WEAK SQUARE SEQUENCES AND SPECIAL ARONSZAJN TREES

JOHN KRUEGER

ABSTRACT. A classical theorem of set theory is the equivalence of the weak square principle $\square_\mu^*$ with the existence of a special Aronszajn tree on μ^+ . We introduce the notion of a weak square sequence on any regular uncountable cardinal, and prove that the equivalence between weak square sequences and special Aronszajn trees holds in general.

Recall the weak square principle $\square_\mu^*$ for an infinite cardinal μ , which asserts the existence of a sequence $\langle \mathcal{C}_\alpha : \alpha \in \mu^+ \cap \text{Lim} \rangle$ satisfying:

- (1) for all $c \in \mathcal{C}_\alpha$, c is a club subset of α with order type at most μ ;
- (2) $|\mathcal{C}_\alpha| \leq \mu$;
- (3) for all $c \in \mathcal{C}_\alpha$, if $\beta \in \text{lim}(c)$ then $c \cap \beta \in \mathcal{C}_\beta$.

For a regular uncountable cardinal κ , a tree $(T, <_T)$ is a κ -tree if it has height κ and all its levels are of size less than κ . For a successor cardinal $\kappa = \mu^+$, a κ -tree $(T, <_T)$ is a *special Aronszajn tree* if T is the union of μ many antichains. Equivalently, T is special if there exists a function $f : T \rightarrow \mu$ such that $t <_T u$ implies $f(t) \neq f(u)$.

The following classical theorem was originally noted by Jensen [2]. Let μ be an infinite cardinal. Then $\square_\mu^*$ is equivalent to the existence of a special Aronszajn tree on μ^+ .

Todorćević [3] introduced a more general definition of a special Aronszajn tree. For a regular uncountable cardinal κ , a tree $(T, <_T)$ of height κ is said to be a *special Aronszajn tree* if there exists a function $g : T \rightarrow T$ satisfying:

- (1) $g(t) <_T t$ for all non-minimal $t \in T$;
- (2) for all $u \in T$, $g^{-1}(\{u\})$ is the union of fewer than κ many antichains.

This definition coincides with the classical definition of a special Aronszajn tree when κ is a successor cardinal.

In this paper we introduce a definition of a weak square sequence which makes sense on any regular uncountable cardinal. We prove that the existence of such a sequence on a regular uncountable cardinal κ is equivalent to the existence of a special Aronszajn tree on κ in the sense of Todorćević.

Notation: Let Lim and Succ denote the classes of limit ordinals and successor ordinals respectively. Let $\text{cof}(\omega)$ denote the class of limit ordinals of countable cofinality, and let $\text{cof}(>\omega)$ denote the class of limit ordinals of uncountable cofinality. For a set of ordinals a , $\text{ot}(a)$ is the order type of a , and $\text{lim}(a)$ is the set of ordinals β such that $\sup(a \cap \beta) = \beta$.

2010 *Mathematical Subject Classification*: 03E05.

Key words and phrases: Weak square sequence, special Aronszajn tree.

A *tree* is a strict partial order $(T, <_T)$ satisfying that for every node $x \in T$, the set $\{y \in T : y <_T x\}$ is well ordered by $<_T$. The *height* of a node $x \in T$, denoted by $\text{ht}(x)$, is the order type of $\{y \in T : y <_T x\}$. Let $T_\alpha = \{x \in T : \text{ht}(x) = \alpha\}$ denote level α of T , for any ordinal α . The *height* of the tree T is the least α such that T_α is empty. For finite sequences u and v , $u \sqsubseteq v$ means that u is an initial segment of v , and $u \sqsubset v$ means that u is a proper initial segment of v .

1. WEAK SQUARE SEQUENCES

The next definition generalizes the idea of a weak square sequence to any regular uncountable cardinal.

Definition 1.1. *Let κ be a regular uncountable cardinal. A sequence $\langle c_\alpha : \alpha \in C \rangle$ is a weak square sequence on κ if:*

- (1) $C \subseteq \kappa \cap \text{Lim}$ is a club;
- (2) for all $\alpha \in C$, c_α is a club subset of α with order type less than α ;
- (3) for every $\xi < \kappa$, $|\{c_\alpha \cap \xi : \alpha \in C\}| < \kappa$.

Note that if there exists a weak square sequence $\langle c_\alpha : \alpha \in C \rangle$ on κ , then κ is non-Mahlo. Indeed, (2) implies that every ordinal in the club C is singular.

The goal of this section is to show that for an infinite cardinal μ , the existence of a weak square sequence on μ^+ in the sense above is equivalent to the classical weak square principle $\square_\mu^*$. The main challenge lies in reducing the order type of the clubs on the sequence.

Let us note that for an infinite cardinal μ , $\square_\mu^*$ is equivalent to the existence of a sequence $\langle c_\alpha : \alpha \in \mu^+ \cap \text{Lim} \rangle$, where each c_α is a club subset of α with order type at most μ , and for every $\xi < \mu^+$, $|\{c_\alpha \cap \xi : \alpha \in \mu^+ \cap \text{Lim}\}| \leq \mu$. For if we have such a sequence, we can define for each limit ordinal α the set $\mathcal{C}_\alpha$ to be the collection of sets of the form $c_\beta \cap \alpha$, where $\beta \in \mu^+ \cap \text{Lim}$ and $\alpha \in \lim(c_\beta)$. Conversely, given $\langle \mathcal{C}_\alpha : \alpha \in \mu^+ \cap \text{Lim} \rangle$, a sequence $\langle c_\alpha : \alpha \in \mu^+ \cap \text{Lim} \rangle$ is obtained as required by choosing c_α to be any member of $\mathcal{C}_\alpha$.

Lemma 1.2. *Let κ be a regular uncountable cardinal. Suppose there exists a weak square sequence on κ . Then there exists a sequence $\langle c_\alpha : \alpha \in \kappa \cap \text{Lim} \rangle$ satisfying:*

- (1) each c_α is a club subset of α ;
- (2) if α is singular then $\text{ot}(c_\alpha) < \alpha$;
- (3) there is a club $C \subseteq \kappa$ such that for all $\alpha \in C$, $\text{ot}(c_\alpha) < \min(c_\alpha)$;
- (4) for all $\alpha \in (\kappa \cap \text{Lim}) \setminus C$, $\min(c_\alpha) > \sup(C \cap \alpha)$;
- (5) for every $\xi < \kappa$, $|\{c_\alpha \cap \xi : \alpha \in \kappa \cap \text{Lim}\}| < \kappa$.

Proof. Fix a sequence $\langle d_\alpha : \alpha \in C \rangle$ satisfying Definition 1.1. We define a sequence $\langle c_\alpha : \alpha \in \kappa \cap \text{Lim} \rangle$ as follows. If $\alpha \in C$, then $\text{ot}(d_\alpha) < \alpha$. So let $c_\alpha = d_\alpha \setminus (\text{ot}(d_\alpha) + 1)$. If $\alpha < \kappa$ is a limit ordinal not in C , then since C is a club, $\sup(C \cap \alpha) < \alpha$. Let c_α be any club subset of α with order type $\text{cf}(\alpha)$ such that $\min(c_\alpha) > \sup(C \cap \alpha)$. Clearly (1)–(4) are satisfied.

We claim that for every $\xi < \kappa$, $|\{c_\alpha \cap \xi : \alpha \in \kappa \cap \text{Lim}\}| < \kappa$. Let $\gamma = \min(C \setminus \xi)$. Then for every limit ordinal $\beta \in \kappa \setminus C$ which is larger than γ , $\min(c_\beta) > \gamma$, so $c_\beta \cap \xi = \emptyset$. It follows that the nonempty members of the set $\{c_\alpha \cap \xi : \alpha \in \kappa \cap \text{Lim}\}$ are in the set

$$\bigcup_{\delta < \xi} \{d_\alpha \cap [\delta, \xi) : \alpha \in C\} \cup \{c_\beta \cap \xi : \beta \in \gamma \setminus C\}.$$

There are fewer than κ many elements in the set on the left by assumption, and clearly there are no more than $|\gamma| < \kappa$ many elements in the set on the right. $\square$

Lemma 1.3. *Let κ be a regular uncountable cardinal. Suppose $\langle c_\alpha : \alpha \in \kappa \cap \text{Lim} \rangle$ is a sequence satisfying:*

- (1) *each c_α is a club subset of α ;*
- (2) *if α is singular then $\text{ot}(c_\alpha) < \alpha$;*
- (3) *for every $\xi < \kappa$, $|\{c_\alpha \cap \xi : \alpha \in \kappa \cap \text{Lim}\}| < \kappa$.*

For each limit ordinal $\alpha < \kappa$, let $f_\alpha : \text{ot}(c_\alpha) \rightarrow c_\alpha$ be the increasing enumeration of c_α . Define a sequence $\langle d_\alpha : \alpha \in \kappa \cap \text{Lim} \rangle$ by letting

$$d_\alpha = \begin{cases} c_\alpha & \text{if } \text{ot}(c_\alpha) = \text{cf}(\alpha) \\ f_\alpha[c_{\text{ot}(c_\alpha)}] & \text{if } \text{ot}(c_\alpha) > \text{cf}(\alpha). \end{cases}$$

Then the sequence $\langle d_\alpha : \alpha \in \kappa \cap \text{Lim} \rangle$ also satisfies properties (1), (2), and (3) above; moreover, in the case that $\text{ot}(c_\alpha) > \text{cf}(\alpha)$, we have $\text{ot}(d_\alpha) < \text{ot}(c_\alpha)$.

Proof. Consider a limit ordinal $\alpha < \kappa$. If $\text{ot}(c_\alpha) = \text{cf}(\alpha)$, then $d_\alpha = c_\alpha$ so (1) and (2) hold for d_α . Suppose $\text{ot}(c_\alpha) > \text{cf}(\alpha)$. Then in particular, α is singular. Since $f_\alpha : \text{ot}(c_\alpha) \rightarrow \alpha$ is normal and cofinal in α , $d_\alpha = f_\alpha[c_{\text{ot}(c_\alpha)}]$ is a club subset of α with order type equal to $\text{ot}(c_{\text{ot}(c_\alpha)})$; but $\text{ot}(c_{\text{ot}(c_\alpha)}) \leq \text{ot}(c_\alpha) < \alpha$. So (1) and (2) hold. For the final comment, assume $\text{ot}(c_\alpha) > \text{cf}(\alpha)$. Note that $\text{cf}(\text{ot}(c_\alpha)) = \text{cf}(\alpha) < \text{ot}(c_\alpha)$, so $\text{ot}(c_\alpha)$ is singular. Therefore $\text{ot}(c_{\text{ot}(c_\alpha)}) < \text{ot}(c_\alpha)$ by (2). So $\text{ot}(d_\alpha) = \text{ot}(c_{\text{ot}(c_\alpha)}) < \text{ot}(c_\alpha)$.

Let $\xi < \kappa$ be given, and we prove $|\{d_\alpha \cap \xi : \alpha \in \kappa \cap \text{Lim}\}| < \kappa$. Note that

$$\{d_\alpha \cap \xi : \alpha \in \kappa \cap \text{Lim}, \text{ot}(c_\alpha) = \text{cf}(\alpha)\} \subseteq \{c_\alpha \cap \xi : \alpha \in \kappa \cap \text{Lim}\},$$

so the set on the left has size less than κ . It remains to show that the set

$$\{d_\alpha \cap \xi : \alpha \in \kappa \cap \text{Lim}, \text{ot}(c_\alpha) > \text{cf}(\alpha)\}$$

has size less than κ .

Consider a limit ordinal α such that $\text{ot}(c_\alpha) > \text{cf}(\alpha)$. Then $d_\alpha = f_\alpha[c_{\text{ot}(c_\alpha)}]$. Since f_α is the increasing enumeration of c_α , clearly $c_\alpha \cap \xi = f_\alpha[\text{ot}(c_\alpha \cap \xi)]$. As $d_\alpha \subseteq c_\alpha$ and f_α is injective, we have $d_\alpha \cap \xi = d_\alpha \cap c_\alpha \cap \xi = f_\alpha[c_{\text{ot}(c_\alpha)}] \cap f_\alpha[\text{ot}(c_\alpha \cap \xi)] = f_\alpha[c_{\text{ot}(c_\alpha)} \cap \text{ot}(c_\alpha \cap \xi)]$. Let $g_\alpha : \text{ot}(c_\alpha \cap \xi) \rightarrow c_\alpha \cap \xi$ be the increasing enumeration of $c_\alpha \cap \xi$. Then $g_\alpha = f_\alpha \upharpoonright \text{ot}(c_\alpha \cap \xi)$. So we have

$$d_\alpha \cap \xi = g_\alpha[c_{\text{ot}(c_\alpha)} \cap \text{ot}(c_\alpha \cap \xi)].$$

Now the function g_α is determined by $c_\alpha \cap \xi$, and there are fewer than κ many possibilities for $c_\alpha \cap \xi$. Once $c_\alpha \cap \xi$ is known, $d_\alpha \cap \xi$ is determined by $c_{\text{ot}(c_\alpha)} \cap \text{ot}(c_\alpha \cap \xi)$, and again there are fewer than κ many possibilities for this set. So there are fewer than κ many possibilities for $d_\alpha \cap \xi$. $\square$

Proposition 1.4. *Let κ be a regular uncountable cardinal. Suppose $\langle c_\alpha : \alpha \in \kappa \cap \text{Lim} \rangle$ is a sequence satisfying:*

- (1) *each c_α is a club subset of α ;*
- (2) *if α is singular then $\text{ot}(c_\alpha) < \alpha$;*
- (3) *for every $\xi < \kappa$, $|\{c_\alpha \cap \xi : \alpha \in \kappa \cap \text{Lim}\}| < \kappa$.*

Then there exists a sequence $\langle d_\alpha : \alpha \in \kappa \cap \text{Lim} \rangle$ satisfying properties (1), (2), and (3), and moreover, each d_α has order type equal to $\text{cf}(\alpha)$.

Proof. By induction we define for each $n < \omega$ a sequence

$$\langle c_\alpha^n : \alpha \in \kappa \cap \text{Lim} \rangle.$$

The inductive hypotheses are that the sequence of c_α^n 's satisfies properties (1), (2), and (3), and moreover, if $\text{ot}(c_\alpha^n) > \text{cf}(\alpha)$, then $\text{ot}(c_\alpha^{n+1}) < \text{ot}(c_\alpha^n)$. Let $c_\alpha^0 = c_\alpha$ for all limit ordinals $\alpha < \kappa$.

Fix $n < \omega$ and suppose that $\langle c_\alpha^n : \alpha \in \kappa \cap \text{Lim} \rangle$ is defined as required. For each α let $f_\alpha^n : \text{ot}(c_\alpha^n) \rightarrow c_\alpha^n$ be the increasing enumeration of c_α^n . Define c_α^{n+1} by

$$c_\alpha^{n+1} = \begin{cases} c_\alpha^n & \text{if } \text{ot}(c_\alpha^n) = \text{cf}(\alpha) \\ f_\alpha^n[c_{\text{ot}(c_\alpha^n)}^n] & \text{if } \text{ot}(c_\alpha^n) > \text{cf}(\alpha). \end{cases}$$

Lemma 1.3 implies that $\langle c_\alpha^{n+1} : \alpha < \kappa \text{ limit} \rangle$ satisfies the inductive hypotheses. This completes the definition.

Now we define the sequence $\langle d_\alpha : \alpha \in \kappa \cap \text{Lim} \rangle$. Consider a limit ordinal $\alpha < \kappa$. Since $\text{ot}(c_\alpha^{n+1}) < \text{ot}(c_\alpha^n)$ provided that $\text{ot}(c_\alpha^n) > \text{cf}(\alpha)$, there must exist a least k such that $\text{ot}(c_\alpha^k) = \text{cf}(\alpha)$. Then by definition, for all $m \geq k$, $c_\alpha^m = c_\alpha^k$. Let $d_\alpha = c_\alpha^k$, which is the eventual value of the club attached to α . Clearly d_α is a club subset of α with order type $\text{cf}(\alpha)$, and in particular, if α is singular then $\text{ot}(c_\alpha) < \alpha$.

To show (3), consider $\xi < \kappa$. Then for all $n < \omega$, $|\{c_\alpha^n \cap \xi : \alpha \in \kappa \cap \text{Lim}\}| < \kappa$. But

$$\{d_\alpha \cap \xi : \alpha \in \kappa \cap \text{Lim}\} \subseteq \bigcup_{n < \omega} \{c_\alpha^n \cap \xi : \alpha \in \kappa \cap \text{Lim}\};$$

so the set on the left is a subset of a countable union of sets each having cardinality less than κ . $\square$

Theorem 1.5. *Let μ be an infinite cardinal. Then $\square_\mu^*$ holds iff there exists a weak square sequence on μ^+ in the sense of Definition 1.1.*

Proof. If $\square_\mu^*$ holds, then as noted above there exists a sequence $\langle c_\alpha : \alpha \in \mu^+ \cap \text{Lim} \rangle$ such that each c_α is a club subset of α with order type at most μ , and for every $\xi < \mu^+$, $|\{c_\alpha \cap \xi : \alpha \in \mu^+ \cap \text{Lim}\}| \leq \mu$. Let C be the club set of limit ordinals α with $\mu < \alpha < \mu^+$. Then $\langle c_\alpha : \alpha \in C \rangle$ satisfies Definition 1.1. Conversely, suppose there exists a weak square sequence on μ^+ . Then by Lemma 1.2 and Proposition 1.4, there exists a sequence $\langle d_\alpha : \alpha \in \kappa \cap \text{Lim} \rangle$ such that each d_α is a club subset of α with order type $\text{cf}(\alpha) \leq \mu$, and for every $\xi < \kappa$, $|\{d_\alpha \cap \xi : \alpha \in \kappa \cap \text{Lim}\}| < \kappa$. Therefore $\square_\mu^*$ holds. $\square$

2. A SPECIAL ARONSZAJN TREE IMPLIES WEAK SQUARE

According to the classical definition, for an infinite cardinal μ , a tree $(T, <_T)$ of height μ^+ is a *special Aronszajn tree* if T is the union of μ many antichains, or equivalently, if there exists a function $f : T \rightarrow \mu$ such that for all $t, u \in T$, $t <_T u$ implies $f(t) \neq f(u)$.

Todorćević [3] introduced a more general definition of a special Aronszajn tree which makes sense for any regular uncountable cardinal. Recall that if $(T, <_T)$ is a tree, a function $g : T \rightarrow T$ is said to be *regressive* if $f(a) <_T a$ for all non-minimal $a \in T$.

Definition 2.1. *Let κ be a regular uncountable cardinal. A tree $(T, <_T)$ with height κ is a special Aronszajn tree if there exists a regressive function $g : T \rightarrow T$ such that for all $b \in T$, the set $g^{-1}(\{b\})$ is the union of fewer than κ many antichains.*

We will sometimes abbreviate “special Aronszajn tree” with “special tree”. A *special Aronszajn tree on κ* means a κ -tree which is special. Note that T is special iff there is a regressive function $g : T \rightarrow T$ such that for all $b \in T$, there is an ordinal $\lambda_b < \kappa$ and a function $f_b : g^{-1}(\{b\}) \rightarrow \lambda_b$ such that for all $t, u \in g^{-1}(\{b\})$, $t <_T u$ implies $f_b(t) \neq f_b(u)$.

The equivalence between the two definitions of “special” for successor cardinals was noted in [3] without proof.

Proposition 2.2 (Todorćević). *Let μ be an infinite cardinal and let $(T, <_T)$ be a tree of height μ^+ . Then T is a special Aronszajn tree in the classical sense iff T satisfies Definition 2.1.*

Proof. The forward direction of the equivalence is trivial; just define a regressive function which maps every node to a minimal node. Now suppose there is a regressive function $g : T \rightarrow T$, and for each $b \in T$, some ordinal $\lambda_b < \mu^+$ and a function $f_b : g^{-1}(\{b\}) \rightarrow \lambda_b$ satisfying that for all $t, u \in g^{-1}(\{b\})$, $t <_T u$ implies $f_b(t) \neq f_b(u)$. Without loss of generality, we can assume $\lambda_b = \mu$ for all b .

We define a function $f : T \rightarrow {}^{<\omega}\mu$ so that $c <_T d$ implies $f(c) \neq f(d)$ for all $c, d \in T$. Clearly this suffices since ${}^{<\omega}\mu$ has size μ . Consider a node $a \in T$. If a is minimal then let $f(a)$ be the empty sequence. Suppose a is not minimal. Define g^k for $k < \omega$ by recursion, letting $g^0(a) = a$, and $g^{k+1}(a) = g(g^k(a))$ if $g^k(a)$ is not minimal. Since g is regressive, we have $\text{ht}(g^1(a)) > \text{ht}(g^2(a)) > \dots > \text{ht}(g^k(a))$. Let m be least such that $g^m(a)$ is minimal. Define $f(a)$ by

$$f(a) = \langle f_{g(a)}(a), f_{g^2(a)}(g(a)), \dots, f_{g^m(a)}(g^{m-1}(a)) \rangle.$$

Suppose for a contradiction $c <_T d$ but $f(c) = f(d)$. Since d is not minimal, $f(c) = f(d)$ is not empty, so c is not minimal either. Let $m > 0$ be least such that $g^m(c)$ is minimal. Since m is the length of the sequence $f(c) = f(d)$, m is also least such that $g^m(d)$ is minimal. As $g^m(c) <_T c <_T d$, $g^m(c) <_T d$, and hence $g^m(c) = g^m(d)$. Let $0 < k \leq m$ be least such that $g^k(c) = g^k(d)$.

Since $g^{k-1}(c) \leq_T c$, $g^{k-1}(d) \leq_T d$, and $c <_T d$, $g^{k-1}(c)$ and $g^{k-1}(d)$ are comparable and not equal. But $g(g^{k-1}(c)) = g^k(c) = g^k(d) = g(g^{k-1}(d))$. Therefore $f_{g^k(c)}(g^{k-1}(c)) \neq f_{g^k(d)}(g^{k-1}(d))$, which contradicts $f(c) = f(d)$. $\square$

Recall the standard fact that for a strongly inaccessible cardinal κ , κ is weakly compact iff there does not exist an Aronszajn tree on κ . Todorćević [3] used his general definition of a special Aronszajn tree to provide an analogue of this result which characterizes Mahlo cardinals.

Theorem 2.3 (Todorćević). *Let κ be a strongly inaccessible cardinal. Then the following are equivalent:*

- (1) κ is a Mahlo cardinal;
- (2) there does not exist a special Aronszajn tree on κ .

We will prove that for a regular uncountable cardinal κ , the existence of a special Aronszajn tree on κ is equivalent to the existence of a weak square sequence on κ . We first show the forward direction; the proof follows along the lines of Section 5.2 in [1], which handles the case when κ is a successor cardinal.

First let us give a simpler characterization of a special Aronszajn tree on κ .

Lemma 2.4. *Let $(T, <_T)$ be a κ -tree, where κ is a regular uncountable cardinal. Then T is special iff there exists a function $g : T \rightarrow \kappa$ such that $g(t) < \text{ht}(t)$ for*

all non-minimal t , and for all $\beta < \kappa$, $g^{-1}(\{\beta\})$ is the union of fewer than κ many antichains.

Proof. For the forward direction, given a regressive $f : T \rightarrow T$ witnessing that T is special, define $g(t) = \text{ht}(f(t))$. Then $g^{-1}(\{\beta\}) = \bigcup \{f^{-1}(\{b\}) : \text{ht}(b) = \beta\}$. Each $f^{-1}(\{b\})$ is the union of fewer than κ many antichains, and there are fewer than κ many such b 's since T is a κ -tree. Hence $g^{-1}(\{\beta\})$ is the union of fewer than κ many antichains. Conversely, given $g : T \rightarrow \kappa$ as described above, define $f(b) = b \upharpoonright g(b)$ for non-minimal b . $\square$

Theorem 2.5. *Let κ be a regular uncountable cardinal. If there exists a special Aronszajn tree on κ , then there exists a weak square sequence on κ .*

Proof. Let $(T, <_T)$ be a κ -tree and suppose that T is special. Fix a function $g : T \rightarrow \kappa$, where $g(t) < \text{ht}(t)$ for all non-minimal t , and for each $\beta < \kappa$ a function $f_\beta : g^{-1}(\{\beta\}) \rightarrow \lambda_\beta$, where $\lambda_\beta < \kappa$, satisfying that for all $c, d \in g^{-1}(\{\beta\})$, $c <_T d$ implies $f_\beta(c) \neq f_\beta(d)$.

For each limit ordinal α we define a family $\mathcal{A}_\alpha$ of cofinal subsets of α . Fix a limit ordinal α . Consider the following property which a node x in T_α may or may not satisfy: there exists $\beta < \alpha$ such that the set

$$\{\text{ht}(y) : y <_T x \wedge g(y) < \beta\}$$

is cofinal in α .

We claim that if α has uncountable cofinality, then this property is true for all $x \in T_\alpha$. Indeed, fix a sequence $\langle \alpha_i : i < \text{cf}(\alpha) \rangle$ which is increasing, continuous, and cofinal in α . Since $g(t) < \text{ht}(t)$ for all non-minimal t , there exists a regressive function $h : \text{cf}(\alpha) \cap \text{Lim} \rightarrow \text{cf}(\alpha)$ so that for all limit ordinals $\gamma < \text{cf}(\alpha)$, if $z <_T x$ has height α_γ , then $g(z) < \alpha_{h(\gamma)}$. Since $\text{cf}(\alpha)$ is regular, there is some $\delta < \text{cf}(\alpha)$ such that $h^{-1}(\{\delta\})$ is stationary in $\text{cf}(\alpha)$. Let $X = \{\alpha_\gamma : \gamma \in h^{-1}(\{\delta\})\}$. Then X is cofinal in α and $X \subseteq \{\text{ht}(y) : y <_T x \wedge g(y) < \alpha_\delta\}$.

For each limit ordinal $\alpha < \kappa$ and each $x \in T_\alpha$, we define a set d_x which is a club in α . Let β_x be the least ordinal such that the set $\{\text{ht}(y) : y <_T x \wedge g(y) < \beta_x\}$ is cofinal in α . Note that $\beta_x \leq \alpha$, and if $\text{cf}(\alpha) > \omega$ then $\beta_x < \alpha$.

The process of defining the club d_x involves defining a limit ordinal $\delta_x \leq \alpha$, and sequences

$$\langle \beta(x, i) : i \in \delta_x \cap \text{Succ} \rangle,$$

$$\langle \alpha(x, i) : i < \delta_x \rangle,$$

$$\langle z(x, i) : i \in \delta_x \cap \text{Succ} \rangle$$

which satisfy:

- (1) $\beta(x, j) \leq \beta(x, i) < \beta_x$ for all successor ordinals $j < i < \delta_x$;
- (2) $\langle \alpha(x, i) : i < \delta_x \rangle$ is an increasing and continuous sequence of ordinals cofinal in α ;
- (3) $z(x, i)$ is the unique node with height $\alpha(x, i)$ such that $z(x, i) <_T x$, for all $i \in \delta_x \cap \text{Succ}$;
- (4) $g(z(x, i)) = \beta(x, i)$ for all $i \in \delta_x \cap \text{Succ}$;
- (5) if $j < i < \delta_x$ are successor ordinals and $\beta(x, j) = \beta(x, i)$, then

$$f_{\beta(x, j)}(z(x, j)) < f_{\beta(x, j)}(z(x, i)).$$

After the construction is complete, we let $d_x = \{\alpha(x, i) : i < \delta_x\}$, which is a club subset of α with order type δ_x .

Let i be given and suppose that the objects above are defined as required for all $j < i$. If $\sup_{j < i} \alpha(x, j) = \alpha$, then let $i = \delta_x$ and we are done. Now assume $\sup_{j < i} \alpha(x, j) < \alpha$. If $i = 0$ then let $\alpha(x, i) = 0$, and if i is a limit ordinal, then let $\alpha(x, i) = \sup_{j < i} \alpha(x, j)$. Suppose that i is a successor ordinal.

Consider the set

$$\{y <_T x : \text{ht}(y) > \alpha(x, i - 1)\}.$$

By the choice of β_x , there exists y in this set such that $g(y) < \beta_x$. Let $\beta(x, i)$ be the least ordinal such that there is $y <_T x$ with height greater than $\alpha(x, i - 1)$ and $g(y) = \beta(x, i)$. Then $\beta(x, i) < \beta_x$. We claim that for all successor ordinals $j < i$, $\beta(x, j) \leq \beta(x, i)$. Since $\alpha(x, j - 1) < \alpha(x, i - 1)$, there exists z in the set $\{y <_T x : \text{ht}(y) > \alpha(x, j - 1)\}$ such that $g(z) = \beta(x, i)$. By the minimality of $\beta(x, j)$, $\beta(x, j) \leq \beta(x, i)$.

To define $\alpha(x, i)$, consider the set

$$\{y <_T x : \text{ht}(y) > \alpha(x, i - 1) \wedge g(y) = \beta(x, i)\}.$$

By the choice of $\beta(x, i)$, this set is nonempty. Moreover, since this set is a chain, $f_{\beta(x, i)}$ is injective on it. Let $z(x, i)$ be the unique element in this set with the minimal value under $f_{\beta(x, i)}$. Then let $\alpha(x, i) = \text{ht}(z(x, i))$.

We claim that if $j < i$ is a successor ordinal and $\beta(x, i) = \beta(x, j)$, then

$$f_{\beta(x, j)}(z(x, j)) < f_{\beta(x, j)}(z(x, i)).$$

For since $\alpha(x, j - 1) < \alpha(x, i - 1)$ and $\beta(x, i) = \beta(x, j)$, the node $z(x, i)$ is in the set

$$\{y <_T x : \text{ht}(y) > \alpha(x, j - 1) \wedge g(y) = \beta(x, j)\}.$$

Since $z(x, j)$ has the minimal value in this set under $f_{\beta(x, j)}$, $f_{\beta(x, j)}(z(x, j)) < f_{\beta(x, j)}(z(x, i))$ as desired.

This completes the construction. Let us consider the order type δ_x of d_x for a node $x \in T$. For any ordinal $\beta < \kappa$, let $\theta(\beta)$ denote the order type of the well-order whose underlying set is

$$\bigcup_{\gamma < \beta} \gamma \times \lambda_\gamma$$

and ordered by lexicographical order $<_{lex}$. Note that $\theta(\beta) < \kappa$. For each $x \in T$, (1) and (5) imply that the function

$$i \mapsto \langle \beta(x, i), f_{\beta(x, i)}(z(x, i)) \rangle$$

which maps from $\delta_x \cap \text{Succ}$ into the well-order $(\bigcup_{\gamma < \beta_x} \gamma \times \lambda_\gamma, <_{lex})$ is monotone increasing. Since δ_x is a limit ordinal, δ_x and $\delta_x \cap \text{Succ}$ have the same order type. It follows that $\delta_x \leq \theta(\beta_x)$.

Let C be the club set of limit ordinals $\alpha < \kappa$ greater than ω such that for all $\beta < \alpha$, $\theta(\beta) < \alpha$. If $\alpha \in C$ has uncountable cofinality and $x \in T_\alpha$, then $\beta_x < \alpha$ and so $\theta(\beta_x) < \alpha$. Therefore $\text{ot}(d_x) = \delta_x \leq \theta(\beta_x) < \alpha$.

Now we prove the following statement: for every limit ordinal $\alpha < \kappa$ and for every node x with height α , if $\xi \in \lim(d_x)$, then letting $w <_T x$ have height ξ , $d_x \cap \xi = d_w$. So let such α , x , ξ , and w be given. Recall that β_w is the least ordinal such that the set $\{\text{ht}(y) : y <_T w \wedge g(y) < \beta_w\}$ is cofinal in ξ . Since $d_x \cap \xi$ is cofinal in ξ and for all $\gamma \in d_x$, $g(\gamma) < \beta_x$, clearly $\beta_w \leq \beta_x$.

Let δ'_w be the least ordinal such that $\{\alpha(x, i) : i < \delta'_w\}$ is cofinal in ξ . We will prove by induction that for all $i < \delta'_w$, $\alpha(x, i) = \alpha(w, i)$. It follows immediately that $\delta'_w = \delta_w$ and $d_x \cap \xi = d_w$.

So let $i < \delta'_w$ be given and suppose that for all $j < i$, $\alpha(x, j) = \alpha(w, j)$. If $i = 0$ then $\alpha(x, 0) = 0 = \alpha(w, 0)$, and if i is a limit ordinal then $\alpha(x, i) = \sup_{j < i} \alpha(x, j) = \sup_{j < i} \alpha(w, j) = \alpha(w, i)$. Suppose i is a successor ordinal.

Recall that $\beta(x, i)$ is the least ordinal such that there is $y <_T x$ with $\text{ht}(y) > \alpha(x, i - 1)$ and $g(y) = \beta(x, i)$. And $z(x, i)$ is the element of the set

$$\{y <_T x : \text{ht}(y) > \alpha(x, i - 1) \wedge g(y) = \beta(x, i)\}.$$

with the $f_{\beta(x, i)}$ least value. Let us show that $\beta(x, i) = \beta(w, i)$. We have $g(z(x, i)) = \beta(x, i) < \alpha(x, i) = \text{ht}(z(x, i)) < \xi$ and $z(x, i) <_T w$. So $z(x, i)$ is a witness to the statement that there is $y <_T w$ such that $\text{ht}(y) > \alpha(w, i - 1)$ and $g(y) = \beta(x, i)$. By minimality it follows that $\beta(w, i) \leq \beta(x, i)$. If $\beta(w, i) < \beta(x, i)$, then there is $y <_T w$ with height greater than $\alpha(w, i - 1) = \alpha(x, i - 1)$ such that $g(y) < \beta(x, i)$. But then $y <_T x$ and we have a contradiction to the minimality of $\beta(x, i)$. So $\beta(x, i) = \beta(w, i)$.

Since $\text{ht}(z(x, i)) < \xi$, $z(x, i) <_T w$. So $z(x, i)$ is in the set

$$\{y <_T w : \text{ht}(y) > \alpha(w, i - 1) \wedge g(y) = \beta(w, i)\}.$$

Since $z(w, i)$ is the element of this set with the least $f_{\beta(w, i)}$ value, $f_{\beta(w, i)}(z(w, i)) \leq f_{\beta(w, i)}(z(x, i))$. On the other hand, $z(w, i)$ is in the set

$$\{y <_T x : \text{ht}(y) > \alpha(x, i - 1) \wedge g(y) = \beta(x, i)\},$$

so for the same reason, $f_{\beta(w, i)}(z(x, i)) \leq f_{\beta(w, i)}(z(w, i))$. Therefore $f_{\beta(w, i)}(z(x, i)) = f_{\beta(w, i)}(z(w, i))$. Since $z(x, i)$ and $z(w, i)$ are both below x , they are comparable. But $f_{\beta(w, i)}$ is injective on chains, so $z(x, i) = z(w, i)$. This completes the proof that $d_x \cap \xi = d_w$.

Now we are ready to define a weak square sequence on κ . Recall that C is a club subset of κ satisfying that for all $\alpha \in C$ with uncountable cofinality and all $x \in T_\alpha$, $\text{ot}(d_x) < \alpha$. Define $\langle c_\alpha : \alpha \in C \rangle$ as follows. For α in C with uncountable cofinality, let $c_\alpha = d_x$ for some $x \in T_\alpha$. For α in C with cofinality ω , let c_α be a cofinal subset of α with order type ω .

It remains to show that for every $\xi < \kappa$, $|\{c_\alpha \cap \xi : \alpha \in C\}| < \kappa$. First note that if $\text{cf}(\alpha) = \omega$, then $c_\alpha \cap \xi$ is either equal to c_α if $\alpha \leq \xi$, or is finite otherwise. Hence $|\{c_\alpha \cap \xi : \alpha \in C \cap \text{cof}(\omega)\}| < \kappa$.

For each $\xi < \kappa$, let $\mathcal{D}_\xi = \{c_\alpha \cap \xi : \alpha \in C \cap \text{cof}(>\omega)\}$. We prove by induction on ξ that $|\mathcal{D}_\xi| < \kappa$. The successor case is easy, so assume that ξ is a limit ordinal. The set $\mathcal{D}_\xi$ splits into two sets:

$$\{c_\alpha \cap \xi : \alpha \in C \cap \text{cof}(>\omega), \sup(c_\alpha \cap \xi) < \xi\}$$

$$\{c_\alpha \cap \xi : \alpha \in C \cap \text{cof}(>\omega), \sup(c_\alpha \cap \xi) = \xi\}.$$

The first set is contained in the union $\bigcup_{\xi' < \xi} \mathcal{D}_{\xi'}$, so has size less than κ by the inductive hypothesis. The second set is a subset of $\{d_w : w \in T_\xi\}$, which has size less than κ since $|T_\xi| < \kappa$. $\square$

3. THE FULL CODE OF A C -SEQUENCE

Fix a regular uncountable cardinal κ . A C -sequence on κ is a sequence $\langle c_\alpha : \alpha < \kappa \rangle$ satisfying:

- (1) $c_0 = \emptyset$;
- (2) $c_{\alpha+1} = \{\alpha\}$;
- (3) if α is a limit ordinal then c_α is a club subset of α .

We will review the *full code* ρ_0 of Todorćević [3], defined from a given C -sequence on κ . We propose that ρ_0 and its corresponding tree $T(\rho_0)$ can be developed most naturally in the context of weak square.

Fix a C -sequence $\langle c_\alpha : \alpha < \kappa \rangle$.

Definition 3.1. Let $\alpha \leq \beta < \kappa$.

- (1) the walk from β to α is the unique sequence $\langle \beta_0, \dots, \beta_n \rangle$ satisfying that $\beta_0 = \beta$, $\beta_{k+1} = \min(c_{\beta_k} \setminus \alpha)$ for $k < n$, and $\beta_n = \alpha$;
- (2) $\rho_0(\alpha, \beta) = \langle \text{ot}(c_{\beta_0} \cap \alpha), \dots, \text{ot}(c_{\beta_{n-1}} \cap \alpha) \rangle$.

In (2) we mean $\rho_0(\alpha, \alpha) = \emptyset$ in the case $\alpha = \beta$. Note that the length of $\rho_0(\alpha, \beta)$ is equal to one less than the length of the walk from β to α . If $\langle \beta_0, \dots, \beta_n \rangle$ is the walk from β to α , then obviously for all $i = 0, \dots, n$, $\langle \beta_i, \dots, \beta_n \rangle$ is the walk from β_i to α . That $\langle \beta_0, \dots, \beta_i \rangle$ is the walk from β to β_i follows from the next lemma.

Lemma 3.2. Let $\alpha \leq \gamma \leq \beta$. Let $\langle \beta_0, \dots, \beta_m \rangle$ be the walk from β to γ . Then the following are equivalent:

- (1) the sequence $\langle \beta_0, \dots, \beta_m \rangle$ is an initial segment of the walk from β to α ;
- (2) γ is in the walk from β to α ;
- (3) for all $i = 0, \dots, m-1$, $c_{\beta_i} \cap [\alpha, \gamma) = \emptyset$.

Proof. (1 $\Rightarrow$ 2) is immediate since $\beta_m = \gamma$. For (3 $\Rightarrow$ 1), it is easy to prove by induction on $i \leq m$ that β_i is the i -th element in the walk from β to α ; namely, $\beta_0 = \beta$, and if β_i is as required for a fixed $i < m$, then $\beta_{i+1} = \min(c_{\beta_i} \setminus \gamma) = \min(c_{\beta_i} \setminus \alpha)$, which is the $i+1$ -st element in the walk from β to α . To show (2 $\Rightarrow$ 3), assume (2) holds and (3) fails. Let $i < m$ be least such that $c_{\beta_i} \cap [\alpha, \gamma) \neq \emptyset$. Then by the implication (3 $\Rightarrow$ 1) just shown, $\langle \beta_0, \dots, \beta_i \rangle$ is an initial segment of the walk from β to α , and the next step of this walk is $\min(c_{\beta_i} \setminus \alpha)$, which is less than γ by the choice of i . This contradicts that γ is in the walk from β to α . $\square$

Lemma 3.3. Let $\alpha \leq \gamma \leq \beta$. Then the following are equivalent:

- (1) $\rho_0(\alpha, \beta) = \rho_0(\gamma, \beta) \hat{\ } \rho_0(\alpha, \gamma)$;
- (2) $\rho_0(\gamma, \beta)$ is an initial segment of $\rho_0(\alpha, \beta)$;
- (3) γ is in the walk from β to α .

Proof. (1 $\Rightarrow$ 2) is immediate. For (2 $\Rightarrow$ 3), let $\langle \beta_0, \dots, \beta_n \rangle$ and $\langle \beta'_0, \dots, \beta'_m \rangle$ be the walks from β to α and from β to γ . If γ is not in the walk from β to α , let $0 < k \leq m$ be least such that $\beta_k \neq \beta'_k$. Then $\beta_k = \min(c_{\beta_{k-1}} \setminus \alpha) < \gamma$. So $c_{\beta_{k-1}} \cap \alpha$ is a proper initial segment of $c_{\beta_{k-1}} \cap \gamma$. Therefore $\rho_0(\alpha, \beta)(k-1) = \text{ot}(c_{\beta_{k-1}} \cap \alpha) < \text{ot}(c_{\beta_{k-1}} \cap \gamma) = \rho_0(\gamma, \beta)(k-1)$. So (2) fails.

Now assume (3). Let $\langle \beta_0, \dots, \beta_n \rangle$ be the walk from β to α . By Lemma 3.2, fix $k < n$ such that $\langle \beta_0, \dots, \beta_k \rangle$ is the walk from β to γ . Also by Lemma 3.2,

Lemmas 3.2, 3.3, and 3.4 are due to Todorćević; they are discussed in Lemmas 2.1.6 and 2.1.16 of [4] in the case $\kappa = \omega_1$.

for all $i \leq k-1$, $c_{\beta_i} \cap [\alpha, \gamma)$ is empty, and therefore $\rho_0(\gamma, \beta)(i) = \text{ot}(c_{\beta_i} \cap \gamma) = \text{ot}(c_{\beta_i} \cap \alpha) = \rho_0(\alpha, \beta)(i)$. So $\rho_0(\gamma, \beta) = \rho_0(\alpha, \beta) \upharpoonright k$. By the definition of ρ_0 and the fact that $\langle \beta_k, \dots, \beta_n \rangle$ is the walk from γ to α , for all $i < n-k$ we have $\rho_0(\alpha, \beta)(k+i) = \text{ot}(c_{\beta_{k+i}} \cap \alpha) = \rho_0(\alpha, \gamma)(i)$. Thus $\rho_0(\alpha, \beta) = \rho_0(\gamma, \beta) \hat{\ } \rho_0(\alpha, \gamma)$. $\square$

Define the *right lexicographical order* $<_r$ on ${}^{<\omega}\kappa$ by letting $t <_r s$ if either s is a proper initial segment of t , or there is k such that $s(k) \neq t(k)$, and the least such k satisfies that $t(k) < s(k)$.

Lemma 3.4. *Let $\alpha < \gamma \leq \beta$. Then $\rho_0(\alpha, \beta) <_r \rho_0(\gamma, \beta)$.*

Proof. Let $\langle \beta_0, \dots, \beta_n \rangle$ and $\langle \beta'_0, \dots, \beta'_m \rangle$ be the walks from β to γ and from β to α respectively. If γ is in the walk from β to α , then by Lemma 3.3 $\rho_0(\gamma, \beta)$ is a proper initial segment of $\rho_0(\alpha, \beta)$, so $\rho_0(\alpha, \beta) <_r \rho_0(\gamma, \beta)$. Otherwise let $k > 0$ be least such that $\beta_k \neq \beta'_k$. Since β_{k-1} is in both walks, $\rho_0(\beta_{k-1}, \beta)$ is an initial segment of both $\rho_0(\gamma, \beta)$ and $\rho_0(\alpha, \beta)$. In particular, the least place where $\rho_0(\gamma, \beta)$ and $\rho_0(\alpha, \beta)$ can differ is at $k-1$. Since $\beta'_k \in c_{\beta_{k-1}} \cap [\alpha, \gamma)$, we have that $c_{\beta_{k-1}} \cap \alpha$ is a proper initial segment of $c_{\beta_{k-1}} \cap \gamma$. Therefore $\rho_0(\alpha, \beta)(k-1) = \text{ot}(c_{\beta_{k-1}} \cap \alpha) < \text{ot}(c_{\beta_{k-1}} \cap \gamma) = \rho_0(\gamma, \beta)(k-1)$. Hence $\rho_0(\alpha, \beta) <_r \rho_0(\gamma, \beta)$. $\square$

In order to construct a special Aronszajn tree from a weak square sequence, we will need to analyze the following situation: suppose $\alpha \leq \beta, \gamma$, where α is a limit ordinal, and for all $\xi < \alpha$, $\rho_0(\xi, \beta) = \rho_0(\xi, \gamma)$. What can be said about the relationship between $\rho_0(\alpha, \beta)$ and $\rho_0(\alpha, \gamma)$? This relationship is described precisely in Proposition 3.6 below. If $\kappa = \omega_1$ then in the situation just described, $\rho_0(\alpha, \beta) = \rho_0(\alpha, \gamma)$. But this is not true in general. For example, it is not true if $\alpha = \beta$ is a limit ordinal, $\alpha < \gamma$, and $c_\alpha = c_\gamma \cap \alpha$.

We make some additional observations about ρ_0 in preparation for Proposition 3.6. Let $\langle \beta_0, \dots, \beta_n \rangle$ be the walk from β to α , where $\alpha \leq \beta$. Then for all $i = 0, \dots, n-2$, $\sup(c_{\beta_i} \cap \alpha) < \alpha$. Namely, if $\sup(c_{\beta_i} \cap \alpha) = \alpha$, then $\alpha \in c_{\beta_i}$, and hence $\alpha = \min(c_{\beta_i} \setminus \alpha)$. This is only possible if $i = n-1$.

Lemma 3.5. *Let $\langle \beta_0, \dots, \beta_n \rangle$ be the walk from β to α , where α is a limit ordinal and $\alpha \leq \beta$. Assume that $\xi < \alpha$ is larger than $\sup(c_{\beta_i} \cap \alpha)$ for all $i = 0, \dots, n-2$. Then $\langle \beta_0, \dots, \beta_{n-1} \rangle$ is an initial segment of the walk from β to ξ , namely, the part of the walk consisting of ordinals above α .*

Proof. The proof is by induction on $k < n$. Assume $\langle \beta_0, \dots, \beta_k \rangle$ is an initial segment of the walk from β to ξ , where $k < n-1$. By assumption, $\sup(c_{\beta_k} \cap \alpha) < \xi$, and hence $\beta_{k+1} = \min(c_{\beta_k} \setminus \alpha) = \min(c_{\beta_k} \setminus \xi)$, which is the next step of the walk from β to ξ . Finally, $\alpha = \min(c_{\beta_{n-1}} \setminus \alpha) \geq \min(c_{\beta_{n-1}} \setminus \xi)$, and $\min(c_{\beta_{n-1}} \setminus \xi)$ is the next step of walk from β to ξ after β_{n-1} . $\square$

Let $\langle \beta_0, \dots, \beta_n \rangle$ be the walk from β to α , where α is a limit ordinal and $\alpha \leq \beta$. Suppose $\sup(c_{\beta_{n-1}} \cap \alpha) < \alpha$. Let $\xi < \alpha$ be larger than $\sup(c_{\beta_i} \cap \alpha)$ for all $i = 0, \dots, n-1$. Then for $i = 0, \dots, n-1$, $\sup(c_{\beta_i} \cap \alpha) < \xi$ implies $c_{\beta_i} \cap [\xi, \alpha) = \emptyset$. By Lemma 3.2, α is in the walk from β to ξ . Therefore $\rho_0(\alpha, \beta)$ is an initial segment of $\rho_0(\xi, \beta)$.

On the other hand, suppose $\sup(c_{\beta_{n-1}} \cap \alpha) = \alpha$. Let $\xi < \alpha$ be larger than $\sup(c_{\beta_i} \cap \alpha)$ for all $i = 0, \dots, n-2$. By Lemma 3.5, $\langle \beta_0, \dots, \beta_{n-1} \rangle$ is an initial segment of the walk from β to ξ . But since $c_{\beta_{n-1}} \cap [\xi, \alpha)$ is nonempty, Lemma 3.2 implies that α is not in the walk from β to ξ . The next step of the walk from β to ξ after β_{n-1} is $\min(c_{\beta_{n-1}} \setminus \xi)$, which is less than α .

Proposition 3.6. *Let $\alpha < \beta, \gamma$ be given, where α is a limit ordinal. Suppose that for all $\xi < \alpha$, $\rho_0(\xi, \beta) = \rho_0(\xi, \gamma)$. Let $\langle \beta_0, \dots, \beta_n \rangle$ be the walk from β to α and let $\langle \gamma_0, \dots, \gamma_m \rangle$ be the walk from γ to α . Let $\alpha_0 = \sup(c_{\beta_{n-1}} \cap \alpha)$ and $\alpha_1 = \sup(c_{\gamma_{m-1}} \cap \alpha)$.*

- (1) *If $\alpha_0 < \alpha$ and $\alpha_1 < \alpha$, then $\rho_0(\alpha, \beta) = \rho_0(\alpha, \gamma)$;*
- (2) *if $\alpha_0 = \alpha_1 = \alpha$, then $\rho_0(\alpha, \beta) = \rho_0(\alpha, \gamma)$;*
- (3) *if $\alpha_0 < \alpha$ and $\alpha_1 = \alpha$, then $\rho_0(\alpha, \gamma) = \rho_0(\alpha, \beta) \hat{\sim} \text{ot}(c_\alpha)$.*

Proof. Note that for all $\xi < \alpha$, $\rho_0(\xi, \beta) = \rho_0(\xi, \gamma)$ implies that the walks from β to ξ and from γ to ξ have the same length.

Suppose $\alpha_0 < \alpha$ and $\alpha_1 < \alpha$. Then for all large enough $\xi < \alpha$, α is in walk from β to ξ and in the walk from γ to ξ . So for all large enough $\xi < \alpha$,

$$\rho_0(\xi, \beta) = \rho_0(\alpha, \beta) \hat{\sim} \rho_0(\xi, \alpha) \text{ and } \rho_0(\xi, \gamma) = \rho_0(\alpha, \gamma) \hat{\sim} \rho_0(\xi, \alpha).$$

Since $\rho_0(\xi, \beta) = \rho_0(\xi, \gamma)$, equating the sequences above and removing the common tails yields $\rho_0(\alpha, \beta) = \rho_0(\alpha, \gamma)$.

Now suppose $\alpha_0 = \alpha_1 = \alpha$. First we show that $n = m$. For all large enough $\xi < \alpha$, $\langle \beta_0, \dots, \beta_{n-1} \rangle$ is an initial segment of the walk from β to ξ , and $\langle \gamma_0, \dots, \gamma_{m-1} \rangle$ is an initial segment of the walk from γ to ξ . Consider a large enough ordinal $\xi \in c_{\beta_{n-1}} \cap \alpha$. Then the walk from β to ξ equals $\langle \beta_0, \dots, \beta_{n-1}, \xi \rangle$, which has length $n + 1$. Since $\rho_0(\xi, \beta) = \rho_0(\xi, \gamma)$, the walk from γ to ξ has length $n + 1$ also. So the walk from γ to γ_{m-1} , namely $\langle \gamma_0, \dots, \gamma_{m-1} \rangle$, has length less than $n + 1$. Hence $m \leq n$. A symmetric argument shows that $n \leq m$.

For all large enough ξ , β_{n-1} is in the walk from β to ξ , and hence $\rho_0(\beta_{n-1}, \beta) \sqsubset \rho_0(\xi, \beta)$ by Lemma 3.3. Similarly for all large enough ξ , $\rho_0(\gamma_{n-1}, \gamma) \sqsubset \rho_0(\xi, \gamma)$. Since $\rho_0(\xi, \beta) = \rho_0(\xi, \gamma)$ and $\rho_0(\beta_{n-1}, \beta)$ and $\rho_0(\gamma_{n-1}, \gamma)$ have the same length, $\rho_0(\beta_{n-1}, \beta) = \rho_0(\gamma_{n-1}, \gamma)$. Since $\rho_0(\alpha, \beta) = \rho_0(\beta_{n-1}, \beta) \hat{\sim} \text{ot}(c_{\beta_{n-1}} \cap \alpha)$ and $\rho_0(\alpha, \gamma) = \rho_0(\gamma_{n-1}, \gamma) \hat{\sim} \text{ot}(c_{\gamma_{n-1}} \cap \alpha)$, it suffices to show that $\text{ot}(c_{\beta_{n-1}} \cap \alpha) = \text{ot}(c_{\gamma_{n-1}} \cap \alpha)$.

Since α is a limit ordinal, it is enough to show that for all large enough $\xi < \alpha$, $\text{ot}(c_{\beta_{n-1}} \cap \xi) = \text{ot}(c_{\gamma_{n-1}} \cap \xi)$. But for all large enough ξ , $\rho_0(\xi, \beta)(n-1) = \text{ot}(c_{\beta_{n-1}} \cap \xi)$ and $\rho_0(\xi, \gamma)(n-1) = \text{ot}(c_{\gamma_{n-1}} \cap \xi)$. Since $\rho_0(\xi, \beta) = \rho_0(\xi, \gamma)$, $\text{ot}(c_{\beta_{n-1}} \cap \xi) = \text{ot}(c_{\gamma_{n-1}} \cap \xi)$.

Finally, suppose that $\alpha_0 < \alpha$ and $\alpha_1 = \alpha$. First we prove that $m = n + 1$. If we take a large enough $\xi \in c_\alpha$, then the walk from β to ξ is equal to $\langle \beta_0, \dots, \beta_n, \xi \rangle$, and $\langle \gamma_0, \dots, \gamma_{m-1} \rangle$ is a proper initial segment of the walk from γ to ξ . Since $\rho_0(\xi, \beta) = \rho_0(\xi, \gamma)$, the walks from β to ξ and from γ to ξ have the same length, namely $n + 2$. Therefore the walk $\langle \gamma_0, \dots, \gamma_{m-1} \rangle$ has length at most $n + 1$, that is, $m \leq n + 1$.

On the other hand, choosing a large enough ξ in $c_{\gamma_{m-1}} \cap \alpha$, $\langle \gamma_0, \dots, \gamma_{m-1}, \xi \rangle$ is the walk from γ to ξ , and α is in the walk from β to ξ . So the walk from γ to ξ has length $m + 1$. Since $\rho_0(\xi, \beta) = \rho_0(\xi, \gamma)$, the walk from β to ξ has length $m + 1$. But the sequence $\langle \beta_0, \dots, \beta_n \rangle$ is a proper initial segment of the walk from β to ξ , so the length of this sequence is less than $m + 1$, that is, $n + 1 \leq m$. So $m = n + 1$.

Now we show that $\rho_0(\alpha, \beta) = \rho_0(\gamma_{m-1}, \gamma)$. Since $m = n + 1$, the walks from β to α and from γ to γ_{m-1} have the same length, so $\rho_0(\alpha, \beta)$ and $\rho_0(\gamma_{m-1}, \gamma)$ have the same length. To show they are equal, it suffices to show they are initial segments of the same sequence. Choose a large enough ξ so that α is in the walk from β to ξ and γ_{m-1} is in the walk from γ to ξ . Then $\rho_0(\alpha, \beta) \sqsubset \rho_0(\xi, \beta)$ and $\rho_0(\gamma_{m-1}, \gamma) \sqsubset \rho_0(\xi, \gamma)$ by Lemma 3.3. Since $\rho_0(\xi, \beta) = \rho_0(\xi, \gamma)$, $\rho_0(\alpha, \beta) = \rho_0(\gamma_{m-1}, \gamma)$.

Now $\rho_0(\alpha, \gamma) = \rho_0(\gamma_{m-1}, \gamma) \hat{\ } \text{ot}(c_{\gamma_{m-1}} \cap \alpha) = \rho_0(\alpha, \beta) \hat{\ } \text{ot}(c_{\gamma_{m-1}} \cap \alpha)$. So to complete the proof, it suffices to show that $\text{ot}(c_{\gamma_{m-1}} \cap \alpha) = \text{ot}(c_\alpha)$. Since α is a limit ordinal, it suffices to show that for all large enough $\xi < \alpha$, $\text{ot}(c_{\gamma_{m-1}} \cap \xi) = \text{ot}(c_\alpha \cap \xi)$. Choose ξ large enough so that α is in the walk from β to ξ and γ_{m-1} is in the walk from γ to ξ . Then $\rho_0(\xi, \beta)(n) = \text{ot}(c_\alpha \cap \xi)$ and $\rho_0(\xi, \gamma)(m-1) = \text{ot}(c_{\gamma_{m-1}} \cap \xi)$. Since $\rho_0(\xi, \beta) = \rho_0(\xi, \gamma)$ and $n = m-1$, we are done. $\square$

4. WEAK SQUARE IMPLIES A SPECIAL ARONSZAJN TREE

We prove now that the existence of a weak square sequence on a regular uncountable cardinal κ implies the existence of a special Aronszajn tree on κ . Fix a C -sequence $\langle c_\alpha : \alpha < \kappa \rangle$, and let ρ_0 be the full code. For each $\beta < \kappa$, define $\rho_{0\beta} : \beta \rightarrow {}^{<\omega}\beta$ by letting $\rho_{0\beta}(\xi) = \rho_0(\xi, \beta)$ for $\xi < \beta$. Recall the tree $T(\rho_0)$ of Todorćević [3]: for each $\alpha < \kappa$, level α of $T(\rho_0)$ consists of functions of the form $\rho_{0\beta} \upharpoonright \alpha$, where $\alpha \leq \beta < \kappa$. For $u, v \in T(\rho_0)$, $u <_{T(\rho_0)} v$ if $v \upharpoonright \text{dom}(u) = u$.

Our goal is to prove that under some additional assumptions on the C -sequence, the tree $T(\rho_0)$ is a special Aronszajn tree. The existence of a C -sequence satisfying these assumptions follows from the existence of a weak square sequence. Our proof is based on the proof of Todorćević [4] that there exists a special Aronszajn tree on κ , for any non-Mahlo strongly inaccessible cardinal κ .¹

It is clear that $T(\rho_0)$ is a tree of height κ . The next lemma will imply that if $|\{c_\beta \cap \xi : \beta < \kappa\}| < \kappa$ for every $\xi < \kappa$, then $T(\rho_0)$ is a κ -tree. The proof is based on the argument in [1] that $\square_\mu^*$ implies the existence of a special Aronszajn tree on μ^+ for any infinite cardinal μ .

Lemma 4.1. *Let $\alpha < \kappa$ be a limit ordinal, and let $\alpha \leq \beta, \gamma$. Let $\langle \beta_0, \dots, \beta_n \rangle$ be the walk from β to α and let $\langle \gamma_0, \dots, \gamma_m \rangle$ be the walk from γ to α . Suppose that the sequences $\langle c_{\beta_0} \cap \alpha, \dots, c_{\beta_n} \cap \alpha \rangle$ and $\langle c_{\gamma_0} \cap \alpha, \dots, c_{\gamma_m} \cap \alpha \rangle$ are equal. Then $\rho_{0\beta} \upharpoonright \alpha = \rho_{0\gamma} \upharpoonright \alpha$.*

Proof. Note that $n = m$. Let $\xi < \alpha$ be given. Let $i \leq n$ be least such that $c_{\beta_i} \cap [\xi, \alpha)$ is nonempty. By Lemma 3.2, β_i is in the walk from β to ξ . The next step of the walk from β to ξ after β_i is $\beta^* = \min(c_{\beta_i} \setminus \xi) < \alpha$. Due to the agreement described in the assumptions, i is also least such that $c_{\gamma_i} \cap [\xi, \alpha)$ is nonempty, γ_i is in the walk from γ to ξ , and $\gamma^* = \min(c_{\gamma_i} \setminus \xi) = \beta^*$ is the next step of the walk from γ to ξ after γ_i . By the agreement we have $\rho_0(\xi, \beta) = \langle \text{ot}(c_{\beta_0} \cap \xi), \dots, \text{ot}(c_{\beta_i} \cap \xi) \rangle \hat{\ } \rho_0(\xi, \beta^*) = \langle \text{ot}(c_{\gamma_0} \cap \xi), \dots, \text{ot}(c_{\gamma_i} \cap \xi) \rangle \hat{\ } \rho_0(\xi, \gamma^*) = \rho_0(\xi, \gamma)$. $\square$

Proposition 4.2. *Suppose the C -sequence $\langle c_\alpha : \alpha < \kappa \rangle$ satisfies that for every $\xi < \kappa$, $|\{c_\beta \cap \xi : \beta < \kappa\}| < \kappa$. Then $T(\rho_0)$ is a κ -tree.*

Proof. Let $\xi < \kappa$ be given and we show that level ξ of the tree $T(\rho_0)$ has size less than κ . Note that it suffices to prove this statement for limit ordinals ξ . For in general, level γ of the tree is equal to $\{\rho_{0\gamma+n} \upharpoonright \gamma : n < \omega\} \cup \{t \upharpoonright \gamma : t \in T(\rho_0)_{\gamma+\omega}\}$.

So let ξ be a limit ordinal. By the previous lemma, for all $\beta \geq \xi$, the function $\rho_{0\beta} \upharpoonright \xi$ is determined from the finite sequence $\langle c_{\beta_0} \cap \xi, \dots, c_{\beta_n} \cap \xi \rangle$, where $\langle \beta_0, \dots, \beta_n \rangle$ is the walk from β to ξ . By assumption, there are fewer than κ many possibilities

¹In that proof it is claimed that for a limit ordinal α and $\alpha \leq \beta, \gamma$, if $\rho_0(\xi, \beta) = \rho_0(\xi, \gamma)$ for all $\xi < \alpha$, then $\rho_0(\alpha, \beta) = \rho_0(\alpha, \gamma)$. This claim appears to be incorrect even with the C -sequence used there. We replace this claim with Proposition 3.6.

for such a sequence. So there are fewer than κ many functions of the form $\rho_{0\beta} \upharpoonright \xi$ for $\beta < \kappa$. $\square$

Assume that there exists a weak square sequence on κ . Then by Lemma 1.2, we can fix a C -sequence $\langle c_\alpha : \alpha < \kappa \rangle$ satisfying the following properties:

- (1) there exists a club $C \subseteq \kappa \cap \text{Lim}$ such that for all α in C , $\text{ot}(c_\alpha) < \min(c_\alpha)$;
- (2) for all $\alpha \in (\kappa \cap \text{Lim}) \setminus C$, $\min(c_\alpha) > \sup(C \cap \alpha)$;
- (3) for every $\xi < \kappa$, $|\{c_\alpha \cap \xi : \alpha < \kappa\}| < \kappa$.

Let ρ_0 be the full code defined from this C -sequence. We will prove that $T(\rho_0)$ is a special Aronszajn tree.

Let $\langle \alpha_0, \dots, \alpha_n \rangle \mapsto \ulcorner \langle \alpha_0, \dots, \alpha_n \rangle \urcorner$ be some coding of finite sequences of ordinals in κ by ordinals in κ . Let D be the club set of ordinals $\alpha \in C$ which are closed under this mapping.

Lemma 4.3. *For all $\alpha \in C$ and $\beta \geq \alpha$, $\text{ot}(c_\beta \cap \alpha) < \alpha$. Hence for all $\alpha \in D$ and $\gamma \geq \alpha$, $\ulcorner \rho_0(\alpha, \gamma) \urcorner < \alpha$.*

Proof. Fix $\alpha \in C$ and $\beta \geq \alpha$. If β is a successor ordinal then $c_\beta \cap \alpha = \emptyset$. Suppose β is a limit ordinal. If β is not in C , then $\alpha \leq \sup(C \cap \beta) < \min(c_\beta)$. Therefore $c_\beta \cap \alpha = \emptyset$. Now suppose that β is in C . If $c_\beta \cap \alpha = \emptyset$ then we are done. Otherwise $\text{ot}(c_\beta \cap \alpha) \leq \text{ot}(c_\beta) < \min(c_\beta) < \alpha$. $\square$

Theorem 4.4. *The tree $T(\rho_0)$ is a special Aronszajn tree.*

Proof. Let $U = \{t \in T(\rho_0) : \text{ht}(t) \in D\}$. We will define a function $g : U \rightarrow \kappa$ satisfying:

- (1) $g(t) < \text{ht}(t)$ for all $t \in U$;
- (2) $t \sqsubset u$ in U implies $g(t) \neq g(u)$.

Let us note that the existence of such a function g implies that $T(\rho_0)$ is special. For in that case, define $h : T(\rho_0) \rightarrow \kappa$ as follows. For $t \in U$, let $h(t) = g(t)$. For $t \in T(\rho_0) \setminus U$, let $h(t) = \sup(D \cap \text{ht}(t))$. Then $h(t) < \text{ht}(t)$ for all non-minimal t . Consider $\nu < \kappa$, and we show that $h^{-1}(\{\nu\})$ is the union of fewer than κ many antichains. If $h(t) = \nu$ and $t \notin U$, then $\nu < \text{ht}(t) < \min(D \setminus \nu + 1)$. There are fewer than κ many such nodes t . Enumerate them as $\{t_i : i < \lambda\}$ where $\lambda < \kappa$. Define $f_\nu : h^{-1}(\{\nu\}) \rightarrow \lambda + 1$ by letting $f_\nu(t_i) = i$ for $i < \lambda$ and $f_\nu(t) = \lambda$ if $h(t) = \nu$ and $t \in U$. If $f_\nu(t) = f_\nu(u)$ then clearly $t, u \in U$. Hence $h(t) = g(t) = \nu$ and $h(u) = g(u) = \nu$, so $t \sqsubset u$ is not possible by the properties of g .

Now we define the function $g : U \rightarrow \kappa$. Consider $t \in T(\rho_0)$ with $\text{ht}(t) \in D$. Let $\alpha = \text{ht}(t)$. Define $A(t, 0)$ as the set of $\beta \geq \alpha$ with $\rho_{0\beta} \upharpoonright \alpha = t$ such that, letting $\langle \beta_0, \dots, \beta_n \rangle$ be the walk from β to α , $\sup(c_{\beta_{n-1}} \cap \alpha) < \alpha$. Define $A(t, 1)$ as the set of $\gamma \geq \alpha$ with $\rho_{0\gamma} \upharpoonright \alpha = t$ such that, letting $\langle \gamma_0, \dots, \gamma_m \rangle$ be the walk from γ to α , $\sup(c_{\gamma_{m-1}} \cap \alpha) = \alpha$. By Proposition 3.6 we have:

- (1) for all $\beta, \beta' \in A(t, 0)$, $\rho_0(\alpha, \beta) = \rho_0(\alpha, \beta')$;
- (2) for all $\gamma, \gamma' \in A(t, 1)$, $\rho_0(\alpha, \gamma) = \rho_0(\alpha, \gamma')$;
- (3) for all $\beta \in A(t, 0)$ and $\gamma \in A(t, 1)$, $\rho_0(\alpha, \gamma) = \rho_0(\alpha, \beta) \hat{\ } \text{ot}(c_\alpha)$.

The definition of $g(t)$ splits into cases. First assume that one of $A(t, 0)$ or $A(t, 1)$ is empty. Fix any $\gamma \geq \alpha$ with $t = \rho_{0\gamma} \upharpoonright \alpha$, and let

$$g(t) = \ulcorner \ulcorner \rho_0(\alpha, \gamma) \urcorner, 0 \urcorner.$$

Note that by (1) and (2) and the case assumption, the definition of $g(t)$ is independent of γ . Secondly, assume that $A(t, 0)$ and $A(t, 1)$ are both nonempty. Fix any $\gamma \in A(t, 1)$, and define

$$g(t) = \ulcorner \langle \ulcorner \rho_0(\alpha, \gamma) \urcorner, 1 \rangle \urcorner.$$

By (2), the definition of $g(t)$ is independent of γ . Note that $g(t) < \text{ht}(t)$ by Lemma 4.3.

To complete the proof, we prove that if $t, u \in U$, then $t \sqsubset u$ implies $g(t) \neq g(u)$. So let $t \sqsubset u$ be given, and let $\alpha = \text{ht}(t)$ and $\delta = \text{ht}(u)$. So $\alpha < \delta$. Assume for a contradiction that $g(t) = g(u)$. Note that $g(t)$ and $g(u)$ are defined by the same case, since the case is coded by a 0 or 1 in the definition of g .

First suppose $g(t)$ and $g(u)$ are defined as in the first case. Fix $\gamma \geq \delta$ such that $u = \rho_{0\gamma} \upharpoonright \delta$. Since $t \sqsubset u$, $t = \rho_{0\gamma} \upharpoonright \alpha$. So

$$\ulcorner \langle \ulcorner \rho_0(\alpha, \gamma) \urcorner, 0 \rangle \urcorner = g(t) = g(u) = \ulcorner \langle \ulcorner \rho_0(\delta, \gamma) \urcorner, 0 \rangle \urcorner.$$

Therefore $\rho_0(\alpha, \gamma) = \rho_0(\delta, \gamma)$. But by Lemma 3.4, $\alpha < \delta$ implies that $\rho_0(\alpha, \gamma) <_r \rho_0(\delta, \gamma)$, and in particular these sequences are different. So we have a contradiction.

Now suppose $g(t)$ and $g(u)$ are defined as in the second case. Fix $\gamma \in A(u, 1)$. Then $u = \rho_{0\gamma} \upharpoonright \delta$ and

$$g(u) = \ulcorner \langle \ulcorner \rho_0(\delta, \gamma) \urcorner, 1 \rangle \urcorner.$$

Since $t \sqsubset u$, $t = \rho_{0\gamma} \upharpoonright \alpha$. Now there are two cases, depending on whether γ is in $A(t, 0)$ or $A(t, 1)$. If $\gamma \in A(t, 1)$, then

$$g(t) = \ulcorner \langle \ulcorner \rho_0(\alpha, \gamma) \urcorner, 1 \rangle \urcorner.$$

But $g(t) = g(u)$ implies $\rho_0(\alpha, \gamma) = \rho_0(\delta, \gamma)$. This contradicts Lemma 3.4.

If $\gamma \in A(t, 0)$, then fix some $\gamma' \in A(t, 1)$. Then

$$g(t) = \ulcorner \langle \ulcorner \rho_0(\alpha, \gamma') \urcorner, 1 \rangle \urcorner.$$

Since $g(t) = g(u)$, we have that $\rho_0(\alpha, \gamma') = \rho_0(\delta, \gamma)$. But by Proposition 3.6(3),

$$\rho_0(\delta, \gamma) = \rho_0(\alpha, \gamma') = \rho_0(\alpha, \gamma) \hat{\smallfrown} \text{ot}(c_\alpha).$$

So $\rho_0(\alpha, \gamma)$ is a proper initial segment of $\rho_0(\delta, \gamma)$, which implies $\rho_0(\delta, \gamma) <_r \rho_0(\alpha, \gamma)$. But by Lemma 3.4, $\alpha < \delta$ implies $\rho_0(\alpha, \gamma) <_r \rho_0(\delta, \gamma)$, and we have a contradiction. $\square$

Remark: If κ is a strongly inaccessible non-Mahlo cardinal, then there exists a weak square sequence on κ . Namely, let C be a club set of singular cardinals, and for each $\alpha \in C$, choose c_α as a club subset of α with order type $\text{cf}(\alpha)$. Then for every $\xi < \kappa$, $|\{c_\alpha \cap \xi : \alpha < \kappa\}| \leq 2^{|\xi|} < \kappa$. We pose the following question: is it consistent that there is a weakly inaccessible non-Mahlo cardinal which does not carry a weak square sequence?

REFERENCES

- [1] J. Cummings. Notes on singular cardinal combinatorics. *Notre Dame J. Formal Logic*, 46(3):251–282, 2005.
- [2] R. Jensen. The fine structure of the constructible hierarchy. *Ann. Math. Logic*, 4:229–308, 1972.
- [3] S. Todorćević. Partitioning pairs of countable ordinals. *Acta Math.*, 159(3-4):261–294, 1987.
- [4] S. Todorćević. *Walks on ordinals and their characteristics*, volume 263 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2007.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH TEXAS, 1155 UNION CIRCLE #311430,
DENTON, TX 76203

E-mail address: `jkrueger@unt.edu`