( B_{\alpha+1} \) then let \( s^i_{\alpha+1}(0) = s^i_\alpha(0) \). If \( \alpha \) is in \( B_{\alpha+1} \) then let

\[
s^i_{\alpha+1}(0) = (s^i_\alpha(0) \upharpoonright j^*_i(\alpha)) \cup (j^*_i(\alpha) \upharpoonright \bigcup j^*_i(H^*_\kappa)).
\]

Note that by the induction hypotheses and the comments above, \( s^i_{\alpha+1}(0) \) is a master condition in \( k(\mathbb{P}^\kappa_{B_{\alpha+1}}) \).

**Lemma 35.** The poset \( \mathbb{P}^\kappa_{B_{\alpha+1}} \) is \( \kappa+1 \)-strategically closed and \( D^\kappa_{B_{\alpha+1}} \) is dense in \( \mathbb{P}^\kappa_{B_{\alpha+1}} \).

**Proof.** Straightforward.

**Lemma 36.** There is a projection mapping \( \pi_{\alpha+1} : D^\kappa_{\alpha+1} \to \mathbb{P}^\kappa_{B_{\alpha+1}} \) such that:

- (a) \( \pi_{\alpha+1} \upharpoonright D^\kappa_{B_{\alpha+1}} \) is the identity,
- (b) For all \( q \) in \( D^\kappa_{\alpha+1} \) and \( \beta < \alpha + 1 \), \( \pi_{\alpha+1}(q) \upharpoonright \beta = \pi_{\beta}(q \upharpoonright \beta) \).

**Proof.** For each \( x \) in \( V[G_\kappa] \) let \( \dot{a}_x \) be a \( \mathbb{P}^\kappa_{B_\delta} \)-name which is forced to be equal to \( \check{x} \) if \( \alpha \) is in \( B_{\alpha+1} \) and is 1 otherwise. Define \( \pi_{\alpha+1}(\dot{p} \upharpoonright \delta) = \pi(\check{p}(\check{a}_x)) \). Statements (a) and (b) follow from induction hypotheses (1) and (2).

This completes the definition for successor stages. Suppose that \( \delta \leq \kappa^+ \) is a limit ordinal and \( \mathbb{P}^\kappa_{B_\delta} \) is forced for \( \alpha < \delta \). Define \( \mathbb{P}^\kappa_{B_\delta} \) as the set of functions \( p \) with domain a subset of \( \delta \) with size no larger than \( \kappa \) such that \( p \upharpoonright \alpha \) is in \( \mathbb{P}^\kappa_{B_\alpha} \) for \( \alpha < \delta \). Let \( B_\delta \) be a \( \mathbb{P}^\kappa_{B_\delta} \)-name for \( \bigcup \{ B_\alpha : \alpha < \delta \} \).

**Lemma 37.** The poset \( \mathbb{P}^\kappa_{B_\delta} \) is \( \kappa+1 \)-strategically closed and \( D^\kappa_{B_\delta} = D^\kappa_{\delta} \cap \mathbb{P}^\kappa_{B_\delta} \) is dense in \( \mathbb{P}^\kappa_{B_\delta} \).

**Proof.** The proof is identical to Propositions 11 and 12.

Fix \( i \) less than \( \kappa \). Define \( q^i_0(0) \) as a name for Player II’s response to the run of the game defined thus far in \( \mathbb{P}^\kappa_{\text{tail}} \). Define \( s^i_0(0) \) as follows. The support of \( s^i_0(0) \) is equal to \( \bigcup \{ \supp(s^i_0(0)) \} : \alpha < \delta \} \). If \( \gamma \) is in \( \supp(s^i_0(0)) \), define \( s^i_\gamma(\gamma) \) by applying the strategy to the run of the game determined on coordinate \( \alpha \).

**Lemma 38.** There is a projection mapping \( \pi_\delta : D^\kappa_\delta \to \mathbb{P}^\kappa_{B_\delta} \) such that:

- (a) \( \pi_\delta \upharpoonright D^\kappa_{B_\delta} \) is the identity,
- (b) for all \( p \) in \( D^\kappa_\delta \) and \( \alpha < \delta \), \( \pi_\delta(p) \upharpoonright \alpha = \pi_{\alpha}(p \upharpoonright \alpha) \).

**Proof.** Define \( \pi_\delta(p) = \bigcup \{ \pi_{\alpha}(p \upharpoonright \alpha) : \alpha < \delta \} \). The induction hypotheses (1) and (2) imply that the definition of \( \pi_\delta \) makes sense and (a) and (b) hold. We show that \( \pi_\delta \) is a projection mapping.

Obviously \( \pi_\delta(1) = 1 \) and \( q \leq p \) implies \( \pi_\delta(q) \leq \pi_\delta(p) \). Suppose that \( q \leq \pi_\delta(p) \) in \( \mathbb{P}^\kappa_{B_\delta} \). We find \( r \leq p \) in \( D^\kappa_\delta \) such that \( \pi_\delta(r) \leq q \). If \( \text{cf}(\delta) \geq \kappa^+ \) then the supports of \( p \) and \( q \) are bounded below \( \delta \), so \( r \) exists by the induction hypotheses.

Suppose that \( \text{cf}(\delta) \leq \kappa \). Let \( (\alpha_i : i < \text{cf}(\delta)) \) be increasing and unbounded in \( \delta \), with \( \alpha_0 = 0 \). We define \( (r_i, r^*_i : i < \text{cf}(\delta)) \) so that \( r_i \) is in \( D^\kappa_{\alpha_i} \) and is below \( p \upharpoonright \alpha_i \),
Let $r_0 = \emptyset$. Suppose that $i < \text{cf}(\delta)$ and $r_i$ is defined such that $\pi_{\alpha_i}(r_i) \leq q \upharpoonright \alpha_i$.

Since $q \leq \pi_\delta(p)$,

$$\pi_{\alpha_i}(r_i) \upharpoonright (q \upharpoonright [\alpha_i, \alpha_{i+1}]) \leq \pi_{\alpha_{i+1}}(r_i \upharpoonright (p \upharpoonright [\alpha_i, \alpha_{i+1}]))$$

in $P_{\alpha_{i+1}}^{\pi_\delta(p)}$. Apply the fact that $\pi_{\alpha_{i+1}}$ is a projection mapping to obtain $r_i^* \leq r_i \upharpoonright (p \upharpoonright [\alpha_i, \alpha_{i+1}])$ in $D_{\alpha_{i+1}}^\kappa$ such that

$$\pi_{\alpha_{i+1}}(r_i^*) \leq \pi_{\alpha_i}(r_i) \upharpoonright (q \upharpoonright [\alpha_i, \alpha_{i+1}]).$$

Now define $r_{i+1}$ as follows. The support of $r_{i+1}$ is the same as the support of $r_i^*$. For $\alpha$ in the support of $r_{i+1}$, we consider the following run of the game. Player I begins by playing condition $r_j(\alpha)$ for the first value of $j$ for which this term is defined. Player II responds according to its strategy with $r_j^*(\alpha)$ defined above. Let $r_{i+1}(\alpha)$ be the condition obtained by applying the strategy to this run of the game.

Suppose $\delta_0 \leq \delta$ is a limit ordinal. Define $r_{\delta_0}$ with support $\bigcup\{\text{sup}(r_i) : i < \delta_0\}$, such that for each $\alpha$ in this support, $r_{\delta_0}(\alpha)$ is obtained by applying the strategy to the run of the game at coordinate $\alpha$.

Let $r$ be an extension of $r_\delta$ in $D_B^\kappa$. Then $r \leq p$ and $\pi_\delta(r) \leq q$. □

This completes the definition of $P_B^\kappa$. By Lemma 2, $P_B^\kappa$ and $P_n^\kappa / P_B^\kappa$ are $\kappa^{++}$-c.c. Suppose that $G_B^\kappa$ is generic for $P_B^\kappa$ over $V[G_\kappa]$.

**Proposition 15.** In $V[G_\kappa * G_B^\kappa]$, $NS \upharpoonright S$ is saturated.

**Proof.** For each set $X \subseteq \kappa^+$, $i < \kappa$, and $\alpha < \kappa^{++}$, let $\Phi(X, i, \alpha)$ be the statement

$$N_i^{\kappa^+}[G_\kappa * G_B^\kappa] \models (P_n^\kappa / P_B^\kappa) * P(\kappa, i) \models q_{\alpha+1}(i) \models \sup j^\kappa_i(\alpha) \notin j^\kappa_i(X).$$

Let $T$ be a stationary subset of $S$. If $a$ is in $T$, then $a \cap \kappa$ is in $A$ and $o(a \cap \kappa)$ is less than $a \cap \kappa$. Therefore the map $a \mapsto o(a \cap \kappa)$ is regressive. By Fodor's Lemma, there exists an index $i$ less than $\kappa$ such that $T \cap S_i$ is stationary. So in order to prove that $NS \upharpoonright S$ is saturated, it suffices to prove that $NS \upharpoonright S_i$ is saturated for all $i$ less than $\kappa$.

Fix $i$ less than $\kappa$ and let $T$ be a stationary subset of $S_i$. Fix $X \subseteq \kappa^+$ such that $T = \{a \in S : \sup a \in X\}$.

We claim that $T$ is non-stationary iff there is $\alpha$ less than $\kappa^{++}$ such that $\Phi(X, i, \alpha)$ holds.

Suppose that $T$ is disjoint from a club set $C$. Since $P_B^\kappa$ is $\kappa^{++}$, there is $\alpha < \kappa^{++}$ such that $C$ and $T$ are in $V[G_\kappa * G_B^\kappa]$. Fix $p$ in $G_\kappa * G_B^\kappa$ which forces that $\hat{T} = \{a \in S : \sup a \in X\}$ and $\hat{C} \cap \hat{T}$ is empty, where $\hat{T}$, $\hat{X}$, and $\hat{C}$ are names for $T$, $X$, and $C$. Since $s_i^0(0)$ is a master condition,

$$N_i^{\kappa^+}[G_\kappa * G_B^\kappa] \models (P_n^\kappa / P_B^\kappa) * P(\kappa, i) * P_i^\text{tail} \models s_i^0(0) \models j^\kappa_i(\hat{C}) \cap j^\kappa_i(\hat{T}) = \emptyset.$$ But it is also forced that $j^\kappa_i(\alpha+\kappa^+) = \bigcup j^\kappa_i (\alpha + \hat{C}) \in j^\kappa_i (\hat{C})$, so $j^\kappa_i(\alpha+\kappa^+)$ is not in $j^\kappa_i(T)$, i.e. sup $j^\kappa_i (\alpha)$ is in $j^\kappa_i(X)$. Therefore $\Phi(X, i, \alpha)$ holds.

Suppose on the other hand that $\Phi(X, i, \alpha)$ holds. Let $Y$ be the closure of $X$ under suprema of subsets with order type $\omega$ or $\omega_1$. Since sup $j^\kappa_i (\alpha)$ does not have cofinality $\omega$ or $\omega_1$, $\Phi(X, i, \alpha)$ implies $\Phi(Y, i, \alpha)$. The poset $P_B^\kappa$ is $\kappa^{++}$-c.c, so there exists $\gamma$ less than $\kappa^{++}$ such that $Y$ is in $V[G_\kappa * G_B^\kappa]$. Moreover, we can choose $\gamma$ so that there is a canonical $D_B^\kappa$-name $\hat{Y}$ for $Y$ and $p$
in $G^*_B$, which forces $\Phi(\hat{Y}, i, \alpha)$ and the other properties of $Y$ stated above. Then there is $\beta < \kappa^+$ greater than $\alpha$ and $\gamma$ such that $\hat{Y} = Y_{\beta}$ and for all $\xi$ less than $\kappa$, $N^c_\xi[G_\kappa \ast G^*_B] = (P^\kappa//P^*_B) \ast \mathcal{P}(\kappa, \xi) \models q^\xi_{\beta+1}(0) \models s^\xi_{\beta}(1) \models \sup j^\kappa_\gamma \kappa^+ \notin j^\kappa_\gamma(Y_{\beta})$.

So $\beta$ is in $B$, and therefore $G^*_B$ adds a $\square^+_\gamma$-sequence and destroys the stationarity of the set $\{a \in S(\kappa, \kappa^+) : \sup a \in Y\}$. But $T$ is contained in this set, so it is non-stationary in $V[G_\kappa \ast G^*_B]$.

Suppose for a contradiction that $NS \upharpoonright S_i$ is not saturated. Let $\langle T_{X_\alpha} : \alpha < \kappa^+ \rangle$ be a sequence of stationary subsets of $S_i$ such that $T_{X_\alpha} \cap T_{X\beta}$ is non-stationary for $\alpha < \beta$. Since $T_{X_\alpha}$ is stationary, there is $p_\alpha$ in $(P^\kappa//P^*_B) \ast \mathcal{P}(\kappa, i)$ and $\gamma$ such that $N^c_\xi[G_\kappa \ast G^*_B] = p_\alpha \models q^\xi_{\gamma+1}(0) \models s^\xi_{\gamma}(1) \models \sup j^\kappa_{\gamma} \kappa^+ \in j^\kappa_{\gamma}(X_\alpha)$.

Then $p_\alpha$ and $p_\beta$ are incompatible for $\alpha < \beta$. For if $q \leq p_\alpha, p_\beta$, then letting $\gamma = \max\{\gamma_\alpha, \gamma_\beta\}$, $N^c_\xi[G_\kappa \ast G^*_B] = q \models q^\xi_{\gamma+1}(0) \models s^\xi_{\gamma}(1) \models \sup j^\kappa_{\gamma} \kappa^+ \in j^\kappa_{\gamma}(X_\alpha \cap X_\beta)$, and therefore $T_{X_\alpha} \cap T_{X_\beta}$ is stationary. We have a contradiction since the poset $(P^\kappa//P^*_B) \ast \mathcal{P}(\kappa, i)$ is $\kappa^+\text{-c.c.}$.

**Proposition 16.** In $V[G_\kappa \ast G^*_B]$ let $\mu$ be a regular cardinal less than $\kappa$. Then the set 
\[ \{a \in S : cf(a \cap \kappa) = \mu\} \]
is stationary.

**Proof.** If $a$ is in $S_i = \{a \in S : o(a \cap \kappa) = \mu\}$, then $cf(a \cap \kappa) = \mu$. Let $C$ be a club subset of $P, \kappa^+$ in $V[G_\kappa \ast G^*_B]$. Since $P^*_B$ is $\kappa^+\text{-c.c.}$, there is $\gamma$ less than $\kappa^+$ such that $C$ is in $V[G_\kappa \ast G^*_B]$.

Let $H$ be generic for $j^\kappa_\mu(P_\kappa) / (P_\kappa * P^*_B)$ over $V[G_\kappa \ast G^*_B]$ and let $h$ be generic for $j^\kappa_\mu(P^*_B)$ over $V[G_\kappa \ast G^*_B * H]$ which contains the condition $s^\kappa_0(0)$. Since $s^\kappa_0(0)$ is a master condition, in $V[G_\kappa \ast G^*_B * H * h]$ we can lift $j^\kappa_\mu$ to

\[ j^\kappa_\mu : V[G_\kappa \ast G^*_B] \rightarrow N^\kappa_\mu[G_\kappa \ast G^*_B * H * h].\]

Recall that $j^\kappa_\mu \kappa^+ = j^\kappa_\mu(S_\mu)$. Also $j^\kappa_\mu \kappa^+ = \bigcup j^\kappa_\mu C$ is in $j^\kappa_\mu(C)$. By elementarity, $S_\mu \cap C$ is non-empty. \qed

**References**


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