DESTROYING STATIONARY SETS

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ABSTRACT. We present a forcing poset for destroying the stationarity of certain subsets of $P_\kappa\kappa^+$. Using this poset along with Prikry forcing techniques we establish some consistency results concerning saturated ideals and $S(\kappa,\kappa^+)$.

In this paper we present a forcing poset for destroying stationary subsets of $P_\kappa\kappa^+$ where κ is strongly inaccessible. We use this forcing poset, along with Prikry forcing techniques, to prove a variety of consistency results concerning saturated ideals and the set $S(\kappa,\kappa^+) = \{a \in P_\kappa\kappa^+ \colon \text{o.t.}(a) = (a \cap \kappa)^+\}$.

Previously Gitik [6] introduced a forcing poset for destroying stationary subsets of $P_\kappa\kappa^+$. He used this poset to construct a model with a non-splitting stationary subset of $P_\kappa\kappa^+$, that is, a stationary set S such that $NS \restriction S$ is κ^+-saturated. However, since Gitik's poset is designed only to destroy stationary sets contained in a fixed inner model of the universe, its applications are limited. Our forcing poset is a modification of Gitik's poset which can destroy stationary sets not necessarily contained in the inner model.

The two main consistency results of the paper can be summarized as follows. Assume the existence of a supercompact cardinal. Then the following two statements are consistent:

(1) The set $S(\kappa,\kappa^+)$ is stationary and for almost all a in $S(\kappa,\kappa^+)$, $\text{cf}(a\cap\kappa) = \omega$.

(2) The ideal $NS \restriction S(\kappa,\kappa^+)$ is κ^+-saturated.

We also prove that (2) is optimal in the sense that whenever $NS \restriction S$ is κ^+-saturated for some set $S \subseteq P_\kappa\kappa^+$, then $S \subseteq S(\kappa,\kappa^+)$ modulo clubs.

The contents of the paper are as follows. Section 1 outlines notation and preliminaries. Section 2 describes some of the forcing posets we use in the paper. Section 3 introduces the set $S(\kappa,\kappa^+)$. Section 4 presents the forcing poset for destroying stationary sets. Section 5 describes how to iterate such posets over different cardinals using Easton support Prikry iterations. Section 6 constructs a model using a Prikry iteration in which almost all a in $S(\kappa,\kappa^+)$ have $\text{cf}(a\cap\kappa) = \omega$. Section 7 constructs a model in which $NS \restriction S(\kappa,\kappa^+)$ is κ^+-saturated.

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1. Preliminaries

We assume that the reader is familiar with iterated forcing, supercompact cardinals, and Prikry forcing; see [2] and [8].

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If κ is regular and $\kappa \subseteq X$, we define $P_\kappa X = \{a \subseteq X\colon |a| < \kappa,\ a \cap \kappa \in \kappa\}$. A subset of $P_\kappa X$ is **club** if it is closed under unions of increasing sequences of length less than κ and is cofinal in $P_\kappa X$. A set is **stationary** if it intersects every club.

If C is a club subset of $P_\kappa X$ then there is a function $f\colon X^{<\omega} \to X$ such that whenever a is in $P_\kappa X$ and is closed under f then a is in C. If A is a directed subset of C with size less than κ then $\bigcup A$ is in C.

When we say that a statement is true **for almost all** a **in** $P_\kappa X$, we mean there is a club set whose elements each satisfy the statement. We write $A = B$ modulo clubs if there is a club set C such that $A \cap C = B \cap C$. A function $f\colon P_\kappa X \to X$ is **regressive** if $f(a)$ is in a for all a.

The ideal of non-stationary subsets of $P_\kappa\lambda$ for $\lambda \geq \kappa$ is denoted by $NS_{\kappa,\lambda}$ or NS. If S is stationary then $NS_{\kappa,\lambda} \restriction S$ denotes the ideal generated by the elements of $NS_{\kappa,\lambda}$ along with the complement of S. An ideal on $P_\kappa\lambda$ is **fine** if it contains the set $\{a \in P_\kappa\lambda\colon \xi \notin a\}$ for every $\xi < \lambda$. An ideal I is **normal** if for every set $S \subseteq P_\kappa\lambda$ not in I and for every regressive function $f\colon S \to \lambda$, there is an $i < \lambda$ such that the set $\{a \in S\colon f(a) = i\}$ is not in I.

By an **ultrafilter** on $P_\kappa\lambda$, we always mean a non-principal, fine ultrafilter.

If I is an ideal on $P_\kappa\lambda$ the set $I^* = \{P_\kappa\lambda \setminus A\colon A \in I\}$ is the **dual filter of** I. The collection of I**-positive sets** $I^+ = \{A \subseteq P_\kappa\lambda\colon A \notin I\}$ is a forcing poset ordered by $A \leq B$ if $A \setminus B$ is in I.

An ideal I is μ**-saturated** if I^+ is μ-c.c. If $I = NS \restriction S$ for some S then I is μ-saturated iff there is no family $\{S_i\colon i < \mu\}$ of stationary subsets of S such that $S_i \cap S_j$ is non-stationary for $i < j$. If κ is weakly inaccessible, a set $S \subseteq P_\kappa\kappa^+$ is **non-splitting** if $NS \restriction S$ is κ^+-saturated. A set $S \subseteq P_\kappa\kappa^+$ is non-splitting iff there does not exist a partition of S into κ^+-many disjoint stationary subsets.

If I is an ideal on $P_\kappa\lambda$, the poset I^+ adds a set U which is an ultrafilter on the Boolean algebra $\mathcal{P}(P_\kappa\lambda)^V$. If M is the ultrapower of V by U in the generic extension by I^+, M is called the **generic ultrapower** and the ultrapower map $j\colon V \to M$ is the **generic embedding**. If I is λ^+-saturated then the generic ultrapower M is well-founded and ${}^\lambda M \subseteq M$.

A cardinal κ is λ**-supercompact** if there is a normal ultrafilter on $P_\kappa\lambda$, or equivalently, there is an elementary embedding $j\colon V \to M$, where M is a transitive inner model, such that crit $j = \kappa$, $j(\kappa) > \lambda$, and ${}^\lambda M \subseteq M$. If U_0 and U_1 are normal fine ultrafilters on $P_\kappa\lambda$ we write $U_0 \lhd U_1$ if U_0 is in the ultrapower of V by U_1. This ordering is called the **Mitchell ordering**, and is well-founded. A cardinal κ is λ-strongly compact if there is a fine ultrafilter on $P_\kappa\lambda$.

The **Magidor ordering** on $P_\kappa\lambda$ is defined by letting $a \underset{\sim}{\subseteq} b$ if $a \subseteq b$ and $|a| < b \cap \kappa$.

We say that $\square_\kappa$ holds if there is a sequence $\langle c_\alpha\colon \alpha < \kappa^+, \alpha \text{ limit}\rangle$ such that for all limit ordinals $\alpha < \kappa^+$:

(1) c_α club in α,

(2) if $\operatorname{cf}(\alpha) < \kappa$ then o.t.$(c_\alpha) < \kappa$,

(3) for all β in $\lim(c_\alpha)$, $c_\alpha \cap \beta = c_\beta$.

Similarly, we say that $\square_{\kappa,<\delta}$ holds if there is a sequence $\langle \mathcal{C}_\alpha\colon \alpha < \kappa^+, \alpha \text{ limit}\rangle$ such that for all limit ordinals $\alpha < \kappa^+$,

(1) $\mathcal{C}_\alpha$ is a non-empty collection of less than δ many club subsets of α,

(2) if $\operatorname{cf}(\alpha) < \kappa$ then every c in $\mathcal{C}_\alpha$ has order type less than κ,

(3) if c is in $\mathcal{C}_\alpha$ then for all β in $\lim(c)$, $c \cap \beta$ is in $\mathcal{C}_\beta$.

If a is a set of ordinals then o.t.(a) is the order type of a, and if o.t.(a) is a limit ordinal then cf(a) is cf(o.t.(a)). Note that $\mathrm{cf}(a) = \mathrm{cf}(\sup a)$.

The expression $\theta \gg \kappa$ means that θ is larger than $2^{2^{|H(\kappa)|}}$. Suppose that N is in $P_\kappa H(\theta)$ and N is an elementary substructure of $\langle H(\theta), \in\rangle$ for some regular $\theta \gg \kappa$. If x is a set in N with size less than κ, then $x \subseteq N$.

We denote the class of singular limit ordinals by Sing and the class of regular cardinals by Reg.

If $F\colon \lambda^{<\omega} \to \lambda$ is a partial function and $A \subseteq \lambda$, we say that F is *Jonsson for* A if A is closed under F, and whenever $B \subsetneq A$ is closed under F, it follows that $|B| < |A|$.

A standard presentation of the theory of forcing uses **partially ordered sets**, that is, orderings which are reflexive, antisymmetric, and transitive. It is well known however that this theory does not actually require antisymmetry. For example, forcing iterations as presented in [2] are not antisymmetric (i.e. it is possible that $q \le p$ and $p \le q$, but $p \ne q$).

Due to the author's treatment of iterated forcing, we use this more general theory of forcing. So we define a **forcing poset** to be an ordering $\langle \mathbb{P}, \le\rangle$ which is reflexive and transitive.

A forcing poset $\langle \mathbb{P}, \le\rangle$ is **separative** if whenever $q \not\le p$, there is $r \le q$ such that r and p are incompatible; equivalently, $q \Vdash p \in \dot{G}$ iff $q \le p$.

We say that forcing posets $\mathbb{P}$ and $\mathbb{Q}$ are **equivalent** if they have the same generic extensions. This is only an intuitive definition, not a formal one, since generic filters do not exist in the universe. We prove that two forcing posets are equivalent in this informal sense by showing how to construct a generic filter for one, given a generic filter for the other. For example, if D is a dense subset of $\mathbb{P}$, then D and $\mathbb{P}$ are equivalent.

For any forcing poset $\mathbb{P}$ there exists a separative poset $\mathbb{P}^*$ which is equivalent to $\mathbb{P}$. Define $\mathbb{P}^*$ as follows. The underlying set of $\mathbb{P}^*$ consists of equivalent classes of $\mathbb{P}$ under the following equivalence relation: p is equivalent to q if for all r, r is compatible with p iff r is compatible with q. Define $[q] \le [p]$ if for all $r \le q$, r is compatible with p. Then $\mathbb{P}^*$ is a separative forcing poset. Moreover, the surjective mapping $i\colon \mathbb{P} \to \mathbb{P}^*$ defined by $i(p) = [p]$ satisfies:

(1) $q \le p$ implies $i(q) \le i(p)$,

(2) If p and q are incompatible in $\mathbb{P}$, then $i(p)$ and $i(q)$ are incompatible in $\mathbb{Q}$.

The equivalence of $\mathbb{P}$ and $\mathbb{P}^*$ will follow from the next lemma.

Lemma 1.1. *Suppose that $\mathbb{P}$ and $\mathbb{Q}$ are forcing posets, $D \subseteq \mathbb{P}$ and $E \subseteq \mathbb{Q}$ are dense sets, and $i\colon D \to E$ is a surjective mapping satisfying:*

(1) $q \le p$ implies $i(q) \le i(p)$,

(2) If p and q are incompatible in $\mathbb{P}$, then $i(p)$ and $i(q)$ are incompatible in $\mathbb{Q}$.

Then $\mathbb{P}$ and $\mathbb{Q}$ are equivalent.

If $\mathbb{P}$ is a separative forcing poset, then there exists a unique complete Boolean algebra r.o.$(\mathbb{P})$ and a mapping $j\colon \mathbb{P} \to \mathrm{r.o.}(\mathbb{P})$ satisfying:

(1) $q < p$ iff $j(q) < j(p)$,

(2) $j``\mathbb{P}$ is a dense subset of r.o.$(\mathbb{P})$.

The algebra r.o.$(\mathbb{P})$ is defined by taking regular cuts of $\mathbb{P}$ ordered by inclusion.

If $\mathbb{P}$ is a forcing poset, define r.o.$(\mathbb{P})$ to be r.o.$(\mathbb{P}^*)$, where $\mathbb{P}^*$ is the separative poset equivalent to $\mathbb{P}$ as defined above. Define $\mathcal{B}(\mathbb{P})$ to be the forcing poset consisting of the Boolean algebra r.o.$(\mathbb{P})$ minus the 0 element. Composing the maps

$\mathbb{P} \to \mathbb{P}^*$ and $\mathbb{P}^* \to$ r.o.$(\mathbb{P}^*)$ described above, there exists a mapping k: $\mathbb{P} \to \mathcal{B}(\mathbb{P})$ satisfying:

(1) $q \leq p$ implies $k(q) \leq k(p)$,

(2) If p and q are incompatible in $\mathbb{P}$, then $k(p)$ and $k(q)$ are incompatible in $\mathcal{B}(\mathbb{P})$,

(3) k"$\mathbb{P}$ is a dense subset of $\mathcal{B}(\mathbb{P})$.

So by Lemma 1.1, $\mathbb{P}$ and $\mathcal{B}(\mathbb{P})$ are equivalent forcing posets.

When we discuss $\mathcal{B}(\mathbb{P})$, we consider it as a forcing poset in the general sense, and avoid using Boolean valued models.

The following genericity criterion for subsets of $\mathcal{B}(\mathbb{P})$ is straightforward: G is a generic filter for $\mathcal{B}(\mathbb{P})$ over V iff G is an ultrafilter on r.o.$(\mathbb{P})$ which contains 1, such that whenever $A \subseteq G$ is in V, $\bigwedge A$ is in G.

Now we discuss projection mappings. A mapping π: $\mathbb{Q} \to \mathbb{P}$ is a **projection mapping** if it satisfies:

(1) $q \leq p$ implies $\pi(q) \leq \pi(p)$,

(2) $\pi(1) = 1$,

(3) if $p \leq \pi(q)$, then there is $r \leq q$ such that $\pi(r) \leq p$.

Conditions (2) and (3) imply that π"$\mathbb{Q}$ is dense in $\mathbb{P}$. If π: $\mathbb{Q} \to \mathbb{P}$ is a projection mapping and G is a generic filter for $\mathbb{Q}$ then π"G generates a generic filter for $\mathbb{P}$.

Suppose that π: $\mathbb{Q} \to \mathbb{P}$ is a projection mapping and G is generic for $\mathbb{P}$. In $V[G]$ define a forcing poset $\mathbb{Q}/\mathbb{P}$ as follows. The underlying set of this poset is $\{q \in \mathbb{Q}\colon \pi(q) \in G\}$. We let $q \leq p$ in $\mathbb{Q}/\mathbb{P}$ iff $q \leq p$ in $\mathbb{Q}$. Now in V let $\mathbb{Q}/\mathbb{P}$ be a $\mathbb{P}$-name for the forcing poset just described.

Define k: $\mathbb{Q} \to \mathbb{P} * (\mathbb{Q}/\mathbb{P})$ by letting $k(q) = \pi(q) * \check{q}$. Note that $k(q)$ really is in $\mathbb{P} * (\mathbb{Q}/\mathbb{P})$, since $\pi(q)$ forces that $\pi(\check{q})$ is in $\dot{G}$. Then the mapping k satisfies:

(1) $q \leq p$ implies $k(q) \leq k(p)$,

(2) If p and q are incompatible in $\mathbb{Q}$, then $k(p)$ and $k(q)$ are incompatible in $\mathbb{P} * (\mathbb{Q}/\mathbb{P})$,

(3) k"$\mathbb{Q}$ is dense in $\mathbb{P} * (\mathbb{Q}/\mathbb{P})$.

So $\mathbb{Q}$ and $\mathbb{P} * (\mathbb{Q}/\mathbb{P})$ are equivalent.

Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing posets, and suppose that there is a $\mathbb{Q}$-name $\dot{G}_{\mathbb{P}}$ satisfying:

(1) $\mathbb{Q}$ forces that $\dot{G}_{\mathbb{P}}$ is a generic filter for $\mathbb{P}$ over V,

(2) for any p in $\mathbb{P}$, there is q in $\mathbb{Q}$ such that q forces that p is in $\dot{G}_{\mathbb{P}}$.

Then since $\mathbb{P}$ and $\mathcal{B}(\mathbb{P})$ are equivalent, there exists a $\mathbb{Q}$-name $\dot{G}$ such that $\mathbb{Q}$ forces that $\dot{G}$ is generic over V for $\mathcal{B}(\mathbb{P})$, and for any a in $\mathcal{B}(\mathbb{P})$ there is q in $\mathbb{Q}$ which forces that a is in $\dot{G}$.

Define π: $\mathbb{Q} \to$ r.o.$(\mathbb{P})$ by

$$\pi(q) = \bigwedge\{a \in \mathcal{B}(\mathbb{P})\colon q \Vdash a \in \dot{G}\}.$$

The set $\{a \in \mathcal{B}(\mathbb{P})\colon q \Vdash a \in \dot{G}\}$ is in the ground model, and q forces that it is contained in $\dot{G}$. By the criterion for genericity of filters on $\mathcal{B}(\mathbb{P})$, q forces that $\pi(q)$ is in $\dot{G}$, and hence is non-zero. Therefore π is a map $\mathbb{Q} \to \mathcal{B}(\mathbb{P})$. It is also a projection mapping.

Suppose that κ is regular and $\mathbb{P}$ is a forcing poset. We say that $\mathbb{P}$ is **$< \kappa$-distributive** if whenever $\{D_i\colon i < \beta\}$ is a family of dense open subsets of $\mathbb{P}$ and $\beta < \kappa$, then $\bigcap D_i$ is dense open. Equivalently, $\mathbb{P}$ is $< \kappa$-distributive if forcing with $\mathbb{P}$ does not add any new sequences of ordinals with order type less than κ.

Suppose that $\mathbb{P}$ is a forcing poset and λ is a cardinal. A **canonical name for a subset of** λ is a $\mathbb{P}$-name of the form

$$\{\langle p, \check{\alpha}\rangle \colon p \in A_\alpha,\ \alpha < \lambda\}$$

where A_α is an antichain for each $\alpha < \lambda$. If p forces that $\dot{A}$ is a subset of λ, then there is a canonical name $\dot{B}$ for a subset of λ such that p forces that $\dot{A} = \dot{B}$. If $\dot{X}$ is a $\mathbb{P}$-name and $\mathbb{P}$ forces that $\dot{f} \colon \lambda \to \dot{X}$ is a bijection, then a **canonical name for a subset of** $\dot{X}$ is a name of the form

$$\{\langle p, \dot{f}(\alpha)\rangle \colon p \in A_\alpha,\ \alpha < \lambda\}$$

where each A_α is an antichain. If p forces that $\dot{A}$ is a subset of $\dot{X}$, then there is a canonical name $\dot{B}$ for a subset of $\dot{X}$ such that p forces that $\dot{A} = \dot{B}$.

Suppose that $\mathbb{P}$ is a forcing poset and p forces $\phi(\dot{X})$ for some $\mathbb{P}$-name $\dot{X}$. Also assume that $\mathbb{P}$ forces that there exists x such that $\phi(x)$ holds. Then there exists a $\mathbb{P}$-name $\dot{Y}$ such that $\mathbb{P}$ forces $\phi(\dot{Y})$ and p forces $\dot{X} = \dot{Y}$.

If H is a subset of a forcing poset $\mathbb{P}$, we say that H **generates** J if J is the set of q in $\mathbb{P}$ such that there is p in H with $p \leq q$.

Suppose that $j \colon M \to N$ is an elementary embedding between transitive models of set theory, $\mathbb{P}$ is a forcing poset in M, G is generic for $\mathbb{P}$ over M, and H is generic for $j(\mathbb{P})$ over N. Then j can be lifted to $j \colon M[G] \to N[H]$ iff $j``G \subseteq H$. In this case, $j(G) = H$. In particular, j can be lifted if there exists a condition s in H such that $s \leq j(p)$ for all p in G. We will use Silver's notation and refer to such a condition s as a **master condition**.

2. Forcing Posets

In this section we describe some of the forcing posets we use in the paper.

Suppose that κ is strongly inaccessible and S is a stationary subset of κ. Define a forcing poset $\mathbb{P}_S$ as follows. A condition in $\mathbb{P}_S$ is a closed bounded subset of $\kappa \setminus S$. We let $q \leq p$ if q end-extends p. Clearly $\mathbb{P}_S$ has size κ.

The poset $\mathbb{P}_S$ is $< \kappa$-distributive iff $\kappa \setminus S$ is **fat**, i.e. for any club C and any ordinal $\beta < \kappa$, $C \setminus S$ contains a closed subset with order type β. If G is a generic filter for $\mathbb{P}_S$ then $\bigcup G$ is a club subset of κ which is disjoint from S. So $\mathbb{P}_S$ destroys the stationarity of S.

The set $\mathrm{Sing} \cap \kappa$ is a fat set. So $\mathbb{P}_S$ is $< \kappa$-distributive whenever $S \subseteq \mathrm{Reg} \cap \kappa$. For more on the poset $\mathbb{P}_S$ see [1].

Gitik [6] generalized the forcing poset $\mathbb{P}_S$ to destroy certain stationary subsets of $P_\kappa \kappa^+$, where κ is strongly inaccessible. Suppose that S is a subset of $P_\kappa \kappa^+$. Define $\mathbb{P}_S$ as follows. A condition in $\mathbb{P}_S$ is a set $\mathbf{x} \subseteq P_\kappa \kappa^+$ disjoint from S with size less than κ, and whenever $\langle a_i \colon i < \beta\rangle$ is $\subseteq$-increasing in $\mathbf{x}$, then $\bigcup a_i$ is in $\mathbf{x}$. Let $\mathbf{y} \leq \mathbf{x}$ if $\mathbf{y}$ end-extends $\mathbf{x}$ in the following sense: $\mathbf{x} \subseteq \mathbf{y}$, and whenever a is in $\mathbf{y} \setminus \mathbf{x}$ and b is in $\mathbf{x}$, a is not a subset of b.

The poset $\mathbb{P}_S$ is always κ^+-c.c. The conditions under which $\mathbb{P}_S$ is $< \kappa$-distributive are somewhat complex. We will consider this situation in detail in Section 4.

Suppose that κ is strongly inaccessible in V. In what follows we will need a generic extension $V \subseteq W$ satisfying the following properties:

(1) κ is strongly inaccessible in W,
(2) V and W have the same limit cardinals,
(3) $\kappa^{+V} = \kappa^{+W}$,

(4) there exists a **club of former regulars** in κ, that is, a club $C \subseteq \kappa$ in W such that every α in C is a regular cardinal in V.
There are a variety of ways to construct such a generic extension. We will use Radin forcing. The following lemma contains all the information about Radin forcing which we will need.

Lemma 2.1. *Suppose that κ is λ-supercompact for some $\lambda \geq 2^\kappa$. Then there exists a forcing poset $\mathbb{R}$, called Radin forcing, which adds a generic club set C to κ consisting of former regulars, and satisfies the following properties:*
(1) $\mathbb{R}$ is κ^+-c.c.,
(2) $\mathbb{R}$ preserves all cardinals and the function $\alpha \mapsto 2^\alpha$,
(3) $\mathbb{R}$ preserves the λ-supercompactness of κ.

See [5] for an exposition of Radin forcing.

A triple $\langle \mathbb{Q}, \leq, \leq^* \rangle$ is called a **Prikry type forcing poset** if $\langle \mathbb{Q}, \leq \rangle$ and $\langle \mathbb{Q}, \leq^* \rangle$ are forcing posets, $q \leq^* p$ implies $q \leq p$, and $\mathbb{Q}$ satisfies the **Prikry property**: for any ϕ in the forcing language for $\langle \mathbb{Q}, \leq \rangle$ and for any p in $\mathbb{Q}$, there exists $q \leq^* p$ such that q decides ϕ. If α is a cardinal we say that $\mathbb{Q}$ is **α-weakly closed** if $\langle \mathbb{Q}, \leq^* \rangle$ is α-closed. We say that $\mathbb{Q}$ satisfies the **direct extension property** if whenever $q, r \leq^* p$, there is $s \leq^* q, r$. It should be made clear that when we say that we force with a Prikry type forcing poset $\mathbb{Q}$, we always mean that we force with $\langle \mathbb{Q}, \leq \rangle$.

Suppose that U is a κ-complete fine ultrafilter on $P_\kappa \lambda$. We define a forcing poset $\mathbb{PR}(U)$ as follows. If U is a normal ultrafilter, $\mathbb{PR}(U)$ will denote the supercompact Prikry forcing. A condition in $\mathbb{PR}(U)$ is a pair $\langle \vec{a}, A \rangle$, where A is in U and $\vec{a}$ is a finite Magidor increasing sequence from $P_\kappa \lambda$. If $p = \langle \vec{a}, A \rangle$ and $q = \langle \vec{b}, B \rangle$ are conditions, let $q \leq p$ if $B \subseteq A$, $\vec{a}$ is an initial segment of $\vec{b}$, and any element of $\vec{b}$ which is not in $\vec{a}$ is in A. If $q \leq p$ and $\vec{a} = \vec{b}$, we let $q \leq^* p$.

Suppose that U is a non-normal ultrafilter. Then we let $\mathbb{PR}(U)$ denote the tree Prikry forcing. A condition in $\mathbb{PR}(U)$ is a pair $\langle t, T \rangle$ satisfying:

(1) T is a collection of finite Magidor-increasing sequences from $P_\kappa \lambda$, ordered by $u \trianglelefteq v$ if u is an initial segment of v.

(2) $\langle T, \trianglelefteq \rangle$ is a tree ordering, that is, if $u \trianglelefteq v$ and v is in T then u is in T.

(3) t is the trunk of T, that is, t is in T and whenever u is in T, either $u \trianglelefteq t$ or $t \trianglelefteq u$.

(4) For every u in T with $t \trianglelefteq u$, the set $\{a \in P_\kappa \lambda \colon u^\frown \langle a \rangle \in T\}$ is in U.

The ordering on $\mathbb{PR}(U)$ is $\langle s, S \rangle \leq \langle t, T \rangle$ if $S \subseteq T$. We let $\langle s, S \rangle \leq^* \langle t, T \rangle$ if $\langle s, S \rangle \leq \langle t, T \rangle$ and $s = t$.

In either case, $\mathbb{PR}(U)$ satisfies the following properties. The poset $\mathbb{PR}(U)$ is a Prikry type forcing poset. If $\{\langle s, S_i \rangle \colon i < \beta\}$ is a family of fewer than κ many conditions with the same first coordinate, then $\langle s, \bigcap S_i \rangle$ is a condition which directly extends each $\langle s, S_i \rangle$. So $\mathbb{PR}(U)$ is κ-weakly closed and satisfies the direct extension property. Since there are only $\lambda^{<\kappa}$ many possible first coordinates for a condition, $\mathbb{PR}(U)$ is $(\lambda^{<\kappa})^+$-c.c.

Suppose that G is a generic filter for $\mathbb{PR}(U)$. Then the set $\bigcup \{a \colon \exists A \ \langle a, A \rangle \in G\}$ is a Magidor-increasing sequence $\langle a_n \colon n < \omega \rangle$ cofinal in $P_\kappa \lambda$, and for any A in U there is $n < \omega$ such that a_m is in A for all $m \geq n$. For each $\kappa \leq \beta \leq \lambda$ with $\mathrm{cf}(\beta) \geq \kappa$, $\beta = \bigcup \{a_n \cap \beta \colon n < \omega\}$, and therefore β has cofinality ω in the generic extension.

If the ultrafilter U is understood, we sometimes write $\mathbb{PR}(\kappa,\lambda)$ or $\mathbb{PR}$ for $\mathbb{PR}(U)$. For more information about $\mathbb{PR}(U)$ see [5].

Suppose that U is a κ-complete fine ultrafilter on $P_\kappa\lambda$. Assume that W is a generic extension of V and $\langle a_n \colon n<\omega\rangle$ is an increasing sequence in W which is cofinal in $P_\kappa\lambda$. Assume that the sequence satisfies the **Prikry genericity criterion**: for every function $h\colon (P_\kappa\lambda)^{<\omega}\to U$ in the ground model there is n such that for all $m\geq n$, a_m is in $h(\langle a_0,\ldots,a_{m-1}\rangle)$. Then the set $\{a_n\colon n<\omega\}$ generates a generic filter for $\mathbb{PR}(U)$ over V. In case U is non-normal, this generic filter is the set

$$\{\langle t,T\rangle\in\mathbb{PR}(U)\colon \langle a_0,\ldots,a_n\rangle\in T \text{ for all } n<\omega\}$$

If U is normal, the generic filter is

$$\{\langle\vec{a},A\rangle\in\mathbb{PR}(U)\colon \vec{a}\text{ is an initial segment of }\langle a_n\colon n<\omega\rangle,\ \forall n\geq|\vec{a}|\ a_n\in A\}.$$

If U is normal, a weaker genericity criterion is sufficient: for all A in U, there is n such that a_m is in A for all $m\geq n$. For a proof see [8].

The following two results summarize a technique due to Gitik which is implicit in [6].

Lemma 2.2. *Suppose that U is a κ-complete fine ultrafilter on $P_\kappa\lambda$ and $\pi\colon\mathbb{PR}(U)\to\mathbb{P}$ is a projection mapping. Then there exists a Prikry type forcing poset $\langle\mathbb{Q},\leq,\leq^*\rangle$ satisfying:*

(1) $\langle\mathbb{Q},\leq\rangle$ and $\langle\mathbb{P},\leq\rangle$ are equivalent posets,

(2) $\mathbb{Q}$ is κ-weakly closed and satisfies the direct extension property.

Proof. Define $\langle\mathbb{Q},\leq,\leq^*\rangle$ as follows. The underlying set of $\mathbb{Q}$ is exactly the underlying set of $\mathbb{PR}(U)$. Define $\leq^*$ on $\mathbb{Q}$ to be exactly $\leq^*$ on $\mathbb{PR}(U)$. It is obvious then that $\langle\mathbb{Q},\leq^*\rangle$ is a forcing poset which is κ-closed, and whenever $q,r\leq^* p$ in $\mathbb{Q}$, there is $s\leq^* q,r$. Define $\leq$ on $\mathbb{Q}$ by letting $q\leq p$ iff $\pi(q)\leq\pi(p)$ holds in $\mathbb{P}$.

Define $i\colon\pi``\mathbb{Q}\to\mathbb{Q}$ by letting $i(a)$ be some q in $\mathbb{Q}$ such that $\pi(q)=a$. Then i is an order preserving and incompatibility preserving surjective mapping between a dense subset of $\mathbb{P}$ and a dense subset of $\mathbb{Q}$. So $\mathbb{Q}$ and $\mathbb{P}$ are equivalent posets.

We show that $\mathbb{Q}$ satisfies the Prikry property. First note that the identity mapping id: $\mathbb{PR}(U)\to\mathbb{Q}$ is a projection mapping. Therefore if H is generic for $\mathbb{PR}(U)$ over V, then H generates a generic filter J for $\langle\mathbb{Q},\leq\rangle$ over V.

Now let p be in $\mathbb{Q}$ and suppose that ϕ is a statement in the forcing language for $\langle\mathbb{Q},\leq\rangle$. Then p is also a condition in $\mathbb{PR}(U)$. Apply the Prikry property in $\mathbb{PR}(U)$ to find $q\leq^* p$ such that q decides over V in $\mathbb{PR}(U)$ the statement $\phi^{V[J]}$. Without loss of generality suppose that q forces $\phi^{V[J]}$. By the definition of $\leq^*$ in $\mathbb{Q}$, we have $q\leq^* p$ in $\mathbb{Q}$.

We claim that q forces ϕ. If not, there is $r\leq q$ in $\mathbb{Q}$ which forces the negation of ϕ. Since $r\leq\mathrm{id}(q)$, apply the fact that id is a projection mapping to get $s\leq q$ in $\mathbb{PR}(U)$ such that $s=\mathrm{id}(s)\leq r$ in $\mathbb{Q}$. Now force with s in $\mathbb{PR}(U)$ to get a generic filter H and generic extensions $V\subseteq V[J]\subseteq V[H]$.

Since $s\leq q$ in $\mathbb{PR}(U)$, $\phi^{V[J]}$ holds in $V[H]$. Therefore ϕ holds in $V[J]$. But also s is in J, and $s\leq r$ in $\mathbb{Q}$ and r forces the negation of ϕ. Therefore ϕ fails in $V[J]$, and we have a contradicton.

□

Lemma 2.3. *Suppose that U is a κ-complete fine ultrafilter on $P_\kappa\lambda$. Let $\mathbb{P}$ be a $<\kappa$-distributive forcing poset such that $|\mathbb{P}|\leq\lambda$, and assume that $\mathbb{P}$ has no* *more than λ-many maximal antichains. Then there exists a projection mapping* $\pi\colon \mathbb{PR}(U) \to \mathcal{B}(\mathbb{P})$.

Therefore there exists a Prikry type forcing poset $\langle \mathbb{Q}, \leq, \leq^* \rangle$ which is κ-weakly closed, satisfies the direct extension property, and $\langle \mathbb{Q}, \leq \rangle$ is equivalent to $\mathcal{B}(\mathbb{P})$, and hence to $\mathbb{P}$.

Proof. It suffices to prove that there exists a $\mathbb{PR}(U)$-name $\dot{G}$ such that $\mathbb{PR}(U)$ forces that $\dot{G}$ is a generic filter for $\mathbb{P}$ over V, and for any p in $\mathbb{P}$, there is q in $\mathbb{PR}(U)$ which forces that p is in $\dot{G}$.

Let $\langle p_i\colon i < \lambda \rangle$ and $\langle A_i\colon i < \lambda \rangle$ enumerate $\mathbb{P}$ and the maximal antichains of $\mathbb{P}$, possibly with repetitions.

Suppose that H is generic for $\mathbb{PR}(U)$ over V. Let $\langle a_n\colon n < \omega \rangle$ be the set $\bigcup \{s\colon \exists S\ \langle s, S \rangle \in H\}$. Then each a_n is in V and $\bigcup \{a_n\colon n < \omega\} = \lambda$.

Define a decreasing sequence $\langle q_n\colon n < \omega \rangle$ in $\mathbb{P}$ inductively as follows. Let q_0 be $p_{\sup a_0}$. Now suppose that q_n is defined. Since a_n has size less than κ, the set $D = \bigcap \{D_i\colon i \in a_n\}$ is dense open in $\mathbb{P}$. So pick q_{n+1} in D below q_n. Clearly the set $\{q_n\colon n < \omega\}$ generates a generic filter for $\mathbb{P}$ over V. Let $\dot{G}$ be a name for G. The definition of q_0 ensures that it is possible for any particular condition to be in $\dot{G}$. $\square$

We will need a generalization of the last lemma.

Lemma 2.4. *Suppose that U is a κ-complete fine ultrafilter on $P_\kappa \lambda$. Let $\mathbb{P}$ be a $< \kappa$-distributive forcing poset with $|\mathbb{P}| \leq \lambda$ and such that that $\mathbb{P}$ has no more than λ many antichains.*

Assume that $\mathbb{P}$ forces that U' is a κ-complete fine ultrafilter on $P_\kappa \mu$, where $2^{\lambda^{<\mu}} \leq \lambda$ and $|\mathbb{P} * \mathbb{PR}(U')| \leq \lambda$. Then there is a projection mapping

$$\pi\colon \mathbb{PR}(U) \to \mathcal{B}(\mathbb{P} * \mathbb{PR}(U')).$$

So there exists a Prikry type forcing poset $\langle \mathbb{Q}, \leq, \leq^* \rangle$ which is κ-weakly closed, satisfies the direct extension property, and $\langle \mathbb{Q}, \leq \rangle$ is equivalent to $\mathbb{P} * \mathbb{PR}(U')$.

Proof. Enumerate $\mathbb{P} * \mathbb{PR}(U')$ as $\langle r_i\colon i < \lambda \rangle$, possibly with repetitions. Let H be generic for $\mathbb{PR}(U)$ over V and let $\langle a_n\colon n < \omega \rangle$ be generic Prikry sequence given by H.

Fix q and $\langle \dot{t}, \dot{T} \rangle$ such that $r_{\sup a_0} = q * \langle \dot{t}, \dot{T} \rangle$. As in Lemma 2.3, contruct a generic filter G_0 for $\mathbb{P}$ over V containing q. Let $\langle h_i\colon i < \lambda \rangle$ enumerate all the functions $h\colon (P_\kappa \mu)^{<\omega} \to U'$ in $V[G_0]$, possibly with repetitions. Now inductively define a Prikry sequence for $\mathbb{PR}(U')$ as follows. Write $t = \langle b_i\colon i \leq k \rangle$. Given b_n for $n \geq k$, choose b_{n+1} which is a Magidor extension of b_n and is in the measure one set

$$\bigcap \{h_i(\langle b_0, \ldots, b_n \rangle)\colon i \in a_n\} \ \cap \ \{a\colon \langle b_0, \ldots, b_n \rangle^\frown a \in T\}.$$

By the Prikry genericity criterion, $\{b_n\colon n < \omega\}$ generates a generic filter G_1 for $\mathbb{PR}(U')$ over $V[G_0]$ which contains $\langle t, T \rangle$. $\square$

3. The Stationarity of $S(\kappa, \kappa^+)$

Now we begin our study by introducing the set $S(\kappa, \kappa^+)$.

Lemma 3.1. *There is a club set of a in $P_\kappa \kappa^+$ such that* $\text{o.t.}(a) \leq (a \cap \kappa)^+$.

Proof. Fix $\theta \gg \kappa$ regular, and let N be an elementary substructure of $\langle H(\theta), \in\rangle$ in $P_\kappa H(\theta)$. Let $a = N \cap \kappa^+$. We claim that $\text{o.t.}(a) \leq (a \cap \kappa)^+$.

We show that for all β in a, $|a \cap \beta| \leq |a \cap \kappa|$. This is clear if $\beta < \kappa$. Suppose that $\beta \geq \kappa$ is in a. Fix a bijection $f_\beta\colon \kappa \to \beta$ in N. By elementarity, for all j in $a \cap \beta$, $f_\beta^{-1}(j)$ is in $a \cap \kappa$. Therefore $f_\beta \restriction a \cap \kappa$ is a bijection of $a \cap \kappa$ onto $a \cap \beta$, and so $|a \cap \beta| = |a \cap \kappa|$. Since each initial segment of a has size at most $|a \cap \kappa|$, clearly $\text{o.t.}(a) \leq (a \cap \kappa)^+$. □

Now define $S(\kappa, \kappa^+)$ as the set of a in $P_\kappa \kappa^+$ such that $\text{o.t.}(a) = (a \cap \kappa)^+$.

In [4] it was established that the stationarity of $S(\kappa, \kappa^+)$ implies an inner model with a measurable. The exact consistency strength of the stationarity of $S(\kappa, \kappa^+)$ is currently an open problem.

Lemma 3.2. *For almost all a in $P_\kappa \kappa^+$, a contains all its limit points below* $\sup a$ *which have cofinality different from* $\text{cf}(a \cap \kappa)$.

Proof. Fix $\theta \gg \kappa$ regular and suppose that N is an elementary substructure of $H(\theta)$ in $P_\kappa H(\theta)$. Let $a = N \cap \kappa^+$. Suppose for a contradiction that α is a limit ordinal in a below $\sup a$, $\text{cf}(\alpha) \neq \text{cf}(a \cap \kappa)$, and α is not in a.

Let β be the least element of a greater than α. We claim that $\text{cf}(\beta) = \kappa$. Fix $g\colon \text{cf}(\beta) \to \beta$ in N increasing and cofinal in β. If $\text{cf}(\beta) < \kappa$, then $\text{cf}(\beta) \subseteq \beta$ so $g``\text{cf}(\beta)$ is a subset of a, which contradicts that a is bounded below β.

For all γ in $a \cap \beta$ there is i in $a \cap \kappa$ such that $g(i) > \gamma$. So $g``(a \cap \kappa)$ is increasing and unbounded below α. Therefore $a \cap \kappa$ and α have the same cofinality, which is a contradiction. □

Theorem 3.3. *If $\square_{\kappa,<\kappa}$ holds then $S(\kappa, \kappa^+)$ is non-stationary.*

Proof. Let $\langle \mathcal{C}_\alpha\colon \alpha < \kappa^+\rangle$ be a $\square_{\kappa,<\kappa}$-sequence. Fix $\theta \gg \kappa^+$ regular. Suppose for a contradiction that $S(\kappa, \kappa^+)$ is stationary. Then there exists N in $P_\kappa H(\theta)$ which is an elementary substructure of the model

$$\langle H(\theta), \in, \langle \mathcal{C}_\alpha\colon \alpha < \kappa^+\rangle\rangle.$$

such that $N \cap \kappa^+$ is in $S(\kappa, \kappa^+)$. Let $a = N \cap \kappa^+$, $\beta = \sup a$, and $\kappa_a = N \cap \kappa$. Note that $\text{cf}(\beta) = \kappa_a^+$.

Let c be some member of $\mathcal{C}_\beta$. Fix a regular cardinal $\delta < \kappa_a$ different from $\text{cf}(\kappa_a)$. By Lemma 3.2, a is closed under suprema of subsets with order type δ. In particular, a is stationary in $\sup a$, and therefore $a \cap \lim(c)$ is unbounded in β. Now c has order type at least $\text{cf}(\beta) = \kappa_a^+$. So there is γ in $a \cap \lim(c)$ such that $\text{o.t.}(c \cap \gamma) \geq \kappa_a$. But $c \cap \gamma$ is in $\mathcal{C}_\gamma$. Since γ is in a, $\mathcal{C}_\gamma$ is a member of N. But $|\mathcal{C}_\gamma| < \kappa$, so $\mathcal{C}_\gamma$ is a subset of N. Therefore $c \cap \gamma$ is in N, and hence $\text{o.t.}(c \cap \gamma)$ is in $N \cap \kappa = \kappa_a$. This contradicts that $\text{o.t.}(c \cap \gamma) \geq \kappa_a$. □

A different proof of this theorem using ill-founded generic ultrapowers appears in [3].

The set $S(\kappa, \kappa^+)$ has some importance in inner model theory. A regular cardinal κ is **subcompact** if for every set $X \subseteq H(\kappa^+)$ there is a cardinal $\delta < \kappa$, a set $a \subseteq H(\delta^+)$, and an elementary embedding

$$i\colon \langle H(\delta^+), \in, a\rangle \to \langle H(\kappa^+), \in, X\rangle$$

such that $\mathrm{crit}(i) = \delta$ and $i(\delta) = \kappa$. Jensen defined subcompactness and proved that if κ is subcompact then $\square_\kappa$ fails. The following result was pointed out to the author by Ernest Schimmerling.

Theorem 3.4. *Suppose that $V = L[\vec{E}]$ where $\vec{E}$ is coherent sequence of extenders. Then the following are equivalent for an inaccessible cardinal κ.*

(1) κ is subcompact,

(2) $S(\kappa, \kappa^+)$ is stationary,

(3) $\square_{\kappa, <\kappa}$ fails.

Proof. The proof of (3) implies (1) appears in [12], and (2) implies (3) was proved in Theorem 3.3. For (1) implies (2), let C be a club subset of $P_\kappa \kappa^+$. Then $C \subseteq H(\kappa^+)$, so there is $\delta < \kappa$, $c \subseteq H(\delta^+)$, and an elementary embedding

$$i \colon \langle H(\delta^+), \in, c\rangle \to \langle H(\kappa^+), \in, C\rangle$$

with critical point δ such that $i(\delta) = \kappa$. Applying elementarity it is straightforward to check that $i``c$ is a directed subset of $C$ with size less than $\kappa$ and with union $i``\delta^+$. Therefore $i``\delta^+$ is in $C \cap S(\kappa, \kappa^+)$. $\square$

In general the stationarity of $S(\kappa, \kappa^+)$ does not imply that κ has any large cardinal property. For example, in the model which we construct in Section 6, $S(\kappa, \kappa^+)$ is stationary, GCH holds, and κ is a non-Mahlo cardinal.

Note: Most of the results presented in this section are already known. The proof of Theorem 3.3 is due to the author. This proof allows for a different approach to destroying stationary subsets of $P_\kappa \kappa^+$ than the one which we use in this paper. For more on this topic see [9] and [10].

4. Destroying Stationary Sets

In this section we present a forcing poset for destroying the stationarity of certain subsets of $P_\kappa \kappa^+$. Assume for the remainder of this section that $V \subseteq W$ are transitive models of set theory with the same ordinals satisfying the properties:

(1) V and W have the same limit cardinals,

(2) κ is strongly inaccessible in W,

(3) $\kappa^{+V} = \kappa^{+W}$,

(4) in W there is a club set $C \subseteq \kappa$ such that every α in C is regular in V.

Now work in W. Consider the following two properties of a set $S \subseteq P_\kappa \kappa^+$:

Property 1: There exists a set $A \subseteq \kappa^+$ such that

$$S = \{a \in S(\kappa, \kappa^+) \colon \sup a \in A\}.$$

Property 2: There exists a partial function $F \colon \kappa^{+<\omega} \to \kappa^+$ in V such that in W,

$$S = \{a \in S(\kappa, \kappa^+) \colon F \text{ is not Jonsson for } a\}.$$

We will prove that if S is a set which satisfies either one of these properties, there exists a κ^+-c.c., $< \kappa$-distributive forcing poset which destroys the stationarity of S. (Note that if a forcing poset is κ^+-c.c. and $< \kappa$-distributive, the sets $P_\kappa \kappa^+$ and $S(\kappa, \kappa^+)$ are the same in the ground model and in the generic extension). If S has Property 1, we will denote this forcing poset by $\mathbb{P}_S$. If S has Property 2, we will denote this poset by $\mathbb{P}_F$.

The two conditions on S work together quite nicely. Namely, after forcing with $\mathbb{P}_F$, every stationary subset of $S(\kappa, \kappa^+)$ has Property 1. For a proof see Proposition 4.9 below.

Before we can define $\mathbb{P}_S$ and $\mathbb{P}_F$, we need the following lemma.

Lemma 4.1. *There exists a function $H\colon \kappa^{+<\omega} \to \kappa^+$ in V such that the following holds in W. For any a in $P_\kappa\kappa^+$ which is closed under H:*

(1) $a \cap \kappa$ is a limit cardinal,
(2) If $|a| > a \cap \kappa$ then $\text{o.t.}(a) = (a \cap \kappa)^+$.
(3) If x is a bounded subset of a and $\text{cf}(x) \neq \text{cf}(a \cap \kappa)$, then $\sup x \in a$.
(4) If a is in $S(\kappa, \kappa^+)$ then $\text{cf}^V(\sup a)$ is not a limit point of C.

Proof. In V fix $\theta \gg \kappa^+$ regular. Let $H^*\colon H(\theta)^{<\omega} \to H(\theta)$ be a Skolem function for the structure $\langle H(\theta), \in\rangle$. Let $H\colon \kappa^{+<\omega} \to \kappa^+$ be a function in V such that whenever $b \subseteq \kappa^+$ in V is closed under H, $\text{cl}_{H^*}(b) \cap \kappa^+ = b$.

Now work in W. Suppose that a is in $P_\kappa\kappa^+$ and is closed under H. Since a is closed under V-cardinal successors and V and W have the same limit cardinals, $a \cap \kappa$ is a limit cardinal. It is easy to check (2) using the proof of Lemma 3.1.

For (3), let x be a bounded subset of a with order type a regular cardinal different from $\text{cf}(a \cap \kappa)$. Let $\beta = \sup x$. Suppose for a contradiction that β is not in a. Let α be the least element of a greater than β. Let N be the H^*-closure of $(a \cap \kappa) \cup \{\alpha\}$ in V. Clearly $N \cap \kappa = a \cap \kappa$. In N fix $f_\alpha\colon \text{cf}^V(\alpha) \to \alpha$ increasing and unbounded in α. If $\text{cf}^V(\alpha) < \kappa$ then $\text{cf}^V(\alpha) \subseteq N$, so $f_\alpha``\text{cf}^V(\alpha)$ is an unbounded subset of $\alpha$ contained in $a$, which is impossible. So $\text{cf}^V(\alpha) = \kappa$. But then $f_\beta``a \cap \kappa$ is increasing and unbounded in $a \cap \alpha$, so $\text{cf}(a \cap \kappa) = \text{cf}(\beta)$, which is a contradiction.

For (4), let a be in $S(\kappa, \kappa^+)$ and suppose for a contradiction that $\text{cf}^V(\sup a)$ is a limit point of C. Let $\kappa_a = a \cap \kappa$, $\beta = \sup a$, and $\delta = \text{cf}^V(\beta)$. By (2) and (3), $\text{o.t.}(a) = \kappa_a^+$ and a is stationary in β. Since $\kappa_a^+ = \text{cf}^W(\beta)$, $\kappa_a^+ \leq \delta$. In V let $\pi\colon \delta \to \beta$ be an increasing and continuous map with range unbounded in β. Since δ is a limit point of C, $C \cap \delta$ is club in δ, and therefore $\pi``(C \cap \delta)$ is club in $\beta$. Since $a$ is stationary in $\beta$, there is $\gamma$ in $C \cap \delta$ with $\gamma > \kappa_a$ such that $\pi(\gamma)$ is in $a$. Note that $\text{cf}^V(\pi(\gamma)) = \gamma$, since $\gamma$ is regular in $V$ and $\sup(\pi``\gamma) = \pi(\gamma)$. But $\pi(\gamma)$ is in a, and therefore $\text{cf}^V(\pi(\gamma)) = \gamma$ is in $a \cap \kappa$. This is a contradiction since $\gamma > a \cap \kappa$. □

For the remainder of the section fix a function $H\colon \kappa^{+<\omega} \to \kappa^+$ as in Lemma 4.1. We will now define our forcing poset $\mathbb{P}$.

If $\mathbf{x}$ and $\mathbf{y}$ are subsets of $P_\kappa\kappa^+$, we say $\mathbf{x}$ **end-extends** $\mathbf{y}$ if $\mathbf{y} \subseteq \mathbf{x}$ and for every a in $\mathbf{x} \setminus \mathbf{y}$ and b in $\mathbf{y}$, a is not a subset of b.

Now suppose that $S \subseteq P_\kappa\kappa^+$ is a set which satisfies either Property 1 or Property 2 above. If S has Property I then let $F\colon \kappa^+ \to \kappa^+$ be the identity map. In this case we will write $\mathbb{P} = \mathbb{P}_S$. If S has Property II we will write $\mathbb{P} = \mathbb{P}_F$. Let E be the club set of a in $P_\kappa\kappa^+$ such that a is closed under H and F.

Definition 4.2. *A condition in $\mathbb{P}$ is a set $\mathbf{x}$ which satisfies the following properties:*

(1) $\mathbf{x} \subseteq E \setminus S$,
(2) $\mathbf{x}$ has size less than κ,
(3) $\mathbf{x}$ is closed under unions of $\subseteq$-increasing sequences,
(4) if a is in $\mathbf{x}$ and u is a set in $E \setminus S$ such that $a \cap \kappa \subseteq u \subseteq a$, then u is in $\mathbf{x}$,
(5) $\mathbf{x}$ contains an element $\max(\mathbf{x})$ such that $b \subseteq \max(\mathbf{x})$ for every b in $\mathbf{x}$.

The ordering on $\mathbb{P}$ is $\mathbf{x} \leq \mathbf{y}$ if $\mathbf{x}$ end-extends $\mathbf{y}$. □

The only significant differences between this definition and the definition of Gitik's poset $\mathbb{P}_S$ given in Section 2 is the addition of clause (4) and the fact that we require the elements of our conditions to be inside the club E. These changes are technical conditions which enable us to prove the $<\kappa$-distributivity of $\mathbb{P}$ without assuming Gitik's requirement that S is a subset of V.

We make some observations. In (4) the set u satisfies that $u \cap \kappa = a \cap \kappa$. If a and b are in E then $a \cap b$ is in E. If a is in E then $\text{o.t.}(a) \leq (a \cap \kappa)^+$. If $\mathbf{x}$ and $\mathbf{y}$ are in $\mathbb{P}$ then $\mathbf{x} \leq \mathbf{y}$ iff $\mathbf{y} \subseteq \mathbf{x}$ and for all a in $\mathbf{x} \setminus \mathbf{y}$, $a \nsubseteq \max(\mathbf{y})$. We can strengthen condition (4) as follows.

Lemma 4.3. *Suppose that $\mathbf{x}$ is a condition in $\mathbb{P}$, a is in $\mathbf{x}$, and u is a set in E such that $a \cap \kappa \subseteq u \subseteq a$. Then u is not in S, therefore u is in $\mathbf{x}$.*

Proof. Recall that $\mathbf{x} \cap S$ is empty so a is not in S. Suppose for a contradiction that u is in S. Then u is in $S(\kappa, \kappa^+)$ and is closed under H. We know that $\text{o.t.}(u) = (u \cap \kappa)^+ = (a \cap \kappa)^+$. Since $\text{o.t.}(a) \leq (a \cap \kappa)^+$ and $u \subseteq a$, $\text{o.t.}(a) = (a \cap \kappa)^+$. Clearly then $\sup a = \sup u$, for otherwise $\text{o.t.}(a)$ would be larger than $\text{o.t.}(u)$. If $\mathbb{P} = \mathbb{P}_S$, then $\sup u$ is in A so a is in S, and we have a contradiction. If $\mathbb{P} = \mathbb{P}_F$, then u is a subset of a closed under F and $|u| = |a|$. But a is not in S, so F is Jonsson for a, therefore $u = a$. But this is impossible since u is in S and a is not in S. $\square$

Clearly $\mathbb{P}$ has size κ^+. We will need the next several lemmas in order to prove that $\mathbb{P}$ is $<\kappa$-distributive. The following lemma will also be useful for constructing master conditions in later sections.

Lemma 4.4. *Suppose that X is a directed subset of $\mathbb{P}$ with size less than κ. Let $\mathbf{z}$ be the closure of $\bigcup X$ under unions of increasing sequences. Then the following statements hold.*

(1) $\mathbf{z}$ is a subset of E with size less than κ which is closed under unions of increasing sequences,

(2) every a in $\mathbf{z}$ is a subset of $\bigcup\{\max(\mathbf{x})\colon \mathbf{x} \in X\}$,

(3) if a is in $\mathbf{z}$ and u is a set in E such that $a \cap \kappa \subseteq u \subseteq a$ then u is in $\mathbf{z}$,

(4) if a is in $\mathbf{z} \setminus \bigcup X$ then for all $\mathbf{x}$ in X, $a \nsubseteq \max(\mathbf{x})$.

(5) $\mathbf{z}$ end-extends $\mathbf{x}$ for all $\mathbf{x}$ in X.

Proof. Let ζ be the least ordinal such that $\mathbf{z} = \bigcup\{\mathbf{z}_i\colon i < \zeta\}$ where $\mathbf{z}_0 = \bigcup X$, $\mathbf{z}_\delta = \bigcup\{\mathbf{z}_i\colon i < \delta\}$ for δ limit, and for each i, $\mathbf{z}_{i+1}$ is the collection of sets a such that there is an increasing sequence $\langle a_i\colon i < \xi\rangle$ of elements of $\mathbf{z}_i$ whose union is a. Statements (1) and (2) are straightforward.

(3) We prove by induction on $i < \zeta$ that if a is in $\mathbf{z}_i$ and u is a set in E such that $a \cap \kappa \subseteq u \subseteq a$ then u is in $\mathbf{z}_i$. This is true for $\mathbf{z}_0$ since X consists of conditions. The limit step is clear. Suppose that the claim is true for $\mathbf{z}_i$ for some $i < \zeta$. Let a be a set in $\mathbf{z}_{i+1}$ and u a set in E such that $a \cap \kappa \subseteq u \subseteq a$. Fix an increasing sequence $\langle a_j\colon j < \xi\rangle$ from $\mathbf{z}_i$ whose union is a. For each $j < \xi$ the set $u \cap a_j$ is in E and $a_j \cap \kappa \subseteq a_j \cap u \subseteq a_j$, so by induction $a_j \cap u$ is in $\mathbf{z}_i$. Therefore $u = \bigcup\{a_j \cap u\colon j < \xi\}$ is in $\mathbf{z}_{i+1}$.

(4) We prove by induction on $0 < i < \zeta$ that for all $\mathbf{x}$ in X, if a is in $\mathbf{z}_i \setminus \bigcup X$ then $a \nsubseteq \max(\mathbf{x})$. Suppose that a is in $\mathbf{z}_1 \setminus \bigcup X$. Write $a = \bigcup\{a_j\colon j < \xi\}$ where each a_j is in $\bigcup X$. Since a is not in $\bigcup X$, not all a_j's are in $\mathbf{x}$ since $\mathbf{x}$ is closed. So fix l with a_l not in $\mathbf{x}$. Then there is some $\mathbf{y}$ in X with a_l in $\mathbf{y}$. Since X is directed, fix $\mathbf{x}^*$ in X which end-extends both $\mathbf{x}$ and $\mathbf{y}$. Then a_l is in $\mathbf{x}^* \setminus \mathbf{x}$, so by the definition of the ordering, $a_l \nsubseteq \max(\mathbf{x})$. Therefore $a \nsubseteq \max(\mathbf{x})$. Now suppose that $0 < i < \zeta$ and the claim holds for $\mathbf{z}_i$. Let a be in $\mathbf{z}_{i+1} \setminus \bigcup X$, and write $a = \bigcup\{a_j\colon j < \xi\}$ where

each a_j is in $\mathbf{z}_i$. If all the a_j's are in $\bigcup X$ then a is in $\mathbf{z}_1$ and we are done by what we just proved. Otherwise some a_l is in $\mathbf{z}_i \setminus \bigcup X$ so by induction $a_l \not\subseteq \max(\mathbf{x})$, so $a \not\subseteq \max(\mathbf{x})$.

(5) Fix $\mathbf{x}$ in X. Then $\mathbf{x} \subseteq \bigcup X \subseteq \mathbf{z}$. Suppose that a is in $\mathbf{z} \setminus \mathbf{x}$. If a is in $\bigcup X$, then since X is directed there is $\mathbf{y} \leq \mathbf{x}$ in X with a in $\mathbf{y}$. By the definition of the ordering, $a \not\subseteq \max(\mathbf{x})$. Otherwise a is in $\mathbf{z} \setminus \bigcup X$, so by (4) $a \not\subseteq \max(\mathbf{x})$. □

The following lemma is the core technical fact for the proof of $< \kappa$-distributivity. We state it abstractly since we will need to use it later in the paper.

Lemma 4.5. *Suppose that $\gamma < \kappa$ is a limit ordinal, $f\colon \kappa \to \kappa^+$ is in V, and $\langle N_i\colon i \leq \gamma\rangle$ is an increasing and continuous sequence of sets with size less than κ which are closed under H and F such that for all $i \leq \gamma$:*

(1) $N_i \cap \kappa = \delta_i$ is in $\lim(C)$,

(2) $f``\delta_i = N_i \cap \kappa^+$.

Suppose that $\langle \mathbf{x}_i\colon i < \gamma\rangle$ is a decreasing sequence in $\mathbb{P}$ such that for each $i < \gamma$:

(3) $\mathbf{x}_i$ is in N_{i+1},

(4) $\max(\mathbf{x}_i) = N_i \cap \kappa^+$.

Let $\mathbf{x}_\gamma$ be the closure of $\bigcup\{\mathbf{x}_i\colon i < \gamma\}$ under unions of increasing sequences. Then the following statements are true:

(A) $\mathbf{x}_\gamma$ is in $\mathbb{P}$,

(B) $\mathbf{x}_\gamma \leq \mathbf{x}_i$ for all i,

(C) $\max(\mathbf{x}_\gamma) = N_\gamma \cap \kappa^+$.

Proof. Apply Lemma 4.4 with $X = \{\mathbf{x}_i\colon i < \gamma\}$ and $\mathbf{z} = \mathbf{x}_\gamma$. Then the only way that $\mathbf{x}_\gamma$ would not be a condition is if $\mathbf{x} \cap S \neq \emptyset$. We also know that $\mathbf{x}_\gamma$ end-extends $\mathbf{x}_i$ for all i. Since $N_\gamma \cap \kappa^+ = \bigcup\{N_i \cap \kappa^+\colon i < \gamma\}$, $N_\gamma \cap \kappa^+$ is in $\mathbf{x}_\gamma$ and by Lemma 4.4 this set is the maximum element of $\mathbf{x}_\gamma$. The only thing left to check is that $\mathbf{x}_\gamma$ is disjoint from S.

Suppose for a contradiction that $\mathbf{x}_\gamma \cap S$ contains an element a. Let $\kappa_a = a \cap \kappa$ and $\beta = \sup a$. Since a is in $E \cap S(\kappa, \kappa^+)$, by Lemma 4.1 we know that o.t.$(a) = \kappa_a^+$, a is closed under suprema of bounded subsets with cofinality different from $\mathrm{cf}^W(\kappa_a)$, and $\mathrm{cf}^V(\beta)$ is not a limit point of C. Since $N_\gamma \cap \kappa^+ = f``\delta_\gamma$, the set $N_\gamma \cap \beta$ is in V and is unbounded in β since it contains a. Therefore since $|N_\gamma \cap \beta|^V \leq \delta_\gamma$, $\mathrm{cf}^V(\beta) \leq \delta_\gamma$. But δ_γ is a limit point of C, therefore $\mathrm{cf}^V(\beta) < \delta_\gamma$. Since $\mathrm{cf}^W(\beta) = \kappa_a^{+W}$, $\kappa_a^{+W} \leq \mathrm{cf}^V(\beta) < \delta_\gamma$.

The set a is difficult to work with because it might not be in V. So what we do is find a set u such that $a \cap \kappa \subseteq u \subseteq a$ and u is covered by a small set d from V. Define an increasing sequence $\langle d_i\colon i < \omega_2^W\rangle$ in V as follows. Let d_0 be any club subset of β in V with order type $\mathrm{cf}^V(\beta)$ and $\kappa_a \subseteq d_0$. Take unions at limits. Suppose that d_i is defined for a fixed $i < \omega_2^W$. Define

$$d_i^* = d_i \cup H``d_i^{<\omega} \cup F``d_i^{<\omega}$$

and let

$$d_{i+1} = d_i^* \cup \lim(d_i^*).$$

(This is where we use the fact that H and F are in V). Let $d = \bigcup\{d_i\colon i < \omega_2^W\}$. Then d is closed under H and F and $|d| = \mathrm{cf}^V(\beta) < \delta_\gamma$. We claim that in W the set d is closed under suprema of any bounded subset x with order type ω or ω_1. For if x is such a set, then there is $i < \omega_2^W$ such that $x \subseteq d_i$. Then $\sup x$ is a limit point of d_i, so $\sup x$ is in $d_{i+1} \subseteq d$.

Now let $u = a \cap d$. Let μ be one of ω or ω_1 which is different from $\mathrm{cf}(\kappa_a)^W$. Then both a and d are closed under suprema of subsets with order type μ. It follows that $u = a \cap d$ is also closed under such suprema, and therefore is an unbounded subset of a. Since u is unbounded in a, it must have order type κ_a^{+W}. Also $\kappa_a = u \cap \kappa$. So u is in $S(\kappa, \kappa^+)$. The set u is in E since it is closed under H and F. But $a \cap \kappa \subseteq u \subseteq a$, so by (3) of Lemma 4.4, u is in $\mathbf{x}_\gamma$.

The rest of the proof splits into two cases. Suppose first that $\mathbb{P} = \mathbb{P}_S$ where $S = \{a \in S(\kappa, \kappa^+)\colon \sup a \in A\}$ for some $A \subseteq \kappa^+$. Since a is in S, $\sup a$ is in A, therefore $\sup u$ is in A so u is in S. Now we get a contradiction as follows. Since $\bigcup X \cap S$ is empty, u is in $\mathbf{x}_\gamma \setminus \bigcup X$. By (4) of Lemma 4.4, for all $l < \gamma$, $u \not\subseteq N_l \cap \kappa^+$. Now the set $N_\gamma \cap \kappa^+ = f``\delta_\gamma$ is in $V$, therefore $d \cap N_\gamma$ is in $V$. Also $|d \cap N_\gamma|^V \leq |d|^V < \delta_\gamma$. Since $\delta_\gamma$ is regular in $V$ and $d \cap N_\gamma \subseteq N_\gamma \cap \kappa^+ = \bigcup\{f``i\colon i < \delta_\gamma\}$, there is $i < \delta_\gamma$ such that $d \cap N_\gamma \subseteq f``i$. Fix $l < \gamma$ with $i < \delta_l$. Then $u = a \cap d \subseteq d \cap N_\gamma \subseteq f``i \subseteq f``\delta_l = N_l \cap \kappa^+$, so $u \subseteq N_l \cap \kappa^+$ and we have a contradiction.

Suppose now that $\mathbb{P} = \mathbb{P}_F$. If u is in S then u is in $\mathbf{x}_\gamma \setminus \bigcup X$, so we can get a contradiction as in the previous paragraph. Suppose that u is not in S. Since a is in S, $a \neq u$, therefore $u \subsetneq a$. Fix an ordinal δ^* in $a \setminus u$. Let v be the closure of $u \cup \{\delta^*\}$ under H and F. Since a is closed under H and F, $v \subseteq a$. Therefore $v \cap \kappa = \kappa_a$. So v is in $P_\kappa \kappa^+$. Since $u \subseteq v$, v is in $S(\kappa, \kappa^+)$, and also v is in E. Since $a \cap \kappa \subseteq v \subseteq a$, by (3) of Lemma 4.4, v is in $\mathbf{x}_\gamma$. Now u is a proper subset of v with the same size as v and u is closed under F. Therefore F is not Jonsson for v, so v is in S. Since $\bigcup X \cap S$ is empty, v is in $\mathbf{x}_\gamma \setminus \bigcup X$. By (4) of Lemma 4.4, for all $l < \gamma$, $v \not\subseteq N_l \cap \kappa^+$.

Let Y be the closure of $(N_\gamma \cap d) \cup \{\delta^*\}$ under H and F. Since $N_\gamma \cap d$ is in V, Y is in V, and $|Y|^V = |N_\gamma \cap d|^V \leq |d|^V < \delta_\gamma$. Since δ^* is in $a \subseteq N_\gamma$, $Y \subseteq N_\gamma \cap \kappa^+ = \bigcup\{f``i\colon i < \delta_\gamma\}$. As above we can find $l < \gamma$ such that $Y \subseteq N_l \cap \kappa^+$. Since $u \cup \{\delta^*\} \subseteq (N_\gamma \cap d) \cup \{\delta^*\}$, we have $v \subseteq Y \subseteq N_l \cap \kappa^+$, which is a contradiction. $\square$

Proposition 4.6. *The forcing poset $\mathbb{P}$ is $< \kappa$-distributive.*

Proof. Suppose that $\{D_i\colon i < \delta\}$ is a family of dense open subsets of $\mathbb{P}$ with $\delta < \kappa$. Let $\mathbf{x}$ be a condition in $\mathbb{P}$. Fix $\theta \gg \kappa^+$ regular, and let $<_\theta$ be a well-ordering of $H(\theta)$. Since κ is strongly inaccessible, there is $M \prec \langle H(\theta), \in\rangle$ such that $|M| = \kappa$, $\kappa \subseteq M$, $M \cap \kappa^+$ is in κ^+, and ${}^{<\kappa}M \subseteq M$. Choose M to contain the sets $\mathbb{P}$, $\mathbf{x}$, C, H, F, E, and $\langle D_i\colon i < \delta\rangle$. Let $\beta = M \cap \kappa^+$. Fix a bijection $f\colon \kappa \to \beta$ in V. Let $\langle z_i\colon i < \kappa\rangle$ enumerate all the elements of M.

Define a chain $\langle M_i\colon i < \kappa\rangle$ of elementary substructures of

$$\mathcal{M} = \langle M, \in, \mathbb{P}, \mathbf{x}, C, H, F, E, \langle D_i\colon i < \delta\rangle, f, <_\theta\rangle$$

as follows. Choose M_0 in $P_\kappa M$ with $M_0 \prec \mathcal{M}$. Take unions at limits. Given M_i, let M_{i+1} be some elementary substructure of $\mathcal{M}$ in $P_\kappa M$ which contains the sequence $\langle M_j\colon j \leq i\rangle$ and also the set z_i. Note that $M_i \cap \kappa$ is in $\lim(C)$ and $M_i \cap \kappa^+$ is in E.

Since z_i is in M_{i+1} for each i, the collection $\{M_i\colon i < \kappa\}$ is a club subset of $P_\kappa M$. Also there is a club set of $i < \kappa$ such that $M_i \cap \kappa = i$. Since $f``\kappa = \beta$, the collection $\{f``\alpha\colon \alpha < \kappa\}$ is club in $P_\kappa \beta$. So it is possible to choose a subsequence $\langle N_i\colon i < \kappa\rangle$ of $\langle M_i\colon i < \kappa\rangle$ such that:

(1) $N_i = M_{\delta_i}$ for some δ_i in $\lim(C)$,
(2) $N_i \cap \kappa = \delta_i$,
(3) $\langle N_j\colon j \leq i\rangle$ is in N_{i+1}.

Since N_i is closed under f and f^{-1},

(4) $f``\delta_i = N_i \cap \kappa^+$.

Since f is in V, $N_i \cap \kappa^+$ is in V. Also $|N_i \cap \kappa^+| = \delta_i = N_i \cap \kappa$, so $N_i \cap \kappa^+$ is not in S.

We define a decreasing sequence of conditions $\langle \mathbf{x}_i \colon i \leq \delta \rangle$ satisfying the following conditions for all $i < \delta$:

(5) $\langle \mathbf{x}_j \colon j \leq i \rangle$ is in N_{i+1},

(6) $\max(\mathbf{x}_i) = N_i \cap \kappa^+$.

Define

$$\mathbf{x}_0 = \mathbf{x} \cup \{u \in E \colon N_0 \cap \kappa \subseteq u \subseteq N_0 \cap \kappa^+\}.$$

Since $\mathbf{x}$ is in N_0 and $|\mathbf{x}| < \kappa$, $\mathbf{x} \subseteq N_0$. So if a is in $\mathbf{x}$ then since $|a| < \kappa$, $a \subseteq N_0$, and $|a| < N_0 \cap \kappa$. Using this fact it is easy to check that $\mathbf{x}_0$ is a condition below $\mathbf{x}$ satisfying the requirements. Suppose that $\mathbf{x}_i$ is defined for a fixed $i < \delta$. Let $\mathbf{y}_i$ be the $<_\theta$-least condition such that $\mathbf{y}_i \leq \mathbf{x}_i$ and $\mathbf{y}_i$ is in D_i. Since D_i is in N_{i+1}, $\mathbf{y}_i$ is in N_{i+1}. Define

$$\mathbf{x}_{i+1} = \mathbf{y}_i \cup \{u \in E \colon N_{i+1} \cap \kappa \subseteq u \subseteq N_{i+1} \cap \kappa^+\}.$$

It is easy to check that $\mathbf{x}_{i+1}$ is a condition which is below $\mathbf{x}_i$ and satisfies the requirements.

Now suppose that $\gamma \leq \delta$ is a limit ordinal and $\langle \mathbf{x}_i \colon i < \gamma \rangle$ is defined. Since $\langle N_i \colon i < \gamma \rangle$ and $\langle D_i \colon i < \gamma \rangle$ are in $N_{\gamma+1}$, the sequence $\langle \mathbf{x}_i \colon i < \gamma \rangle$ can be defined in $N_{\gamma+1}$. Let $\mathbf{x}_\gamma$ be the closure of $\bigcup \{\mathbf{x}_i \colon i < \gamma\}$ under unions of increasing sequences. By Lemma 4.5, $\mathbf{x}_\gamma$ is a condition in $N_{\gamma+1}$ below each $\mathbf{x}_i$ and $\max(\mathbf{x}_\gamma) = N_\gamma \cap \kappa^+$.

This completes the construction. The condition $\mathbf{x}_\delta$ is in $\bigcap \{D_i \colon i < \delta\}$ and is below $\mathbf{x}$. So $\mathbb{P}$ is $< \kappa$-distributive. □

The proof of the κ^+-chain condition is basically the same as with Gitik's poset.

Proposition 4.7. *The forcing poset $\mathbb{P}$ is κ^+-c.c.*

Proof. Suppose for a contradiction that $\langle \mathbf{x}_i \colon i < \kappa^+ \rangle$ is an antichain in $\mathbb{P}$. By the Δ-system lemma, we can assume without loss of generality that there is a set d such that $\max(\mathbf{x}_i) \cap \max(\mathbf{x}_j) = d$ for $i < j$. Since κ is strongly inaccessible, there exist $i < j$ such that $\mathcal{P}(d) \cap \mathbf{x}_i = \mathcal{P}(d) \cap \mathbf{x}_j$.

Choose a set a in $E \cap (P_\kappa \kappa^+ \setminus S(\kappa, \kappa^+))$ which contains $\max(\mathbf{x}_i) \cup \max(\mathbf{x}_j)$, and $|a| > |\max(\mathbf{x}_i) \cup \max(\mathbf{x}_j)|$. Define $X = \{u \in E \setminus S \colon a \cap \kappa \subseteq u \subseteq a\}$, and let $\mathbf{y} = \mathbf{x}_i \cup \mathbf{x}_j \cup X$. Note that if u is in X then $|u| = |a|$, since $|a| = a \cap \kappa$. Clearly $\mathbf{y}$ is a condition. But since $\mathbf{x}_i$ and $\mathbf{x}_j$ are incompatible, $\mathbf{y}$ is not below one of them.

Without loss of generality assume that $\mathbf{y}$ is not below $\mathbf{x}_i$, i.e. $\mathbf{y}$ does not end-extend $\mathbf{x}_i$. Fix b in $\mathbf{y} \setminus \mathbf{x}_i$ and c in $\mathbf{x}_i$ such that $b \subseteq c$. If b is in X, then $|b| = |a| > |\max(\mathbf{x}_i)| \geq |c|$, which is impossible. So b is in $\mathbf{x}_j \setminus \mathbf{x}_i$. Hence $b \subseteq \max(\mathbf{x}_j) \cap c \subseteq \max(\mathbf{x}_j) \cap \max(\mathbf{x}_i) = d$. Therefore b is in $\mathcal{P}(d) \cap \mathbf{x}_j = \mathcal{P}(d) \cap \mathbf{x}_i$. So b is in $\mathbf{x}_i$, which is a contradiction. □

Proposition 4.8. *The forcing poset $\mathbb{P}$ forces that S is non-stationary.*

Proof. Fix a name $\dot{D}$ such that $\mathbb{P}$ forces that $\bigcup \dot{G}_{\mathbb{P}} = \dot{D}$. We claim that $\mathbb{P}$ forces that $\dot{D}$ is a club subset of $P_\kappa \kappa^+$ disjoint from S. Clearly $\dot{D}$ is disjoint from S. To show that $\dot{D}$ is cofinal, suppose that $\mathbf{x} \Vdash \dot{a} \in P_\kappa \kappa^+$. Since $\mathbb{P}$ is $< \kappa$-distributive, there is $\mathbf{x}_0 \leq \mathbf{x}$ and a such that $\mathbf{x}_0 \Vdash \check{a} = \dot{a}$. Let b be any set in $P_\kappa\kappa^+ \setminus S(\kappa, \kappa^+)$ which is closed under H and F such that

$$|\bigcup \mathbf{x}_0|^+ \cup \max(\mathbf{x}_0) \cup a \subseteq b.$$

Define

$$\mathbf{y} = \mathbf{x}_0 \cup \{u \in E\colon b \cap \kappa \subseteq u \subseteq b\}.$$

Then $\mathbf{y}$ is a condition refining $\mathbf{x}_0$ and $y \Vdash a \subseteq b \in \dot{D}$. So $\dot{D}$ is cofinal.

Suppose that $\mathbf{x} \Vdash \langle \dot{a}_i\colon i < \beta\rangle$ is an increasing sequence from $\dot{D}$ with size less than κ. Since $\mathbb{P}$ is $< \kappa$-distributive there is $\mathbf{y} \leq \mathbf{x}$ and a family $\{a_i\colon i < \beta\}$ such that for each $i < \beta$, $\mathbf{y} \Vdash \check{a}_i = \dot{a}_i$. Note that since $\mathbf{y}$ forces that a_i is in $\dot{D}$, a_i must be in $\mathbf{y}$. For otherwise we can extend $\mathbf{y}$ to a condition $\mathbf{z}$ for which it is impossible to add a_i. Since $\mathbf{y}$ is closed, $\bigcup\{a_i\colon i < \beta\}$ is in $\mathbf{y}$. So $\mathbf{y}$ forces that $\bigcup\{\dot{a}_i\colon i < \beta\}$ is in $\dot{D}$. $\square$

We now verify the comments made prior to Lemma 4.1 concerning the relationship between $\mathbb{P}_F$ and $\mathbb{P}_S$.

Proposition 4.9. *Suppose that $F\colon \kappa^{+<\omega} \to \kappa^+$ is a partial function in V, and in W, $\mathbb{P}$ is a $\mathbb{P}_F$-name such that $\mathbb{P}_F * \mathbb{P}$ is $< \kappa$-distributive and κ^+-c.c. Then $\mathbb{P}_F * \mathbb{P}$ forces that for every stationary set $T \subseteq S(\kappa, \kappa^+)$, there is $A \subseteq \kappa^+$ such that*

$$T = \{a \in S(\kappa, \kappa^+)\colon \sup a \in A\}$$

modulo clubs.

Proof. Let $G_F * G_{\mathbb{P}}$ be generic for $\mathbb{P}_F * \mathbb{P}$ over W. Suppose that T is a stationary subset of $S(\kappa, \kappa^+)$ in $W[G_F * G_{\mathbb{P}}]$. Let D be the club set of a in $P_\kappa\kappa^+$ such that a is in $\bigcup G_F$ and is closed under F and H. Let

$$A = \{\alpha < \kappa^+\colon \exists a \in T \cap D \ \sup a = \alpha\}$$

We show that a is in $T \cap D$ iff a is in $S(\kappa, \kappa^+) \cap D$ and $\sup a$ is in A. Note that if a is in $T \cap D$ then $\sup a$ is in A by definition of A.

Suppose that a is in $S(\kappa, \kappa^+) \cap D$ and $\sup a$ is in A. Fix b in $T \cap D$ such that $\sup a = \sup b = \beta$. Let $\kappa_a = a \cap \kappa$. Now go back to W. By Lemma 4.1, $\text{o.t.}(a) = \kappa_a^+ = \text{cf}(\beta) = \text{o.t.}(b)$. So $b \cap \kappa = \kappa_a$. Since a and b are closed under suprema of bounded subsets with cofinality different from $\text{cf}(\kappa_a)$, $a \cap b$ is an unbounded subset of β and so $|a \cap b| = |a| = |b| = \kappa_a^+$. But $a \cap b$ is closed under H and F. Since a and b are in $\bigcup G_F \cap S(\kappa, \kappa^+)$, F is Jonsson for a and b. Since $a \cap b \subseteq a$, $a \cap b = a$, and similarly $a \cap b = b$. So $a = b$, and therefore a is in T. $\square$

5. Iterated Forcing

In the sections which follow we will need to iterate forcing posets for destroying stationary sets over different cardinals. These posets are distributive, but not strategically closed. The usual methods of Easton support iterations require that the posets be strategically closed. To overcome this problem we use Prikry forcing techniques.

In Section 2 we described how to turn a distributive forcing poset into a Prikry type forcing poset. So we can iterate the poset for destroying stationary sets by using iterations of Prikry type forcing posets.

Magidor [11] first showed how to iterate Prikry forcing using conditions with full support. Gitik [6] used this style of iteration to construct a model with a non-splitting subset of $P_\kappa\kappa^+$. However, lifting an elementary embedding with such an iteration is problematic. To overcome this difficulty, Gitik [7] invented a method for iterating Prikry type forcing posets using conditions with Easton support. We will use Easton support Prikry iterations in this paper.

In this section we describe a general schema for iterating Prikry type forcing posets with Easton support, and give a sample application.

An **Easton support Prikry iteration** is an iterated forcing

$$\langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \colon \alpha < \kappa \rangle,$$

for some ordinal κ, satisfying the following properties:

(1) There exists a set $A \subseteq \kappa$ consisting of strongly inaccessible cardinals such that if $\mathbb{Q}_\alpha$ is non-trivial, then α is in A,

(2) $\mathbb{P}_\alpha$ forces that $|\mathbb{Q}_\alpha| < \min(A \setminus (\alpha + 1))$,

(3) $\mathbb{P}_\alpha$ forces that $\langle \mathbb{Q}_\alpha, \leq, \leq^* \rangle$ is a Prikry type forcing poset,

(4) $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{Q}_\alpha$,

(5) If α is a limit ordinal, then $\mathbb{P}_\alpha$ consists of functions p with domain an Easton subset of $\alpha \cap A$ such that $p \restriction \beta$ is in $\mathbb{P}_\beta$ for all β less than α,

(6) If α is a limit ordinal and p, q are in $\mathbb{P}_\alpha$, then $q \leq p$ if $q \restriction \beta \leq p \restriction \beta$ for β less than α, and there is a finite set a such that for all β in $\mathrm{dom}(p) \setminus a$, $q \restriction \beta \Vdash q(\beta) \leq^* p(\beta)$.

(7) If α is a limit ordinal, $q \leq^* p$ in $\mathbb{P}_\alpha$ if $q \leq p$ and the finite set a in (6) is empty; i.e. for all β in $\mathrm{supp}(p)$, $q \restriction \beta \Vdash q(\beta) \leq^* p(\beta)$,

(8) $q^\frown \dot{b} \leq^* p^\frown \dot{a}$ in $\mathbb{P}_{\beta+1}$ if $q \leq^* p$ in $\mathbb{P}_\beta$ and $q \Vdash \dot{b} \leq^* \dot{a}$.

Such an iteration satisfies the Prikry property; see [7] for a proof. We say that the iteration above is **defined on** A. If p is a condition in $\mathbb{P}_\alpha$, the **support of** p, denoted by $\mathrm{supp}(p)$, is the domain of p as a function.

If β is less than α, then $\mathbb{P}_\alpha$ factors into $\mathbb{P}_\beta * \mathbb{Q}_\beta * \mathbb{P}_{\beta,\alpha}$. Suppose that $\mathbb{Q}_\gamma$ is forced to be γ-weakly closed for all γ greater than β; then $\langle \mathbb{P}_{\beta,\alpha}, \leq^* \rangle$ is forced to be $\min(A \setminus (\beta + 1))$-weakly closed. Therefore $\mathbb{P}_{\beta,\alpha}$ does not add any bounded subsets to $\min(A \setminus (\beta + 1))$.

The following result follows from the usual proof of the corresponding fact for Easton support iterations.

Proposition 5.1. *If $|\mathbb{P}_\beta| < \alpha$ for all β less than α and α is a Mahlo cardinal, then $\mathbb{P}_\alpha$ is α-c.c.*

The following lemma describes in more detail the sort of iteration we will use later.

Lemma 5.2. *Suppose that $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \colon \alpha < \kappa \rangle$ is an Easton support Prikry iteration defined on a set A. Assume that for all α in A:*

(1) $\mathbb{P}_\alpha$ forces that $\mathbb{Q}_\alpha$ is α-weakly closed and satisfies the direct extension property,

(2) there is μ_α and λ_α such that

$$\alpha \leq \mu_\alpha < 2^{(\mu_\alpha^{<\alpha})} \leq \lambda_\alpha^+ < \min(A \setminus (\alpha + 1)),$$

(3) there is a normal ultrafilter on $P_\alpha\lambda_\alpha$ with $j_\alpha \colon V \to M_\alpha = \mathrm{Ult}(V, U_\alpha)$ such that $\min(j_\alpha(A) \setminus (\alpha + 1)) > \lambda_\alpha$.

Then for each α in A, $\mathbb{P}_\alpha$ forces that α is μ_α-strongly compact.

Proof. If α is not a limit point of A, then $|\mathbb{P}_\alpha| < \alpha$, so the result follows.

Suppose that α is a limit point of A. Then

$$j_\alpha(\mathbb{P}_\alpha) = \mathbb{P}_\alpha * \mathbb{Q}_\alpha * \mathbb{P}_{\mathrm{tail}}$$

where $\mathbb{P}_{\mathrm{tail}}$ is forced to be $\min(j_\alpha(A) \setminus (\alpha+1))$-weakly closed. Let G_α be generic for $\mathbb{P}_\alpha$ over V.

Since $\mathbb{P}_\alpha$ is α-c.c., every member of $(P_\alpha\mu_\alpha)^{V[G_\alpha]}$ is covered by a set in $(P_\alpha\mu_\alpha)^V$. Since α is strongly inaccessible in $V[G_\alpha]$, $|(P_\alpha\mu_\alpha)^{V[G_\alpha]}| = (\mu_\alpha^{<\alpha})^V$. So we can enumerate all canonical $\mathbb{P}_\alpha$-names for subsets of $P_\alpha\mu_\alpha$ as $\langle \dot{X}_i \colon i < \lambda_\alpha^+\rangle$, possibly with repetitions. By the closure of M_α, every initial segment of $\langle j_\alpha(\dot{X}_i) \colon i < \lambda_\alpha^+\rangle$ is in M_α.

Now j_α“$G_\alpha = G_\alpha$, so if H is generic for $\mathbb{Q}_\alpha * \mathbb{P}_{\mathrm{tail}}$ over $V[G_\alpha]$, we can lift j_α to $j_\alpha \colon V[G_\alpha] \to M_\alpha[G_\alpha * H]$ in $V[G_\alpha * H]$. Let G be generic for $\mathbb{Q}_\alpha$ over $V[G_\alpha]$. In $V[G_\alpha * G]$ define a $\leq^*$-decreasing sequence $\langle q_i \colon i < \lambda_\alpha^+\rangle$ in $\mathbb{P}_{\mathrm{tail}}$ as follows. Let $q_0 = 1$, and given q_i, let $q_{i+1} \leq^* q_i$ such that q_{i+1} decides the statement j_α“$\mu_\alpha \in j_\alpha(\dot{X}_i)$. For limit $\delta < \lambda_\alpha^+$ apply the weak closure of $\mathbb{P}_{\mathrm{tail}}$ to obtain q_δ. Note that each initial segment of the sequence is in $M_\alpha[G_\alpha * G]$.

Now $\mathbb{Q}_\alpha$ is α-weakly closed and satisfies the direct extension property (if $\mathbb{Q}_\alpha$ is the trivial poset, this is true vacuously). In $V[G_\alpha]$ define U^* by letting $X \in U^*$ iff there is $s \leq^* 1$ in $\mathbb{Q}_\alpha$ and $\dot{X}_i$ such that $\dot{X}_i^{G_\alpha} = X$ and

$$M_\alpha[G_\alpha] \models s \Vdash_{\mathbb{Q}_\alpha} q_{i+1} \Vdash_{\mathbb{P}_{\mathrm{tail}}} j_\alpha\text{“}\mu_\alpha \in j_\alpha(\dot{X}_i).$$

We show that U^* is a fine ultrafilter on $P_\alpha\mu_\alpha$.

First we prove that the definition does not depend on i. Suppose $X_i^{G_\alpha} = X_j^{G_\alpha} = X$, where $i < j$. Fix p in G_α which forces that $\dot{X}_i = \dot{X}_j$. Now $j_\alpha(p) = p^\frown 1$ in $j_\alpha(\mathbb{P}_\alpha)$, and p is in G_α. Therefore

$$M_\alpha[G_\alpha] \models \mathbb{Q}_\alpha * \mathbb{P}_{\mathrm{tail}} \Vdash j_\alpha(\dot{X}_i) = j_\alpha(\dot{X}_j).$$

Apply the Prikry property of $\mathbb{Q}_\alpha$ to find $s_i, s_j \leq^* 1$ such that s_i decides which way q_{i+1} decides the statement j_α“$\mu_\alpha \in j_\alpha(\dot{X}_i)$, and similarly for s_j and $\dot{X}_j$. By the direct extension property, s_i and s_j are compatible, and $j_\alpha(\dot{X}_i) = j_\alpha(\dot{X}_j)$. So clearly s_i and s_j must decide their respective statements the same way.

Similar arguments show that U^* is a fine ultrafilter. We show that U^* is α-complete. Suppose that p is in G, $\dot{a}$ is a name for a subset of λ_α^+ with size less than α, and p forces that $\langle \dot{X}_i \colon i \in \dot{a}\rangle$ is a partition of $P_\alpha\mu_\alpha$. Then $j_\alpha(p) = p^\frown 1$ forces in $j_\alpha(\mathbb{P}_\alpha)$ that $j_\alpha(\langle \dot{X}_i \colon i \in \dot{a}\rangle) = \langle j_\alpha(\dot{X}_i) \colon i \in a\rangle$ is a partition of $j_\alpha(P_\alpha\mu_\alpha)$. So

$$M_\alpha[G_\alpha] \models \mathbb{Q} * \mathbb{P}_{\mathrm{tail}} \Vdash \langle j_\alpha(\dot{X}_i) \colon i \in a\rangle \text{ is a partition of } j_\alpha(P_\alpha\mu_\alpha).$$

Apply the α-weak closure of $\mathbb{Q}_\alpha$ to obtain s, a direct extension of 1 in $\mathbb{Q}_\alpha$, which decides which way that q_{i+1} decides the statement j_α“$\mu \in j_\alpha(\dot{X}_i)$, for all i in a. Clearly there must be some i in a such that

$$M_\alpha[G_\alpha] \models s \Vdash_{\mathbb{Q}_\alpha} q_{i+1} \Vdash_{\mathbb{P}_{\mathrm{tail}}} j_\alpha\text{“}\mu \in j_\alpha(\dot{X}_i).$$

So $\dot{X}_i^{G_\alpha}$ is in U^*. □

We now give a sample application of the forcing poset from Section 4 and the Prikry techniques above to prove the following theorem.

Theorem 5.3. *Suppose that GCH holds and there exists a κ^{++}-supercompact cardinal κ. Then there exists a model in which κ is κ^+-supercompact and almost all a in $S(\kappa,\kappa^+)$ satisfy that $a \cap \kappa$ is strongly inaccessible.*

There are limits to this sort of result; see Proposition 5.7 below. In the next section we give a more elaborate construction of a model in which almost all a in $S(\kappa,\kappa^+)$ satisfy that $\mathrm{cf}(a \cap \kappa) = \omega$.

Start with a model V in which κ is κ^{++}-supercompact and GCH holds. Let $\mathbb{R}$ be the Radin forcing for adding a club C of V-regulars to κ, while preserving the κ^{++}-supercompactness of κ. Let $G_{\mathbb{R}}$ be generic for $\mathbb{R}$ over V and let $W = V[G_{\mathbb{R}}]$.

We would like to force with the poset $\mathbb{P}_S$ which destroys the stationarity of the set of a in $S(\kappa,\kappa^+)$ such that $a \cap \kappa$ is not strongly inaccessible. But in order to preserve the κ^+-supercompactness of κ, or even to preserve the stationarity of $S(\kappa,\kappa^+)$, we need to iterate posets like $\mathbb{P}_S$ below κ.

Let A be the set of $\beta < \kappa$ in $\lim(C)$ such that β is β^+-supercompact. We define an Easton support Prikry iteration

$$\langle \mathbb{P}_\beta, \dot{\mathbb{Q}}_\beta \colon \beta < \kappa \rangle.$$

Suppose that $\mathbb{P}_\beta$ is defined for some fixed β in $A \cup \{\kappa\}$. Let $\dot{S}_\beta$ be a $\mathbb{P}_\beta$-name for the set of a in $S(\beta,\beta^+)$ such that $a \cap \beta$ is not strongly inaccessible. Let $\dot{B}_\beta$ be a $\mathbb{P}_\beta$-name for the set of $\alpha < \beta^+$ such that $\mathrm{cf}(\alpha) = \mu^+$ for some $\mu < \kappa$ which is not strongly inaccessible.

We claim that $\mathbb{P}_\beta$ forces that

$$\dot{S}_\beta = \{a \in S(\beta,\beta^+) \colon \sup a \in \dot{B}_\beta\}.$$

For suppose that a is in S_β. Then a is in $S(\beta,\beta^+)$ and $a \cap \beta$ is not inaccessible. Since $\mathrm{cf}(\sup a) = (a \cap \beta)^+$, $\sup a$ is in B_β. On the other hand, suppose that a is in $S(\kappa,\kappa^+)$ and $\sup a$ is in B_β. Then $\mathrm{o.t.}(a) = (a \cap \beta)^+$, so $\mathrm{cf}(\sup a) = (a \cap \beta)^+$. Since $\sup a$ is in B_β, $a \cap \beta$ is not inaccessible.

So $\dot{S}_\beta$ is of the form required to define the $< \beta$-distributive, β^+-c.c. forcing poset $\mathbb{P}_{S_\beta}$ from Definition 4.2.

By Lemma 5.2, $\mathbb{P}_\beta$ forces that β is β^+-strongly compact. The forcing poset $\mathbb{P}_{S_\beta}$ is $< \beta$-distributive, has size β^+, and has no more than β^+-many maximal antichains. By Lemma 2.3 there exist a Prikry type forcing poset $\langle \mathbb{Q}_\beta, \leq, \leq^* \rangle$ which is β-weakly closed, has the direct extension property, and $\langle \mathbb{Q}, \leq \rangle$ is equivalent to $\mathbb{P}_{S_\beta}$. So at stage β we force with $\langle \mathbb{Q}_\beta, \leq \rangle$.

This completes the definition of the iteration. Let $\dot{S} = \dot{S}_\kappa$. Clearly $\mathbb{P}_\kappa * \mathbb{P}_S$ forces that for almost all a in $S(\kappa,\kappa^+)$, $a \cap \kappa$ is strongly inaccessible. It remains to check that $\mathbb{P}_\kappa * \mathbb{P}_S$ preserves the κ^+-supercompactness of κ.

Fix an elementary embedding $j \colon W \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \kappa^{++}$, and ${}^{\kappa^{++}}M \subseteq M$. By GCH, any ultrafilter on $P_\kappa \kappa^+$ has size κ^{++} and therefore is in M. So M models that κ is κ^+-supercompact and κ is in $j(A)$. Enumerate all canonical $\mathbb{P}_\kappa * \mathbb{P}_S$-names for subsets of $P_\kappa \kappa^+$ as $\langle \dot{X}_i \colon i < \kappa^{++} \rangle$. Write

$$j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{Q}_\kappa * \mathbb{P}_{\mathrm{tail}},$$

where $\mathbb{Q}_\kappa$ is a Prikry type forcing poset which is equivalent to $\mathbb{P}_S$ and $\mathbb{P}_{\mathrm{tail}}$ is forced to be κ^{+3}-weakly closed. Let $G_\kappa * G_S * G_{\mathrm{tail}}$ be generic for $\mathbb{P}_\kappa * \mathbb{P}_S * \mathbb{P}_{\mathrm{tail}}$ over W. In $W[G_\kappa * G_S * G_{\mathrm{tail}}]$ the map j can be extended to

$$j \colon W[G_\kappa] \to M[G_\kappa * G_S * G_{\mathrm{tail}}].$$

Let $M_1 = M[G_\kappa * G_S * G_{\text{tail}}]$.

Now we construct a master condition. In M_1 define

$$t = \bigcup j``G_S \cup \{u \in j(E) \colon \kappa \subseteq u \subseteq j``\kappa^+\}$$

Lemma 5.4. *The set t is a condition in $j(\mathbb{P}_S)$ and $t \leq j(\mathbf{x})$ for all $\mathbf{x}$ in G_S.*

Proof. It suffices to prove that t is the closure of $\bigcup j``G$ in M_1 under unions of $\subseteq$-increasing sequences. For if this is true, then Lemma 4.4 and the fact that κ is strongly inaccessible in M_1 imply that $t \cap j(S) = \emptyset$ and t is a condition in $j(\mathbb{P}_S)$ which is below $j(\mathbf{x})$ for all $\mathbf{x}$ in G_S.

Clearly the set $\{u \in j(E) \colon \kappa \subseteq u \subseteq j``\kappa^+\}$ is itself closed. Since $\mathbb{P}_{\text{tail}}$ is $\kappa^{+3}$-weakly closed, $\bigcup G_S$ remains a club subset of $P_\kappa \kappa^+$ in $M_1$. Therefore $\bigcup j``G_S$ is closed under unions of $\subseteq$-increasing sequences with length less than κ. On the other hand, suppose that $\langle j(a_i) \colon i < \kappa \rangle$ is $\subset$-increasing. Then the sequence of ordinals

$$\langle j(a_i) \cap j(\kappa) \colon i < \kappa \rangle = \langle a_i \cap \kappa \colon i < \kappa \rangle$$

must be unbounded in κ, since $\text{o.t.}(a_i) = (a_i \cap \kappa)^+$ and the a_i's are getting larger. Therefore

$$u = \bigcup \{j(a_i) \colon i < \kappa\}$$

must satisfy that $\kappa \subseteq u \subseteq j``\kappa^+$. So the closure of $\bigcup j``G$ is contained in t.

To show that the closure is equal to t, by Lemma 4.4 (3) it suffices to show that for any u in E with $\kappa \subseteq u \subseteq j``\kappa^+$, there is v in the closure with $u \subseteq v$. Suppose first that $|u| = \kappa$. Let $\langle \alpha_i \colon i < \kappa \rangle$ enumerate u. Inductively define a $\subseteq$-increasing sequence $\langle a_i \colon i < \kappa \rangle$ from $\bigcup G_S$ by choosing a_{i+1} so that $j(a_{i+1})$ contains α_i and i as members. Then $u \subseteq \bigcup j(a_i)$ and $\bigcup j(a_i)$ is in the closure of $\bigcup j``G_S$. In particular this proves that all initial segments of $j``\kappa^+$ which are in E are in the closure of $\bigcup j``G_S$, and we can write $j``\kappa^+$ as the union of such initial segments. Hence $j``\kappa^+$ is in the closure of $\bigcup j``G_S$ and t is equal to this closure.

□

Lemma 5.5. *In $W[G_\kappa * G_S]$, κ is κ^+-supercompact.*

Proof. For each $i < \kappa^{++}$, let D_i be the dense set of conditions in $j(\mathbb{P}_S)$ which decide the statement $j``\kappa^+ \in j(\dot{X}_i)$. Since $j(\mathbb{P}_{S_\kappa})$ is $< j(\kappa)$-distributive and $\kappa^{++} < j(\kappa)$, $D = \bigcap D_i$ is a dense set. So fix $s \leq t$ in D. Let $\dot{s}$ be a $\mathbb{P}_{\text{tail}}$-name for s. That is, $\dot{s}$ is a name such that $\mathbb{P}_{\text{tail}}$ forces that $\dot{s}$ decides the statements $j``\kappa^+ \in j(\dot{X}_i)$, and $s \leq j(\mathbf{x})$ for all $\mathbf{x}$ in G_S.

Apply the weak-closure of $\mathbb{P}_{\text{tail}}$ to get a condition q in $\mathbb{P}_{\text{tail}}$ such that for all $i < \kappa^{++}$, q decides the statement

$$\dot{s} \Vdash_{j(\mathbb{P}_S)} j``\kappa^+ \in j(\dot{X}_i).$$

Define U in $W[G_\kappa * G_S]$ by letting X_i be in U if

$$q \Vdash_{\mathbb{P}_{\text{tail}}} \dot{s} \Vdash_{j(P_S)} j``\kappa^+ \in j(\dot{X}_i).$$

We claim that U is a normal ultrafilter on $P_\kappa \kappa^+$. The proof that U is well-defined and is a fine ultrafilter on $P_\kappa \kappa^+$ is essentially the same as the proof of Lemma 5.2.

For normality, suppose that $p^\frown \mathbf{x}$ is in $G_\kappa * G_S$ and forces that $\dot{f} \colon P_\kappa \kappa^+ \to \kappa^+$ is a regressive function. Then $j(p^\frown \mathbf{x}) = p^\frown 1^\frown 1^\frown j(\mathbf{x})$, which is above the condition $p^\frown 1^\frown q^\frown s$. So

$$M[G_\kappa * G_S] \models q \Vdash s \Vdash j(\dot{f}) \colon P_{j(\kappa)} j(\kappa^+) \to j(\kappa^+) \text{ is regressive.}$$

Applying the weak closure of $\mathbb{P}_{\rm tail}$ and distributivity of $j(\mathbb{P}_{S_\kappa})$, we can choose $q_1{}^\frown s_1 \leq q{}^\frown s$ and $\beta < \kappa^+$ such that

$$q_1 \Vdash s_1 \Vdash j(\dot{f})(j``\kappa^+) = j(\beta).$$

Let $\dot{X}_i$ be a name for the set $\{a \in P_\kappa\kappa^+ \colon f(a) = \beta\}$. Then

$$q_1 \Vdash s_1 \Vdash j``\kappa^+ \in j(\dot{X}_i).$$

Since $q{}^\frown\dot{s}$ decides the statement $j``\kappa^+ \in j(\dot{X}_i)$, clearly

$$q \Vdash \dot{s} \Vdash j``\kappa^+ \in j(\dot{X}_i)$$

and X_i is in U. Therefore U is normal. $\square$

This completes the proof of Theorem 5.3.

We cannot replace "strongly inaccessible" in the last result with "supercompact", as we see next.

Lemma 5.6. *If β is β^+-supercompact, then there is a stationary set of a in $S(\beta,\beta^+)$ such that $a \cap \beta$ is not $(a\cap\beta)^+$-supercompact.*

Proof. Let S be the set of a in $S(\beta,\beta^+)$ such that $a\cap\beta$ is not $(a\cap\beta)^+$-supercompact. Fix U a normal ultrafilter on $P_\beta\beta^+$ which is minimal in the Mitchell ordering of normal ultrafilter on $P_\beta\beta^+$. Let $j\colon V \to M = Ult(V,U)$. Since M contains all subsets of $P_\beta\beta^+$ and U is minimal, M models that β is not β^+-supercompact. So $j``\beta^+$ is in $j(S)$. Suppose that $C$ is club subset of $P_\beta\beta^+$. Then $j``C$ is a directed subset of $j(C)$ with size less than $j(\beta)$, so $\bigcup j``C = j``\beta^+$ is in $j(C) \cap j(S)$. By elementarity, $S \cap C$ is non-empty. So S is stationary. $\square$

Proposition 5.7. *Suppose that $S(\kappa,\kappa^+)$ is stationary. Then there is a stationary set of a in $S(\kappa,\kappa^+)$ such that $a\cap\kappa$ is not $(a\cap\kappa)^+$-supercompact.*

Proof. Suppose that $S(\kappa,\kappa^+)$ is stationary. Let C be a club subset of $P_\kappa\kappa^+$. Fix a function $f\colon \kappa^{+<\omega} \to \kappa^+$ such that any set a in $P_\kappa\kappa^+$ which is closed under f is in C. We will find a set a in $S(\kappa,\kappa^+)$ which is closed under f such that $a\cap\kappa$ is not $(a\cap\kappa)^+$-supercompact.

Fix b in $S(\kappa,\kappa^+)$ which is closed under f. If $b\cap\kappa$ is not $(b\cap\kappa)^+$-supercompact then we are done. Otherwise $b \cap\kappa = \kappa_b$ is κ_b^+-supercompact. Let $\pi\colon \kappa_b^+ \to b$ be the unique order preserving bijection. Note that crit $\pi = \kappa_b$. Define $f_b\colon \kappa_b^{+<\omega} \to \kappa_b^+$ by letting

$$f_b(\alpha_0,\ldots,\alpha_n) = \pi^{-1}(f(\pi(\alpha_0),\ldots,\pi(\alpha_n))).$$

Since κ_b is κ_b^+-supercompact, by Lemma 5.6 there is c in $P_{\kappa_b}\kappa_b^+$ such that c is closed under f_b, o.t.$(c) = (c\cap\kappa_b)^+$, and $c\cap\kappa_b$ not $(c\cap\kappa_b)^+$-supercompact. Let $a = \pi``c$. It is straightforward to check that $a$ is closed under $f$. Then $a\cap\kappa = \pi``c\cap\kappa = c\cap\kappa_b$ and o.t.$(a) =$ o.t.$(c) = (c\cap\kappa_b)^+$. So a is in $S(\kappa,\kappa^+)\cap C$ and $a\cap\kappa$ is not $(a\cap\kappa)^+$-supercompact. $\square$

6. $S(\kappa,\kappa^+)$ and Cofinalities

In this section we give another application of the forcing poset from Section 4 to prove the following theorem.

Theorem 6.1. *Suppose that GCH holds and κ is κ^{+4}-supercompact. Then there exists a model in which κ is a non-Mahlo inaccessible cardinal, GCH holds, $S(\kappa,\kappa^+)$ is stationary, and for almost all a in $S(\kappa,\kappa^+)$, $\mathrm{cf}(a\cap\kappa) = \omega$.*

Start with a model V in which κ is κ^{+4}-supercompact and GCH holds. Let W be a generic extension of V by a Radin forcing $\mathbb{R}$ which preserves the κ^{+4}-supercompactness of κ and adds a club set $C \subseteq \kappa$ such that every α in C is regular in V.

Let A be the set of $\beta < \kappa$ in $\lim(C)$ such that β is β^+-supercompact. We define an Easton support Prikry iteration $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_i \colon i < \kappa \rangle$ such that $\mathbb{Q}_\alpha$ is non-trivial iff α is in A. We will split the definition of $\mathbb{Q}_\alpha$ into three cases. So partition A into disjoint sets as follows. Let A_1 be the set of β in A such that β is β^+-supercompact but not β^{+2}-supercompact, and let A_2 be the set of β in A such that β is β^{+2}-supercompact but not β^{+3}-supercompact. Let A_3 be the set of β in A such that β is β^{+3}-supercompact.

Suppose that $\mathbb{P}_\beta$ is defined for a fixed β in A. Let $\dot{S}_\beta$ be a $\mathbb{P}_\beta$-name for the set of a in $S(\beta, \beta^+)$ such that $a \cap \kappa$ is a singular ordinal with uncountable cofinality. Let $\mathbb{P}_{S_\beta}$ be a $\mathbb{P}_\beta$-name for the forcing poset for destroying the stationarity of $\dot{S}_\beta$.

The definition of $\mathbb{Q}_\beta$ depends on which set A_i the ordinal β is in. Suppose that β is in A_1. By Lemma 5.2, $\mathbb{P}_\beta$ forces that β is β^+-strongly compact. By Lemma 2.3 there is a Prikry type forcing poset $\dot{\mathbb{Q}}_\beta$ which is β-weakly closed, has the direct extension property, and is equivalent to $\mathbb{P}_{S_\beta}$.

Suppose that β is in A_2. By the same argument as in Lemma 5.5, $\mathbb{P}_\beta * \mathbb{P}_{S_\beta}$ forces that β is β^{++}-supercompact. Apply Lemma 2.4 to choose a Prikry type forcing poset $\dot{\mathbb{Q}}_\beta$ which is β-weakly closed, satisfies the direct extension property, and is equivalent to $\mathbb{P}_{S_\beta} * \mathbb{PR}(U)$, where U is a name for a normal ultrafilter on $P_\beta \beta^+$.

Suppose that β is in A_3. We will prove that $\mathbb{P}_\beta * \mathbb{P}_{S_\beta}$ forces that there is a β-complete ultrafilter U_β on β which contains the set $A_2 \cap \beta$. We then define $\mathbb{Q}_\beta$ as a Prikry type forcing poset which is equivalent to $\mathbb{P}_{S_\beta} * \mathbb{PR}(U_\beta)$.

Lemma 6.2. *The poset $\mathbb{P}_\beta * \mathbb{P}_{S_\beta}$ forces that there is a β-complete ultrafilter U_β on β which contains the set $A_2 \cap \beta$.*

Proof. Let $j \colon W \to M = \mathrm{Ult}(W, U)$ for some normal ultrafilter U on $P_\beta \beta^{+3}$ which is minimal in the Mitchell ordering. Note that β is in $j(A_2)$. Write

$$j(\mathbb{P}_\beta) = \mathbb{P}_\beta * \dot{\mathbb{Q}}_\beta * \mathbb{P}_{\text{tail}}$$

where $\dot{\mathbb{Q}}_\beta$ is a Prikry type forcing poset equivalent to $\mathbb{P}_{S_\beta} * \mathbb{PR}(\beta, \beta^+)$. Let $G_\beta * G_S$ be generic for $\mathbb{P}_\beta * \mathbb{P}_{S_\beta}$.

Let $\dot{s}$ be a $\mathbb{PR}(\beta, \beta^+) * \mathbb{P}_{\text{tail}}$-name for the closure under unions of $\subseteq$-increasing sequences of the set

$$\bigcup j``G_S \cup \{u \in j(E) \colon \beta \subseteq u \subseteq j``\beta^+\}$$

in the generic ultrapower, where G_S is the generic filter for $\mathbb{P}_{S_\beta}$. We will show that $\dot{s}$ is forced to be a master condition.

Let $G_\beta * G_S * H * G_{\text{tail}}$ be generic for $j(\mathbb{P}_\beta)$ over M. Let $M_1 = M[G_\beta * G_S * H * G_{\text{tail}}]$. Extend j to $j \colon W[G_\beta] \to M_1$. The poset $\mathbb{P}_{\text{tail}}$ is sufficiently weakly closed, so s is in $M[G_\beta * G_S * H]$. Since $j``G_S$ is a directed subset of $j(P_S)$, $s$ satisfies all the conditions of Lemma 4.4, and in particular end-extends each member of $j``G_S$. So in order to check that s is a condition in $j(\mathbb{P}_S)$, we need to show that s is disjoint from $j(S)$.

Write

$$s^* = \{a \in s \colon a \cap j(\beta) < \beta\}.$$

Note that

$$s = s^* \cup \{u \in j(E)\colon \beta \subseteq u \subseteq j``\beta^+\}.$$

Since $\mathrm{cf}(j``\beta^+ \cap j(\beta)) = \mathrm{cf}(\beta) = \omega$ in M_1, it suffices to show that s^* is disjoint from $j(S)$.

Define $\{t_i\colon i \in On\}$ as follows. Let $t_0 = \bigcup j``G_S$, and $t_\delta = \bigcup\{t_i\colon i < \delta\}$ for limit $\delta \leq \zeta$. For each i, let t_{i+1} be the collection of sets a such that a is the union of an increasing sequence $\langle a_j\colon j < \xi\rangle$ of elements from t_i with length less than β. Let ζ be the first ordinal such that $t_{\zeta+1} = t_\zeta$. Then $s^* = \bigcup\{t_i\colon i < \zeta\}$. We only need to take unions of sequences with length less than β in the successor case since any strictly increasing sequence with length at least β has a union which contains β.

Sublemma 6.3. *If x is a set in $M[G_\beta]$ with size less than β and $y \subseteq x$ in M_1, then y is in $M[G_\beta]$.*

Proof. Since $\mathbb{P}_{S_\beta}$ is $<\beta$-distributive and $\mathbb{PR}(\beta,\beta^+) * \mathbb{P}_{\mathrm{tail}}$ is β-weakly closed over $M[G_\beta * G_S]$, $\mathbb{P}_{S_\beta} * \mathbb{PR}(\beta,\beta^+) * \mathbb{P}_{\mathrm{tail}}$ does not add bounded subsets to β over $M[G_\beta]$. By considering a bijection between x and its cardinality in $M[G_\beta]$, it is clear that the poset does not add subsets to x. □

Sublemma 6.4. $s^* \cap W[G_\beta] = t_0$.

Proof. Since $G_S \subseteq W[G_\beta]$ and $j \upharpoonright \beta^+$ is in $W[G_\beta]$, clearly $t_0 \subseteq s^* \cap W[G_\beta]$. We prove by induction on $i < \zeta$ that $t_i \cap W[G_\beta] \subseteq t_0$. The claim is obvious for t_0 and for limit stages. Suppose that $t_i \cap W[G_\beta] \subseteq t_0$ for a fixed $i < \xi$, and let a be in $t_{i+1} \cap W[G_\beta]$. Fix an increasing sequence $\langle a_j\colon j < \xi\rangle$ from t_i whose union is a. Since a is in $W[G_\beta]$, it is in $M[G_\beta]$, so each a_j is in $M[G_\beta] \cap t_i \subseteq W[G_\beta] \cap t_i \subseteq t_0$. Since $\mathcal{P}(a)$ is in $M[G_\beta]$, so also is the set $\{a_j\colon j < \xi\}$. But t_0 is a closed subset of $P_\beta(j``\beta^+)$ in $M[G_\beta * G_S]$, so $a = \bigcup\{a_j\colon j < \xi\}$ is in t_0. □

Sublemma 6.5. *The set s^* is disjoint from $j(S)$.*

Proof. It suffices to prove that whenever a is in s^* and $\mathrm{o.t.}(a) = (a \cap j(\beta))^+$, then either $\mathrm{cf}(a \cap j(\beta)) = \omega$ or $a \cap j(\beta)$ is strongly inaccessible. Suppose that a is in s^* and $\mathrm{o.t.}(a) = (a \cap j(\beta))^+$. Let $b = j^{-1}[a]$. Note that $b \cap \beta = a \cap j(\beta)$, so it suffices to show that $b \cap \beta$ is either strongly inaccessible or has cofinality ω. Consider the case when a is in $W[G_\beta]$. Then by the last sublemma, a is in t_0, so clearly $a = j(b)$ and b is in $\bigcup G_S$. Then b is in $S(\beta,\beta^+) \cap \bigcup G_S$, therefore $b \cap \beta = a \cap j(\beta)$ either has cofinality ω or is strongly inaccessible.

Suppose that a is not in $W[G_\beta]$. Let $\mu = b \cap \beta$ and let $\gamma = \sup b$. Note that μ^+ is the same in $M[G_\beta]$ and M_1. We claim that $\mathrm{cf}(\gamma)^{M[G_\beta]} = \mu^+$. Since $\mathbb{P}_{\mathrm{tail}}$ is weakly closed enough that it does not add subsets to β^+, it does not change the cofinality of γ. But $\mathbb{P}_{S_\beta} * \mathbb{PR}(\beta,\beta^+)$ only changes cofinalities to ω. This is true because $\mathbb{P}_{S_\beta}$ preserves all cofinalities, since it is $<\beta$-distributive and β^+-c.c.; also $\mathbb{PR}(\beta,\beta^+)$ does not add bounded subsets to β, changes the cofinality of β and β^+ to ω, and is β^{++}-c.c. So γ must have cofinality μ^+ in $M[G_\beta]$.

Define an increasing sequence $\langle d_i\colon i < \mu^+\rangle$ in $M[G_\beta]$ as follows. Let d_0 be any club subset of γ with order type μ^+ and $\mu \subseteq d_0$. Take unions at limits. Given d_i, define

$$d_i^* = d_i \cup H``d_i^{<\omega},$$

and define

$$d_{i+1} = d_i^* \cup \lim(d_i^*).$$

Now let $d = \bigcup\{d_i : i < \mu^+\}$. Then d is closed under H, has size μ^+, and $d \cap \gamma$ is a club subset of γ which contains μ. Note that b is closed under H. Let $u = d \cap b$. Then u is closed under H. Since u is a subset of d, $|d| < \beta$, and d is in $M[G_\beta]$, u is in $M[G_\beta]$. The set $u = b \cap d$ has order type μ^+ and so is in $S(\beta, \beta^+)$. Also

$$\mu = a \cap j(\beta) \subseteq j(u) \subseteq a$$

and $j(u)$ is in $j(E)$, so by Lemma 4.4 (3), $j(u)$ is in s^*. Since $j \restriction \beta^+$ is in $W[G_\beta]$, $j(u) = j``u$ is in $s^* \cap W[G_\beta]$. So by the last sublemma, $j(u)$ is in $t_0 = \bigcup j``G_S$.Therefore u is in $\bigcup G_S \cap S(\beta, \beta^+)$, and so either $\mu = u \cap \beta$ is strongly inaccessible or has cofinality ω. □

Enumerate all canonical $\mathbb{P}_\beta * \mathbb{P}_{S_\beta}$-names for subsets of β as $\langle \dot{X}_i : i < \beta^+\rangle$. Let $G_\beta * G_S$ be generic for $\mathbb{P}_\beta * \mathbb{P}_{\dot{S}_\beta}$ over W. Let $\dot{s}$ be a name for the master condition defined above.

Since $j(\mathbb{P}_{S_\beta})$ is $j(\beta)$-distributive, let $\dot{t}$ be a name for a refinement of $\dot{s}$ which decides the statement $\beta \in j(\dot{X}_i)$ for every $i < \beta^+$. Let q be a condition in $\mathbb{P}_{\mathrm{tail}}$ which decides each statement $s \Vdash \beta \in j(\dot{X}_i)$. Define U_β in $W[G_\beta * G_{S_\beta}]$ as the set of X_i such that there is a condition p in $\mathbb{PR}(\beta, \beta^+)$ which is a direct extension of 1 and

$$p \Vdash q \Vdash t \Vdash \beta \in j(\dot{X}_i).$$

Recall that whenever A is a collection of less than β many direct extensions of 1 in $\mathbb{PR}(\beta, \beta^+)$, there is p such that $p \leq^* p_0$ for all p_0 in A. Using this fact it is straightforward to check that U_β is a β-complete ultrafilter and contains $A_2 \cap \beta$. This completes the proof of the lemma. □

Let $\mathbb{P}^{\mathrm{sing}}_\beta$ denote the poset $\mathbb{P}_{\mathrm{Sing} \cap \beta}$ from Section 2 for adding a club set to $\mathrm{Sing} \cap \beta$ using closed, bounded subsets ordered by end-extension.

Lemma 6.6. *The poset $\mathbb{P}_\beta * \mathbb{P}_{S_\beta}$ forces that there is a projection mapping*

$$\mathbb{PR}(U_\beta) \to \mathcal{B}(\mathbb{P}^{\mathrm{sing}}_\beta),$$

where $\mathbb{P}^{\mathrm{sing}}_\beta$ is a name for the poset for adding a club to the set $\beta \cap \mathrm{Sing}$.

Proof. Let $G_\beta * G_S$ be generic for $\mathbb{P}_\beta * \mathbb{P}_{S_\beta}$ over W. We describe a canonical way to define a generic for $\mathbb{P}^{\mathrm{sing}}_\beta$ over $W[G_\beta * G_S]$ from a generic for $\mathbb{PR}(U_\beta)$.

Working in the model $W[G_\beta * G_S]$, enumerate $\mathbb{P}^{\mathrm{sing}}_\beta$ as $\langle c_i : i < \beta\rangle$. Let H be generic for $\mathbb{PR}(U_\beta)$ and let $\langle \beta_n : n < \omega\rangle$ be the generic Prikry sequence given by H.

We define by induction a sequence $\langle d_n, \alpha_n : n < \omega\rangle$ satisfying the following induction hypotheses:

(1) each α_n is in A_2,

(2) $\langle \alpha_n : n < \omega\rangle$ is a subsequence of $\langle \beta_n : n < \omega\rangle$,

(3) d_n is in $\mathbb{P}^{\mathrm{sing}}_{\alpha_n}$ in the model $W[G_{\alpha_n} * G_{S_{\alpha_n}}]$, where $G_{\alpha_n} * G_{S_{\alpha_n}}$ is the generic filter for $\mathbb{P}_{\alpha_n} * \mathbb{P}_{S_{\alpha_n}}$ given by G_β,

(4) d_{n+1} is an end-extension of d_n,

Let $d_0 = c_{\beta_0}$. Let α_0 be the least element in the Prikry sequence such that $\alpha_0 > \max(d_0)$ and for all $\alpha \geq \alpha_0$ in the Prikry sequence, α is in A_2. Since d_0 is a bounded subset of α_0 and $\mathbb{P}_\beta = \mathbb{P}_{\alpha_0} * \mathbb{P}_{S_{\alpha_0}} * \mathbb{PR}(\alpha_0, \alpha_0^+) * \mathbb{P}_{\mathrm{tail}}$, clearly d_0 is in $W[G_{\alpha_0} * G_{S_{\alpha_0}}]$.

Suppose that d_n and α_n are defined. Since α_n is in A_2, $\mathbb{P}_{\alpha_n+1} = \mathbb{P}_{\alpha_n} * \mathbb{P}_{S_{\alpha_n}} * \mathbb{PR}(\alpha_n, \alpha_n^+)$. Let $G_{\alpha_n+1} = G_{\alpha_n} * G_{S_{\alpha_n}} * H_{\alpha_n}$ be the generic for $\mathbb{P}_{\alpha_n+1}$ given by G_β.

Since $\mathbb{P}^{\text{sing}}_{\alpha_n}$ is $< \alpha_n$-distributive and has size α_n, there is a generic filter $G^{\text{sing}}_{\alpha_n}$ for $\mathbb{P}^{\text{sing}}_{\alpha_n}$ over $W[G_{\alpha_n} * G_{S_{\alpha_n}}]$ in the model $W[G_{\alpha_n+1}]$ which contains the condition d_n. Define

$$d_{n+1} = \bigcup G^{\text{sing}}_{\alpha_n} \cup \{\alpha_n\}.$$

Let α_{n+1} be the least element of the Prikry sequence greater than α_n. Since α_n has cofinality ω in $W[G_{\alpha_{n+1}}]$, d_{n+1} is a condition in $\mathbb{P}^{\text{sing}}_{\alpha_{n+1}}$. This completes the induction.

Let $d = \bigcup\{d_n : n < \omega\}$ and let

$$G^{\text{sing}} = \{d \cap (\gamma + 1) : \gamma < \beta\}.$$

We prove that G^{sing} is generic for $\mathbb{P}^{\text{sing}}_{\beta}$ over $W[G_\beta * G_S]$.

Let D be a dense subset of $\mathbb{P}^{\text{sing}}_{\beta}$ in $W[G_\beta * G_S]$. Define

$$B = \{\alpha \in A_2 \colon\ D \cap \mathbb{P}^{\text{sing}}_{\alpha} \text{ is a dense subset of } \mathbb{P}^{\text{sing}}_{\alpha} \text{ in } W[G_\alpha * G_{S_\alpha}]\ \}.$$

We claim that B is in U. It suffices to prove that in $M[G_\beta * G_S]$:

$$\mathbb{PR}(\beta, \beta^+) \Vdash q \Vdash s \Vdash \beta \in j(B),$$

where q and s are the names we constructed to define U_β. So let $G * G_{\text{tail}} * g$ be generic for $\mathbb{PR}(\beta, \beta^+) * \mathbb{P}_{\text{tail}} * j(S_\beta)$ such that q is in G_{tail} and t is in g. Since t is a master condition, we can extend j to

$$j \colon W[G_\beta * G_{\dot{S}_\beta}] \to M[G_\beta * G_S * G * G_{\text{tail}} * g].$$

Since $j \restriction \mathbb{P}^{\text{sing}}_{\beta}$ is the identity, $j(D) \cap \mathbb{P}^{\text{sing}}_{\beta} = D$. But D is a dense subset of $\mathbb{P}^{\text{sing}}_{\beta}$ in $M[G_\beta * G_{\dot{S}_\beta}]$, so by the definition of B we have $\beta \in j(B)$, and B is in U_β.

Since a tail of the Prikry sequence is contained in B, fix α_n such that $D \cap \mathbb{P}^{\text{sing}}_{\alpha_n}$ is a dense subset of $\mathbb{P}^{\text{sing}}_{\alpha_n}$ in $W[G_{\alpha_n} * G_{S_{\alpha_n}}]$. Since $G^{\text{sing}}_{\alpha_n}$ is generic, fix a in $D \cap G^{\text{sing}}_{\alpha_n}$. Then clearly a is in $D \cap G^{\text{sing}}$. So indeed G is generic. □

Note: The proof of Lemma 6.6 is similar to a method used by Gitik in [7].

This completes the definition of the iteration. Let $\dot{S}$ be a $\mathbb{P}_\kappa$-name for the set of a in $S(\kappa, \kappa^+)$ such that $a \cap \kappa$ is a singular ordinal with uncountable cofinality. We force with the poset $\mathbb{P}_\kappa * \mathbb{P}_S * \mathbb{P}^{\text{sing}}$, where $\mathbb{P}^{\text{sing}}$ adds a club through $\kappa \cap \text{Sing}$. Let $G_\kappa * G_S * G^{\text{sing}}$ be generic for $\mathbb{P}_\kappa * \mathbb{P}_S * \mathbb{P}^{\text{sing}}$ over W. Let $D = \bigcup G_S$, which is a club subset of $P_\kappa \kappa^+$. By definition of S, if a is in $S(\kappa, \kappa^+) \cap D$, then $a \cap \kappa$ is either inaccessible or has cofinality ω. But κ is non-Mahlo in $W[G_\kappa * G_S * G^{\text{sing}}]$, so there is a club of singulars in κ. It follows that for almost all a in $S(\kappa, \kappa^+) \cap D$, $a \cap \kappa$ is singular, and therefore has cofinality ω. The only thing left to prove is that $S(\kappa, \kappa^+)$ remains stationary.

Fix an elementary embedding $j \colon W \to M$ in W with $\text{crit} j = \kappa$, $j(\kappa) > \kappa^{+4}$, and ${}^{\kappa^{+4}}M \subseteq M$. By GCH κ is a κ^{+3}-supercompact cardinal in M, so κ is in $j(A_3)$. Write

$$j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \mathbb{Q}_\kappa * \mathbb{P}_{\text{tail}}$$

where $\mathbb{P}_{\text{tail}}$ is forced to be κ^{+5}-weakly closed and $\dot{\mathbb{Q}}_\kappa$ is equivalent to $\mathbb{P}_S * \mathbb{PR}(U_\kappa)$, where U_κ is an ultrafilter on κ which contains A_2. Since κ is in $j(A_3)$, by Lemma 6.6 $\mathbb{P}_\kappa * \mathbb{P}_S$ forces that there is a projection mapping

$$\mathbb{PR}(U_\kappa) \to \mathcal{B}(\mathbb{P}^{\text{sing}}).$$

Let $G_\kappa * G_S * G^{\text{sing}} * G^* * G_{\text{tail}}$ be generic over W for the poset

$$\mathbb{P}_\kappa * \mathbb{P}_S * \mathbb{P}^{\text{sing}} * (\mathbb{PR}(U_\kappa)/\mathcal{B}(\mathbb{P}_{\text{sing}})) * \mathbb{P}_{\text{tail}}.$$

Let $M_1 = M[G_\kappa * G_S * G^{\text{sing}} * G^* * G_{\text{tail}}]$. Extend j to $j \colon W[G_\kappa] \to M_1$.

We would like to further extend j to $W[G_\kappa * G_S * G^{\text{sing}}]$. In order to do this we construct a master condition in M_1 for $j(\mathbb{P}_S * \mathbb{P}^{\text{sing}})$. Note that $j(c) = c$ for all c in G^{sing}. Define s_1 by

$$s_1 = \bigcup G^{\text{sing}} \cup \{\kappa\}.$$

Since κ has cofinality ω in M_1, it is straightforward to check that s_1 is a closed bounded subset of $j(\kappa) \cap \text{Sing}$ which end-extends $j(c) = c$ for all c in G^{sing}. Define s_0 as the closure under unions of increasing sequences in M_1 of the set

$$\bigcup j``G_S \cup \{u \in j(E) \colon \kappa \subseteq u \subseteq j``\kappa^+\}$$

Exactly the same argument as in the proof of Lemma 6.2 shows that s_0 is a master condition in $j(\mathbb{P}_S)$.

So the condition $\langle s_0, s_1 \rangle$ is a master condition. Let g be a generic for $j(\mathbb{P}_S * \mathbb{P}^{\text{sing}})$ over $W[G_\kappa * G_S * G^{\text{sing}} * G^* * G_{\text{tail}}]$ which contains $\langle s_0, s_1 \rangle$. Then j extends to

$$j \colon W[G_\kappa * G_S * G^{\text{sing}}] \to M_1[g].$$

We show that $S(\kappa, \kappa^+)$ is stationary in $W[G_\kappa * G_S * G^{\text{sing}}]$. Suppose that C is a club subset of $P_\kappa \kappa^+$. Since κ^{+W} is a cardinal in $M_1[g]$, $j``\kappa^+$ is in $j(S(\kappa, \kappa^+))$. Also $j``C$ is a directed subset of $j(C)$ with size less than $j(\kappa)$, therefore

$$\bigcup j``C = j``\kappa^+ \in j(C) \cap j(S(\kappa, \kappa^+)).$$

By elementarity, $S(\kappa, \kappa^+) \cap C$ is non-empty. So $S(\kappa, \kappa^+)$ is stationary.

This completes the proof of Theorem 6.1.

7. $S(\kappa, \kappa^+)$ can be Non-Splitting

In this section we prove the following theorem.

Theorem 7.1. *Suppose GCH holds and there exists a κ^{+3}-supercompact cardinal κ. Then there exists a model in which κ is strongly inaccessible and $NS \restriction S(\kappa, \kappa^+)$ is κ^+-saturated.*

We will need the following result, which is a variation of the proof due to Tryba and Woodin [13] that successors of regular cardinals are not Jonsson cardinals.

Proposition 7.2. *Suppose that β is a regular cardinal. Then there is a partial function $F \colon \beta^{+<\omega} \to \beta^+$ such that whenever $\mathbb{Q}$ is a forcing poset which preserves stationary subsets of β^+, then $\mathbb{Q}$ forces that F is Jonsson for β^+.*

Proof. Partition $\beta^+ \cap \text{cof}(\beta)$ into disjoint stationary sets $\langle S_i \colon i < \beta^+ \rangle$. For each limit ordinal α below β^+, fix a set d_α closed and unbounded in α with order type $\text{cf}(\alpha)$. Define a partial function F as follows:

(1) $F(\gamma + 1) = \gamma$,

(2) $F(\xi, \xi)$ is the unique i such that ξ is in S_i,

(3) $F(\alpha, \gamma) = \min(d_\alpha \setminus \gamma)$ when $\gamma < \alpha$ and α is a limit ordinal.

Let G be generic for $\mathbb{Q}$ over V. In $V[G]$ suppose that A is a subset of β^+ with size β^+ and is closed under F. By the assumption on $\mathbb{Q}$, each S_i is still stationary in $V[G]$. We claim that for all $i < \beta^+$, the set $A \cap S_i$ is non-empty. This is enough, since if ξ is in $A \cap S_i$ then $F(\xi, \xi) = i$ is in A, so $\beta^+ \subseteq A$.

Since S_i is stationary in $V[G]$, fix ξ in $S_i \cap \lim(A)$. We show that ξ is in A. Otherwise let α be the least element of A greater than ξ. By (1), α is a limit ordinal. By (3), d_α is unbounded in ξ. Now $\text{o.t.}(d_\alpha) \leq \beta$, and since $d_\alpha \cap \xi$ is a proper initial segment of d_α, $\text{o.t.}(d_\alpha \cap \xi) < \beta$. But $d_\alpha \cap \xi$ is in V, so $\text{cf}^V(\xi) < \beta$. This is a contradiction since ξ is in $S_i \subseteq \text{cof}^V(\beta)$. □

Using this proposition we can obtain some information about non-splitting subsets of $P_\kappa \kappa^+$.

Corollary 7.3. *Suppose that S is a stationary subset of $P_\kappa \kappa^+$ such that $NS \restriction S$ is κ^+-saturated. Then there is a club set C such that:*
1. *$S \cap C \subseteq S(\kappa, \kappa^+)$,*
2. *for all $a \subsetneq b$ in $S \cap C$, $|a| < b \cap \kappa$.*

Proof. Let $F \colon \kappa^{+<\omega} \to \kappa^+$ be a partial function as in Proposition 7.2. Let S^* be the set of a in S such that $a \cap \kappa$ is a limit cardinal, $\text{o.t.}(a) = (a \cap \kappa)^+$, and F is Jonsson for a. Suppose that a and b are in S^* and $a \subsetneq b$. Then since a is closed under F and F is Jonsson for b, $|a| < |b| = (b \cap \kappa)^+$. But $|a|$ is a successor cardinal and $b \cap \kappa$ is a limit cardinal, therefore $|a| < b \cap \kappa$. So it suffices to show that $T = S \setminus S^*$ is non-stationary.

Suppose for a contradiction that T is stationary. Let U be a generic filter for $(NS \restriction S)^+$ with T in U. Let $j \colon V \to M$ be the generic elementary embedding induced by U. Since $(NS \restriction S)^+$ is κ^+-c.c., by Proposition 7.2 the function F is still Jonsson for κ^+ in $V[U]$. It is straightforward to check that this implies that $j(F)$ is Jonsson for $j``\kappa^+$ in $M$, and therefore $j``\kappa^+$ is in $j(S^*)$. So $j``\kappa^+$ is not in $j(T)$, which contradicts the fact that T is in U. □

The remainder of the section will be devoted to proving Theorem 7.1 by constructing a model in which $NS \restriction S(\kappa, \kappa^+)$ is κ^+-saturated. Start with a model V in which GCH holds and κ is a κ^{+3}-supercompact cardinal. Let W be a generic extension of V by a Radin forcing $\mathbb{R}$ which preserves the κ^{+3}-supercompactness of κ and adds a club set C to κ with order type κ such that every α in C is regular in V. For each strongly inaccessible $\beta < \kappa$ choose partial functions H_β and F_β in V as in Lemma 4.1 and Proposition 7.2.

Let B be the set of $\beta < \kappa$ in $\lim(C)$ such that β is β^{++}-supercompact. We define an Easton support Prikry iteration $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_i \colon i < \kappa \rangle$ as follows. Suppose that $\mathbb{P}_\beta$ is defined for some $\beta < \kappa$. If β is not in B, then let $\dot{\mathbb{Q}}_\beta$ be trivial.

Suppose that β is in $B \cup \{\kappa\}$. Let G_β be generic for $\mathbb{P}_\beta$ over W. We describe a forcing poset

$$\mathbb{P}_{F_\beta} * \mathbb{P}^\beta$$

in $W[G_\beta]$ and then define a Prikry type forcing poset $\mathbb{Q}_\beta$ which is equivalent to it. This will complete the definition of $\mathbb{P}_\kappa * \mathbb{P}_{F_\kappa} * \mathbb{P}^\kappa$. Let $\mathbb{P}_{F_\beta}$ be the $< \beta$-distributive, β^+-c.c. forcing poset from Definition 4.2 for destroying the stationarity of the set of a in $S(\beta, \beta^+)$ such that F_β is not Jonsson for a. Let G_F be generic for $\mathbb{P}_{F_\beta}$ over $W[G_\beta]$.

Working in $W[G_\beta * G_F]$, fix a surjective function $f \colon \beta^{++} \to \beta^{++} \times \beta^{++}$ such that

$$\forall \alpha, i, j < \beta^{++} \ (\ f(\alpha) = \langle i, j \rangle \Longrightarrow i \leq \alpha \ ).$$

Let E be the set of a in $P_\beta\beta^+$ which are closed under H_β and F_β. We define a $<\beta$-support forcing iteration

$$\langle \mathbb{P}^\beta_\alpha, \dot{\mathbb{Q}}^\beta_\alpha \colon \alpha < \beta^{++}\rangle$$

and a family of names

$$\{\dot{X}^\alpha_i \colon \alpha, i < \beta^{++}\}$$

by recursion on $\alpha < \beta^{++}$. The following properties will hold for each α:

(1) $\mathbb{P}^\beta_\alpha$ is $<\beta$-distributive,

(2) Let D^β_α denote the set of q in $\mathbb{P}^\beta_\alpha$ such that for all γ in $\mathrm{supp}(q)$ there is $\mathbf{x}$ in the ground model $W[G_\beta * G_F]$ such that $q(\gamma) = \check{\mathbf{x}}$. Then D^β_α is dense.

(3) $|D^\beta_\alpha| < \beta^{++}$ if $\alpha < \beta^{++}$.

Note that $D^\beta_i \subseteq D^\beta_j$ for $i < j$.

First we consider the successor case. Suppose that $\mathbb{P}^\beta_\alpha$ is defined and satisfies the required properties for a fixed $\alpha < \beta^{++}$. Since D^β_α has size less than β^{++}, we can enumerate all canonical D^β_α-names for stationary subsets of $S(\beta,\beta^+) \cap \bigcup G_F$ as $\langle \dot{X}^\alpha_i \colon i < \beta^{++}\rangle$. Write $f(\alpha) = \langle i, j\rangle$. Then $i \le \alpha$, so $\dot{X}^i_j$ is defined. By Proposition 4.9 and the recursion hypotheses, $\dot{X}^i_j$ is of the form required to define the $<\beta$-distributive, β^+-c.c. poset $\mathbb{P}_{\dot{X}^i_j}$ from Definition 4.2 which destroys the stationarity of $\dot{X}^i_j$. So let $\dot{\mathbb{Q}}^\beta_\alpha = \mathbb{P}_{\dot{X}^i_j}$. It is straightforward to check that (1), (2), and (3) hold for $\mathbb{P}^\beta_{\alpha+1}$.

Now we consider the limit case. Suppose that $\alpha \le \beta^{++}$ is a limit ordinal and $\mathbb{P}^\beta_\gamma$ is defined as required for all $\gamma < \alpha$. Define $\mathbb{P}^\beta_\alpha$ by taking conditions with $<\beta$-support. That is, a condition in $\mathbb{P}^\beta_\alpha$ is a sequence p with domain a subset of α with size less than β such that $p \restriction \gamma$ is in $\mathbb{P}^\beta_\gamma$ for all $\gamma < \alpha$.

Now we verify properties (1), (2), and (3) for $\mathbb{P}^\beta_\alpha$. First we prove (2) by showing that D^β_α is dense.

Lemma 7.4. *Property (2) holds for* $\mathbb{P}^\beta_\alpha$.

Proof. Let p be a condition. If $\mathrm{cf}(\alpha) \ge \beta$ then there is $\gamma < \alpha$ such that $\mathrm{supp}(p) \subseteq \gamma$. Then p is in $\mathbb{P}^\beta_\gamma$, so by the recursion hypotheses there is $q \le p$ in D^β_γ. But then q is in D^β_α.

Suppose on the other hand that $\mathrm{cf}(\alpha) < \beta$. Choose an increasing sequence $\langle \alpha_i \colon i < \mathrm{cf}(\alpha)\rangle$ unbounded in α, with $\alpha_0 = 0$. Exactly as in Proposition 4.6 construct an increasing and continuous sequence of sets $\langle N_i \colon i \le \mathrm{cf}(\alpha)\rangle$ closed under H and F such that for some function $f \colon \beta \to \beta^+$ in V, for all $i \le \mathrm{cf}(\alpha)$:

(1) $N_i \cap \beta = \delta_i$ is in $\lim(C)$,

(2) $f``\delta_i = N_i \cap \beta^+$.

Choose N_0 to contain as elements the sets $\mathbb{P}^\beta_\alpha$, p, C, H, E, $\langle \alpha_i \colon i < \mathrm{cf}(\alpha)\rangle$, and $\langle D^\beta_i \colon i < \alpha\rangle$. We construct a decreasing sequence of conditions $\langle p_i \colon i \le \mathrm{cf}(\alpha)\rangle$ satisfying:

(3) $\langle p_j \colon j \le i\rangle$ is in N_{i+1},

(4) $p_i \restriction \alpha_i$ is in $D^\beta_{\alpha_i}$,

(5) for all γ in $\mathrm{supp}(p_i) \cap \alpha_i$, $N_i \cap \beta^+ = \max(p_i(\gamma))$,

(6) $p_i(\xi) = p(\xi)$ for all $\alpha_i \le \xi < \alpha$.

Suppose that p_i is given satisfying the requirements. Let p^*_i be the $<_\theta$-least extension of $p_i \restriction \alpha_{i+1}$ in $D^\beta_{\alpha_{i+1}}$. Define p_{i+1} as follows. For each ξ in the support of p_i^*, define

$$p_{i+1}(\xi) = p_i^*(\xi) \cup \{u \in E\colon N_{i+1} \cap \beta \subseteq u \subseteq N_{i+1} \cap \beta^+\}.$$

For each ξ in $\mathrm{supp}(p_i) \setminus \alpha_{i+1}$, let $p_{i+1}(\xi) = p_i(\xi)$. It is straightforward to check that p_{i+1} satisfies the inductive requirements.

Suppose that $\gamma \leq \mathrm{cf}(\alpha)$ is a limit ordinal and p_i is defined for all $i < \gamma$. Define p_γ as follows. The support of p_γ is $\bigcup\{\mathrm{supp}(p_i)\colon i < \gamma\}$. Let ξ be in this support. If $\xi \geq \alpha_\gamma$ then let $p_\gamma(\xi) = p(\xi)$. Otherwise there exists $i < \gamma$ such that $\langle p_j(\xi)\colon i \leq j < \gamma\rangle$ is a sequence of canonical names for sets in the ground model. Apply Lemma 4.5 to this sequence to obtain $p_\gamma(\xi)$.

This completes the construction. The condition $p_{\mathrm{cf}(\alpha)}$ refines p and is in D^β_α. Therefore property (2) holds. $\square$

Now we verify property (1) by showing that $\mathbb{P}^\beta_\alpha$ is β-distributive. This proof is basically the same as the proof of property (2).

Lemma 7.5. *Property (1) holds for $\mathbb{P}^\beta_\alpha$.*

Proof. Suppose that $\langle D_i\colon i < \delta\rangle$ is a sequence of less than β many dense subsets of $\mathbb{P}^\beta_\alpha$. Suppose that p is a condition. Choose an increasing and continuous sequence $\langle N_i\colon i \leq \delta\rangle$ of sets closed under H and F such that for all $i \leq \delta$:

(1) $N_i \cap \beta = \delta_i$ is in $\lim(C)$,

(2) $f``\delta_i = N_i \cap \beta^+$.

Choose N_0 to contain as elements the sets $\mathbb{P}^\beta_\alpha$, p, C, H, E, $\langle \alpha_i\colon i < \mathrm{cf}(\alpha)\rangle$, and $\langle D_i\colon i < \delta\rangle$. We construct a decreasing sequence of conditions $\langle p_i\colon i \leq \delta\rangle$ satisfying:

(3) $\langle p_j\colon j \leq i\rangle$ is in N_{i+1},

(4) p_{i+1} is in $D^\beta_\alpha \cap D_i$,

(5) for all γ in $\mathrm{supp}(p_i)$, $N_i \cap \beta^+ = \max(p_i(\gamma))$.

The construction is straightforward and uses Lemma 4.5 for the limit stages. $\square$

For property (3), first suppose that $\mathrm{cf}(\alpha) \geq \beta$. In this case $D^\beta_\alpha = \bigcup\{D^\beta_i\colon i < \alpha\}$, so clearly $|D^\beta_\alpha| < \beta^{++}$ if $\alpha < \beta^{++}$. Otherwise fix an increasing sequence $\langle \alpha_i\colon i < \mathrm{cf}(\alpha)\rangle$ unbounded in α. Then D^β_α is the set of p in $\mathbb{P}^\beta_\alpha$ such that $p \restriction \alpha_i$ is in $D^\beta_{\alpha_i}$ for each i. Since each $|D^\beta_{\alpha_i}| < \beta^{++}$, $|D^\beta_\alpha| \leq (\beta^+)^{\mathrm{cf}(\alpha)} = \beta^+$.

This completes the definition of the forcing iteration and the proof of properties (1), (2), and (3). Let $\mathbb{P}^\beta = \mathbb{P}^\beta_{\beta^{++}}$ and $D^\beta = \bigcup\{D^\beta_i\colon i < \beta^{++}\}$. Then $\mathbb{P}_{F_\beta} * \mathbb{P}^\beta$ is $< \beta$-distributive and has size β^{++}, and D^β is dense in $\mathbb{P}^\beta$.

Lemma 7.6. *The poset $\mathbb{P}^\beta$ is β^+-c.c.*

Proof. Suppose for a contradiction that $\mathbb{P}^\beta$ is not β^+-c.c. Then there is an antichain $\langle p_i\colon i < \beta^+\rangle$ in D^β. Without loss of generality, for each i and for each α in the support of p_i, $p_i(\alpha) = \check{\mathbf{x}}^i_\alpha$ for some $\mathbf{x}^i_\alpha$ in $W[G_\beta * G_F]$. Applying the Δ-system lemma we can assume that there is $d \subseteq \beta^{++}$ with size less than β such that $\mathrm{supp}(p_i) \cap \mathrm{supp}(p_j) = d$ for $i < j$. Let $a_i = \bigcup\{\max(\mathbf{x}^i_\alpha)\colon \alpha \in \mathrm{supp}(p_i)\}$. Again by the Δ-system lemma we can assume there is $e \subseteq \beta^+$ with size less than β such that $a_i \cap a_j = e$ for $i < j$. Now there are less than β many possibilities for a sequence $\langle \mathcal{P}(e) \cap \mathbf{x}^i_\alpha\colon \alpha \in d\rangle$, so we can assume that all such sequences are equal for different values of i.

Consider any $i < j$. Since p_i and p_j are incompatible, then exactly as in the proof of Proposition 4.7 there is α in d and (without loss of generality) a in $\mathbf{x}^i_\alpha \setminus \mathbf{x}^j_\alpha$

with $a \subseteq \max \mathbf{x}^j_\alpha$. But then $a \subseteq a_i \cap a_j = e$. So a is in $\mathcal{P}(e) \cap \mathbf{x}^i_\alpha$, therefore a is in $\mathcal{P}(e) \cap \mathbf{x}^j_\alpha \subseteq \mathbf{x}^j_\alpha$, which is a contradiction. $\square$

Therefore $\mathbb{P}^\beta$ has no more than β^{++}-many antichains. By Lemma 2.3 there exists a β-weakly closed Prikry type forcing poset $\mathbb{Q}_\beta$ which satisfies the direct extension property and is equivalent to $\mathbb{P}_{F_\beta} * \mathbb{P}^\beta$.

This completes the definition of $\mathbb{P}_\kappa * \mathbb{P}_{F_\kappa} * \mathbb{P}^\kappa$. Let $F = F_\kappa$. We will now define a complete suborder $\mathbb{P}_F * \mathbb{P}^\kappa_A$ of $\mathbb{P}_F * \mathbb{P}^\kappa$ and prove that this suborder forces that $NS \restriction S(\kappa,\kappa^+)$ is κ^+-saturated.

In order to define $\mathbb{P}^\kappa_A$, we need to consider a supercompact elementary embedding. Let $j\colon W \to M$ be an elementary embedding with $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \kappa^{+3}$, and ${}^{\kappa^{+3}}M \subseteq M$. Then κ is in $j(B)$, so write

$$j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa * \mathbb{P}_{\mathrm{tail}}$$

where $\dot{\mathbb{Q}}_\kappa$ is equivalent to $\mathbb{P}_F * \mathbb{P}^\kappa$. Let G_κ be generic for $\mathbb{P}_\kappa$ over W. In the model $W[G_\kappa]$, let $\langle \dot{X}_i\colon i < \kappa^{++}\rangle$ be the sequence of names defined by $\dot{X}_i = \dot{X}^\alpha_\beta$, where $f(i) = \langle \alpha, \beta\rangle$ in the definition of $\mathbb{P}^\kappa$. So for all $i < \kappa^{++}$, $\mathbb{P}_F * \mathbb{P}^\kappa_i$ forces that $\dot{\mathbb{Q}}^\kappa_i = \mathbb{P}_{\dot{X}_i}$.

First let us consider how to extend j by constructing a master condition for $\mathbb{P}_F$. Let $G_F * G^\kappa * G_{\mathrm{tail}}$ be generic for $\mathbb{P}_F * \mathbb{P}^\kappa * \mathbb{P}_{\mathrm{tail}}$ over $W[G_\kappa]$. Extend j to

$$j\colon W[G_\kappa] \to M[G_\kappa * G_F * G^\kappa * G_{\mathrm{tail}}].$$

Write $M_1 = M[G_\kappa * G_F * G^\kappa * G_{\mathrm{tail}}]$. We define a master condition t_F for G_F and j. Let

$$t_F = \bigcup j``G_F \cup \{u \in j(E)\colon \kappa \subseteq u \subseteq j``\kappa^+\}.$$

Since $\bigcup G_F$ is a club subset of $P_\kappa \kappa^+$ in $M[G_\kappa * G_F]$, by the distributivity of $\mathbb{P}^\kappa$ and the weak closure of $\mathbb{P}_{\mathrm{tail}}$, $\bigcup j``G_F$ is a club subset of $P_\kappa(j``\kappa^+)$ in M_1. So t_F is closed under unions of increasing sequences. Also $\mathbb{R} * j(\mathbb{P}_\kappa)$ does not destroy stationary subsets of κ^+, since $\mathbb{R}$ is κ^+-c.c. and $j(\mathbb{P}_\kappa) = \mathbb{P}_{\kappa+1} * \mathbb{P}_{\mathrm{tail}}$, where $\mathbb{P}_{\kappa+1}$ is κ^+-c.c. and $\mathbb{P}_{\mathrm{tail}}$ does not add subsets to κ^+. So by Proposition 7.2, $j(F)$ is Jonsson for $j``\kappa^+$ in $M_1$. But every member of $t_F \cap j(S(\kappa,\kappa^+))$ which is not in $\bigcup j``G_F$ must be unbounded in $j``\kappa^+$ and hence by the Jonsson property is equal to $j``\kappa^+$. Therefore $j(F)$ is Jonsson for a for every a in $t_F \cap j(S(\kappa,\kappa^+))$. So t_F is a master condition.

Still working in M_1, fix a regular $\theta \gg j(\kappa)$ and a well-ordering $<_\theta$ of $H(\theta)$. As in Proposition 4.6, construct an increasing sequence $\langle N_i\colon i < j(\kappa)\rangle$ of elementary substructures of $H(\theta)$ in $P_{j(\kappa)}H(\theta)$ such that for some function $f\colon j(\kappa) \to j(\kappa^+)$ in V:

(1) $N_i \cap j(\kappa) = \delta_i$ is in $j(\lim(C))$,
(2) $N_i \cap j(\kappa^+) = f``\delta_i$,
(3) $\langle N_j\colon j \le i\rangle$ is in N_{i+1}.

Choose N_0 to contains the sets $j(\mathbb{P}_\kappa)$, G_F, G^κ, $j(F)$, $j(H)$, $j(E)$, t_F, and $\langle j(\dot{X}_i)\colon i < \kappa^{++}\rangle$. For each $i < \kappa^{++}$ let G^κ_i be the generic filter for $\mathbb{Q}^\kappa_i$ given by G^κ. Now go back to $W[G_\kappa * G_F]$ and identify the objects just defined with their $\mathbb{P}^\kappa * \mathbb{P}_{\mathrm{tail}}$-names.

Working in $W[G_\kappa * G_F]$ we define a complete suborder $\mathbb{P}^\kappa_A$ of $\mathbb{P}^\kappa$ by inductively defining a $< \kappa$-support forcing iteration

$$\langle \mathbb{P}^\kappa_{A_i}, \dot{\mathbb{Q}}^\kappa_{A_i}\colon i < \kappa^{++}\rangle.$$

For each i, A_{i+1} is a $\mathbb{P}^\kappa_{A_i}$-name for a subset of $i+1$, and $\dot{\mathbb{Q}}^\kappa_{A_i}$ is non-trivial iff i is in A_{i+1}. If $\dot{\mathbb{Q}}^\kappa_{A_i}$ is non-trivial then $\dot{\mathbb{Q}}^\kappa_{A_i} = \mathbb{P}_{\dot{X}_i}$. Let $D^\kappa_{A_i} = \mathbb{P}^\kappa_{A_i} \cap D^\kappa_i$. Then $D^\kappa_{A_i}$ is dense in $\mathbb{P}^\kappa_{A_i}$ and there is a projection mapping

$$D^\kappa_i \to D^\kappa_{A_i}.$$

Since the posets are $<\kappa$-distributive, the poset $\mathbb{P}_{\dot{X}_i}$ is the same in the generic extensions by $\mathbb{P}^\kappa_i$ and $\mathbb{P}^\kappa_{A_i}$. At the same time we construct sequences

$$\langle \dot{s}_i{}^\frown \dot{x}_i \colon i \leq \kappa^{++} \rangle$$

and

$$\langle \dot{q}_i \colon i \leq \kappa^{++} \rangle.$$

Let $D^\kappa = D^\kappa_{\kappa^{++}} = \bigcup\{D^\kappa_i \colon i < \kappa^{++}\}$. Our induction hypothesis is that $\mathbb{P}^\kappa$ forces:

(1) q_i is in $\mathbb{P}_{\text{tail}}$, and $q_j \leq^* q_i$ for all $i < j$,

(2) $\mathbb{P}_{\text{tail}}$ forces that $\dot{s}_i{}^\frown \dot{\mathbf{x}}_i$ is in $j(\mathbb{P}_F * \mathbb{P}^\kappa_{A_i})$, $\dot{\mathbf{x}}_i$ is in $j(D^\kappa_{A_i})$, and $\dot{s}_j{}^\frown \dot{\mathbf{x}}_j \leq \dot{s}_i{}^\frown \dot{\mathbf{x}}_i$ for $i < j$,

(3) $\mathbb{P}_{\text{tail}}$ forces that $\langle \dot{s}_j{}^\frown \dot{\mathbf{x}}_j \colon j \leq i\rangle$ is in N_{i+1},

(4) $\mathbb{P}_{\text{tail}}$ forces that for all α in $\operatorname{supp}(\dot{\mathbf{x}}_i)$, $\max(\dot{\mathbf{x}}_i(\alpha)) = N_i \cap j(\kappa^+)$, and also $\max(\dot{s}_i) = N_i \cap j(\kappa^+)$.

Let $A_0 = \emptyset$ and let $\dot{q}_0$, $\dot{\mathbf{x}}_0$ be names for the maximum conditions in $\mathbb{P}_{\text{tail}}$ and $j(\mathbb{P}^\kappa)$. Let $\dot{s}_0 = \dot{t}_F$. Suppose that $i < \kappa^{++}$ and A_i, $\langle \dot{s}_0{}^\frown \dot{\mathbf{x}}_j \colon j \leq i\rangle$, and $\langle \dot{q}_j \colon j \leq i\rangle$ are defined satisfying the induction hypotheses. Consider the name $\dot{X}_i$. If this is not a $\mathbb{P}^\kappa_{A_i}$-name then let $A_{i+1} = A_i$ and let $\dot{q}_{i+1} = \dot{q}_i$. Let $\dot{\mathbf{x}}_{i+1}$ be a name for the condition with the same support as $\dot{\mathbf{x}}_i$ such that for all α in $\operatorname{supp}(\dot{\mathbf{x}}_i)$,

$$\dot{\mathbf{x}}_{i+1}(\alpha) = \dot{\mathbf{x}}_i(\alpha) \cup \{u \in j(E) \colon N_{i+1} \cap j(\kappa) \subseteq u \subseteq N_{i+1} \cap j(\kappa^+)\},$$

and similarly define $\dot{s}_{i+1}$. Since $|N_{i+1} \cap j(\kappa^+)| = N_{i+1} \cap j(\kappa)$, $\dot{\mathbf{x}}_{i+1}(\alpha)$ is disjoint from $j(\dot{X}_\alpha)$. Using the induction hypotheses it is easy to check that $\dot{s}_{i+1}{}^\frown \dot{\mathbf{x}}_{i+1}$ is a condition in $j(\mathbb{P}_F * \mathbb{P}^\kappa_{A_{i+1}})$ which refines $\dot{s}_i{}^\frown \dot{\mathbf{x}}_i$.

Suppose that $\dot{X}_i$ is a $\mathbb{P}^\kappa_{A_i}$-name. By the induction hypothesis $\dot{s}_i{}^\frown \dot{\mathbf{x}}_i$ is in $j(\mathbb{P}_F * \mathbb{P}^\kappa_{A_i})$, so let $\dot{s}^*_i * \dot{\mathbf{x}}^*_i$ be a name for the $<_\theta$-least refinement of $\dot{s}_i{}^\frown \dot{\mathbf{x}}_i$ in $j(\mathbb{P}_F * \mathbb{P}^\kappa_{A_i})$ such that $\dot{\mathbf{x}}_i$ is in $j(D^\kappa_{A_i})$ and $\dot{s}^*_i{}^\frown \dot{\mathbf{x}}^*_i$ decides the statement $j``\kappa^+ \in j(\dot{X}_i)$. Choose a name $\dot{q}_{i+1}$ for a direct extension of $\dot{q}_i$ such that $\dot{q}_{i+1}$ decides the statement

$$\dot{s}^*_i * \dot{x}^*_i \Vdash j``\kappa^+ \in j(\dot{X}_i).$$

Let A_{i+1} be a $\mathbb{P}^\kappa_{A_i}$-name for the set which is equal to A_i unless

$$\mathbb{P}^\kappa / \mathbb{P}^\kappa_{A_i} \Vdash \dot{q}_{i+1} \Vdash \dot{s}^*_i \Vdash \dot{x}^*_i \Vdash j``\kappa^+ \notin j(\dot{X}_i),$$

in which case $A_{i+1} = A_i \cup \{i\}$. Now define a $\mathbb{P}^\kappa * \mathbb{P}_{\text{tail}}$-name $\dot{s}_{i+1} * \dot{\mathbf{x}}_{i+1}$ as follows. If i is not in A_{i+1} then define $\dot{\mathbf{x}}_{i+1}$ as in the previous case, namely, as a name for the condition with the same support as $\dot{\mathbf{x}}_i$ and for all α in $\operatorname{supp}(\dot{\mathbf{x}}_i)$,

$$\dot{\mathbf{x}}_{i+1}(\alpha) = \dot{\mathbf{x}}_i(\alpha) \cup \{u \in j(E) \colon N_{i+1} \cap j(\kappa) \subseteq u \subseteq N_{i+1} \cap j(\kappa^+)\};$$

define $\dot{s}_{i+1}$ similarly. If i is in A_{i+1}, then define

$$t_i = \bigcup j``G^\kappa_i \cup \{u \in j(E) \colon \kappa \subseteq u \subseteq j``\kappa^+\}.$$

Since $s^*_i * \mathbf{x}^*_i$ forces that $j``\kappa^+ \notin j(\dot{X}_i)$ and $j(F)$ is Jonsson for $j``\kappa^+$, $s^*_i * \mathbf{x}^*_i$ forces that t_i is a condition in $j(\mathbb{P}_{\dot{X}_i})$. Let $\dot{s}_{i+1} * \mathbf{x}_{i+1}$ be obtained by taking the condition

$\dot{s}_i^* * \mathbf{x}_i^{*\frown} t_i$ and adding to each coordinate the elements of

$$\{u \in j(E) \colon N_{i+1} \cap j(\kappa) \subseteq u \subseteq N_{i+1} \cap j(\kappa^+)\}.$$

Suppose that $\delta \leq \kappa^{++}$ is a limit ordinal. Let A_δ denote $\bigcup\{A_i \colon i < \delta\}$. Since $\langle q_i \colon i < \delta \rangle$ is forced to be $\leq^*$-decreasing in the $j(\kappa)$-weakly closed poset $\mathbb{P}_{\text{tail}}$, we can choose a name $\dot{q}_\delta$ for a condition in $\mathbb{P}_{\text{tail}}$ which directly refines each $\dot{q}_i$. By the inductive requirements of $\dot{s}_i{}^\frown \dot{\mathbf{x}}_i$ for $i < \delta$, apply Lemma 4.5 to obtain $\dot{s}_\delta * \dot{\mathbf{x}}_\delta$ refining each $\dot{s}_i * \dot{\mathbf{x}}_i$.

This completes the definition of $\mathbb{P}_A^\kappa$. Let $\dot{s}_F = \dot{s}_{\kappa^{++}}$, $\dot{\mathbf{x}} = \dot{\mathbf{x}}_{\kappa^{++}}$ and $\dot{q} = \dot{q}_{\kappa^{++}}$. It is straightforward to check that $\dot{s}_F * \dot{\mathbf{x}}$ is forced to be a master condition for $G_F * G_A^\kappa$ and j. Let $D_A^\kappa = D^\kappa \cap \mathbb{P}_A^\kappa$. Note that D_A^κ is a dense subset of $\mathbb{P}_A^\kappa$.

Suppose that G_A^κ is generic for $\mathbb{P}_A^\kappa$ over $W[G_\kappa * G_F]$. By the properties of $\mathbb{P}_A^\kappa$, any set $X \subseteq P_\kappa \kappa^+$ in $W[G_\kappa * G_F * G_A^\kappa]$ is in $W[G_\kappa * G_F * G_{A_i}^\kappa]$ for some $i < \kappa^{++}$. Using this fact it is straightforward to check that if X is a subset of $S(\kappa, \kappa^+)$ in $W[G_\kappa * G_F * G_A^\kappa]$, then there is i such that $X = X_i$, $\dot{X}_i$ is a D_i^κ-name, and if X is non-stationary then we can choose such an i so that X is non-stationary in $W[G_\kappa * G_F * G_{A_i}^\kappa]$.

Lemma 7.7. *The poset $\mathbb{P}_F * \mathbb{P}_A^\kappa$ forces that $NS \restriction S(\kappa, \kappa^+)$ is κ^+-saturated.*

Proof. Let $G_F * G_A$ be generic for $\mathbb{P}_F * \mathbb{P}_A^\kappa$ over $W[G_\kappa]$. For any $\mathbb{P}_A^\kappa$-name $\dot{X}$, let $\Phi(\dot{X})$ be the statement

$$\dot{q} \Vdash \dot{s}_F \Vdash \dot{x} \Vdash j``\kappa^+ \notin j(\dot{X}),$$

First we show that for every set $X \subseteq S(\kappa, \kappa^+)$ in $W[G_\kappa * G_F * G_A]$, X is in $NS \restriction S(\kappa, \kappa^+)$ iff there is a $\mathbb{P}_A^\kappa$-name $\dot{X}$ for X such that $\mathbb{P}^\kappa / \mathbb{P}_A^\kappa$ forces $\Phi(\dot{X})$. Suppose that $\mathbb{P}^\kappa / \mathbb{P}_A^\kappa$ forces $\Phi(\dot{X})$. Fix p in G_A^κ such that $p * 1$ forces $\Phi(\dot{X})$ in $\mathbb{P}^\kappa$. Then we can choose $\dot{Y}$ such that $\mathbb{P}^\kappa$ forces $\Phi(\dot{Y})$ and $p * 1$ forces $\dot{X} = \dot{Y}$. Therefore $\dot{Y}$ is a name for X. Fix i such that $\dot{X}_i$ is a $D_{A_i}^\kappa$-name and $\Vdash \dot{Y} = \dot{X}_i$. Then clearly i is in A_{i+1} and so $\dot{\mathbb{Q}}_{A_i}^\kappa = \mathbb{P}_{\dot{Y}_i}$. So $\mathbb{P}_A^\kappa$ destroys X and X is non-stationary. The other direction is straightforward.

If X is in $(NS \restriction S(\kappa, \kappa^+))^+$ then there is p_X in $\mathbb{P}^\kappa / \mathbb{P}_A^\kappa$ such that

$$p_X \Vdash \dot{q} \Vdash \dot{s}_F \Vdash \dot{\mathbf{x}} \Vdash j``\kappa^+ \in j(\dot{X}).$$

Suppose that $X \cap Y$ is disjoint from a club D. If $r \leq p_X, p_Y$ then

$$r \Vdash \dot{q} \Vdash \dot{s}_F \Vdash \dot{\mathbf{x}} \Vdash j``\kappa^+ \in j(\dot{X}) \cap j(\dot{Y}) \cap j(D)$$

which is a contradiction. So p_X and p_Y are incompatible. Since $\mathbb{P}^\kappa / \mathbb{P}_A^\kappa$ is κ^+-c.c., $NS \restriction S(\kappa, \kappa^+)$ is κ^+-saturated. $\square$

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