PARAMETRIZED MEASURING AND CLUB GUESSING

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Abstract. We introduce Strong Measuring, a maximal strengthening of J. T. Moore’s Measuring principle, which asserts that every collection of fewer than continuum many closed bounded subsets of \( \omega_1 \) is measured by some club subset of \( \omega_1 \). The consistency of Strong Measuring with the negation of CH is shown, solving an open problem from [2] about parametrized measuring principles. Specifically, we prove that Strong Measuring follows from MR

Also, Strong Measuring is shown to be consistent with the continuum being arbitrarily large.

Club guessing principles at \( \omega_1 \) are well–studied natural weakenings of Jensen’s \( \Diamond \) principle. Presented in a general form, they assert the existence of a sequence \( \vec{C} = \langle c_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle \), where each \( c_\alpha \) is a club of \( \alpha \), such that \( \vec{C} \) guesses clubs of \( \omega_1 \) in some suitable sense. \( \vec{C} \) guessing a club \( D \) of \( \omega_1 \) usually means that there is some (equivalently, stationarily many) \( \delta \in D \) such that \( c_\delta \cap D \) is a suitably large subset of \( c_\delta \); for example, we could require that \( c_\delta \subseteq D \), which is called club guessing, or that \( c_\delta \cap D \) is cofinal in \( \delta \), which we call very weak club guessing.

Unlike the case of their versions at cardinals higher than \( \omega_1 \), for which there are non–trivial positive ZFC theorems (see, for example, [14]), club guessing principles at \( \omega_1 \) are independent of ZFC. On the one hand, all of these principles obviously follow from \( \Diamond \), and hence they hold in \( L \), and they can always be forced by countably closed forcing. On the other hand, classical forcing axioms at the level of \( \omega_1 \), such as the Proper Forcing Axiom (PFA), imply the failure of even the weakest of these principles. It should nevertheless be noted that Martin’s Axiom + \( \neg \text{CH} \) is compatible with Club Guessing. This is because Martin’s Axiom can always be forced by a c.c.c. forcing, and the fact that every club of \( \omega_1 \) in a generic extension via a c.c.c. forcing contains a club of \( \omega_1 \) from the ground model implies that a club–guessing sequence from the ground model remains club–guessing in the extension. (On the other hand, of course, the negation of \( \text{CH} \) violates \( \Diamond \).

Measuring is a particularly strong failure of Club Guessing due to J. T. Moore ([8]). Let \( X \) and \( Y \) be countable subsets of \( \omega_1 \) with the same supremum \( \delta \). We say that \( X \) measures \( Y \) if there exists \( \beta < \delta \) such that \( X \setminus \beta \) is either contained in, or disjoint from, \( Y \). Measuring is the statement that for any sequence \( \langle c_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle \), where each \( c_\alpha \) is a closed subset of \( \alpha \), there exists a club \( D \subseteq \omega_1 \) such that for all limit points \( \delta \) of \( D \), \( D \cap \delta \) measures \( c_\delta \).

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Measuring can be viewed as a strong negation of Club Guessing since, as is easy to see, it implies the failure of Very Weak Club Guessing. Measuring follows from the Mapping Reflection Principle (MRP), and therefore from PFA, and it can be forced over any model of ZFC.

From Measuring as a vantage point, one can attempt to consider even stronger failures of Club Guessing. In this vein, the following parametrized family of strengthenings of Measuring was considered in [2].

Definition. For a cardinal $\kappa$, let $\text{Measuring}_{<\kappa}$ denote the statement that whenever $\vec{C} = \langle C_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle$ is a sequence such that each $C_\alpha$ is a family of fewer than $\kappa$ many closed subsets of $\alpha$, there exists a club $D \subseteq \omega_1$ with the property that for every limit point $\delta$ of $D$ and every $c \in C_\delta$, $D \cap \delta$ measures $c$. For a cardinal $\lambda$, let $\text{Measuring}_{<\lambda^+}$ denote $\text{Measuring}_{<\lambda^+}$.

In the situation given by the above definition, we say that $D$ measures $\vec{C}$. We also define Strong Measuring to be the statement $\text{Measuring}_{<2^\omega}$.

In the present article we contribute to the body of information on Measuring and related strong failures of Club Guessing (see also [8], [3], [5], [6], and [2]). One of the questions left unresolved in [2] is whether $\text{Measuring}_{\omega_1}$ is consistent at all. Answering this question was the motivation for the work in the present article. Our main results are the following.

1. Strong Measuring + $\neg$CH is consistent. In fact, this statement follows from MRP + Martin’s Axiom for the class of $\sigma$-centered posets, and also from BPFA.

2. Strong Measuring is consistent with the continuum being arbitrarily large.

We also show the failure, in ZFC, of $\text{Measuring}_{\kappa}$, where $\kappa$ is among some of the classical cardinal characteristics of the continuum.

1. Background

We review some background material and notation which is needed for understanding the paper. Let $c$ denote the cardinality of the continuum $2^\omega$. A set $S \subseteq [\omega]^\omega$ is a splitting family if for any infinite set $x \subseteq \omega$, there exists $A \in S$ such that $A$ splits $x$ in the sense that both $x \cap A$ and $x \setminus A$ are infinite. The splitting number $s$ is the least cardinality of some splitting family. Given functions $f, g : \omega \to \omega$, we say that $g$ dominates $f$ if for all $n < \omega$, $f(n) < g(n)$. A family $B \subseteq \omega^\omega$ is bounded if there exists a function $g \in \omega^\omega$ which dominates every member of $B$, and otherwise it is unbounded. The bounding number $b$ is the least cardinality of some unbounded family. Both cardinal characteristics $s$ and $b$ are uncountable.

Let $P$ be a forcing poset. A set $X \subseteq P$ is centered if every finite subset of $X$ has a lower bound. We say that $P$ is $\sigma$-centered if it is a union of countably many centered sets. Martin’s Axiom for $\sigma$-centered forcings (MA($\sigma$-centered)) is the statement that for any $\sigma$-centered forcing $P$ and any collection of fewer than $\epsilon$ many dense subsets of $P$, there exists a filter on $P$ which meets each dense set in the collection. More generally, let $m(\sigma$-centered) be the least cardinality of a collection of dense subsets of some $\sigma$-centered forcing poset for which there does not exist a filter which meets each dense set in the collection. Note that MA($\sigma$-centered) is equivalent to the statement that $m(\sigma$-centered) equals $\epsilon$.

The Bounded Proper Forcing Axiom (BPFA) is the statement that whenever $P$ is a proper forcing and $\langle A_i : i < \omega_1 \rangle$ is a sequence of maximal antichains of $P$.
each of size at most \( \omega_1 \), then there exists a filter on \( \mathbb{P} \) which meets each \( A_i \) ([9]).

We note that BPFA implies \( \varepsilon = \omega_2 \) ([12, Section 5]). It easily follows that BPFA implies Martin’s Axiom, and in particular, implies MA(\( \sigma \)-centered). The forcing axiom BPFA is equivalent to the statement that for any proper forcing poset \( \mathbb{P} \) and any \( \Sigma_1 \) statement \( \Phi \) with a parameter from \( H(\omega_2) \), if \( \Phi \) holds in a generic extension by \( \mathbb{P} \), then \( \Phi \) holds in the ground model ([7]).

An open stationary set mapping for an uncountable set \( X \) and regular cardinal \( \theta > \omega \) is a function \( \Sigma \) whose domain is the collection of all countable elementary substructures \( M \) of \( H(\theta) \) with \( X \in M \), such that for all such \( M \), \( \Sigma(M) \) is an open, \( M \)-stationary subset of \([X]^\omega\). By open we mean in the Ellentuck topology on \([X]^\omega\), and \( M \)-stationary means meeting every club subset of \([X]^\omega\) which is a member of \( M \) (see [12] for the complete details). In this article, we are only concerned with these ideas in the simplest case that \( X = \omega_1 \) and for each \( M \in \text{dom}(\Sigma) \), \( \Sigma(M) \subseteq \omega_1 \). In this case, being open is equivalent to being open in the topology on \( \omega_1 \) with basis the collection of all open intervals of ordinals, and being \( M \)-stationary is equivalent to meeting every club subset of \( \omega_1 \) in \( M \).

For an open stationary set mapping \( \Sigma \) for \( X \) and \( \theta \), a \( \Sigma \)-reflecting sequence is an \( \varepsilon \)-increasing and continuous sequence \( \langle M_i : i < \omega_1 \rangle \) of countable elementary substructures of \( H(\theta) \) containing \( X \) as a member satisfying that for all limit ordinals \( \delta < \omega_1 \), there exists \( \beta < \delta \) so that for all \( \beta \leq \xi < \delta \), \( M_\xi \cap X \in \Sigma(M_\delta) \). The Mapping Reflection Principle (MRP) is the statement that for any open stationary set mapping \( \Sigma \), there exists a \( \Sigma \)-reflecting sequence. We will use the fact that for any open stationary set mapping \( \Sigma \), there exists a proper forcing which adds a \( \Sigma \)-reflecting sequence ([12, Section 3]). Consequently, MRP follows from PFA.

2. Parametrized Measuring and Club Guessing

Let \( X \) and \( Y \) be countable subsets of \( \omega_1 \) with the same supremum \( \delta \). We say that \( X \) measures \( Y \) if there exists \( \beta < \delta \) such that \( X \setminus \beta \) is either contained in, or disjoint from, \( Y \). Measuring is the statement that for any sequence \( \langle c_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle \), where each \( c_\alpha \) is a closed and cofinal subset of \( \alpha \), there exists a club \( D \subseteq \omega_1 \) such that for all limit points \( \alpha \) of \( D \), \( D \cap \alpha \) measures \( c_\alpha \).

The next two results are due to J. T. Moore ([8]).

**Theorem 2.1.** MRP implies Measuring.

**Theorem 2.2.** BPFA implies Measuring.

We now describe parametrized forms of measuring which were introduced in [2]. Let \( \vec{C} = \langle C_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle \) be a sequence such that each \( C_\alpha \) is a collection of closed and cofinal subsets of \( \alpha \). A club \( D \subseteq \omega_1 \) is said to measure \( \vec{C} \) if for all \( \alpha \in \text{lim}(D) \), for all \( c \in C_\alpha \), \( D \cap \alpha \) measures \( c \).

**Definition 2.3.** For a cardinal \( \kappa \), let \( \text{Measuring}_{< \kappa} \) denote the statement that whenever \( \vec{C} = \langle C_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle \) is a sequence such that each \( C_\alpha \) is a collection of fewer than \( \kappa \) many closed and cofinal subsets of \( \alpha \), then there exists a club \( D \subseteq \omega_1 \) which measures \( \vec{C} \). For a cardinal \( \lambda \), let \( \text{Measuring}_{< \lambda^+} \) denote \( \text{Measuring}_{< \lambda^+} \).

Observe that the principle Measuring is the same as \( \text{Measuring}_{1} \). If \( \kappa < \lambda \), then clearly \( \text{Measuring}_{< \lambda} \) implies \( \text{Measuring}_{< \kappa} \). It is easy to see that \( \text{Measuring}_{< \kappa} \) is false.

**Definition 2.4.** Strong Measuring is the statement that \( \text{Measuring}_{< \kappa} \) holds.
Since the intersection of countably many clubs in $\omega_1$ is club, Measuring easily implies Measuring$s$. In particular, Measuring together with CH implies Strong Measuring. We will prove in Section 3 the consistency of Strong Measuring together with $\neg$CH. We also observe at the end of that section that Measuring does not imply Measuring$\omega_1$.

**Proposition 2.5 ([2]).** Measuring$s$ is false.

**Proof.** Fix a splitting family $S$ of cardinality $s$. For each limit ordinal $\alpha < \omega_1$, fix a function $f_\alpha : \omega \to \alpha$ which is increasing and cofinal in $\alpha$. For each $A \in S$, let $c_{\alpha,A} = \bigcup \{(f_\alpha(n), f_\alpha(n + 1)) : n \in A\}$, which is clearly closed and cofinal in $\alpha$. Let $C_\alpha := \{c_{\alpha,A} : A \in S\}$. Then $\vec{C} := \langle C_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle$ is a sequence such that for each $\alpha$, $C_\alpha$ is a collection of at most $s$ many closed and cofinal subsets of $\alpha$.

Let $D \subseteq \omega_1$ be a club. Fix $\alpha \in \text{lim}(D)$. We will show that there exists a member of $C_\alpha$ which $D \cap \alpha$ does not measure. Define $x := \{n < \omega : D \cap (f_\alpha(n), f_\alpha(n + 1)] \neq \emptyset\}$. Since $\alpha \in \text{lim}(D)$, $x$ is infinite. As $S$ is a splitting family, we can fix $A \in S$ which splits $x$. So both $x \cap A$ and $x \setminus A$ are infinite. We claim that $D \cap \alpha$ does not measure $c_{\alpha,A}$.

Suppose for a contradiction that for some $\beta < \alpha$, $(D \cap \alpha) \setminus \beta$ is either a subset of, or disjoint from, $c_{\alpha,A}$. Since $A \cap x$ is infinite, we can fix $n \in A \cap x$ such that $f_\alpha(n) > \beta$. Then $n \in x$ implies that $D \cap (f_\alpha(n), f_\alpha(n + 1)] \neq \emptyset$, and $n \in A$ implies that $(f_\alpha(n), f_\alpha(n + 1)] \subseteq c_{\alpha,A}$. It follows that $(D \cap \alpha) \setminus \beta$ meets $c_{\alpha,A}$. By the choice of $\beta$, this implies that $(D \cap \alpha) \setminus \beta$ is a subset of $c_{\alpha,A}$. But $x \setminus A$ is also infinite, so we can fix $m \in x \setminus A$ such that $f_\alpha(m) > \beta$. Then $m \in x$ implies that $D \cap (f_\alpha(m), f_\alpha(m + 1)] \neq \emptyset$, and $n \notin A$ implies that $(f_\alpha(m), f_\alpha(m + 1)]$ is disjoint from $c_{\alpha,A}$. Thus, there is a member of $(D \cap \alpha) \setminus \beta$ which is not in $c_{\alpha,A}$, which is a contradiction. \qed

We will prove later in this section that Measuring$s$ is also false.

We now turn to parametrized club guessing. We recall some standard definitions. Consider a sequence $\vec{L} = \langle L_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle$, where each $L_\alpha$ is a cofinal subset of $\alpha$ with order type $\omega$ (that is, a ladder system). We say that $\vec{L}$ is a club guessing sequence, weak club guessing sequence, or very weak club guessing sequence, respectively, if for every club $D \subseteq \omega_1$, there exists a limit ordinal $\alpha < \omega_1$ such that:

1. $L_\alpha \subseteq D$,  
2. $L_\alpha \setminus D$ is finite, or  
3. $L_\alpha \cap D$ is infinite, respectively.

We say that Club Guessing, Weak Club Guessing, or Very Weak Club Guessing holds, respectively, if there exists a club guessing sequence, a weak club guessing sequence, or a very weak club guessing sequence, respectively. It is well known that Measuring implies the failure of Very Weak Club Guessing (see Proposition 2.8 below).

**Definition 2.6.** Let $\vec{L} = \langle L_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle$ be a sequence where each $L_\alpha$ is a non-empty collection of cofinal subsets of $\alpha$ with order type $\omega$. The sequence $\vec{L}$ is said to be a club guessing sequence, weak club guessing sequence, or very weak club guessing sequence, respectively, if for every club $D \subseteq \omega_1$, there exists a limit ordinal $\alpha < \omega_1$ and some $L \in L_\alpha$ such that:

1. $L \subseteq D$,  
2. $L \setminus D$ is finite, or
Definition 2.7. For a cardinal \( \kappa \), let \( CG_{\lt \kappa} \), \( WCG_{\lt \kappa} \), and \( VWCG_{\lt \kappa} \), respectively, be the statements that there exists a club guessing sequence, weak club guessing sequence, or very weak club guessing sequence \( (L_\alpha : \alpha \in \omega_1 \cap \text{Lim}) \), respectively, such that for each \( \alpha \), \( |L_\alpha| < \kappa \). Let \( CG_\kappa \), \( WCG_\kappa \), and \( VWCG_\kappa \) denote the statements \( CG_{\lt \kappa+} \), \( WCG_{\lt \kappa+} \), and \( VWCG_{\lt \kappa+} \), respectively.

Clearly, if \( \kappa < \lambda \), then \( CG_{\lt \kappa} \) implies \( CG_{\lt \lambda} \), and similarly with \( WCG \) and \( VWCG \). Observe that Club Guessing, Weak Club Guessing, and Very Weak Club Guessing are equivalent to \( CG_1 \), \( WCG_1 \), and \( VWCG_1 \), respectively. Obviously, \( CG_\kappa \) is true. The weakest forms of club guessing principles which are not provable in \( ZFC \) are when the index is \( < \kappa \).

Proposition 2.8. For any cardinal \( \kappa \geq 2 \), \( Measuring_{\lt \kappa} \) implies the failure of \( VWCG_{\lt \kappa} \).

Proof. Suppose for a contradiction that \( Measuring_{\lt \kappa} \) and \( VWCG_{\lt \kappa} \) both hold. Fix a very weak club guessing sequence \( \vec{E} = (E_\alpha : \alpha \in \omega_1 \cap \text{Lim}) \) such that each \( L_\alpha \) has cardinality less than \( \kappa \). Observe that for each \( \alpha \), every member of \( L_\alpha \) is vacuously a closed subset of \( \alpha \) since it has order type \( \omega \).

By \( Measuring_{\lt \kappa} \), there exists a club \( D \subseteq \omega_1 \) which measures \( \vec{E} \). Let \( E \) be the club set of indecomposable limit ordinals \( \alpha \) in \( \text{lim}(D) \) such that \( \text{ot}(D \cap \alpha) = \alpha \). Since \( \vec{E} \) is a very weak club guessing sequence, there exists a limit ordinal \( \alpha \) and \( L \in L_\alpha \) such that \( L \cap E \) is infinite. In particular, \( \alpha \) is a limit point of \( E \), and hence of \( D \).

Since \( D \) measures \( \vec{E} \) and \( L \in L_\alpha \), \( D \cap \alpha \) measures \( L \). So we can fix \( \beta < \alpha \) such that \( (D \cap \alpha) \setminus \beta \) is either a subset of, or disjoint from, \( L \). Now \( L \cap E \), and hence \( L \cap D \), is infinite. As \( L \) has order type \( \omega \), this implies that \( L \cap D \) is cofinal in \( \alpha \). By the choice of \( \beta \), \( (D \cap \alpha) \setminus \beta \) must be a subset of \( L \). But since \( \alpha \in E \), \( \text{ot}(D \cap \alpha) = \alpha \) and \( \alpha \) is indecomposable, which implies that \( \text{ot}((D \cap \alpha) \setminus \beta) = \alpha \). As \( \alpha > \omega \), this is impossible since \( (D \cap \alpha) \setminus \beta \) is a subset of \( L \) and \( L \) has order type \( \omega \).

In particular, since Strong Measuring is consistent, so is the failure of \( VWCG_{\lt \kappa} \). (The consistency of \( \neg VWCG_{\lt \kappa} \) together with \( \kappa \) arbitrarily large was previously shown in [4].)

Proposition 2.9 ([Hrušák [5]]). \( VWCG_\kappa \) is true.

Proof. Fix an unbounded family \( \{r_\alpha : \alpha < b \} \) in \( \omega^\omega \). For each limit ordinal \( \delta < \omega_1 \), fix a cofinal subset \( C_\delta \) of \( \delta \) with order type \( \omega \) and a bijection \( h_\delta : \omega \to \delta \). Let \( C_\delta(n) \) denote the \( n \)-th member of \( C_\delta \) for all \( n < \omega \). For all limit ordinals \( \delta < \omega_1 \) and \( \alpha < b \), define

\[
A_\delta^\alpha := C_\delta \cup \bigcup \{h_\delta[A_\delta^\alpha(n)] \setminus C_\delta(n) : n < \omega \}.
\]

It is easy to check that for all \( \delta < \omega_1 \) and \( \alpha < b \), \( A_\delta^\alpha \) has order type \( \omega \) and \( \text{sup}(A_\delta^\alpha) = \delta \). Given a club \( C \subseteq \omega_1 \), let \( \delta \) be a limit point of \( C \) and let \( g_{C,\delta} : \omega \to \omega \) be the function given by

\[
g_{C,\delta}(n) = \min \{m < \omega : h_\delta(m) \in C \setminus C_\delta(n) \}.
\]

Now let \( \alpha < b \) be such that \( r_\alpha(n) > g_{C,\delta}(n) \) for infinitely many \( n \). It then follows that \( |A_\delta^\alpha \cap C| = \omega \).
By Propositions 2.8 and 2.9, the following is immediate.

**Corollary 2.10.** Measuring \( p \) is false.

An obvious question is whether the parametrized versions of club guessing are actually the same as the usual ones. We conclude this section by showing that they are not.

Recall that a forcing poset \( P \) is \( \omega^\omega \)-bounding if every function in \( \omega^\omega \cap V^P \) is dominated by a function in \( \omega^\omega \cap V \).

**Lemma 2.11** (Hrušák). Assume that VWCG fails. Let \( P \) be any \( \omega_1 \)-c.c., \( \omega^\omega \)-bounding forcing. Then \( P \) forces that VWCG fails.

**Proof.** Since \( P \) is \( \omega_1 \)-c.c. and \( \omega^\omega \)-bounding, a standard argument shows that whenever \( p \in P \) and \( p \) forces that \( b \in \omega^\omega \), then there exists a function \( b^* \in \omega^\omega \) such that \( p \) forces that \( b^* \) dominates \( b \).

Let us show that whenever \( p \in P \), \( \delta < \omega_1 \), and \( p \) forces that \( \check{X} \) is a cofinal subset of \( \delta \) of order type \( \omega \), then there exists a set \( Y \) with order type \( \omega \) such that \( p \) forces that \( \check{X} \subseteq Y \). To see this, fix a bijection \( f : \omega \to \delta \) and a strictly increasing sequence \( \langle \alpha_n : n < \omega \rangle \) cofinal in \( \alpha \) with \( \alpha_0 = 0 \). We claim that there exists a \( P \)-name \( \dot{b} \) for a function from \( \omega \) to \( \omega \) such that \( p \) forces that for all \( n < \omega \), \( b(n) \) is the least \( m < \omega \) such that \( \check{X} \cap [\alpha_n, \alpha_{n+1}) \subseteq f[m] \). This is true since \( p \) forces that \( \check{X} \) has order type \( \omega \) and hence that \( \check{X} \cap [\alpha_n, \alpha_{n+1}) \) is finite for all \( n < \omega \). Fix a function \( b^* : \omega \to \omega \) such that \( p \) forces that \( b^* \) dominates \( \dot{b} \). Now let

\[
Y := \bigcup \{ f[\check{b}^*(n)] \cap [\alpha_n, \alpha_{n+1}) : n < \omega \}.
\]

It is easy to check that \( Y \) has order type \( \omega \) and \( p \) forces that \( \check{X} \subseteq Y \).

Now we are ready to prove the proposition. So suppose that \( p \in P \) forces that \( \langle \check{X}_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle \) is a very weak club guessing sequence. By the previous paragraph, for each limit ordinal \( \alpha < \omega_1 \) we can find a cofinal subset \( Y_\alpha \) of \( \alpha \) with order type \( \omega \) such that \( p \) forces that \( \check{X}_\alpha \subseteq Y_\alpha \). We claim that \( \langle Y_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle \) is a very weak club guessing sequence in the ground model, which completes the proof. So consider a club \( C \subseteq \omega_1 \). Then \( C \) is still a club in \( V^P \). Fix \( q \leq p \) and a limit ordinal \( \alpha < \omega_1 \) such that \( q \) forces that \( \check{X}_\alpha \cap C \) is infinite. Then clearly \( q \) forces that \( Y_\alpha \cap C \) is infinite, so in fact, \( Y_\alpha \cap C \) is infinite. \( \square \)

**Proposition 2.12.** It is consistent that \( \neg \text{VWCG} \) and \( \text{CG}_{\omega_1} \) both hold.

**Proof.** Let \( V \) be a model in which CH holds and VWCG fails. Such a model was shown to exist by Shelah [13]. Let \( P \) be an \( \omega_1 \)-c.c., \( \omega^\omega \)-bounding forcing poset which adds at least \( \omega_2 \) many reals; for example, random real forcing with product measure is such a forcing. We claim that in \( V^P \), \( \text{CG}_{\omega_1} \) holds but VWCG fails. By Lemma 2.11, VWCG is false in \( V^P \). In \( V \), define \( \check{L} = \langle L_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle \) by letting \( L_\alpha \) be the collection of all cofinal subsets of \( \alpha \) with order type \( \omega \). Since CH holds, the cardinality of each \( L_\alpha \) is \( \omega_1 \). If \( C \) is a club subset of \( \omega_1 \) in \( V^P \), then since \( P \) is \( \omega_1 \)-c.c., there is a club \( D \subseteq \omega_1 \) in \( V \) such that \( D \subseteq C \). In \( V \), fix \( d \subseteq D \) with order type \( \omega \), and let \( \alpha := \sup(d) \). Then \( d \in L_\alpha \) and \( d \subseteq C \). Thus, \( \check{L} \) witnesses that \( \text{CG}_{\omega_1} \) holds in \( V^P \). \( \square \)
3. The Consistency of Strong Measuring and \( \neg \text{CH} \)

As we previously mentioned, Measuring is equivalent to Measuring\(_{\omega_1}\), and therefore under CH, Measuring is equivalent to Strong Measuring. In this section we establish the consistency of Strong Measuring with the negation of CH. More precisely, we will prove that MRP together with MA(\(\sigma\)-centered) implies Strong Measuring, and BPFA implies Strong Measuring. Recall that both MRP and BPFA imply that \( \epsilon = \omega_2 \) ([12]).

A set \( M \) is suitable if for some regular cardinal \( \theta > \omega_1 \), \( M \) is a countable elementary substructure of \( H(\theta) \). We will follow the conventions introduced in Section 1 that the properties “open” and “M-stationary” refer to open and M-stationary subsets of \( \omega_1 \) (where \( \omega_1 \) is considered as a subspace of \([\omega_1]^\omega \)).

**Proposition 3.1.** Assume that \( M \) is suitable. Let \( \delta := M \cap \omega_1 \). Suppose that \( \mathcal{Y} \) is a collection of open subsets of \( \delta \) such that for any finite set \( a \subseteq \mathcal{Y} \), \( \bigcap a \) is \( M \)-stationary. Then there exists a \( \sigma \)-centered forcing \( \mathbb{P} \) and a collection \( \mathcal{D} \) of dense subsets of \( \mathbb{P} \) of size at most \( |\mathcal{Y}| + \omega \) such that whenever \( G \) is a filter on \( \mathbb{P} \) in some outer model \( W \) of \( V \) with \( \omega_1^W = \omega_1^V \) which meets each member of \( \mathcal{D} \), then there exists a set \( z \subseteq \delta \) in \( W \) which is open, \( M \)-stationary, and satisfies that for all \( x \in \mathcal{Y} \), \( z \setminus X \) is bounded in \( \delta \).

**Proof.** Define a forcing poset \( \mathbb{P} \) to consist of conditions which are pairs \((x, a)\), where \( x \) is an open and bounded subset of \( \delta \) in \( M \) and \( a \) is a finite subset of \( \mathcal{Y} \). Let \((y, b) \leq (x, a)\) if \( y \) is an end-extension of \( x, a \subseteq b \), and \( y \setminus x \subseteq \bigcap a \).

Since \( M \) is countable, there are only countably many possibilities for the first component of a condition. If \((x, a_0), \ldots, (x, a_n)\) are finitely many conditions with the same first component, then easily \((x, a_0 \cup \ldots \cup a_n)\) is a condition in \( \mathbb{P} \) which is below each of the conditions \((x, a_0), \ldots, (x, a_n)\). It follows that \( \mathbb{P} \) is \( \sigma \)-centered.

For each \( x \in \mathcal{Y} \), let \( D_X \) denote the set of conditions \((x, a)\) such that \( x \in a \). Observe that \( D_X \) is dense. For every club \( C \) of \( \omega_1 \) which is a member of \( M \), let \( E_C \) denote the set of conditions \((x, a)\) such that \( x \cap C \) is non-empty. We claim that \( E_C \) is dense. Let \((x, a)\) be a condition. Since \( \bigcap a \) is \( M \)-stationary and \( \lim(C) \setminus (\sup(x) + 1) \) is a club subset of \( \omega_1 \) in \( M \), we can find a limit ordinal \( \alpha \) in \( C \cap (\bigcap a) \) which is in the interval \((\sup(x), \delta)\). Since \( \alpha \in \bigcap a \) and \( \bigcap a \) is open, we can find \( \beta < \gamma < \delta \) such that \( \alpha \in (\beta, \gamma) \subseteq \bigcap a \). As \( \sup(x) + 1 < \alpha \), without loss of generality \( \sup(x) < \beta \).

By elementarity, the interval \( b := (\beta, \gamma) \) is in \( M \). It follows that \((x \cup b, a)\) is a condition, \( x \cup b \) end-extends \( x \), and \((x \cup b) \setminus x = b \subseteq \bigcap a \). Thus, \((x \cup b, a) \leq (x, a)\), and since \( \alpha \in C \), \((x \cup b, a) \in E_C \).

Let \( \mathcal{D} \) denote the collection of all dense sets of the form \( D_X \) where \( X \in \mathcal{Y} \), or \( E_C \) where \( C \) is a club subset of \( \omega_1 \) belonging to \( M \). Then \(|\mathcal{D}| \leq |\mathcal{Y}| + \omega \). Let \( G \) be a filter on \( \mathbb{P} \) in some outer model \( W \) with \( \omega_1^W = \omega_1^V \) which meets each dense set in \( \mathcal{D} \). Define \( z := \bigcup \{x : \exists a \ (x, a) \in G\} \). Note that since \( z \) is a union of open sets, it is open (using the fact that being open is absolute between \( V \) and \( W \)). For each club \( C \subseteq \omega_1 \) which lies in \( M \), there exists a condition \((x, a)\) which belongs to \( G \cap E_C \), and thus \( x \cap C \neq \emptyset \). Therefore, \( z \cap C \neq \emptyset \). Hence, \( z \) is \( M \)-stationary.

It remains to show that for all \( x \in \mathcal{Y} \), \( z \setminus x \) is bounded in \( \delta \). Consider \( X \in \mathcal{Y} \). Then we can fix \((x, a) \in G \cap D_X\), which means that \( X \in a \). Now the definition of the ordering on \( \mathbb{P} \) together with the fact that \( G \) is a filter easily implies that \( z \setminus x \subseteq X \). Therefore, \( z \setminus X \subseteq x \), and hence \( z \setminus X \) is bounded in \( \delta \). \( \square \)
Corollary 3.2. Assume that $M$ is suitable. Let $\delta := M \cap \omega_1$. Suppose that $\mathcal{Y}$ is a collection of less than $m(\sigma\text{-centered})$ many open subsets of $\delta$ such that for any finite set $a \subseteq \mathcal{Y}$, $\bigcap a$ is $M$-stationary. Then there exists a set $z \subseteq \delta$ which is open, $M$-stationary, and satisfies that for all $X \in \mathcal{Y}$, $z \setminus X$ is bounded in $\delta$.

**Proof.** Fix a $\sigma$-centered forcing $\mathbb{P}$ and a collection $\mathcal{D}$ of dense subsets of $\mathbb{P}$ of size at most $|\mathcal{Y}| + \omega$ as described in Proposition 3.1. Since $m(\sigma\text{-centered})$ is uncountable, $|\mathcal{D}| < m(\sigma\text{-centered})$. Hence, there exists a filter $G$ on $\mathbb{P}$ which meets each dense set in $\mathcal{D}$. By Proposition 3.1, there exists a set $z \subseteq \delta$ which is open, $M$-stationary, and satisfies that for all $X \in \mathcal{Y}$, $z \setminus X$ is bounded in $\delta$. □

Proposition 3.3. Let $\tilde{C} = (C_\alpha : \alpha \in \omega_1 \cap \text{Lim})$ be a sequence such that each $C_\alpha$ is a collection of less than $m(\sigma\text{-centered})$ many closed and cofinal subsets of $\alpha$. Then there exists an open stationary set mapping $\Sigma$ such that, if $W$ is any outer model with the same $\omega_1$ in which there exists a $\Sigma$-reflecting sequence, then there exists in $W$ a club subset of $\omega_1$ which measures $\tilde{C}$.

**Proof.** For each limit ordinal $\alpha < \omega_1$, let $\mathcal{D}_\alpha := \{\alpha \setminus c : c \in C_\alpha\}$. Observe that each $\mathcal{D}_\alpha$ is a collection of fewer than $m(\sigma\text{-centered})$ many open subsets of $\alpha$.

We will define $\Sigma$ to have domain the collection of all countable elementary substructures $M$ of $H(\omega_2)$. Consider such an $M$ and we define $\Sigma(M)$. Note that $M$ is suitable. Let $\delta := M \cap \omega_1$. We consider two cases. In the first case, there does not exist a member of $\mathcal{D}_\delta$ which is $M$-stationary. Define $\Sigma(M) := \delta$, which is clearly open and $M$-stationary.

In the second case, there exists some member of $\mathcal{D}_\delta$ which is $M$-stationary. A straightforward application of Zorn’s lemma implies that there exists a non-empty set $\mathcal{Y}_M \subseteq \mathcal{D}_\delta$ such that for any $a \in [\mathcal{Y}_M]^{< \omega}$, $\bigcap a$ is $M$-stationary, and moreover, $\mathcal{Y}_M$ is a maximal subset of $\mathcal{D}_\delta$ with this property. Since $\mathcal{Y}_M \subseteq \mathcal{D}_\delta$, $|\mathcal{Y}_M| < m(\sigma\text{-centered})$. So the collection $\mathcal{Y}_M$ satisfies the assumptions of Corollary 3.2. It follows that there exists a set $z_M \subseteq \delta$ which is open, $M$-stationary, and satisfies that for all $X \in \mathcal{Y}_M$, $z_M \setminus X$ is bounded in $\delta$. Now define $\Sigma(M) := z_M$.

This completes the definition of $\Sigma$. Consider an outer model $W$ of $V$ with the same $\omega_1$, and assume that in $W$ there exists a $\Sigma$-reflecting sequence $(M_\delta : \delta < \omega_1)$. Let $\alpha_\delta := M_\delta \cap \omega_1$ for all $\delta < \omega_1$. Let $D$ be the club set of $\delta < \omega_1$ such that $\alpha_\delta = \delta$. We claim that $D$ measures $\tilde{C}$.

Consider $\delta \in \text{lim}(D)$. Then $\delta = \alpha_\delta = M_\delta \cap \omega_1$. Let $M := M_\delta$. We first claim that if $c \in C_\delta$ and $\delta \setminus c$ is not $M$-stationary, then for some $\beta < \delta$, $(D \cap \delta) \setminus \beta \subseteq c$. Fix a club subset $E$ of $\omega_1$ in $M$ which is disjoint from $\delta \setminus c$. By the continuity of the $\Sigma$-reflecting sequence, there exists $\beta < \delta$ such that $E \subseteq M_\beta$. We claim that $(D \cap \delta) \setminus \beta \subseteq c$. Let $\xi \in (D \cap \delta) \setminus \beta$. Then $E \in M_\xi$, and hence by elementarity, $\xi = M_\xi \cap \omega_1 \in E$. Since $E$ is disjoint from $\delta \setminus c$, $\xi \in c$. We split the argument according to the two cases in the definition of $\Sigma(M)$. In the first case, there does not exist a member of $\mathcal{D}_\delta$ which is $M$-stationary. Consider $c \in C_\delta$. Then $\delta \setminus c$ is not $M$-stationary. By the previous paragraph, there exists $\beta < \delta$ such that $(D \cap \delta) \setminus \beta \subseteq c$.

In the second case, there exists a member of $\mathcal{D}_\delta$ which is $M$-stationary. Consider $c \in C_\delta$. Then $X := \delta \setminus c \in \mathcal{D}_\delta$. We consider two possibilities. First, assume that $X$ is in $\mathcal{Y}_M$. By the choice of $\mathcal{Y}_M$ and $z_M$, we know that $z_M \setminus X$ is bounded in $\delta$. So fix $\beta_0 < \delta$ so that $z_M \setminus \beta_0 \subseteq X$. By the definition of being a $\Sigma$-reflecting sequence, there exists $\beta_1 < \delta$ so that for all $\beta_1 \leq \xi < \delta$, $M_\xi \cap \omega_1 \in \Sigma(M) = z_M$. 

Let \( \beta := \max\{\beta_1, \beta_2\} \). Consider \( \xi \in (D \cap \delta) \setminus \beta \). Then \( \xi \geq \beta_1 \) implies that
\[
\xi = M_\xi \cap \omega_1 \in \mathcal{Z}_M. \quad \text{So} \quad \xi \in z_M \setminus \beta_2 \subseteq X \setminus \delta \setminus c.
\]
Secondly, assume that \( X \) is not in \( \mathcal{Y}_M \). By the maximality of \( \mathcal{Y}_M \), there exists a set \( a \in [\mathcal{Y}_M]^{<\omega} \) such that \( X \cap \bigcap \alpha a \) is not \( M \)-stationary. Fix a club \( E \) in \( M \) which is disjoint from \( X \cap \bigcap \alpha a \). By the continuity of the \( \Sigma \)-reflecting sequence, there exists \( \beta < \delta \) such that \( E \in M_{\beta} \). Consider \( \xi \in (D \cap \delta) \setminus \beta \). Then \( E \in M_{\xi} \), which implies that \( \xi = M_\xi \cap \omega_1 \in E \). Thus, \( \xi \) is not in \( X \cap \bigcap \alpha a \). On the other hand, letting \( a = \{X_0, \ldots, X_n\} \), for each \( i \leq n \) the previous paragraph implies that there exists \( \beta_i < \delta \) such that \( (D \cap \delta) \setminus \beta_i \subseteq X_i \). Let \( \beta^* \) be an ordinal in \( \delta \) which is larger than \( \beta \) and \( \beta_i \) for all \( i \leq n \). Consider \( \xi \in (D \cap \delta) \setminus \beta^* \). Then by the choice of \( \beta \), \( \xi \notin X \cap \bigcap \alpha a \). By the choice of the \( \beta_i \)’s, \( \xi \notin \bigcap \alpha a \). Therefore, \( \xi \notin X = \delta \setminus c \), which means that \( \xi \notin c \). Thus, \( (D \cap \delta) \setminus \beta^* \subseteq c \).

**Corollary 3.4.** Assume \( \text{MRP} \) and \( \text{MA}(\sigma\text{-centered}) \). Then Strong Measuring holds.

**Proof.** Let \( \tilde{C} = (C_\alpha : \alpha \in \omega_1 \cap \text{Lim}) \) be a sequence such that each \( C_\alpha \) is a collection of fewer than \( c \) many closed and cofinal subsets of \( \alpha \). We claim that there exists a club subset of \( \omega_1 \) which measures \( \tilde{C} \). By \( \text{MA}(\sigma\text{-centered}) \), \( m(\sigma\text{-centered}) \) equals \( c \). So each \( C_\alpha \) has size less than \( m(\sigma\text{-centered}) \).

By Proposition 3.3, there exists an open stationary set mapping \( \Sigma \) such that, if \( W \) is any outer model with the same \( \omega_1 \) in which there exists a \( \Sigma \)-reflecting sequence, then there exists in \( W \) a club subset of \( \omega_1 \) which measures \( \tilde{C} \). Applying \( \text{MRP} \), there exists a \( \Sigma \)-reflecting sequence in \( V \). Thus, in \( V \) there exists a club subset of \( \omega_1 \) which measures \( \tilde{C} \).

**Corollary 3.5.** Assume \( \text{BPFA} \). Then Strong Measuring holds.

**Proof.** Let \( \tilde{C} = (C_\alpha : \alpha \in \omega_1 \cap \text{Lim}) \) be a sequence such that each \( C_\alpha \) is a collection of fewer than \( c = \omega_2 \) many closed and cofinal subsets of \( \alpha \). We claim that there exists a club subset of \( \omega_1 \) which measures \( \tilde{C} \). Since \( c = \omega_2 \), \( \tilde{C} \) is a member of \( H(\omega_2) \).

Thus, the existence of a club subset of \( \omega_2 \) which measures \( \tilde{C} \) is expressible as a \( \Sigma_1 \) statement involving a parameter in \( H(\omega_2) \). By \( \text{BPFA} \), it suffices to show that there exists a proper forcing which forces that such a club exists.

Now \( \text{BPFA} \) implies Martin’s Axiom, and in particular, that \( m(\sigma\text{-centered}) \) is equal to \( c \). So each \( C_\alpha \) has size less than \( m(\sigma\text{-centered}) \). By Proposition 3.3, there exists an open stationary set mapping \( \Sigma \) such that, if \( W \) is any outer model with the same \( \omega_1 \) in which there exists a \( \Sigma \)-reflecting sequence, then there exists in \( W \) a club subset of \( \omega_1 \) which measures \( \tilde{C} \). By [12, Section 3], there exists a proper forcing \( P \) which adds a \( \Sigma \)-reflecting sequence, so in \( V^P \) there is a club subset of \( \omega_1 \) which measures \( \tilde{C} \).

We now sketch a proof that \( \text{MRP} \) alone does not imply Strong Measuring. In particular, Measuring does not imply Strong Measuring. Start with a model of \( \text{CH} \) in which there exists a supercompact cardinal \( \kappa \). Construct a forcing iteration \( P \) in the standard way to obtain a model of \( \text{MRP} \). To do this, fix a Laver function \( f : \kappa \rightarrow V_\kappa \). Then define a countable support forcing iteration \( \langle P_\alpha, \dot{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle \) as follows. Given \( P_\alpha \), consider \( f(\alpha) \). If \( f(\alpha) \) happens to be a \( P_\alpha \)-name for some open stationary set mapping, then let \( \dot{Q}_\alpha \) be a \( P_\alpha \)-name for a proper forcing which adds an \( f(\alpha) \)-reflecting sequence. Otherwise let \( \dot{Q}_\alpha \) be a \( P_\alpha \)-name for \( \text{Col}(\omega_1, \omega_2) \). Now define \( P := P_\kappa \). Arguments similar to those in the standard construction of a model of \( \text{PFA} \) can be used to show that \( P \) forces \( \text{MRP} \).
The forcing for adding a \( \Sigma \)-reflecting sequence for a given open stationary set mapping does not add reals ([12, Section 3]). In particular, it is vacuously \( \omega^\omega \)-bounding. The property of being proper and \( \omega^\omega \)-bounding is preserved under countable support forcing iterations ([1, Theorem 3.5]), so \( \mathbb{P} \) is also \( \omega^\omega \)-bounding. In particular, \( V \cap \omega^\omega \) is an unbounded family in \( V^\mathbb{P} \), and it has size \( \omega_1 \) since CH holds in \( V \). It follows that the bounding number \( b \) is equal to \( \omega_1 \). But by Corollary 2.9, Measuring, is false. So \( \mathbb{P} \) forces that Measuring is false. As \( c = \omega_2 \) in \( V^\mathbb{P} \), Strong Measuring fails in \( V^\mathbb{P} \).

We also note that Strong Measuring plus \( c = \omega_2 \) is consistent with the existence of an \( \omega_1 \)-Suslin tree. Namely, both the forcing for adding a \( \Sigma \)-reflecting sequence for a given open stationary set mapping \( \Sigma \), as well as any \( \sigma \)-centered forcing, preserve Suslin trees ([11]). And the property of being proper and preserving a given Suslin tree is preserved under countable support forcings iterations ([10]). So starting with a model in which there exists an \( \omega_1 \)-Suslin tree \( S \) and a supercompact cardinal \( \kappa \), we can iterate forcing similar to the argument in the preceding paragraphs to produce a model of MA(\( \sigma \)-centered) plus MRP in which \( S \) is an \( \omega_1 \)-Suslin tree. By Corollary 3.4, Strong Measuring holds in that model.

4. Strong Measuring with Continuum Arbitrarily Large

In this section we prove the consistency of Strong Measuring with arbitrarily large continuum. The main result is the following.

**Theorem 4.1.** Assume CH. Let \( \kappa \) be a regular cardinal such that \( 2^{<\kappa} = \kappa \) and \( \kappa^{\omega_1} = \kappa \). Then there exists a forcing poset \( \mathbb{P} \) with the following properties:

1. \( \mathbb{P} \) is proper and \( \omega_2 \)-Knaster;
2. \( \mathbb{P} \) forces Strong Measuring and \( 2^\mu = \kappa \) for every infinite cardinal \( \mu < \kappa \).

The forcing witnessing the theorem is a natural variation of the main forcing from [6] which was used to prove the consistency of Measuring together with arbitrarily large continuum. Due to the similarities of these forcing constructions, to avoid repetition we refer the reader to [6] for the complete details on some of the more technical parts of the proof.

We note that the forcing construction from [6] can be easily adapted to yield a model of Measuring, Martin’s Axiom, and \( c \) arbitrarily large. Therefore, if Measuring and Strong Measuring are equivalent under Martin’s Axiom, which is conceivable in light of Corollary 3.4, then we would immediately get a forcing satisfying the conclusion of Theorem 4.1. We do not know, however, whether this equivalence is true.

Given a set \( N \), if \( N \cap \omega_1 \) is an ordinal, then we denote this ordinal by \( \delta_N \) and call \( \delta_N \) the height of \( N \). Given \( T \subseteq H(\kappa) \) and \( N \in [H(\theta)]^{<\kappa} \), we will tend to write \( (N, T) \) instead of \( (N, T \cap N) \). We will need the following notion of symmetric system from [3].

**Definition 4.2.** Let \( T \subseteq H(\kappa) \) and let \( \mathcal{N} \) be a finite set of countable subsets of \( H(\kappa) \). We will say that \( \mathcal{N} \) is a \( T \)-symmetric system if the following holds:

A) for every \( N \in \mathcal{N} \), \( (N, \in, T) \) is an elementary substructure of \( (H(\kappa), \in, T) \);
B) given distinct \( N, N' \) in \( \mathcal{N} \), if \( \delta_N = \delta_N' \), then there is a unique isomorphism

\[ \Psi_{N,N'} : (N, \in, T) \rightarrow (N', \in, T) ; \]

and \( \Psi_{N,N'} \) is the identity on \( N \cap N' \);
(C) for all $N, N', M$ in $N$, if $M \in N$ and $\delta_N = \delta_{N'}$, then $\Psi_{N,N'}(M) \in N'$;
(D) for all $N, M$ in $N$, if $\delta_M < \delta_N$, then there is some $N' \in N$ such that $\delta_{N'} = \delta_N$ and $M \in N'$.

The next three lemmas are proved in [3].

**Lemma 4.3.** Let $T \subseteq H(\kappa)$ and $N$ and $N'$ be countable elementary substructures of $(H(\kappa), \in, T)$. Suppose $N \in N$ is a $T$-symmetric system and

$$\Psi : (N, \in, T) \rightarrow (N', \in, T)$$

is an isomorphism. Then $\Psi[N]$ is also a $T$-symmetric system.

**Lemma 4.4.** Let $T \subseteq H(\kappa)$, $N$ be a $T$-symmetric system, and $N \in N$. Then the following statements hold:

1. $N \cap N$ is a $T$-symmetric system.
2. Suppose that $N^* \in N$ is a $T$-symmetric system such that $N \cap N \subseteq N^*$. Let

$$\mathcal{M} = N \cup \bigcup \{ \Psi_{N,N'}[N^*] : N' \in N, \delta_{N'} = \delta_N \}. $$

Then $\mathcal{M}$ is the $\subseteq$-minimal $T$-symmetric system $W$ such that $N \cup N^* \subseteq W$.

Given $T \subseteq H(\kappa)$ and $T$-symmetric systems $N_0, N_1$, let us write $N_0 \cong N_1$ iff

- $|N_0| = |N_1| = m$, and
- there are enumerations $(N_0^i)_{i<m}$ and $(N_1^i)_{i<m}$ of $N_0$ and $N_1$, respectively, together with an isomorphism between

$$\langle \bigcup N_0, \in, T, N_0^0 \rangle_{i<m}$$

and

$$\langle \bigcup N_1, \in, T, N_1^1 \rangle_{i<m}$$

which is the identity on $(\bigcup N_0) \cap (\bigcup N_1)$.

**Lemma 4.5.** Let $T \subseteq H(\kappa)$ and let $N_0$ and $N_1$ be $T$-symmetric systems. Suppose $N_0 \cong N_1$. Then $N_0 \cup N_1$ is a $T$-symmetric system.

In contexts where the predicate $T$ is irrelevant, we will also refer to $T$-symmetric systems simply as symmetric systems.

Let $(\Phi_\beta)_{\beta < \kappa}$ be a sequence of subsets of $H(\kappa)$ defined by letting $\Phi_\beta$ code, in some fixed uniform way, the satisfaction predicate for the structure $(H(\kappa), \in, \Phi_\alpha)_{\alpha < \beta}$. Also, given $\beta < \kappa$, let $M_\beta$ be the collection of countable elementary submodels of $(H(\kappa), \in, \Phi_\beta)$.

Given an ordinal $\beta < \kappa$, will call an ordered pair of the form $(N, \gamma)$, where

- $N$ is a countable elementary submodel of $H(\kappa)$,
- $\gamma \leq \beta$, and
- $N \in N_\alpha$ for every $\alpha \in N \cap \gamma$

a model with marker (at most $\beta$).

Given an ordinal $\beta < \kappa$ and a collection $\Delta$ of models with markers, we let $N^\Delta_\beta$ denotes the set

$$\{ N : (N, \gamma) \in \Delta, \beta \in N, \beta \leq \gamma \}$$

Given an ordered pair $q = (F, \Delta)$ where $\Delta$ is a collection of models with markers, and given an ordinal $\beta$, we let

$$N^q_\beta := \{ N : (N, \beta) \in \Delta, \beta \in N \}$$
Also, if \( \mathbb{P} \) is a forcing poset consisting of ordered pairs as above, we let \( \mathcal{N}_\beta^G \) be a \( \mathbb{P} \)-name for
\[
\bigcup \{ \mathcal{N}_\beta^r : r \in \dot{G} \},
\]
where \( \dot{G} \) is the canonical \( \mathbb{P} \)-name for the generic filter.

Given an ordinal \( \beta < \kappa \) and a collection \( \Delta \) of models with markers at most \( \beta \), we will say that \( \Delta \) is a \( (\Phi_\alpha)_{\alpha \leq \beta} \)-tower of symmetric systems if the following holds.

(A) For every \( \alpha \leq \beta \), \( \mathcal{N}_\alpha^\Delta \) is a \( \Phi_\alpha \)-symmetric system.

(B) For all \( (N_0, \gamma_0), (N_1, \gamma_1), (M, \gamma) \) in \( \Delta \) and \( \alpha < \beta \), both in \( N_0 \cap N_1 \cap \min\{\gamma_0, \gamma_1, \gamma\} \), if \( \delta_{N_0} = \delta_{N_1} \) and \( M \in N_0 \), then there is some \( \gamma^* > \delta \) such that \( (\Psi_{N_0, N_1}(M), \gamma^*) \in \Delta \).

Note that if \( \Delta \) is a \( (\Phi_\alpha)_{\alpha \leq \beta} \)-tower of symmetric systems, then \( \mathcal{N}_{\alpha+1}^\Delta \) is a symmetric system for every \( \alpha < \beta \).

Our forcing \( \mathbb{P} \) which witnesses Theorem 4.1 will be \( \mathbb{P}_\beta \), where \( \langle \mathbb{P}_\beta : \beta \leq \kappa \rangle \) is the sequence of posets to be defined next. In the following definition, and throughout this section, if \( q \) is an ordered pair \( (F, \Delta) \), we will denote \( F \) and \( \Delta \) by \( F_q \) and \( \Delta_q \), respectively.

Let \( \beta \leq \kappa \) and suppose \( \mathbb{P}_\alpha \) has been defined for all \( \alpha < \beta \). Conditions in \( \mathbb{P}_\beta \) are ordered pairs \( q = (F, \Delta) \) with the following properties.

(1) \( F \) is a finite function with \( \text{dom}(F) \subseteq \beta \).

(2) \( \Delta \) is a finite \( (\Phi_\alpha)_{\alpha \leq \beta} \)-tower of symmetric systems.

(3) For every \( \alpha < \beta \), the restriction of \( q \) to \( \alpha \),
\[
q \upharpoonright \alpha := (F \upharpoonright \alpha, \{(N, \min\{\alpha, \gamma\}) : (N, \gamma) \in \Delta\}),
\]
is a condition in \( \mathbb{P}_\alpha \).

(4) Suppose \( \beta = \alpha + 1 \) and \( \alpha \) is an even ordinal. Let \( \dot{Y}_\alpha^\kappa \) be a \( \mathbb{P}_\alpha \)-name for a function with domain \( D^\alpha := \{ \delta_N : N \in \mathcal{N}_\beta^\kappa \} \) sending \( \delta \in D^\alpha \) to a canonically chosen surjection \( \dot{Y}_\alpha^\kappa : \kappa \to \dot{Z}_\alpha^\kappa \) of a canonically chosen \( \subseteq \)-maximal collection \( \dot{Z}_\alpha^\kappa \) of closed subsets of \( \delta \) with the property that for some \( N \) (equivalently, for every) \( N \in \mathcal{N}_\beta^\kappa \) such that \( \delta_N = \delta \),
\[
\left( \bigcap \{ C : C \subseteq \dot{Z}_\alpha^\kappa \} \right) \cap \{ \delta_M : M \in N \cap \mathcal{N}_\alpha^{G_\alpha} \cap \mathcal{M}_\beta, a \in M \} \neq \emptyset
\]
for each \( a \in N \) and each finite \( C \subseteq \dot{Z}_\alpha^\kappa \).

If \( \alpha \in \text{dom}(F) \), then \( F(\alpha) \) is a finite function with domain \( D^\alpha \) such that for every \( \delta \in D^\alpha \), \( F(\alpha)(\delta) \) is an ordered pair \( (x_{\alpha, \delta}, C_{\alpha, \delta}) \), where
\begin{itemize}
  \item[(a)] \( x_{\alpha, \delta} \) is an open subset of \( \delta \) bounded in \( \delta \) such that \( x_{\alpha, \delta} \in N \) for some (equivalently, for every) \( N \in \mathcal{N}_\beta^\kappa \) such that \( \delta_N = \delta \), and
  \item[(b)] \( q \upharpoonright \alpha \) forces that \( C_{\alpha, \delta} \) is a finite subset of \( \kappa \).
\end{itemize}

(5) Suppose \( \beta = \alpha + 1 \) and \( \alpha = \alpha_0 + 1 \) is an odd ordinal. If \( \alpha \in \text{dom}(F) \), then \( F^*(\alpha) = (I_\alpha^\beta, b_\alpha^\beta, O_\alpha^\beta) \) is as follows.
\begin{itemize}
  \item[(a)] \( I_\alpha^\beta \) is a finite collection of pairwise disjoint closed intervals \([\delta_0, \delta_1]\) for \( \delta_0 \leq \delta_1 < \omega_1 \).
\end{itemize}
Lemma 4.6. For every \( b_\beta^\delta \) is a function with \( \text{dom}(b_\beta^\delta) \subseteq \{ \min(I) : I \in \mathcal{I}_\beta^\alpha \} \) and \( b_\beta^\delta(\delta) < \delta \) for every \( \delta \in \text{dom}(b_\beta^\delta) \).

(c) \( \mathcal{O}^\beta_\alpha \subseteq N^q_{\alpha^x} \cap \mathcal{M}_\beta \) is a \( \Phi_\beta \)-symmetric system.

(d) \( \{ \min(I) : I \in \mathcal{I}_\beta^\alpha \} = \{ \delta_N : N \in \mathcal{O}^\beta_\alpha \} \)

(e) For every \( \delta \in \text{dom}(b_\beta^\delta) \) and every \( I \in \mathcal{I}_\beta^\alpha \), if \( b_\beta^\delta(\delta) < \min(I) < \delta \), then \( q \upharpoonright \alpha \upharpoonright \min(I) \in \check{X}_\delta^\alpha \),

where \( \check{X}_\delta^\alpha \) is a name for

\[ \bigcup \{ x_{\alpha, \beta, \delta}^r : r \in \hat{G}_\alpha, \alpha_0 \in \text{dom}(F_r) \} \]

(f) If \( N \in N^\beta_\alpha \), then \( N \in \mathcal{O}^\beta_\alpha \) and \( \delta_N = \min(I) \) for some \( I \in \mathcal{I}_\beta^\alpha \).

Given \( P_\beta \)-conditions \( q_i = (F_i, \Delta_i) \), for \( i = 0, 1 \), \( q_1 \) extends \( q_0 \) if and only if:

1. \( \text{dom}(F_0) \subseteq \text{dom}(F_1) \) and the following holds for all \( \alpha \in \text{dom}(F_0) \).
   (a) If \( \alpha \) is even, then \( \text{dom}(F_0(\alpha)) \subseteq \text{dom}(F_1(\alpha)) \), and for every \( \delta \in \text{dom}(F_0(\alpha)) \),
      (i) \( x_{\alpha, \beta, \delta}^q \) is an initial segment of \( x_{\alpha, \beta, \delta}^{q_0} \).
      (ii) \( c_{\alpha, \beta, \delta}^{q_0} \subseteq c_{\alpha, \beta, \delta}^q \), and
      (iii) \( q_1 \upharpoonright \alpha \upharpoonright \min(I) \models \{ X_\delta^q(\xi) : \xi \in c_{\alpha, \beta, \delta}^{q_0} \} = \emptyset \).
   (b) If \( \alpha \) is odd, then
      (i) for every \( I \in \mathcal{I}_\beta^\alpha \) there is some \( I' \in \mathcal{I}_\beta^{\alpha_1} \) such that \( I \subseteq I' \) and \( \min(I) = \min(I') \),
      (ii) \( b_\beta^\alpha \subseteq b_\beta^{\alpha_1} \), and
      (iii) \( \mathcal{O}^{q_0}_\alpha \subseteq \mathcal{O}^{q_1}_{\alpha_1} \).

2. \( \Delta_0 \subseteq \Delta_1 \)

4.1. Main properties of \( \langle \mathbb{P}_\beta : \beta \leq \kappa \rangle \). In this section we list the main facts about our construction, which together prove Theorem 4.1. We will skip some of these proofs as they are essentially the same as corresponding proofs in [6], but we will include the proof that our forcing is proper and forces Strong Measuring.

Our first lemma follows immediately from the choice of \( (\Phi_\beta)^{\beta \prec \kappa} \). This lemma will be repeatedly used, without specific mention, in the proof of Lemma 4.10.

Lemma 4.6. For every \( \beta < \kappa \), \( \mathbb{P}_\beta \) is definable in \( (H(\kappa), \in, \Phi_{\beta+1}) \). Moreover, this definition is uniform in \( \beta \).

Our next lemma shows that our construction is a forcing iteration, in the sense that \( \mathbb{P}_\alpha \) is a complete suborder of \( \mathbb{P}_\beta \) whenever \( \alpha < \beta \).

Lemma 4.7. Let \( \alpha \leq \beta \leq \kappa \). If \( q = (F_q, \Delta_q) \in \mathbb{P}_\alpha \), \( r = (F_r, \Delta_r) \in \mathbb{P}_\beta \), and \( q \subseteq \alpha \upharpoonright \beta \upharpoonright \alpha \), then

\[ r \upharpoonright \alpha \uparrow q := (F_q \cup (F_r \upharpoonright (\alpha, \beta)), \Delta_q \cup \Delta_r) \]

is a condition in \( \mathbb{P}_\beta \) extending \( r \). Hence, \( \mathbb{P}_\alpha \) is a complete suborder of \( \mathbb{P}_\beta \).

The next lemma can be easily proved by a standard \( \Delta \)-system argument making use of \( \text{CH} \) (and Lemma 4.5).

Lemma 4.8. For every ordinal \( \alpha \leq \kappa \), \( \mathbb{P}_\alpha \) is \( \omega_2 \)-Knaster.

The following easy lemma shows that the required cardinal arithmetic holds after forcing with \( \mathbb{P}_\kappa \).

Lemma 4.9. \( \mathbb{P}_\kappa \) forces that \( 2^\mu = \kappa \) for every infinite cardinal \( \mu < \kappa \).
In the above lemma, one proves $2^\omega \geq \kappa$ in the $P_\kappa$-extension by showing that this forcing adds $\kappa$-many Cohen reals. In order to show $2^\omega \leq \kappa$ for every infinite cardinal $\mu < \kappa$ one runs an easy counting argument of nice names for subsets of $\mu$ using the $\omega_2$-c.c. of the forcing.

Given $\alpha < \kappa$, a condition $q \in P_\alpha$, and a countable elementary substructure $N$ of $H(\kappa)$, we will say that $q$ is $(N, P_\alpha)$-pre-generic in case $(N, \alpha) \in \Delta_\eta$. Also, given a countable elementary substructure $N$ of $H(\kappa)$ and a $P_\alpha$-condition $q$, we will say that $q$ is $(N, P_\alpha)$-generic if $q$ forces $G_\alpha \cap A \cap N \neq \emptyset$ for every maximal antichain $A$ of $P_\alpha$ such that $A \in N$. Note that this is more general than the standard notion of $(N, P)$-genericity, for a forcing notion $P$, which applies only if $P \in N$. In fact, in our situation $P_\alpha$ is of course never a member of $N$ if $N \subseteq H(\kappa)$.

Given $\beta \leq \kappa$, a $P_\beta$-condition $q$ and $N \in \text{dom}(\Delta_\eta)$, let $q \upharpoonright N$ denote the pair $(F, \Delta)$ where $F$ and $\Delta$ are as follows:

1. $F$ is the function with domain $\text{dom}(F_q) \cap N$ such that for each $\alpha \in \text{dom}(F_q) \cap N$,
   a. if $\alpha$ is even, then $\text{dom}(F(\alpha)) = \text{dom}(F(\alpha)) \upharpoonright \delta_N$ and for each $\delta \in \text{dom}(F(\alpha)) \upharpoonright \delta_N$, $(F(\alpha))(\delta) = (x_{\alpha, \delta}^q, C_{\alpha, \delta}^q \cap N)$, and
   b. if $\alpha$ is odd, then $F(\alpha) = (\mathcal{I}_\alpha \cap N, b_\alpha^q \upharpoonright \delta_N, C_\alpha^q \cap N)$.

2. $\Delta = \Delta_q \cap N$.

The properness of $P_\beta$, for every $\beta \leq \kappa$, follows immediately from the following lemma.

**Lemma 4.10.** Suppose $\beta < \kappa$ and $N \in M_{\beta+1}$. Then the following holds.

1. $(1)_\beta$ For every $q \in N$ there is some $q' \leq_\beta q$ such that $q'$ is $(N, P_\beta)$-pre-generic.
2. $(2)_\beta$ If $q \in P_\beta$ is $(N, P_\beta)$-pre-generic, then $q$ is $(N, P_\beta)$-generic.

**Proof.** The proof is by induction on $\beta$. The case $\beta = 0$ follows immediately from Lemmas 4.4 and 4.6.

Suppose $\beta = \alpha + 1$, with $\alpha$ an even ordinal. We start with the verification of $(1)_\beta$. By the induction hypothesis we may find an $(N, P_\alpha)$-pre-generic condition $r \in P_\alpha$ extending $q \upharpoonright \alpha$. But then it is clear that if $q' = r \wedge_\alpha q$, then $(F_{q'}, \Delta_{q'} \cup \{(N, \beta)\})$ is an $(N, P_\beta)$-pre-generic extension of $q$.

Let us now show $(2)_\beta$. Suppose $q$ is $(N, P_\beta)$-pre-generic, $A$ is a maximal antichain of $P_\beta$ in $N$, and $q$ extends $r \in A$. It suffices to prove that $r \in N$, and for this it is of course enough to show that there is some $r^* \in A \cap N$ compatible with $q$. We may assume that $\alpha \in \text{dom}(F_q)$, as otherwise the proof is an easier version of the present proof. Let $G$ be a $P_\alpha$-generic filter containing $q \upharpoonright \alpha$ and let us work in $N[G]$. Thanks to Lemma 4.6 we may then find a condition $q' \in N[G] \cap P_\beta$ such that

1. $q'$ extends some condition $r^* \in A \cap N[G]$,
2. $q'$ extends $q \upharpoonright N$, and such that
3. $q' \upharpoonright \alpha \in G$.

(In an abuse of notation, we define ‘$q'$ extends $q \upharpoonright N$’, even when $q \upharpoonright N$ is not an actual condition, if the relevant inclusions in the definition of extension hold between the objects building up $q'$ and $q \upharpoonright N$.)

Since $q' \upharpoonright \alpha \in G$, we may find a common extension $q''$ of $q' \upharpoonright \alpha$ and $q$. But then, since $q''$ extends $q \upharpoonright N$, it is straightforward to verify that $q''$, $q'$ and $q$ can be amalgamated into a $\mathcal{Q}_\beta$-condition. Finally, since $N[G] \cap H(\kappa)^{V} = N$ by the
genericity of \( q \upharpoonright \alpha \) (thanks to the induction hypothesis), we know that \( r^* \in N \), which ﬁnishes the proof in this case.

Suppose next that \( \beta = \alpha + 1 \) with \( \alpha \) odd. For the proof of (1)\( _\beta \) we will assume that \( \alpha \in \text{dom}(F_q) \) as otherwise the construction of the \((N,\mathbb{P}_\beta)\)-pre-generic extension of \( q \) is slightly simpler. As in the previous case, by induction hypothesis we may extend \( q \upharpoonright \alpha \) to an \((N,\mathbb{P}_\alpha)\)-pre-generic \( r \in \mathbb{P}_\alpha \). Then

\[
(F_r \cup \{(\alpha, (\mathcal{I}, b, O))\}, \Delta_r \cup \Delta_q \cup \{(N, \beta)\})
\]

is an \((N,\mathbb{P}_\beta)\)-pre-generic extension of \( q \), where

- \( \mathcal{I} = \mathcal{I}_\alpha \cup \{[\delta_N, \delta_N] \} \),
- \( b = b^\alpha_\delta \), and
- \( O = O^\alpha_\delta \cup \{N\} \).

Let us move on to the proof of (2)\( _\beta \). Suppose \( q \) is \((N,\mathbb{P}_\beta)\)-pre-generic, \( A \) is a maximal antichain of \( \mathbb{P}_\beta \) in \( N \), \( q \) extends \( r \in A \), and \( G \) is a \( \mathbb{P}_\alpha \)-generic ﬁlter such that \( q \upharpoonright \alpha \in G \). As in the proof in the previous case, it will be enough to show that there is some \( r^* \in A \cap N[G] \) compatible with \( q \). Also as in that proof, we may assume that \( \alpha \in \text{dom}(F_q) \) as otherwise the proof is easier.

Let \( \alpha_0 \) be such that \( \alpha = \alpha_0 + 1 \). Let \( G_{\alpha_0} = G \upharpoonright \mathbb{P}_{\alpha_0} \), and for every \( \delta < \omega_1 \) let \( X_\delta = (X^\alpha_\delta)^{G_{\alpha_0}} \). Let \( \psi \) be a function with domain \( A \times \omega_1 \) such that for each \( r^* \in A \) and each \( \eta < \omega_1 \), \( \psi(r^*, \eta) \) is as follows.

1. If there is a condition \( q' \in \mathbb{P}_\beta \) such that
   (a) \( q' \upharpoonright \alpha_0 \in G_{\alpha_0} \),
   (b) \( q' \) extends \( r^* \),
   (c) \( q' \) extends \( q \upharpoonright N \),
   (d) \( \alpha \in \text{dom}(F_{q'}) \), and
   (e) \( \{\min(I) : I \in \mathcal{I}_\delta \} \cap \eta = \{\min(I) : I \in \mathcal{I}_\delta \} \cap \delta_N \),
   then \( \psi(r^*, \eta) \) is such a condition \( q' \).

2. If there is no condition \( q' \) in \( \mathbb{P}_\beta \) as above, then \( \psi(r^*, \eta) = \emptyset \).

Note that \( \psi \in N[G_{\alpha_0}] \) since \( A, q \upharpoonright N, \{\min(I) : I \in \mathcal{I}_\delta \} \cap \delta_N \in N \). Let \( \dot{\psi} \) be a \( \mathbb{P}_{\alpha_0} \)-name for \( \psi \).

By a density argument as in the proof of Proposition 3.3, for every \( \delta \), every \( N' \in N^G \) such that \( \delta_N' = \delta \), and every \( a \in N' \) there is some \( M \in N' \cap N^{\mathbb{P}_{\alpha_0}} \cap M_\alpha \) such that \( a \in M \) and \( \delta_M \in X_\delta \). Hence, working in \( N[G] \) and using the openness of all relevant \( X_\delta \), we may now ﬁnd some \( M \in N \cap N^{\mathbb{P}_{\alpha_0}} \cap M_\alpha \) containing all relevant objects—and this includes \( \dot{\psi} \)—such that \( \delta_M \in X_\delta \) for every \( \delta \geq \delta_N \) such that \( b^\alpha_\delta(\delta) < \delta_N \). Again by openness of all relevant \( X_\delta \), we may ﬁx some \( \eta < \delta_M \) such that \( \eta, \delta_M \subseteq X_\delta \) for every \( \delta \geq \delta_N \) in \( \text{dom}(b^\alpha_\delta) \) such that \( b^\alpha_\delta(\delta) < \delta_N \).

Note that \( r \in A \) is such that \( \psi(r, \eta) \neq \emptyset \). Hence, working in \( M[G_{\alpha_0}] \), by correctness of this model in the structure \((H(\kappa))[G_{\alpha_0}], e)\) we may ﬁnd some \( r^* \in A \cap M[G_{\alpha_0}] \) such that \( q' = \dot{\psi}(r^*, \eta) \neq \emptyset \). As in the proof in the previous case, we may then ﬁx \( q'' \in \mathbb{P}_\alpha \) extending \( q' \upharpoonright \alpha \) and \( q \upharpoonright \alpha \). But then, by the choice of \( \phi(r^*, \eta) \), it is clear that all of \( q, q' \) and \( q'' \) can be amalgamated into a condition.

Finally, let us consider the case in which \( \beta \) is a non-zero limit ordinal. The proof of (1)\( _\beta \) is straightforward: we just need to pick \( \alpha < \beta \) such that \( \text{dom}(F_q) \subseteq \alpha \) and ﬁnd an \((N,\mathbb{P}_\alpha)\)-pre-generic \( q' \in \mathbb{P}_\alpha \) extending \( q \upharpoonright \alpha \), which exists by induction hypothesis. Then the ordered pair \((F_{q'}, \Delta_q \cup \Delta_{q'} \cup \{(N, \beta)\})\) is an \((N,\mathbb{P}_\beta)\)-pre-generic extension of \( q \).
We will now finish the proof of the lemma by proving (2)$_\beta$ in this case. As usual, suppose $A \cap N$ is a maximal antichain of $\mathbb{P}_\beta$ and $q$ is an $(N, \mathbb{P}_\beta)$-pre-generic condition extending some $r \in A$. We will show that there is some $r^* \in A \cap N$ compatible with $q$. When $\text{cf}(\alpha) = \omega$ or $\text{cf}(\alpha) > \omega_1$, the proof is easy.

In the first case we pick $\alpha \in N \cap N$ above $\text{dom}(F_0)$ and, working in a $N[G]$, for a $\mathbb{P}_\alpha$-generic $G$ such that $q \upharpoonright \alpha \in G$, find some $q' \in N[G] \cap \mathbb{P}_\beta$ such that

- $q'$ extends $q \upharpoonright N$,
- $q'$ extends some $r^* \in A$, and
- $\text{dom}(F_{q'}) \subseteq \alpha$.

It then follows that there is some $q'' \in \mathbb{P}_\alpha$ extending $q' \upharpoonright \alpha$ and $q \upharpoonright \alpha$ and, since $\text{dom}(F_{q''}) \subseteq \alpha$, there is a common extension of $q$, $q'$, and $q''$.

The case when $\text{cf}(\beta) > \omega_1$ follows immediately from the induction hypothesis thanks to the fact that $|A| \leq \omega_1$ (by the $\omega_2$-c.c. of $\mathbb{P}_\beta$).

We are thus left with the case when $\text{cf}(\beta) = \omega_1$. In this case we pick some $\bar{\alpha} < \alpha$, $\alpha \in N \cap \beta$, such that the following holds.

1. $\text{dom}(F_\alpha) \cap [\bar{\alpha}, \sup(N \cap \beta)] = \emptyset$;
2. $M \cap M' \cap \sup(N \cap \beta) \subseteq \bar{\alpha}$ whenever $M, M' \in \text{dom}(\Delta)$ are such that $\delta_M = \delta_{M'}$ and $M \cap M' \cap \sup(N \cap \beta)$ is bounded in $\sup(N \cap \beta)$.
3. $M \cap M' \cap [\bar{\alpha}, \alpha) \neq \emptyset$ whenever $M, M' \in \text{dom}(\Delta)$ are such that $\delta_M = \delta_{M'}$ and $M \cap M' \cap \sup(N \cap \beta)$ is unbounded in $\sup(N \cap \beta)$.

Let $G$ be a $\mathbb{P}_\alpha$-generic filter such that $q \upharpoonright \alpha \in G$ and let us work in $N[G]$. We may then find some $q' \in \mathbb{P}_\beta \cap N[G]$ with the following properties.

- $q'$ extends $q \upharpoonright N$.
- $q' \upharpoonright \alpha \in G$.
- $\text{dom}(F_\alpha) \cap [\bar{\alpha}, \alpha] \subseteq \alpha$.

As usual, there is a common extension $q''$ of $q' \upharpoonright \alpha$ and $q \upharpoonright \alpha$. Finally, thanks to the choice of $\bar{\alpha}$ and $\alpha$, and using the fact that $\Delta_q$ is a $(\Phi_\alpha)_{\alpha \leq \beta}$-tower of symmetric systems, it is easy to check that $q$, $q'$, and $q''$ can be amalgamated into a condition in $\mathbb{P}_\beta$. This concludes the proof of the lemma. $\Box$

The following lemma shows that if $\alpha < \kappa$ is an even ordinal, $G$ is $\mathbb{P}_{\alpha + 2}$-generic, $G_\alpha = G \cap \mathbb{P}_\alpha$, and $D$ is the set of ordinals of the form $\min(I)$, where $I \in \mathcal{I}^q_{\alpha + 1}$ for some $q \in G$ such that $\alpha + 1 \in \text{dom}(F_q)$, then $D$ is a club of $\omega_1$ measuring the ground model $V[G_\alpha]$ (in the relevant sense).

**Lemma 4.11.** Let $\alpha < \kappa$ be an even ordinal, $G$ a $\mathbb{P}_{\alpha + 2}$-generic filter, $G_\alpha = G \cap \mathbb{P}_\alpha$, and

$$D = \{\min(I) : I \in \mathcal{I}^q_{\alpha + 1}, \text{ for some } q \in G \text{ such that } \alpha + 1 \in \text{dom}(F_q)\}$$

Then $D$ is a club of $\omega_1$ measuring $\langle C_\delta : \delta < \omega_1 \rangle$, where $C_\delta$ is, for each $\delta$, the set of closed subsets of $\delta$ in $V[G_\alpha]$.

**Proof.** It is immediate to see that $D$ is unbounded in $\omega_1$, and a standard argument shows that it is also closed. Hence, it remains to argue that $D$ measures $\langle C_\delta : \delta < \omega_1 \rangle$.

Suppose $\delta^* < \omega_1$ is a limit point of $D$ and $C^{* \delta^*} \subseteq C^\alpha$. Let $\mathcal{Y}_\alpha = (\mathcal{Y}_\alpha^\delta)_{\alpha \in \omega}$ and $X^\alpha_{\delta^*} = (X^\alpha_{\delta^*})_{\alpha \in \omega}$. If $C^* \in \mathcal{Y}_\alpha(\delta^*)$, then an argument as in the proof of Proposition
3.3 shows that a tail of $X_\alpha^\omega$ is disjoint from $C^\ast$. Also, by the definition of $D$, a tail of $D \cap \delta^\ast$ is contained in $X_\alpha^\omega$, and therefore a tail of $D \cap \delta^\ast$ is disjoint from $C^\ast$.

We are left with the case when $C^\ast \notin \mathcal{Y}^\alpha(\delta^\ast)$. This means that there is a finite $\mathcal{C} \subset \mathcal{Y}^\alpha(\delta^\ast)$, a model $N \in \mathcal{N}_{\alpha+1}^\omega$, and some $\alpha \in N$, such that $\delta_N = \delta^\ast$ and

$$\left(\bigcap \{\delta \setminus C : C \in \mathcal{C}\}\right) \cap \{\delta_M : M \in N \cap \mathcal{N}_{\alpha+1}^\omega \cap \mathcal{M}_{\alpha+1}, a \in M\} \subseteq C^\ast$$

Let $\eta < \delta^\ast$ be such that

- $\eta < \delta_M$ for some $M_0 \in N \cap \mathcal{N}_{\alpha+1}^\omega \cap \mathcal{M}_{\alpha+1}$ such that $a \in M_0$, and such that
- $X_\alpha^\omega \setminus \eta \subseteq \bigcap \{\delta^\ast \setminus C : C \in \mathcal{C}\}$

It suffices to show that $D \cap [\eta, \delta^\ast) \subseteq C^\ast$. But if $\delta \in D \cap [\eta, \delta^\ast)$, then $\delta = \delta_M$ for some $M \in \mathcal{O}^{\alpha+1}$, where

$$\mathcal{O}^{\alpha+1} = \bigcup \{\mathcal{O}_{\alpha+1}^\delta : q \in G, \alpha + 1 \in \text{dom}(F_q)\}$$

Since $N$, $M_0$ and $M$ are all in $\mathcal{O}^{\alpha+1}$ and $\mathcal{O}^{\alpha+1}$ is a $\Phi_{\alpha+1}$-symmetric system, it follows that there is some $M' \in \mathcal{O}^{\alpha+1} \cap N$ such that $M_0 \subseteq M'$ and $\delta_{M'} = \delta_M$. But then $\delta = \delta_M = \delta_{M'} \subseteq C^\ast$ since $a \in M_0 \subseteq M'$, $M' \in N$, and $M' \in N_{\alpha+1} \cap \mathcal{M}_{\alpha+1}$. \qed

Finally, the following lemma concludes the proof of Theorem 4.1.

**Lemma 4.12.** $\mathbb{P}_\kappa$ forces Strong Measuring.

**Proof.** Let $G$ be a $\mathbb{P}_\kappa$-generic filter and let $\bar{C} = \langle C_\delta : \delta < \omega_1 \rangle \in V[G]$ be such that for every $\delta$, $C_\delta$ is a collection of less than $\kappa$-many closed subsets of $\delta$. By the $\omega_2$-c.c. we may then find some even $\alpha_0 < \kappa$ such that $\bar{C} \subseteq V[G_{\alpha_0}]$, where $G_{\alpha_0} = G \cap \mathbb{P}_\alpha$ for all $\alpha < \kappa$. By Lemma 4.11 in $V[G_{\alpha_0}]$ there is a club $D$ measuring $C_\delta^\omega : \delta < \omega_1$, where $C_\delta^\omega$ is, for each $\delta$, the set of closed subsets of $\delta$ in $V[G_{\alpha_0}]$. But then $D$ measures $\bar{C}$ since $C_\delta \subseteq C_\delta^\omega$ for each $\delta$. \qed

## 5. Measuring Without the Axiom of Choice

Another natural way to strengthen Measuring is to allow, in the sequence to be measured, not just closed sets, but also sets of higher Borel complexity. This line of strengthenings of Measuring was also considered in [2]. For completeness, we are including here the corresponding observations.

The version of Measuring where one considers sequences $\bar{X} = \langle X_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle$, with each $X_\alpha$ an open subset of $\alpha$ in the order topology, is of course equivalent to Measuring. A natural next step would therefore be to consider sequences in which each $X_\alpha$ is a countable union of closed sets. This is obviously the same as allowing each $X_\alpha$ to be an arbitrary subset of $\alpha$. Let us call the corresponding statement Measuring$^+$:

**Definition 5.1.** Measuring$^+$ holds if and only if for every sequence $\bar{X} = \langle X_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle$, if $X_\alpha \subseteq \alpha$ for each $\alpha$, then there is some club $D \subseteq \omega_1$ such that for every limit point $\delta \in D$ of $D$, $D \cap \delta$ measures $X_\delta$.

It is easy to see that Measuring$^+$ is false in ZFC. In fact, given a stationary and co-stationary $S \subseteq \omega_1$, there is no club of $\omega_1$ measuring $\bar{X} = \langle S \cap \alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle$. The reason is that if $D$ is any club of $\omega_1$, then both $D \cap S \cap \delta$ and $(D \cap \delta) \setminus S$ are cofinal subsets of $\delta$ for each $\delta$ in the club of limit points in $\omega_1$ of both $D \cap S$ and $D \setminus S$. 

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The status of $\text{Measuring}^*$ is more interesting in the absence of the Axiom of Choice. Let $C_{\omega_1} = \{X \subseteq \omega_1 : C \subseteq X \text{ for some club } C \text{ of } \omega_1\}$. 

**Observation 5.2.** ($\text{ZF} + C_{\omega_1}$ is a normal filter on $\omega_1$) Suppose $\bar{X} = \langle X_\delta : \delta \in \omega_1 \cap \text{Lim} \rangle$ is such that

1. $X_\delta \subseteq \delta$ for each $\delta$.
2. For each club $C \subseteq \omega_1$,
   - (a) there is some $\delta \in C$ such that $C \cap X_\delta \neq \emptyset$, and
   - (b) there is some $\delta \in C$ such that $(C \cap \delta) \setminus X_\delta \neq \emptyset$.

Then there is a stationary and co-stationary subset of $\omega_1$ definable from $\bar{X}$.

**Proof.** We have two possible cases. The first case is that for all $\alpha < \omega_1$, either

- $W_0^\alpha = \{\delta < \omega_1 : \alpha \notin X_\delta\}$ is in $C_{\omega_1}$, or
- $W_1^\alpha = \{\delta < \omega_1 : \alpha \in X_\delta\}$ is in $C_{\omega_1}$.

For each $\alpha < \omega_1$, let $W_\alpha^\epsilon$ be $W_\alpha^\epsilon$ for the unique $\epsilon \in \{0, 1\}$ such that $W_\alpha^\epsilon \in C_{\omega_1}$, and let $W^* = \Delta_{\alpha<\omega_1} W_\alpha \in C_{\omega_1}$. Then $X_{\delta_0} = X_{\delta_1} \cap \delta_0$ for all $\delta_0 < \delta_1$ in $W^*$. It then follows, by (2), that $S = \bigcup_{\delta \in W^*} X_\delta$, which of course is definable from $\bar{C}$, is a stationary and co-stationary subset of $\omega_1$. Indeed, suppose $C \subseteq \omega_1$ is a club, and let us fix a club $D \subseteq W^*$. There is then some $\delta = C \cap D$ and some $\alpha = C \cap D \cap X_\delta$. But then $\alpha \subseteq S$ since $\delta \subseteq W^*$ and $\alpha \subseteq W^* \cap X_\delta$. There is also some $\delta = C \cap D$ and some $\alpha = C \cap D$ such that $\alpha \notin X_\delta$, which implies that $\alpha \notin S$ by a symmetrical argument, using the fact that $X_{\delta_0} = X_{\delta_1} \cap \delta_0$ for all $\delta_0 < \delta_1$ in $W^*$.

The second possible case is that there is some $\alpha < \omega_1$ with the property that both $W_0^\alpha$ and $W_1^\alpha$ are stationary subsets of $\omega_1$. But now we can let $S$ be $W_0^\alpha$, where $\alpha$ is first such that $W_0^\alpha$ is stationary and co-stationary. \hfill $\square$

It is worth comparing the above observation with Solovay's classic result that an $\omega_1$-sequence of pairwise disjoint stationary subsets of $\omega_1$ is definable from any given ladder system on $\omega_1$ (working in the same theory).

**Corollary 5.3.** ($\text{ZF} + C_{\omega_1}$ is a normal filter on $\omega_1$) The following are equivalent.

1. $C_{\omega_1}$ is an ultrafilter on $\omega_1$;
2. $\text{Measuring}^*$;
3. For every sequence $\langle X_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle$, if $X_\alpha \subseteq \alpha$ for each $\alpha$, then there is a club $C \subseteq \omega_1$ such that either
   - $C \cap \delta \subseteq X_\delta$ for every $\delta \in C$, or
   - $C \cap X_\delta = \emptyset$ for every $\delta \in C$.

**Proof.** (3) trivially implies (2), and by the observation (1) implies (3). Finally, to see that (2) implies (1), note that the argument right after the definition of $\text{Measuring}^*$ uses only $\text{ZF}$ together with the regularity of $\omega_1$ and the negation of (1). \hfill $\square$

In particular, the strong form of $\text{Measuring}^*$ given by (3) in the above observation follows from $\text{ZF}$ together with the Axiom of Determinacy.

We finish this digression into set theory without the Axiom of Choice by observing that any attempt to parametrize $\text{Measuring}^*$, in the same vein as we did with $\text{Measuring}$, gives rise to principles vacuously equivalent to $\text{Measuring}^*$ itself, at least when the parametrization is done with the alephs.
Specifically, given an aleph $\kappa$, let us define $\text{Measuring}^*_\kappa$ as the statement that for every sequence $\langle X_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle$, if each $X_\alpha$ is a set of cardinality at most $\kappa$ consisting of subsets of $\alpha$, then there is a club $D \subseteq \omega_1$ such that for every limit point $\delta \in D$ of $D$, $D \cap \delta$ measures $X$ for all $X \in X_\delta$. Then $\text{Measuring}^*_\kappa$ is clearly equivalent to $\text{Measuring}^*_\omega$ under $\text{ZF}$ together with the normality of $\mathcal{C}_{\omega_1}$ and the Axiom of Choice for countable families of subsets of $\omega_1$ (which follows from $\text{ZF} + \text{AD}$). On the other hand, working in $\text{ZF} + \mathcal{C}_{\omega_1}$, a normal filter on $\omega_1$, we have that $\text{Measuring}^*_\omega$ follows vacuously from $\text{Measuring}^*_\kappa$ simply because under $\text{Measuring}^*_\kappa$ there is no sequence $\langle X_\alpha : \alpha \in \omega_1 \cap \text{Lim} \rangle$ as in the definition of $\text{Measuring}^*_\omega$, and such that $|X_\alpha| = \omega_1$ for some $\alpha$; indeed, $\text{Measuring}^*_\kappa$ implies, over this base theory, that $\mathcal{C}_{\omega_1}$ is an ultrafilter (Corollary 5.3), and if $\mathcal{C}_{\omega_1}$ is an ultrafilter then there is no $\omega_1$-sequence of distinct reals, whereas the existence of a family of size $\omega_1$ consisting of subsets of some fixed countable ordinal clearly implies that there is such a sequence. This fact was pointed out by Asaf Karagila.

We conclude the article with two natural questions.

**Question 5.4.** Is $\text{Measuring}^p_\kappa$ false?

**Question 5.5.** Are $\text{Measuring}$ and $\text{Strong Measuring}$ equivalent statements assuming Martin’s Axiom?

**References**


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