ON THE WEAK REFLECTION PRINCIPLE

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Abstract. The *Weak Reflection Principle* for ω_2, or $\mathrm{WRP}(\omega_2)$, is the statement that every stationary subset of $P_{\omega_1}(\omega_2)$ reflects to an uncountable ordinal in ω_2. The *Reflection Principle* for ω_2, or $\mathrm{RP}(\omega_2)$, is the statement that every stationary subset of $P_{\omega_1}(\omega_2)$ reflects to an ordinal in ω_2 with cofinality ω_1. Let κ be a κ^+-supercompact cardinal and assume $2^\kappa = \kappa^+$. Then there exists a forcing poset $\mathbb{P}$ which collapses κ to become ω_2, and $\Vdash_{\mathbb{P}} \mathrm{WRP}(\omega_2) \wedge \neg\mathrm{RP}(\omega_2)$.

In this paper we will be concerned with reflection of stationary subsets of $P_{\omega_1}(\omega_2)$. Recall that for an uncountable ordinal α, a set $S \subseteq P_{\omega_1}(\alpha) = \{x \subseteq \alpha : |x| < \omega_1\}$ is *stationary* if for any function $F : [\alpha]^{<\omega} \to \alpha$, there is a set b in S which is closed under F. Let $S \subseteq P_{\omega_1}(\omega_2)$ be a stationary set. We say that S *reflects to* α, where α is an uncountable ordinal in ω_2, if $S \cap P_{\omega_1}(\alpha)$ is stationary in $P_{\omega_1}(\alpha)$.

The *Weak Reflection Principle* for ω_2, or $\mathrm{WRP}(\omega_2)$, is the statement that for every stationary set $S \subseteq P_{\omega_1}(\omega_2)$, there is an uncountable ordinal α in ω_2 such that S reflects to α. This principle is a special case of the stronger *Weak Reflection Principle* introduced in [1]. The statement $\mathrm{WRP}(\omega_2)$ has a number of interesting combinatorial consequences, including $2^\omega \leq \omega_2$ ([4], [6]), $\neg\square(\omega_2)$ ([7]), and every stationary subset of $\omega_2 \cap \mathrm{cof}(\omega)$ reflects to an ordinal in ω_2 with cofinality ω_1.

A related principle is the *Reflection Principle* for ω_2, or $\mathrm{RP}(\omega_2)$, which asserts that every stationary subset of $P_{\omega_1}(\omega_2)$ reflects to an ordinal in ω_2 with cofinality ω_1. The standard models for obtaining $\mathrm{WRP}(\omega_2)$ (for example, by Lévy collapsing a large cardinal to become ω_2) all satisfy $\mathrm{RP}(\omega_2)$. Also, it tends to be easier to work with $\mathrm{RP}(\omega_2)$ than with $\mathrm{WRP}(\omega_2)$. Thus it is a natural question, and one which has been open for some time, whether $\mathrm{WRP}(\omega_2)$ implies $\mathrm{RP}(\omega_2)$.

A standard argument shows that if $\Diamond_A$ holds for every stationary set $A \subseteq \omega_2 \cap \mathrm{cof}(\omega)$, then for any stationary set $S \subseteq P_{\omega_1}(\omega_2)$, there is a stationary set $T \subseteq S$ which does not reflect to any uncountable ordinal in ω_2 with cofinality ω. So assuming the existence of such diamonds, $\mathrm{WRP}(\omega_2)$ implies $\mathrm{RP}(\omega_2)$. Classically, the existence of such diamonds were known to follow from GCH; more recently, Shelah [5] has proven they are a consequence of $2^{\omega_1} = \omega_2$. Thus assuming $2^{\omega_1} = \omega_2$, $\mathrm{WRP}(\omega_2)$ implies $\mathrm{RP}(\omega_2)$ (originally, this result was proven in [2] by a different argument).

The Weak Reflection Principle for ω_2 is equiconsistent with a weakly compact cardinal. However, Sakai [3] has shown that a kind of local reflection is consistent from ZFC. Specifically, assuming GCH and $\square_{\omega_1}$, there is a generic extension in which there exists a stationary set $S \subseteq P_{\omega_1}(\omega_2)$ such that every stationary subset of S reflects to an uncountable ordinal in ω_2 with cofinality ω.

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We had the idea to combine parts of Sakai's argument with the classical methods for constructing models of the Weak Reflection Principle, in an attempt to show that $\mathrm{WRP}(\omega_2)$ does not imply $\mathrm{RP}(\omega_2)$. This paper is the culmination of those efforts. We will prove that if κ is a κ^+-supercompact cardinal and $2^\kappa = \kappa^+$, then there is a forcing poset $\mathbb{P}$ which collapses κ to become ω_2, and forces $\mathrm{WRP}(\omega_2) \wedge \neg\mathrm{RP}(\omega_2)$.

1. Background Material

We begin by recording some definitions and results which will be used in the paper.

By a *forcing poset*, we mean an order which is reflexive and transitive (we do not assume it is antisymmetric or separative). Let $\mathbb{P}$ be a forcing poset, and let μ be a regular uncountable cardinal. We say $\mathbb{P}$ has the *μ-chain condition*, or is *μ-c.c.*, if for any sequence $\langle p_i : i < \mu \rangle$ of conditions in $\mathbb{P}$, there are $i < j$ such that p_i and p_j are compatible. We say $\mathbb{P}$ is *μ-Knaster* if for any sequence $\langle p_i : i < \mu \rangle$ of conditions in $\mathbb{P}$, there is a set $Z \subseteq \mu$ with $|Z| = \mu$ such that for all $i < j$ in Z, p_i and p_j are compatible. Clearly if $\mathbb{P}$ is μ-Knaster, then $\mathbb{P}$ is μ-c.c.

We say $\mathbb{P}$ is μ-distributive if whenever $\langle D_i : i < \nu \rangle$ is a sequence of dense open subsets of $\mathbb{P}$, where $\nu < \mu$, then $\bigcap\{D_i : i < \nu\}$ is dense. This property implies that if G is a generic filter on $\mathbb{P}$ over V, then for any set $x \subseteq V$ in $V[G]$ such that $V[G] \models |x| < \mu$, x is in V. (Assuming $\mathbb{P}$ is separative, these two properties are equivalent, but they are not equivalent in general.)

A map $i : \mathbb{P} \to \mathbb{Q}$, where $\mathbb{P}$ and $\mathbb{Q}$ are forcing posets, is a *complete embedding* if:

(1) $q \le p$ in $\mathbb{P}$ implies $i(q) \le i(p)$ in $\mathbb{Q}$,
(2) if $i(p)$ and $i(q)$ are compatible in $\mathbb{Q}$, then p and q are compatible in $\mathbb{P}$,
(3) for all q in $\mathbb{Q}$, there is p in $\mathbb{P}$ such that whenever $t \le p$ in $\mathbb{P}$, $i(t)$ and q are compatible in $\mathbb{Q}$.

Suppose $i : \mathbb{P} \to \mathbb{Q}$ is a complete embedding. If H is a generic filter on $\mathbb{Q}$ over V, then $i^{-1}(H)$ is a generic filter on $\mathbb{P}$ over V, and $V[i^{-1}(H)] \subseteq V[H]$. Note that if $\mathbb{P} \subseteq \mathbb{Q}$ and i is the inclusion map, then $i^{-1}(H) = \{p \in \mathbb{P} : p \in H\} = H \cap \mathbb{P}$.

A complete embedding $i : \mathbb{P} \to \mathbb{Q}$ provides a way to translate $\mathbb{P}$-names to $\mathbb{Q}$-names. Define $i(\dot{x})$, for each $\mathbb{P}$-name $\dot{x}$, by recursion, letting $i(\dot{x}) = \{(i(p), i(\dot{y})) : (p, y) \in \dot{x}\}$. If H is a generic filter on $\mathbb{Q}$ over V, then $i(\dot{x})^H = \dot{x}^{i^{-1}(H)}$. Note that if i is the inclusion map, then for any $\mathbb{P}$-name $\dot{x}$, $i(\dot{x}) = \dot{x}$. So if H is a generic filter on $\mathbb{Q}$ over V, then $\dot{x}^H = \dot{x}^{H \cap \mathbb{P}}$.

A map $i : \mathbb{P} \to \mathbb{Q}$ is a *dense embedding* if

(1) $q \le p$ in $\mathbb{P}$ implies $i(q) \le i(p)$ in $\mathbb{Q}$,
(2) if $i(p)$ and $i(q)$ are compatible in $\mathbb{Q}$, then p and q are compatible in $\mathbb{P}$,
(3) $i[\mathbb{P}]$ is dense in $\mathbb{Q}$.

Every dense embedding is a complete embedding. If i is a dense embedding, then $\mathbb{P}$ and $\mathbb{Q}$ are forcing equivalent, in the sense that they have the same generic extensions. We will consider a particular kind of dense embedding, as follows. Say a map $i : \mathbb{P} \to \mathbb{Q}$ is an *isomorphism* if i is a bijection, and $q \le p$ in $\mathbb{P}$ iff $i(q) \le i(p)$ in $\mathbb{Q}$. If $i : \mathbb{P} \to \mathbb{Q}$ is an isomorphism of $\mathbb{P}$ onto a dense subset of $\mathbb{Q}$, then i is a dense embedding.

Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing posets with maximum elements $1_{\mathbb{P}}$ and $1_{\mathbb{Q}}$ respectively. A map $\pi : \mathbb{Q} \to \mathbb{P}$ is a *projection mapping* if:

(1) $\pi(1_{\mathbb{Q}}) = 1_{\mathbb{P}}$,

(2) $q \leq p$ in $\mathbb{Q}$ implies $\pi(q) \leq \pi(p)$ in $\mathbb{P}$,
(3) if $t \leq \pi(q)$ in $\mathbb{P}$, then there is $r \leq q$ in $\mathbb{Q}$ such that $\pi(r) \leq t$ in $\mathbb{P}$.

Suppose $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection mapping. If H is a generic filter on $\mathbb{Q}$ over V, then $\pi[H]$ generates a generic filter on $\mathbb{P}$ over V. Furthermore, the projection mapping π allows a factorization of $\mathbb{Q}$ into a two-step iteration $\mathbb{P} * (\mathbb{Q}/\mathbb{P})$ as follows. Let G be a generic filter on $\mathbb{P}$ over V. In $V[G]$, define $\mathbb{Q}/\mathbb{P}$ as the forcing poset with underlying set $\{q \in \mathbb{Q} : \pi(q) \in G\}$, ordered as in $\mathbb{Q}$. Then the map $i : \mathbb{Q} \to \mathbb{P} * (\mathbb{Q}/\mathbb{P})$ defined by $i(q) = (\pi(q), \check{q})$ is an isomorphism of $\mathbb{Q}$ onto a dense subset of $\mathbb{P} * (\mathbb{Q}/\mathbb{P})$. Moreover, if K is a generic filter on $\mathbb{Q}$ over V, and J is the filter on $\mathbb{P}$ generated by $\pi[K]$, then K is a generic filter on $\mathbb{Q}/\mathbb{P}$ over $V[J]$, and $V[K] = V[J][K]$.

Lemma 1.1. *Suppose $\pi : \mathbb{Q} \to \mathbb{P}$ is a projection mapping. If D is a dense subset of $\mathbb{Q}$, then $\mathbb{P}$ forces $D \cap (\mathbb{Q}/\mathbb{P})$ is a dense subset of $\mathbb{Q}/\mathbb{P}$.*

Proof. Let G be a generic filter on $\mathbb{P}$ over V. Then $\mathbb{Q}/\mathbb{P} = \{q \in \mathbb{Q} : \pi(q) \in G\}$. Let p be in $\mathbb{Q}/\mathbb{P}$, and we find $r \leq p$ in $D \cap (\mathbb{Q}/\mathbb{P})$. Note that $\pi(p) \in G$.

Let $D' = \{u \leq \pi(p) : \exists r \leq p \ (r \in D \ \wedge \ \pi(r) = u)\}$. We claim D' is dense below $\pi(p)$. So let $t \leq \pi(p)$ in $\mathbb{P}$ be given. By the definition of a projection mapping, fix $q \leq p$ in $\mathbb{Q}$ such that $\pi(q) \leq t$. Since D is dense, choose $r \leq q$ in D. Let $u = \pi(r)$. Then $u = \pi(r) \leq \pi(q) \leq t$, so $u \leq t$. Also $r \leq q \leq p$, so $r \leq p$, $r \in D$, and $\pi(r) = u$. So $u \leq t$ and u is in D'.

Since $\pi(p)$ is in G and D' is dense below $\pi(p)$, fix u in $D' \cap G$. Then there is $r \leq p$ in D such that $\pi(r) = u$. Since $\pi(r) = u$ is in G, r is in $D \cap (\mathbb{Q}/\mathbb{P})$. $\square$

Suppose $\mathbb{P}$ and $\mathbb{Q}$ are forcing posets. The Product Lemma says that for any set $G \subseteq \mathbb{P} \times \mathbb{Q}$, G is a generic filter on $\mathbb{P} \times \mathbb{Q}$ over V iff there is a generic filter $G_{\mathbb{P}}$ on $\mathbb{P}$ over V, and a generic filter $G_{\mathbb{Q}}$ on $\mathbb{Q}$ over $V[G_{\mathbb{P}}]$, such that $G = G_{\mathbb{P}} \times G_{\mathbb{Q}}$. Moreover, $V[G] = V[G_{\mathbb{P}}][G_{\mathbb{Q}}]$, where $G = G_{\mathbb{P}} \times G_{\mathbb{Q}}$. Since $\mathbb{P} \times \mathbb{Q}$ is isomorphic to $\mathbb{Q} \times \mathbb{P}$ by the map $(p, q) \mapsto (q, p)$, it follows that if $G_{\mathbb{P}}$ is a generic filter on $\mathbb{P}$ over V, and $G_{\mathbb{Q}}$ is a generic filter on $\mathbb{Q}$ over $V[G_{\mathbb{P}}]$, then $G_{\mathbb{P}}$ is a generic filter on $\mathbb{P}$ over $V[G_{\mathbb{Q}}]$, and $V[G_{\mathbb{P}}][G_{\mathbb{Q}}] = V[G_{\mathbb{Q}}][G_{\mathbb{P}}]$.

We will apply the Product Lemma in the following way. Consider a generic extension $V[G][H]$, where G is a generic filter on $\mathbb{P}$ over V, and H is a generic filter on $\mathbb{Q}$ over $V[G]$. If it happens that the forcing poset $\mathbb{Q}$ is already in V, then by the Product Lemma, $V[G][H] = V[G \times H] = V[H \times G] = V[H][G]$. So $V[G][H] = V[H][G]$.

Let X be a set. A *nice $\mathbb{P}$-name for a subset of X* is a $\mathbb{P}$-name of the form $\bigcup\{A_x \times \{\check{x}\} : x \in X\}$, where for each x in X, A_x is an antichain of $\mathbb{P}$. A basic fact about $\mathbb{P}$-names is that if $p \Vdash \dot{A} \subseteq X$, then there is a nice $\mathbb{P}$-name $\dot{B}$ for a subset of X such that $p \Vdash \dot{A} = \dot{B}$. Nice names are useful for controlling the continuum function. Let Z be the collection of all nice $\mathbb{P}$-names for a subset of X. If G is a generic filter on $\mathbb{P}$ over V, then in $V[G]$, the map $\dot{A} \mapsto \dot{A}^G$ is a surjective function from Z onto $P(X)$. Hence $V[G] \models |P(X)| \leq |Z|$. Usually the chain condition and the size of $\mathbb{P}$ are used to put an upper bound on the size of Z.

Suppose $f : X \to Y$ is a function and a is a subset of X. If we say a is closed under f, this implicitly assumes that a is non-empty. If f is a function and $A \subseteq \mathrm{dom}(f)$, we write $f[A]$ for the pointwise image of A, $\{f(a) : a \in A\}$.

For any set A, let $P_{\omega_1}(A) = \{a \subseteq A : |a| < \omega_1\}$. Suppose $\omega_1 \subseteq A$. A set $C \subseteq P_{\omega_1}(A)$ is *club* if (1) whenever $\langle a_n : n < \omega \rangle$ is a sequence of sets in C such that $a_n \subseteq a_{n+1}$ for all $n < \omega$, then $\bigcup\{a_n : n < \omega\} \in C$, and (2) for all b in $P_{\omega_1}(A)$, there is c in C with $b \subseteq c$. A set $S \subseteq P_{\omega_1}(A)$ is *stationary* if for every club set $C \subseteq P_{\omega_1}(A)$, $S \cap C$ is non-empty.

A useful characterization of stationarity is given in terms of functions. A set $S \subseteq P_{\omega_1}(A)$ is stationary iff for any function $F : [A]^{<\omega} \to A$, there is a set a in S which is closed under F. Fodor's Lemma says that whenever $S \subseteq P_{\omega_1}(A)$ is stationary, and $F : S \to A$ satisfies that $F(b) \in b$ for all b in S, then there is a stationary set $S' \subseteq S$ and x in A such that $F(a) = x$ for all $a \in S'$.

When we say a cardinal θ is *much larger* than a set X, we mean that $P(P(P(X)))$ is in $H(\theta)$. We will frequently consider countable elementary substructures of $H(\theta)$, where $\theta \geq \omega_2$ is a regular cardinal. Let $N \prec H(\theta)$ be countable. We claim that for any uncountable ordinal ν in N, $N \cap \nu$ is not in N. For if $N \cap \nu$ is in N, then since $\nu \not\subseteq N \cap \nu$, let ξ be the least ordinal in $\nu \setminus (N \cap \nu)$. Then by elementarity, ξ is in N, which is a contradiction.

Let $N \prec H(\theta)$ be countable. Suppose $C \in N$ is a club subset of $P_{\omega_1}(A)$. Then $N \cap A \in C$. Suppose $\mathbb{P}$ is a forcing poset in N. If G if a generic filter on $\mathbb{P}$ over V, then $N[G] = \{\dot{x}^G : \dot{x}$ is a $\mathbb{P}$-name in $N\}$ is an elementary substructure of $H(\theta)^{V[G]}$. Assume $\mathbb{P}$ preserves ω_1, $\dot{C}$ is a $\mathbb{P}$-name in N, and p forces $\dot{C}$ is club in $P_{\omega_1}(A)$. If G is a generic filter on $\mathbb{P}$ over V and $p \in G$, then $C = \dot{C}^G$ is in $N[G]$, and hence $N[G] \cap A \in C$.

A condition q is *N-generic* (for $\mathbb{P}$) if $q \Vdash N[\dot{G}] \cap V = N$, where $\dot{G}$ is the canonical $\mathbb{P}$-name for the generic filter. Suppose $p \Vdash \dot{C}$ is club in $P_{\omega_1}(A)$, where $\dot{C} \in N$, and p is N-generic. Then by the last paragraph, $p \Vdash N[\dot{G}] \cap A \in \dot{C}$. But since p is N-generic, $p \Vdash N[\dot{G}] \cap A = N \cap A$. So $p \Vdash N \cap A \in \dot{C}$. A descending sequence of conditions $\langle p_n : n < \omega \rangle$ in $\mathbb{P} \cap N$ is *N-generic* if for every dense open set $D \subseteq \mathbb{P}$ in N, there is $n < \omega$ such that $p_n \in D \cap N$. It is always possible to construct an N-generic sequence, by simply enumerating the dense open subsets of $\mathbb{P}$ which are in N in order type ω, and defining a descending sequence of conditions in $N \cap \mathbb{P}$ by recursion, meeting each dense open set in the enumeration. If $\langle p_n : n < \omega \rangle$ is an N-generic sequence, and $q \leq p_n$ for all $n < \omega$, then q is an N-generic condition.

We will use the following special case of the Δ-System Lemma. Assume CH. Then for any sequence $\langle x_i : i < \omega_2 \rangle$ of countable sets, there is a set $Z \subseteq \omega_2$ with $|Z| = \omega_2$, and a countable set b, such that for all $i < j$ in Z, $x_i \cap x_j = b$.

A cardinal κ is *λ-supercompact*, where $\kappa \leq \lambda$, if there exists an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$.

Suppose $j : N \to M$ is an elementary embedding, where N and M are transitive inner models of ZFC. Suppose $\mathbb{P}$ is a forcing poset in N, G is a generic filter on $\mathbb{P}$ over N, and H is a generic filter on $j(\mathbb{P})$ over M. Then j can be extended to $j : N[G] \to M[H]$ such that $j(G) = H$ iff $j[G] \subseteq H$ (namely, let $j(\dot{x}^G) = j(\dot{x})^H$).

Let $M \subseteq N$ be inner models of ZFC, suppose λ is a regular uncountable cardinal of N, and assume $M^{<\lambda} \cap N \subseteq M$. Let $\mathbb{P}$ be a forcing poset in M, and assume N models that $\mathbb{P}$ is λ-c.c. If G is a generic filter on $\mathbb{P}$ over N, then $M[G]^{<\lambda} \cap N[G] \subseteq M[G]$.

2. Outline of Proof

The objective of this paper is to define a forcing poset $\mathbb{P}$ such that

$$\Vdash_{\mathbb{P}} \mathrm{WRP}(\omega_2) \wedge \neg \mathrm{RP}(\omega_2).$$

Specifically, we will define a forcing poset $\mathbb{P}$ which satisfies that whenever K is a generic filter on $\mathbb{P}$ over V, then in $V[K]$ there is a set S satisfying:

(1) S is a stationary subset of $P_{\omega_1}(\omega_2)$,
(2) S does not reflect to any ordinal in $\omega_2 \cap \operatorname{cof}(\omega_1)$,
(3) every stationary subset of S reflects to an uncountable ordinal in $\omega_2 \cap \operatorname{cof}(\omega)$,
(4) every stationary subset of $P_{\omega_1}(\omega_2) \setminus S$ reflects to an ordinal in $\omega_2 \cap \operatorname{cof}(\omega_1)$.

Clearly statements (1) and (2) imply that $\neg\mathrm{RP}(\omega_2)$ holds in $V[K]$. Statements (3) and (4) imply $\mathrm{WRP}(\omega_2)$ holds in $V[K]$. For if U is a stationary subset of $P_{\omega_1}(\omega_2)$, then either $U \cap S$ is stationary, or $U \setminus S$ is stationary. In the first case, (3) implies there is an uncountable ordinal α in $\omega_2 \cap \operatorname{cof}(\omega)$ such that $U \cap S$ reflects to α. In the second case, (4) implies there is an ordinal α in $\omega_2 \cap \operatorname{cof}(\omega_1)$ such that $U \setminus S$ reflects to α. In either case, there is an uncountable ordinal α in ω_2 such that U reflects to α.

We will define $\mathbb{P}$ in a model V in which there is a cardinal κ which is κ^+-supercompact and $2^\kappa = \kappa^+$. The forcing poset $\mathbb{P}$ will be of the form

$$\textsc{Coll}(\omega_1, {<}\kappa) * \dot{\mathbb{P}}_{\omega_2} * \dot{\mathbb{Q}}.$$

So in the first step, we Lévy collapse κ to become ω_2. Let G be a generic filter on $\textsc{Coll}(\omega_1, {<}\kappa)$ over V. Note that in $V[G]$, $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, and $2^{\omega_2} = \omega_3$.

In $V[G]$ we define a forcing poset $\mathbb{P}_{\omega_2}$ which adds a generic stationary set $S \subseteq P_{\omega_1}(\omega_2)$ which does not reflect to any ordinal in $\omega_2 \cap \operatorname{cof}(\omega_1)$.

Definition 2.1. *Let $\mathbb{P}_{\omega_2}$ be the forcing poset consisting of pairs (X, F) satisfying:*

(1) *X is a countable subset of $P_{\omega_1}(\omega_2) \setminus \{\emptyset\}$,*
(2) *F is a function whose domain is a countable subset of $\omega_2 \cap \operatorname{cof}(\omega_1)$,*
(3) *for all ξ in $\operatorname{dom}(F)$, $F(\xi)$ is a $\subseteq$-increasing and continuous sequence $\langle a_i^\xi : i \leq \gamma_\xi \rangle$ of countable subsets of ξ, for some $\gamma_\xi < \omega_1$,*
(4) *for all ξ in $\operatorname{dom}(F)$ and $i \leq \gamma_\xi$, a_i^ξ is not in X,*
(5) *for all ξ in $\operatorname{dom}(F)$, $a_{\gamma_\xi}^\xi \subseteq \bigcup X$.*

Define an order on $\mathbb{P}_{\omega_2}$ by letting $(Y, K) \leq (X, F)$ if:

(a) *$X \subseteq Y$,*
(b) *for all b in $Y \setminus X$, $b \not\subseteq \bigcup X$,*
(c) *$\operatorname{dom}(F) \subseteq \operatorname{dom}(K)$,*
(d) *for all ξ in $\operatorname{dom}(F)$, $F(\xi)$ is an initial segment of $K(\xi)$.*

The forcing poset $\mathbb{P}_{\omega_2}$ is ω_1-distributive. Since $2^\omega = \omega_1$ holds in $V[G]$, $\mathbb{P}_{\omega_2}$ is ω_2-c.c. It is easy to see that $\mathbb{P}_{\omega_2}$ forces $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, and $2^{\omega_2} = \omega_3$.

Let H be a generic filter on $\mathbb{P}_{\omega_2}$ over V. Define $S = \bigcup\{X : \exists F\ (X, F) \in H\}$. Then S is a stationary subset of $P_{\omega_1}(\omega_2)$. For each ν in $\omega_2 \cap \operatorname{cof}(\omega_1)$, the union of the collection of sequences $\{F(\nu) : \exists X\ (X, F) \in H,\ \nu \in \operatorname{dom}(F)\}$ is an increasing and continuous sequence $\langle a_i^\nu : i < \omega_1 \rangle$, with union equal to ν, which is disjoint from S. It follows that S does not reflect to any ordinal in $\omega_2 \cap \operatorname{cof}(\omega_1)$. Moreover, by upwards absoluteness, S does not reflect to any ordinal in $\omega_2 \cap \operatorname{cof}(\omega_1)$ in any outer model of $V[G * H]$ with the same ω_1 and ω_2.

In $V[G * H]$, we define a forcing iteration $\mathbb{Q}$, which is designed to destroy the stationarity of any subset of S which does not reflect to an uncountable ordinal in $\omega_2 \cap \operatorname{cof}(\omega)$. This forcing poset is ω_1-distributive and ω_2-c.c. in $V[G * H]$. Let I be a generic filter on $\mathbb{Q}$ over $V[G * H]$. Then in $V[G * H * I]$, S is a stationary subset of

$P_{\omega_1}(\omega_2)$ which does not reflect to any ordinal in $\omega_2 \cap \mathrm{cof}(\omega_1)$, and moreover, every stationary subset of S reflects to an uncountable ordinal in $\omega_2 \cap \mathrm{cof}(\omega)$.

Finally, let $K = G * H * I$. Statements (1), (2), and (3) hold in $V[K]$, as we have already observed. It remains to show (4), that every stationary subset of $P_{\omega_1}(\omega_2) \setminus S$ reflects to an ordinal in $\omega_2 \cap \mathrm{cof}(\omega_1)$.

In the ground model V, let $j : V \to M$ be an elementary embedding with critical point κ such that $j(\kappa) > \kappa^+$ and $M^{\kappa^+} \subseteq M$. Let $\overline{G} * \overline{H} * \overline{I}$ be a generic filter on $j(\textsc{Coll}(\omega_1, {<}\kappa) * \dot{\mathbb{P}}_{\omega_2} * \dot{\mathbb{Q}})$ over V. Let $G = \overline{G} \cap \textsc{Coll}(\omega_1, {<}\kappa)$ and $H = \overline{H} \cap \mathbb{P}_{\omega_2}$. Then G is a generic filter on $\textsc{Coll}(\omega_1, <\kappa)$ over V, and H is a generic filter on $\mathbb{P}_{\omega_2}$ over $V[G]$. In $V[\overline{G} * \overline{H}]$, we can extend j to $j : V[G * H] \to M[\overline{G} * \overline{H}]$ such that $j(G * H) = \overline{G} * \overline{H}$. Also let $G' = \overline{G} \cap \textsc{Coll}(\omega_1, [\kappa, j(\kappa)))$. Note that $M[\overline{G}] = M[G][G']$.

In the model $M[\overline{G}]$, $j(\mathbb{P}_{\omega_2})$ factors as a two-step iteration

$$\mathbb{P}_{\omega_2} * \mathbb{P}(\kappa, S, j(\kappa)).$$

(The definition of $\mathbb{P}(\kappa, S, j(\kappa))$ will be given in the next section; it is a generalization of Definition 2.1.) Since $j(\mathbb{P}_{\omega_2})$ is forcing equivalent to $\mathbb{P}_{\omega_2} * \mathbb{P}(\kappa, S, j(\kappa))$, there exists a generic filter H' on $\mathbb{P}(\kappa, S, j(\kappa))$ over $M[\overline{G}][H]$ such that $M[\overline{G}][\overline{H}] = M[\overline{G}][H][H']$.

In the model $M[\overline{G} * \overline{H}]$, there exists a projection mapping $\pi : j(\mathbb{Q}) \to \mathbb{Q}$. This allows us to factor $j(\mathbb{Q})$ as $\mathbb{Q} * (j(\mathbb{Q})/\mathbb{Q})$. Let I be the filter on $\mathbb{Q}$ generated by $\pi[\overline{I}]$. Then $M[\overline{G} * \overline{H}][\overline{I}] = M[\overline{G} * \overline{H}][I][\overline{I}]$. In the model $V[\overline{G} * \overline{H} * \overline{I}]$, we extend j to $j : V[G * H * I] \to M[\overline{G} * \overline{H} * \overline{I}]$ such that $j(G * H * I) = \overline{G} * \overline{H} * \overline{I}$. By several applications of the Product Lemma, the model $M[\overline{G} * \overline{H} * \overline{I}]$ can be shown to be equal to $M[G * H * I][G'][H'][\overline{I}]$.

Suppose T is a stationary subset of $P_{\omega_1}(\omega_2) \setminus S$ in the model $V[G * H * I]$. Since the forcing poset $\textsc{Coll}(\omega_1, {<}\kappa) * \dot{\mathbb{P}}_{\omega_2} * \dot{\mathbb{Q}}$ is κ-c.c., $M[G * H * I]^{\kappa^+} \cap V[G * H * I] \subseteq M[G * H * I]$. So in $M[G * H * I]$, T is a stationary subset of $P_{\omega_1}(\kappa)$. The model $M[\overline{G} * \overline{H} * \overline{I}]$ is a generic extension of $M[G * H * I]$ by the forcing poset $\textsc{Coll}(\omega_1, [\kappa, j(\kappa))) * \mathbb{P}(\kappa, S, j(\kappa)) * (j(\mathbb{Q})/\mathbb{Q})$. It turns out that this forcing poset preserves the stationarity of any stationary subset of $P_{\omega_1}(\kappa)$ in $M[G * H * I]$ which is disjoint from S. Thus, T remains stationary in $M[\overline{G} * \overline{H} * \overline{I}]$.

In the model $M[\overline{G} * \overline{H} * \overline{I}]$, $j(T)$ reflects to κ, since $j(T) \cap P_{\omega_1}(\kappa) = T$ and T is stationary in $P_{\omega_1}(\kappa)$. In this model, κ is an ordinal in $j(\kappa) = \omega_2$ with cofinality ω_1. By the elementarity of j, in $V[G * H * I]$ the set T reflects to an ordinal in ω_2 with cofinality ω_1.

3. The forcing poset $\mathbb{P}(\alpha, A, \beta)$

In the previous section we defined the forcing poset $\mathbb{P}_{\omega_2}$, which adds a stationary subset of $P_{\omega_1}(\omega_2)$ which does not reflect to any ordinal in ω_2 with cofinality ω_1. For the purpose of extending an elementary embedding after forcing with $\mathbb{P}_{\omega_2}$, we will need to consider a more general version of this forcing poset.

Definition 3.1. *Let $\alpha < \beta \leq \omega_2$ be ordinals, where β has uncountable cofinality. Let A be a subset of $P_{\omega_1}(\alpha)$. If $\mathrm{cf}(\alpha) = \omega_1$, also assume $P_{\omega_1}(\alpha) \setminus A$ is stationary in $P_{\omega_1}(\alpha)$. Define a forcing poset $\mathbb{P}(\alpha, A, \beta)$ as follows. A condition in $\mathbb{P}(\alpha, A, \beta)$ is a pair (X, F) satisfying:*

(1) *X is a countable subset of $P_{\omega_1}(\beta)$,*

(2) *for all b in X, b is not a subset of α,*
(3) *F is a function whose domain is a countable subset of $[\alpha, \beta) \cap \mathrm{cof}(\omega_1)$,*
(4) *for all ξ in $\mathrm{dom}(F)$, $F(\xi)$ is a $\subseteq$-increasing and continuous sequence $\langle a_i^\xi : i \le \gamma_\xi \rangle$ of countable subsets of ξ, for some $\gamma_\xi < \omega_1$,*
(5) *for all ξ in $\mathrm{dom}(F)$ and $i \le \gamma_\xi$, a_i^ξ is not in $X \cup A$,*
(6) *for all ξ in $\mathrm{dom}(F)$, $a_{\gamma_\xi}^\xi \subseteq \bigcup X$.*

Define an order on $\mathbb{P}(\alpha, A, \beta)$ by letting $(Y, K) \le (X, F)$ if:
(a) *$X \subseteq Y$,*
(b) *for all b in $Y \setminus X$, $b \nsubseteq \bigcup X$,*
(c) *$\mathrm{dom}(F) \subseteq \mathrm{dom}(K)$,*
(d) *for all ξ in $\mathrm{dom}(F)$, $F(\xi)$ is an initial segment of $K(\xi)$.*

Note that property (2) implies every set in X is non-empty. Also, for any condition (X, F) in $\mathbb{P}(\alpha, A, \beta)$, $X \cap A$ is empty. Namely, if b is in X, then b is not a subset of α by (2). But A is a subset of $P_{\omega_1}(\alpha)$.

If (X, F) is a condition, then X is an approximation of a generic stationary set $S \subseteq P_{\omega_1}(\beta)$. The function F in (X, F) approximates an array of clubs which witnesses that S does not reflect to any ordinal in $[\alpha, \beta)$ with cofinality ω_1.

We will only be interested in two particular cases of the definition. The first case is when $\alpha = 0$ and $A = \emptyset$. In this case, we abbreviate $\mathbb{P}(\alpha, A, \beta)$ as $\mathbb{P}_\beta$. For example, the forcing poset $\mathbb{P}_{\omega_2}$ described in the previous section is equal to $\mathbb{P}(0, \emptyset, \omega_2)$.

The second case of interest occurs in the following context. Let $\nu < \omega_2$ be an ordinal with cofinality ω_1. Let H be a generic filter on $\mathbb{P}_\nu$ over V. In $V[H]$, define $A = \bigcup\{X : \exists F\ (X, F) \in H\}$. So A is a subset of $P_{\omega_1}(\nu)$. As we will prove later, $P_{\omega_1}(\nu) \setminus A$ is stationary in $P_{\omega_1}(\nu)$. So it is meaningful to define $\mathbb{P}(\nu, A, \omega_2)$ in the model $V[H]$. As it turns out, $\mathbb{P}_{\omega_2}$ is forcing equivalent to the two-step iteration $\mathbb{P}_\nu * \mathbb{P}(\nu, A, \omega_2)$. This factorization is the motivation for Definition 3.1.

Notation 3.2. *Let $\beta \le \omega_2$ be an ordinal with uncountable cofinality. Then $\mathbb{P}_\beta$ denotes $\mathbb{P}(0, \emptyset, \beta)$.*

The proof of the next lemma is straightforward.

Lemma 3.3. *Let $\beta_1 < \beta_2 \le \omega_2$ be ordinals with uncountable cofinality. Then $\mathbb{P}_{\beta_1} \subseteq \mathbb{P}_{\beta_2}$. If p and q are in $\mathbb{P}_{\beta_1}$, then $q \le p$ in $\mathbb{P}_{\beta_1}$ iff $q \le p$ in $\mathbb{P}_{\beta_2}$.*

Lemma 3.4. *Let $p = (X, F)$ be a condition in $\mathbb{P}(\alpha, A, \beta)$, and let d be a set in $P_{\omega_1}(\beta)$ such that d is not a subset of α and d is not a subset of $\bigcup X$. Then $q = (X \cup \{d\}, F)$ is a condition in $\mathbb{P}(\alpha, A, \beta)$ and $q \le p$.*

Proof. Properties (1), (2), (3), (4), and (6) of Definition 3.1 are immediate. For (5), let ξ be in $\mathrm{dom}(F)$ and let $F(\xi) = \langle a_i^\xi : i \le \gamma_\xi \rangle$. Let $i \le \gamma_\xi$. Then a_i^ξ is not in $X \cup A$. Also a_i^ξ is a subset of $\bigcup X$. Since d is not a subset of $\bigcup X$, a_i^ξ is not equal to d. So a_i^ξ is not in $(X \cup \{d\}) \cup A$. It is immediate that $q \le p$. □

Lemma 3.5. *Let b be a set in $P_{\omega_1}(\beta)$. Then there are densely many conditions (Y, K) in $\mathbb{P}(\alpha, A, \beta)$ such that $b \subseteq \bigcup Y$.*

Proof. Let (X, F) be a condition. Choose a countable set $d \subseteq \beta$ such that $b \cup (\bigcup X) \cup \{\alpha\} \subsetneq d$. By Lemma 3.4, $(X \cup \{d\}, F)$ is a condition in $\mathbb{P}(\alpha, A, \beta)$ which is below (X, F), and clearly $b \subseteq \bigcup(X \cup \{d\})$. □

Lemma 3.6. *Let ξ be an ordinal in $[\alpha, \beta) \cap \mathrm{cof}(\omega_1)$, let $\pi < \omega_1$, and let b be a countable subset of ξ. Then there are densely many conditions in $\mathbb{P}(\alpha, A, \beta)$ of the form (Y, K), where ξ is in $\mathrm{dom}(K)$, and $K(\xi) = \langle b_i^\xi : i \leq \gamma_\xi \rangle$, where $\pi \leq \gamma_\xi$ and $b \subseteq b_{\gamma_\xi}^\xi$.*

Proof. Let (X, F) be a condition in $\mathbb{P}(\alpha, A, \beta)$. We choose sets c and d as follows, depending on whether $\xi > \alpha$ or $\xi = \alpha$. First suppose $\xi > \alpha$. Since ξ is uncountable and $\bigcup X$ is countable, we can choose countable sets c and d such that $((\bigcup X) \cap \xi) \cup b \cup \{\alpha\} \subsetneq c \subsetneq d \subseteq \xi$. Now suppose $\xi = \alpha$. Then $\mathrm{cf}(\alpha) = \omega_1$, so $P_{\omega_1}(\alpha) \setminus A$ is stationary in $P_{\omega_1}(\alpha)$. So we can choose c in $P_{\omega_1}(\alpha) \setminus A$ such that $((\bigcup X) \cap \alpha) \cup b \subsetneq c$. Now let $d = c \cup \{\alpha\}$.

Define $Y = X \cup \{d\}$. By the choice of d, $\alpha \in d$ and $d \not\subseteq \bigcup X$. So by Lemma 3.4, (Y, F) is a condition in $\mathbb{P}(\alpha, A, \beta)$ which is below (X, F).

Now we define a function K. Let K have domain equal to $\mathrm{dom}(F) \cup \{\xi\}$. For all ν in $\mathrm{dom}(K)$ different from ξ, let $K(\nu) = F(\nu)$. To define $K(\xi)$, we consider the two cases of whether ξ is in $\mathrm{dom}(F)$ or not. Suppose ξ is not in $\mathrm{dom}(F)$. Then let $K(\xi)$ be the sequence $\langle b_i^\xi : i \leq \pi \rangle$, where $b_i^\xi = c$ for all $i \leq \pi$. Suppose ξ is in $\mathrm{dom}(F)$. Let $F(\xi) = \langle a_i^\xi : i \leq \overline{\gamma}_\xi \rangle$. Let $\gamma_\xi = \max(\overline{\gamma}_\xi, \pi) + 1$. Now define $K(\xi) = \langle b_i^\xi : i \leq \gamma_\xi \rangle$, where $b_i^\xi = a_i^\xi$ for $i \leq \overline{\gamma}_\xi$, and $b_j^\xi = c$ for $\overline{\gamma}_\xi < j \leq \gamma_\xi$. It is routine to verify that (Y, K) is a condition in $\mathbb{P}(\alpha, A, \beta)$ which is below (X, F), and (Y, K) clearly satisfies the conclusion of the lemma. $\square$

We will frequently consider descending sequences of conditions in $\mathbb{P}(\alpha, A, \beta)$. So it will be useful to establish some notation and basic facts about such sequences.

Suppose $\langle (X_n, F_n) : n < \omega \rangle$ is a descending sequence of conditions in $\mathbb{P}(\alpha, A, \beta)$. Let us define what we will call the infinum of the sequence of functions $\langle F_n : n < \omega \rangle$. Let K have domain equal to $\bigcup \{\mathrm{dom}(F_n) : n < \omega\}$, which is a countable subset of $[\alpha, \beta) \cap \mathrm{cof}(\omega_1)$. Consider ξ in the domain of K. If $n < m$ and ξ is in the domain of F_n, then since $(X_m, F_m) \leq (X_n, F_n)$, $F_n(\xi)$ is an initial segment of $F_m(\xi)$. So if we take the union of all sequences $F_n(\xi)$, where $\xi \in \mathrm{dom}(F_n)$, we obtain an increasing and continuous sequence of the form $\langle a_i^\xi : i < \gamma_\xi \rangle$, where $\gamma_\xi < \omega_1$. If γ_ξ is a successor ordinal, let $K(\xi)$ be the sequence $\langle a_i^\xi : i \leq \gamma_\xi - 1 \rangle$. If γ_ξ is a limit ordinal, define $a_{\gamma_\xi}^\xi = \bigcup \{a_i^\xi : i < \gamma_\xi\}$, and let $K(\xi) = \langle a_i^\xi : i \leq \gamma_\xi \rangle$. Then $K(\xi)$ is an increasing and continuous sequence of countable subsets of ξ. We refer to K as the *infinum* of the sequence $\langle F_n : n < \omega \rangle$.

Lemma 3.7. *Let θ be a regular cardinal much larger than $\mathbb{P}(\alpha, A, \beta)$, and suppose N is a countable elementary substructure of $H(\theta)$ such that $\mathbb{P}(\alpha, A, \beta)$ is in N. Let $\langle (X_n, F_n) : n < \omega \rangle$ be an N-generic sequence for $\mathbb{P}(\alpha, A, \beta)$.*

Let $X' = \bigcup \{X_n : n < \omega\}$ and let K be the infinum of $\langle F_n : n < \omega \rangle$. For each ξ in $\mathrm{dom}(K)$, write $K(\xi)$ as $\langle a_i^\xi : i \leq \gamma_\xi \rangle$. Then:

(1) $\bigcup X' = N \cap \beta$,
(2) $\mathrm{dom}(K) = N \cap [\alpha, \beta) \cap \mathrm{cof}(\omega_1)$,
(3) *for all ξ in $\mathrm{dom}(K)$, $\gamma_\xi = N \cap \omega_1$ and $a_{\gamma_\xi}^\xi = N \cap \xi$.*

Proof. For each $n < \omega$, X_n is in N. Since X_n is countable, $X_n \subseteq N$. It follows that $X' = \bigcup \{X_n : n < \omega\} \subseteq N$. So if b is in X', then b is in N, and therefore since b is countable, $b \subseteq N \cap \beta$. It follows that $\bigcup X' \subseteq N \cap \beta$. On the other hand, if ν is in $N \cap \beta$, then by Lemma 3.5 there is a dense set D of conditions (X, F) such that

ν is in $\bigcup X$. By elementarity, D is in N. So fix $n < \omega$ such that (X_n, F_n) is in D. Then ν is in $\bigcup X_n \subseteq \bigcup X'$, so ν is in $\bigcup X'$. Thus $\bigcup X' = N \cap \beta$. Statements (2) and (3) can be proven by similar arguments, using Lemma 3.6. $\square$

The next result is our main tool for constructing lower bounds of descending sequences.

Proposition 3.8. *Let θ be a regular cardinal much larger than $\mathbb{P}(\alpha, A, \beta)$. Let N be a countable elementary substructure of $H(\theta)$ such that $\mathbb{P}(\alpha, A, \beta)$ is in N, and moreover, if $\mathrm{cf}(\alpha) = \omega_1$, then $N \cap \alpha$ is not in A. Let $\langle (X_n, F_n) : n < \omega \rangle$ be an N-generic sequence for $\mathbb{P}(\alpha, A, \beta)$.*

Let $X' = \bigcup\{X_n : n < \omega\}$ and let K be the infinum of $\langle F_n : n < \omega \rangle$. Suppose that Y is a countable subset of $P_{\omega_1}(\beta)$ such that $X' \subseteq Y$, and for all b in $Y \setminus X'$, either $b = N \cap \beta$, or there is an uncountable ordinal ν in $(\alpha, \beta) \cap N \cap \mathrm{cof}(\omega)$ such that $b = N \cap \nu$. Then (Y, K) is a condition in $\mathbb{P}(\alpha, A, \beta)$ and $(Y, K) \leq (X_n, F_n)$ for all $n < \omega$.

Proof. We prove that (Y, K) is a condition in $\mathbb{P}(\alpha, A, \beta)$. Note that properties (1), (3), and (4) of Definition 3.1 are obvious, and (2) follows from the fact that α is in N. To prove (5) and (6), consider ξ in $\mathrm{dom}(K)$. Let $K(\xi) = \langle a_i^\xi : i \leq \gamma_\xi \rangle$. By Lemma 3.7, $\bigcup Y = N \cap \beta$, $\xi \in N$, and $a^\xi_{\gamma_\xi} = N \cap \xi$. So $a^\xi_{\gamma_\xi} = N \cap \xi \subseteq N \cap \beta = \bigcup Y$, proving (6).

To prove (5), let $i \leq \gamma_\xi$, and we show a_i^ξ is not in $Y \cup A$. First suppose $i < \gamma_\xi$. Then a_i^ξ is in N. Note that every set in $Y \setminus X'$ is not in N. So if a_i^ξ is in Y, then it is in X'. But if a_i^ξ is in X', then we can find $n < \omega$ such that a_i^ξ is in X_n and also appears on the sequence $F_n(\xi)$, contradicting that (X_n, F_n) is a condition. So a_i^ξ is not in Y. For some $n < \omega$, a_i^ξ is on the sequence $F_n(\xi)$, which implies by definition that a_i^ξ is not in A. So a_i^ξ is not in $Y \cup A$.

Now assume $i = \gamma_\xi$. Then by Lemma 3.7, $a^\xi_{\gamma_\xi} = N \cap \xi$. In particular, $a^\xi_{\gamma_\xi}$ is not in N. So $a^\xi_{\gamma_\xi}$ is not in X'. If $\xi = \alpha$, then α has cofinality ω_1. By the assumptions on N, $N \cap \xi = N \cap \alpha = a^\xi_{\gamma_\xi}$ is not in A. If $\xi > \alpha$, then $\alpha \in N \cap \xi = a^\xi_{\gamma_\xi}$, and therefore $a^\xi_{\gamma_\xi}$ is not in A. So we have proven that $a^\xi_{\gamma_\xi}$ is not in $X' \cup A$.

Let b be in $Y \setminus X'$, and we show b is not equal to $a^\xi_{\gamma_\xi}$. First suppose $b = N \cap \beta$. Then $\xi \in N \cap \beta = b$, whereas $a^\xi_{\gamma_\xi}$ is a subset of ξ. So b is not equal to $a^\xi_{\gamma_\xi}$. Now suppose $b = N \cap \nu$, for some uncountable ordinal ν in $(\alpha, \beta) \cap N \cap \mathrm{cof}(\omega)$. Since ν has cofinality ω and ξ has cofinality ω_1, either $\xi < \nu$ or $\nu < \xi$. If $\xi < \nu$, then ξ is in $N \cap \nu = b$, whereas $a^\xi_{\gamma_\xi} \subseteq \xi$, so $a^\xi_{\gamma_\xi}$ is not equal to b. If $\nu < \xi$, then ν is in $N \cap \xi = a^\xi_{\gamma_\xi}$, whereas $b \subseteq \nu$, so $a^\xi_{\gamma_\xi}$ is not equal to b. This completes the proof that (Y, K) is in $\mathbb{P}(\alpha, A, \beta)$.

Let $n < \omega$ be given, and we show $(Y, K) \leq (X_n, F_n)$. Properties (a), (c), and (d) of Definition 3.1 are immediate. Suppose b is in $Y \setminus X_n$, and we show $b \nsubseteq \bigcup X_n$. If b is in X', then there is $m > n$ such that b is in $X_m \setminus X_n$. Since $(X_m, F_m) \leq (X_n, F_n)$, $b \nsubseteq \bigcup X_n$. Now suppose b is not in X'. Then $b = N \cap \nu$, where either $\nu = \beta$, or ν is an uncountable ordinal in $(\alpha, \beta) \cap N \cap \mathrm{cof}(\omega)$. Suppose for a contradiction that $b \subseteq \bigcup X_n$. Then $b = b \cap \nu \subseteq (\bigcup X_n) \cap \nu \subseteq N \cap \nu = b$. So $(\bigcup X_n) \cap \nu = N \cap \nu$. But $\bigcup X_n$ and ν are in N, so by elementarity, $(\bigcup X_n) \cap \nu$ is in N. So $N \cap \nu$ is in N, which is impossible. So indeed $b \nsubseteq \bigcup X_n$. $\square$

Theorem 3.9. *The forcing poset $\mathbb{P}(\alpha, A, \beta)$ is ω_1-distributive.*

Proof. Let $\{D_n : n < \omega\}$ be a collection of dense open subsets of $\mathbb{P}(\alpha, A, \beta)$. Let (X, F) be a condition. We find a condition $(Y, K) \leq (X, F)$ in $\bigcap\{D_n : n < \omega\}$.

Fix a regular cardinal θ much larger than $\mathbb{P}(\alpha, A, \beta)$. Choose a countable elementary substructure N of $H(\theta)$ which contains as members the objects $\mathbb{P}(\alpha, A, \beta)$, (X, F), and $\{D_n : n < \omega\}$, and moreover, if $\mathrm{cf}(\alpha) = \omega_1$, then $N \cap \alpha$ is not in A. This is possible since if $\mathrm{cf}(\alpha) = \omega_1$, then $P_{\omega_1}(\alpha) \setminus A$ is stationary in $P_{\omega_1}(\alpha)$.

Choose an N-generic sequence $\langle (X_n, F_n) : n < \omega\rangle$ with $(X_0, F_0) = (X, F)$. Let $Y = \bigcup\{X_n : n < \omega\}$ and let K be the infinum of the sequence $\langle F_n : n < \omega\rangle$. By Proposition 3.8, (Y, K) is a condition and $(Y, K) \leq (X_n, F_n)$ for all $n < \omega$. In particular, $(Y, K) \leq (X, F)$. Let $n < \omega$ be given. Then D_n is in N. So for some m, (X_m, F_m) is in D_n. Since D_n is open, (Y, K) is in D_n. Thus (Y, K) is in $\bigcap\{D_n : n < \omega\}$. □

Theorem 3.10. *If $2^\omega = \omega_1$, then $\mathbb{P}(\alpha, A, \beta)$ is ω_2-c.c.*

Proof. We will prove that $\mathbb{P}(\alpha, A, \beta)$ is ω_2-Knaster. Let $\langle (X_j, F_j) : j < \omega_2\rangle$ be a sequence of conditions in $\mathbb{P}(\alpha, A, \beta)$. For each $j < \omega_2$ and $\xi \in \mathrm{dom}(F_j)$, let $F_j(\xi) = \langle a(j)^\xi_i : i \leq \gamma^j_\xi\rangle$. We will thin out this sequence of conditions in five steps.

First, apply the Δ-System Lemma to the sequence of countable sets $\langle \mathrm{dom}(F_j) : j < \omega_2\rangle$ to obtain an unbounded set $Z_0 \subseteq \omega_2$ and a countable set B such that for all $j < k$ in Z_0, $\mathrm{dom}(F_j) \cap \mathrm{dom}(F_k) = B$.

Secondly, an easy counting argument using CH shows there are at most ω_1 many possibilities for a sequence of the form $\langle \gamma^j_\xi : \xi \in B\rangle$, for j in Z_0. So we can fix an unbounded set $Z_1 \subseteq Z_0$ and a sequence $\langle \gamma_\xi : \xi \in B\rangle$ such that for all $j \in Z_1$ and $\xi \in B$, $\gamma^j_\xi = \gamma_\xi$.

Thirdly, another straightforward counting argument shows there are at most ω_1 many possibilities for a sequence of the form $\langle a(j)^\xi_i : \xi \in B, i \leq \gamma_\xi\rangle$, for $j \in Z_1$. Fix an unbounded set $Z_2 \subseteq Z_1$ and a sequence $\langle a^\xi_i : \xi \in B, i \leq \gamma_\xi\rangle$ such that for all $j \in Z_2$, $\xi \in B$, and $i \leq \gamma_\xi$, $a(j)^\xi_i = a^\xi_i$.

Fourthly, apply the Δ-system lemma to the sequence of countable sets $\langle \bigcup X_j : j \in Z_2\rangle$ to obtain an unbounded set $Z_3 \subseteq Z_2$ and a countable set E such that for all $j < k$ in Z_3, $(\bigcup X_j) \cap (\bigcup X_k) = E$.

Fifthly, a routine counting argument shows there are at most ω_1 possibilities for a set of the form $P(E) \cap X_j$, for $j \in Z_3$. So we can choose an unbounded set $Z_4 \subseteq Z_3$ so that for all $j < k$ in Z_4, $P(E) \cap X_j = P(E) \cap X_k$.

Claim 3.11. *Let j and k be distinct ordinals in Z_4, and suppose b is in $X_j \setminus X_k$. Then b is not a subset of $\bigcup X_k$.*

Proof. Suppose for a contradiction b is in $X_j \setminus X_k$ and $b \subseteq \bigcup X_k$. Since b is in X_j, $b \subseteq \bigcup X_j$. So $b \subseteq (\bigcup X_j) \cap (\bigcup X_k) = E$. Therefore b is in $P(E) \cap X_j$. But $P(E) \cap X_j = P(E) \cap X_k$. So b is in X_k, which is a contradiction. □

Let $j < k$ be in Z_4, and we prove (X_j, F_j) and (X_k, F_k) are compatible. Define (Y, K) as follows. Let $Y = X_j \cup X_k$. Let $\mathrm{dom}(K) = \mathrm{dom}(F_j) \cup \mathrm{dom}(F_k)$. If ξ is in $\mathrm{dom}(F_j) \setminus \mathrm{dom}(F_k)$, let $K(\xi) = F_j(\xi)$, and if ξ is in $\mathrm{dom}(F_k) \setminus \mathrm{dom}(F_j)$, let $K(\xi) = F_k(\xi)$. If ξ is $\mathrm{dom}(F_j) \cap \mathrm{dom}(F_k) = B$, then $F_j(\xi)$ and $F_k(\xi)$ are both equal to $\langle a^\xi_i : i \leq \gamma_\xi\rangle$, and we define $K(\xi)$ to be this sequence. This completes the definition of (Y, K).

We prove that (Y, K) is a condition. Properties (1), (2), (3), and (4) of Definition 3.1 are immediate. For (5) and (6), suppose ξ is in $\mathrm{dom}(K)$. Let $K(\xi) = \langle a_i^\xi : i \leq \gamma_\xi \rangle$. Consider $i \leq \gamma_\xi$. Since a_i^ξ appears on either the sequence $F_j(\xi)$ or $F_k(\xi)$, a_i^ξ is either a subset of $\bigcup X_j$ or $\bigcup X_k$. These sets are in turn subsets of $\bigcup Y$. So $a_i^\xi \subseteq \bigcup Y$. This verifies (6).

It remains to show that a_i^ξ is not in $Y \cup A$. Since a_i^ξ appears on either $F_j(\xi)$ or $F_k(\xi)$, it is not in A. Now we show a_i^ξ is not in Y. First assume ξ is in $\mathrm{dom}(F_j) \setminus \mathrm{dom}(F_k)$. Then $K(\xi) = F_j(\xi)$. So a_i^ξ appears on the sequence $F_j(\xi)$. Since (X_j, F_j) is a condition, a_i^ξ is not in X_j and $a_i^\xi \subseteq \bigcup X_j$. If for a contradiction a_i^ξ is in Y, then a_i^ξ is in $X_k \setminus X_j$. By Claim 3.11, a_i^ξ is not a subset of $\bigcup X_j$, which is a contradiction. The case when ξ is in $\mathrm{dom}(F_k) \setminus \mathrm{dom}(F_j)$ is the same, with the roles of j and k reversed. Now assume ξ is in $\mathrm{dom}(F_j) \cap \mathrm{dom}(F_k) = B$. Then a_i^ξ appears on the sequence $F_j(\xi) = F_k(\xi)$. Since (X_j, F_j) and (X_k, F_k) are conditions, a_i^ξ is neither in X_j nor X_k. Therefore a_i^ξ is not in $Y = X_j \cup X_k$. This completes the proof that (Y, K) is in $\mathbb{P}(\alpha, A, \beta)$.

For proving that $(Y, K) \leq (X_j, F_j), (X_k, F_k)$, properties (a), (c), and (d) of Definition 3.1 are immediate, and (b) follows from Claim 3.11. □

4. The Set $S(\alpha, A, \beta)$

Fix for the remainder of the section α, β, and A as described in the assumptions of Definition 3.1. Specifically, $\alpha < \beta \leq \omega_2$, β has uncountable cofinality, A is a subset of $P_{\omega_1}(\alpha)$, and if $\mathrm{cf}(\alpha) = \omega_1$, then $P_{\omega_1}(\alpha) \setminus A$ is stationary in $P_{\omega_1}(\alpha)$. Also assume $2^\omega = \omega_1$.

Notation 4.1. *Let $\dot{S}(\alpha, A, \beta)$ be a $\mathbb{P}(\alpha, A, \beta)$-name such that*

$$\Vdash \dot{S}(\alpha, A, \beta) = \bigcup \{X : \exists F \ (X, F) \in \dot{H}\},$$

where $\dot{H}$ is the canonical name for the generic filter.

Notation 4.2. *In the case $\alpha = 0$ and $A = \emptyset$, let $\dot{S}_\beta$ denote $\dot{S}(0, \emptyset, \beta)$.*

Consider a generic filter H on $\mathbb{P}(\alpha, A, \beta)$ over V. Since $\mathbb{P}(\alpha, A, \beta)$ is ω_1-distributive and ω_2-c.c., all cardinals and cofinalities are preserved in $V[H]$. Also by distributivity, $P_{\omega_1}(\beta)$ is computed the same in V and $V[H]$. The set $S(\alpha, A, \beta)$ is a subset of $P_{\omega_1}(\beta)$ in $V[H]$. Let ξ be in $[\alpha, \beta) \cap \mathrm{cof}(\omega_1)$. By Lemma 3.6, if we take the union of all sequences of the form $F(\xi)$, where $(X, F) \in H$ for some X, and $\xi \in \mathrm{dom}(F)$, then we obtain an increasing and continuous sequence $\langle a_i^\xi : i < \omega_1 \rangle$ cofinal in $P_{\omega_1}(\xi)$.

Lemma 4.3. *In $V[H]$, for all ξ in $[\alpha, \beta) \cap \mathrm{cof}(\omega_1)$, the set $\{a_i^\xi : i < \omega_1\}$ is a club subset of $P_{\omega_1}(\xi)$ which is disjoint from $S(\alpha, A, \beta)$. So $S(\alpha, A, \beta)$ does not reflect to any ordinal in $[\alpha, \beta)$ with cofinality ω_1.*

Proof. If b is in $S(\alpha, A, \beta)$, fix (X, F) in H such that $b \in X$. Suppose for a contradiction b is equal to a_i^ξ for some ξ in $[\alpha, \beta) \cap \mathrm{cof}(\omega_1)$ and $i < \omega_1$. Then there is $(Y, K) \leq (X, F)$ such that b appears on the sequence $F(\xi)$. But then b is in Y and on the sequence $F(\xi)$, which contradicts property (5) of Definition 3.1. □

Note that the statement that $S(\alpha, A, \beta)$ does not reflect to any ordinal in $[\alpha, \beta) \cap \mathrm{cof}(\omega_1)$ is upwards absolute between $V[H]$ and any outer model W of $V[H]$ with the same ω_1.

Proposition 4.4. *The forcing poset $\mathbb{P}(\alpha, A, \beta)$ forces that $\dot{S}(\alpha, A, \beta)$ is a stationary subset of $P_{\omega_1}(\beta)$.*

Proof. Let (X, F) be a condition in $\mathbb{P}(\alpha, A, \beta)$ and suppose (X, F) forces $\dot{C}$ is a club subset of $P_{\omega_1}(\beta)$. We will find $(Y, K) \leq (X, F)$ and a set z such that $(Y, K) \Vdash z \in \dot{S}(\alpha, A, \beta) \cap \dot{C}$.

Fix a regular cardinal θ much larger than $\mathbb{P}(\alpha, A, \beta)$ and $\dot{C}$. Let N be a countable elementary substructure of $H(\theta)$ which contains as members the objects $\mathbb{P}(\alpha, A, \beta)$, (X, F), and $\dot{C}$, and moreover, if $\mathrm{cf}(\alpha) = \omega_1$, then $N \cap \alpha$ is in $P_{\omega_1}(\alpha) \setminus A$. Let $\langle (X_n, F_n) : n < \omega \rangle$ be an N-generic sequence with $(X_0, F_0) = (X, F)$.

Let $X' = \bigcup \{X_n : n < \omega\}$. Define $Y = X' \cup \{N \cap \beta\}$. Let K be the infinum of the sequence $\langle F_n : n < \omega \rangle$. By Proposition 3.8, (Y, K) is a condition in $\mathbb{P}(\alpha, A, \beta)$, and $(Y, K) \leq (X_n, F_n)$ for all $n < \omega$. In particular, $(Y, K) \leq (X, F)$ and (Y, K) is an N-generic condition. Since $N \cap \beta \in Y$, $(Y, K) \Vdash N \cap \beta \in \dot{S}(\alpha, A, \beta)$.

Since $\dot{C}$ is in N, $\mathbb{P}(\alpha, A, \beta)$ forces that $N[\dot{H}] \cap \beta \in \dot{C}$, where $\dot{H}$ is the canonical $\mathbb{P}(\alpha, A, \beta)$-name for the generic filter. As (Y, K) is N-generic, $(Y, K) \Vdash N[\dot{H}] \cap \beta = N \cap \beta$. So $(Y, K) \Vdash N \cap \beta \in \dot{S}(\alpha, A, \beta) \cap \dot{C}$. $\square$

The next easy lemma will be used frequently throughout the paper.

Lemma 4.5. *Let (X, F) be a condition in $\mathbb{P}(\alpha, A, \beta)$. Then*

$$(X, F) \Vdash \dot{S}(\alpha, A, \beta) \cap P(\bigcup X) = X.$$

Proof. Clearly $(X, F) \Vdash X \subseteq \dot{S}(\alpha, A, \beta) \cap P(\bigcup X)$. Let H be a generic filter on $\mathbb{P}(\alpha, A, \beta)$ with (X, F) in H. Suppose for a contradiction there is x in $S(\alpha, A, \beta) \cap P(\bigcup X)$ which is not in X. Fix a condition (Y, K) in H such that x is in Y. Choose (Z, L) in H which is below (X, F) and (Y, K). Then x is in $Z \setminus X$ and $x \subseteq \bigcup X$, which contradicts that $(Z, L) \leq (X, F)$. $\square$

Lemma 4.6. *The forcing poset $\mathbb{P}(\alpha, A, \beta)$ forces that for any countable set $B \subseteq \beta$,*

$$|\dot{S}(\alpha, A, \beta) \cap P(B)| \leq \aleph_0.$$

Proof. If B is a countable subset of β in a generic extension by $\mathbb{P}(\alpha, A, \beta)$, then since $\mathbb{P}(\alpha, A, \beta)$ is ω_1-distributive, B is in the ground model. There are densely many conditions of the form (X, F), where $B \subseteq \bigcup X$. If (X, F) is such a condition, then by Lemma 4.5, (X, F) forces that any subset of B which is in $\dot{S}(\alpha, A, \beta)$ is in $\dot{S}(\alpha, A, \beta) \cap P(\bigcup X) = X$. So $(X, F) \Vdash \dot{S}(\alpha, A, \beta) \cap P(B) \subseteq X$. Since X is countable, $(X, F) \Vdash |\dot{S}(\alpha, A, \beta) \cap P(B)| \leq \aleph_0$. $\square$

The next technical, unmotivated lemma will be crucial to several arguments in later sections.

Lemma 4.7. *Let $\theta \geq \omega_2$ be a regular cardinal.*

1. *$\mathbb{P}(\alpha, A, \beta)$ forces there are stationarily many N in $P_{\omega_1}(H(\theta))$ such that $\dot{S}(\alpha, A, \beta) \cap P(N \cap \beta) \subseteq N$.*

(2) *Assume* $\mathrm{cf}(\alpha) = \omega_1$, *and let* T *be a stationary subset of* $P_{\omega_1}(\alpha)$ *which is disjoint from* A. *Then* $\mathbb{P}(\alpha, A, \beta)$ *forces there are stationarily many* N *in* $P_{\omega_1}(H(\theta))$ *such that* $\dot{S}(\alpha, A, \beta) \cap P(N \cap \beta) \subseteq N$ *and* $N \cap \alpha \in T$.

Proof. Let (X, F) be a condition in $\mathbb{P}(\alpha, A, \beta)$, and suppose (X, F) forces $\dot{C}$ is a club subset of $P_{\omega_1}(H(\theta))$. (By $H(\theta)$, we mean $H(\theta)$ as computed in the generic extension.) Let χ be a regular cardinal much larger than $\mathbb{P}(\alpha, A, \beta)$, θ, and $\dot{C}$. Let M be a countable elementary substructure of $H(\chi)$ which contains as members the objects $\mathbb{P}(\alpha, A, \beta)$, (X, F), θ, and $\dot{C}$, and if $\mathrm{cf}(\alpha) = \omega_1$, then $M \cap \alpha \in P_{\omega_1}(\alpha) \setminus A$. In addition, in case (2), also assume $M \cap \alpha$ is in T.

Let $\langle (X_n, F_n) : n < \omega \rangle$ be an M-generic sequence with $(X_0, F_0) = (X, F)$. Let $Y = \bigcup \{X_n : n < \omega\}$ and let K be the infinum of the sequence $\langle F_n : n < \omega \rangle$. By Proposition 3.8, (Y, K) is in $\mathbb{P}(\alpha, A, \beta)$, and $(Y, K) \leq (X_n, F_n)$ for all $n < \omega$. In particular, $(Y, K) \leq (X, F)$ and (Y, K) is M-generic. Since $\bigcup Y = M \cap \beta$, Lemma 4.5 implies $(Y, K) \Vdash \dot{S}(\alpha, A, \beta) \cap P(M \cap \beta) = Y$.

Let H be a generic filter on $\mathbb{P}(\alpha, A, \beta)$ with $(Y, K) \in H$. Since (Y, K) is M-generic, $M[H] \cap V = M$. Also $S(\alpha, A, \beta) \cap P(M \cap \beta) = Y$. Let $N = M[H] \cap H(\theta)^{V[H]}$. We show that N is as desired.

Since θ is in M, $H(\theta)^{V[H]}$ is in $M[H]$, so by standard arguments, N is an elementary substructure of $H(\theta)^{V[H]}$. As $\dot{C}$ is in M, N is in C. And since (Y, K) is M-generic, $N \cap \beta = M[H] \cap \beta = M \cap \beta$, and $N \cap \alpha = M[H] \cap \alpha = M \cap \alpha$. If $\mathrm{cf}(\alpha) = \omega_1$, then $N \cap \alpha = M \cap \alpha$ is in T.

It suffices to show $S(\alpha, A, \beta) \cap P(N \cap \beta) \subseteq N$. Let b be in $S(\alpha, A, \beta) \cap P(N \cap \beta)$. Since $N \cap \beta = M \cap \beta$, b is in $S(\alpha, A, \beta) \cap P(M \cap \beta) = Y$. So b is in Y. But $Y = \bigcup \{X_n : n < \omega\}$. So fix $n < \omega$ such that b is in X_n. Then since $X_n \in M$ and X_n is countable, $X_n \subseteq M$, so b is in M. Hence b is in $M[H]$. Since $\beta \leq \theta$, b is in $M[H] \cap H(\theta) = N$. $\square$

Corollary 4.8. *Assume that* $\mathrm{cf}(\alpha) = \omega_1$, *and let* T *be a stationary subset of* $P_{\omega_1}(\alpha)$ *which is disjoint from* A. *Then* $\mathbb{P}(\alpha, A, \beta)$ *forces that* T *is stationary in* $P_{\omega_1}(\alpha)$.

Proof. Let H be a generic filter on $\mathbb{P}(\alpha, A, \beta)$ over V. In $V[H]$ let C be a club subset of $P_{\omega_1}(\alpha)$. Fix a regular cardinal θ in $V[H]$ much larger than ω_2 and C. Note that in V, $\theta \geq \omega_2$ is regular. By Lemma 4.7, we can find a countable elementary substructure M of $H(\theta)$ with C in M such that $M \cap \alpha \in T$. Since C is in M, $M \cap \alpha \in C \cap T$. $\square$

Corollary 4.9. *The forcing poset* $\mathbb{P}(\alpha, A, \beta)$ *forces that* $P_{\omega_1}(\beta) \setminus \dot{S}(\alpha, A, \beta)$ *is stationary in* $P_{\omega_1}(\beta)$.

Proof. Let H be a generic filter on $\mathbb{P}(\alpha, A, \beta)$ over V. In $V[H]$ consider a function $F : [\beta]^{<\omega} \to \beta$. Fix a regular cardinal θ much larger than β. By Lemma 4.7, we can choose a countable elementary substructure M of $H(\theta)$ with F in M such that $S(\alpha, A, \beta) \cap P(M \cap \beta) \subseteq M$. By elementarity, $M \cap \beta$ is closed under F. Since $M \cap \beta$ is in $P(M \cap \beta)$ but not in M, $M \cap \beta$ is not in $S(\alpha, A, \beta)$. Thus $P_{\omega_1}(\beta) \setminus S(\alpha, A, \beta)$ is stationary in $P_{\omega_1}(\beta)$. $\square$

We end this section by proving a general lemma regarding subsets of $P_{\omega_1}(\beta)$ which do not reflect to any ordinal of cofinality ω_1. We will use this result in Section 12.

Lemma 4.10. *Let $\beta \leq \omega_2$ be an ordinal with uncountable cofinality. Suppose S is a subset of $P_{\omega_1}(\beta)$ which satisfies the property that for all ν in $\beta \cap \operatorname{cof}(\omega_1)$, S does not reflect to ν. Then there is a club set $C \subseteq P_{\omega_1}(\beta)$ such that for all b in C, for all ν in $b \cap \operatorname{cof}(\omega_1)$, $b \cap \nu$ is not in S.*

Proof. Suppose for a contradiction there is a stationary set $U \subseteq P_{\omega_1}(\beta)$ such that for all b in U, there is an ordinal ν in $b \cap \operatorname{cof}(\omega_1)$ such that $b \cap \nu$ is in S. By Fodor's Lemma, fix a stationary set $U' \subseteq U$ and an ordinal ν in $\beta \cap \operatorname{cof}(\omega_1)$ such that for all b in U', ν is in b and $b \cap \nu$ is in S.

We prove that $S \cap P_{\omega_1}(\nu)$ is stationary in $P_{\omega_1}(\nu)$, which contradicts our assumption on S. So consider a function $F : [\nu]^{<\omega} \to \nu$. Since U' is stationary, fix b in U' such that b is closed under F. Then $b \cap \nu$ is closed under F. Since b is in U', $b \cap \nu$ is in S. □

5. Factorization

Fix for the remainder of the section ordinals $\nu < \beta \leq \omega_2$ with uncountable cofinality. Assume $2^\omega = \omega_1$. Recall that $\mathbb{P}_\nu$ and $\dot{S}_\nu$ abbreviate $\mathbb{P}(0, \emptyset, \nu)$ and $\dot{S}(0, \emptyset, \nu)$. In this section we prove that $\mathbb{P}_\beta$ factors as the two-step iteration $\mathbb{P}_\nu * \mathbb{P}(\nu, \dot{S}_\nu, \beta)$. This factorization will be important for the purpose of extending an elementary embedding.

Notation 5.1. *Let (Y, K) be a condition in $\mathbb{P}_\beta$. For each ξ in $\operatorname{dom}(K)$, let $K(\xi) = \langle a_i^\xi : i \leq \gamma_\xi \rangle$. Define:*

(1) $Y_\nu = \{b \in Y : b \subseteq \nu\}$,
(2) $K_\nu = K \restriction \nu$,
(3) $Y^\nu = \{b \in Y : b \nsubseteq \nu\}$,
(4) $K^\nu = K \restriction [\nu, \beta)$.

We define a dense suborder $\mathbb{P}_{\nu,\beta}$ of $\mathbb{P}_\beta$. The definition is designed so that if (Y, K) is in $\mathbb{P}_{\nu,\beta}$, then (Y_ν, K_ν) is in $\mathbb{P}_\nu$, and $(Y_\nu, K_\nu) \Vdash_{\mathbb{P}_\nu} (Y^\nu, K^\nu) \in \mathbb{P}(\nu, \dot{S}_\nu, \beta)$.

Definition 5.2. *Let $\mathbb{P}_{\nu,\beta}$ be the suborder of $\mathbb{P}_\beta$ consisting of conditions (Y, K) such that $\bigcup Y_\nu = (\bigcup Y^\nu) \cap \nu$.*

Lemma 5.3. *The suborder $\mathbb{P}_{\nu,\beta}$ is dense in $\mathbb{P}_\beta$.*

Proof. Let (X, F) be a condition in $\mathbb{P}_\beta$. Since $\bigcup X$ is countable and ν is uncountable, we can choose a countable set e such that $(\bigcup X) \cap \nu \subsetneq e \subseteq \nu$. Now choose f, a countable subset of β, such that $f \cap \nu = e$, $\nu \in f$, $\bigcup X \subseteq f$, and $f \setminus \nu \nsubseteq \bigcup X$.

Let $Y = X \cup \{e, f\}$. By two applications of Lemma 3.4, (Y, F) is a condition in $\mathbb{P}_\beta$ which is below (X, F). We show (Y, F) is in $\mathbb{P}_{\nu,\beta}$.

The set e is in Y_ν, and e is the maximum element of Y_ν. So $\bigcup Y_\nu = e$. Also the set f is the maximum element of Y, and since f is in Y^ν, f is the maximum element of Y^ν. It follows that $\bigcup Y^\nu = f$. So $\bigcup Y_\nu = e = f \cap \nu = (\bigcup Y^\nu) \cap \nu$. □

Lemma 5.4. *Suppose (Y, K) is in $\mathbb{P}_{\nu,\beta}$. For each ξ in $\operatorname{dom}(K)$, let $K(\xi) = \langle a_i^\xi : i \leq \gamma_\xi \rangle$. Then:*

(1) $\bigcup Y = \bigcup Y^\nu$,
(2) $\bigcup Y_\nu = (\bigcup Y) \cap \nu$,
(3) *for all ξ in $\operatorname{dom}(K)$, $a_{\gamma_\xi}^\xi \subseteq \bigcup Y^\nu$,*
(4) *for all ξ in $\operatorname{dom}(K)$ and $i \leq \gamma_\xi$, if $a_i^\xi \subseteq \nu$, then $a_i^\xi \subseteq \bigcup Y_\nu$.*

Proof. (1) Clearly $\bigcup Y^\nu \subseteq \bigcup Y$. Suppose α is in b, where b is in Y. If b is in Y^ν, then α is in $\bigcup Y^\nu$. Otherwise b is in Y_ν, so α is in $\bigcup Y_\nu$, which is equal to $(\bigcup Y^\nu)\cap\nu$. So α is in $\bigcup Y^\nu$. (2) $\bigcup Y_\nu = (\bigcup Y^\nu) \cap \nu$ by definition of $\mathbb{P}_{\nu,\beta}$, which is equal to $(\bigcup Y) \cap \nu$ by (1). (3) By Definition 3.1, $a^\xi_{\gamma_\xi} \subseteq \bigcup Y$, and $\bigcup Y = \bigcup Y^\nu$ by (1). (4) If $a^\xi_i \subseteq \nu$, then by (3), $a^\xi_i \subseteq (\bigcup Y^\nu) \cap \nu = \bigcup Y_\nu$. □

Proposition 5.5. (I) *If (Y, K) is in $\mathbb{P}_{\nu,\beta}$, then (Y_ν, K_ν) is in $\mathbb{P}_\nu$.*

(II) *Suppose $(Z, L) \leq (Y, K)$ in $\mathbb{P}_\beta$. If (Z, L) and (Y, K) are both in $\mathbb{P}_{\nu,\beta}$, then $(Z_\nu, L_\nu) \leq (Y_\nu, K_\nu)$ in $\mathbb{P}_\nu$.*

(III) *If (Y, K) is in $\mathbb{P}_{\nu,\beta}$, then $(Y_\nu, K_\nu) \Vdash_{\mathbb{P}_\nu} (Y^\nu, K^\nu) \in \mathbb{P}(\nu, \dot S_\nu, \beta)$.*

(IV) *Suppose $(Z, L) \leq (Y, K)$ in $\mathbb{P}_\beta$. If (Z, L) and (Y, K) are both in $\mathbb{P}_{\nu,\beta}$, then $(Z_\nu, L_\nu) \Vdash_{\mathbb{P}_\nu} (Z^\nu, L^\nu) \leq (Y^\nu, K^\nu)$ in $\mathbb{P}(\nu, \dot S_\nu, \beta)$.*

Proof. (I) We verify the properties of Definition 3.1 in the case of $\mathbb{P}_\nu = \mathbb{P}(0, \emptyset, \nu)$. Properties (1), (2), (3), and (4) are immediate. To show (5) and (6), consider ξ in $\mathrm{dom}(K_\nu) = \mathrm{dom}(K) \cap \nu$. Let $K_\nu(\xi) = K(\xi) = \langle a^\xi_i : i \leq \gamma_\xi\rangle$. Let $i \leq \gamma_\xi$. Since (Y, K) is in $\mathbb{P}_\beta$, a^ξ_i is not in Y. But $Y_\nu \subseteq Y$, so a^ξ_i is not in Y_ν. Also a^ξ_i is obviously not in $\emptyset$. This proves (5). For (6), we know that $a^\xi_{\gamma_\xi} \subseteq \bigcup Y$, and $a^\xi_{\gamma_\xi} \subseteq \xi \subseteq \nu$. So by Lemma 5.4(2), $a^\xi_{\gamma_\xi} \subseteq (\bigcup Y) \cap \nu = \bigcup Y_\nu$.

(II) Properties (a), (c), and (d) of Definition 3.1 are immediate. For (b), suppose b is in $Z_\nu \setminus Y_\nu$. Then b is in Z and $b \subseteq \nu$. Since b is not in Y_ν, b is not in Y. So b is in $Z \setminus Y$. As $(Z, L) \leq (Y, K)$, $b \not\subseteq \bigcup Y$. But $\bigcup Y_\nu \subseteq \bigcup Y$, so $b \not\subseteq \bigcup Y_\nu$.

(III) It is immediate that $\mathbb{P}_\nu$ forces that (Y^ν, K^ν) satisfies properties (1), (2), (3), and (4) of Definition 3.1. For (5) and (6), consider ξ in $\mathrm{dom}(K^\nu) = \mathrm{dom}(K)\cap[\nu, \beta)$. Let $K^\nu(\xi) = K(\xi) = \langle a^\xi_i : i \leq \gamma_\xi\rangle$. Let $i \leq \gamma_\xi$. Since (Y, K) is in $\mathbb{P}_\beta$, a^ξ_i is not in Y, so a^ξ_i is not in Y^ν. We need to show $(Y_\nu, K_\nu) \Vdash_{\mathbb{P}_\nu} a^\xi_i \notin \dot S_\nu$. If a^ξ_i is not a subset of ν, then this is clear. Suppose $a^\xi_i \subseteq \nu$. Then by Lemma 5.4(4), $a^\xi_i \subseteq \bigcup Y_\nu$. By Lemma 4.5, $(Y_\nu, K_\nu) \Vdash_{\mathbb{P}_\nu} \dot S_\nu \cap P(\bigcup Y_\nu) = Y_\nu$. But a^ξ_i is not in Y, and hence is not in Y_ν, and a^ξ_i is in $P(\bigcup Y_\nu)$. So $(Y_\nu, K_\nu) \Vdash a^\xi_i \notin \dot S_\nu$. This proves (5). By Lemma 5.4(3), $a^\xi_{\gamma_\xi} \subseteq \bigcup Y^\nu$, proving (6).

(IV) Since properties (a), (b), (c), and (d) of Definition 3.1 are absolute between V and a generic extension by $\mathbb{P}_\nu$, it suffices to show these properties hold in V. Properties (a), (c), and (d) are immediate. For (b), suppose b is in $Z^\nu \setminus Y^\nu$. Then b is Z and $b \not\subseteq \nu$. But then b is not in Y, since otherwise b would be in Y^ν. So b is in $Z \setminus Y$. Since $(Z, L) \leq (Y, K)$, $b \not\subseteq \bigcup Y$. As $\bigcup Y^\nu \subseteq \bigcup Y$, $b \not\subseteq \bigcup Y^\nu$. □

If (Y, K) is in $\mathbb{P}_{\nu,\beta}$, then (Y_ν, K_ν) is in $\mathbb{P}_\nu$. By Lemma 3.3, $\mathbb{P}_\nu \subseteq \mathbb{P}_\beta$, so (Y_ν, K_ν) is in $\mathbb{P}_\beta$ as well.

Lemma 5.6. *Let (Y, K) be in $\mathbb{P}_{\nu,\beta}$. Then $(Y, K) \leq (Y_\nu, K_\nu)$ in $\mathbb{P}_\beta$.*

Proof. Properties (a), (c), and (d) of Definition 3.1 are immediate. For (b), if b is in $Y \setminus Y_\nu$, then $b \not\subseteq \nu$. But $\bigcup Y_\nu \subseteq \nu$. So $b \not\subseteq \bigcup Y_\nu$. □

Lemma 5.7. *Let (X, F) be a condition in $\mathbb{P}_\nu$. Suppose that Y, K are sets such that $(X, F) \Vdash_{\mathbb{P}_\nu} (\check Y, \check K) \in \mathbb{P}(\nu, \dot S_\nu, \beta)$. Let $Z = X \cup Y$ and let $L = F \cup K$. Then (Z, L) is in $\mathbb{P}_\beta$. Moreover, $Z_\nu = X$, $L_\nu = F$, $Z^\nu = Y$, and $L^\nu = K$.*

Proof. Properties (1), (2), (3), and (4) of Definition 3.1 are immediate. For (5) and (6), let ξ be in $\mathrm{dom}(L)$. Let $L(\xi) = \langle a^\xi_i : i \leq \gamma_\xi\rangle$. If ξ is in $\mathrm{dom}(F)$, then $a^\xi_{\gamma_\xi} \subseteq \bigcup X \subseteq \bigcup Z$, and if ξ is in $\mathrm{dom}(K)$, then $a^\xi_{\gamma_\xi} \subseteq \bigcup Y \subseteq \bigcup Z$. This proves (6).

To show (5), let $i \leq \gamma_\xi$, and we show a_i^ξ is not in $Z \cup \emptyset = Z = X \cup Y$. First assume ξ is in $\mathrm{dom}(K)$. Since $(X,F) \Vdash_{\mathbb{P}_\nu} (\check{Y},\check{K}) \in \mathbb{P}(\nu,\dot{S}_\nu,\beta)$, a_i^ξ is not in Y. By Definition 3.1(5), $(X,F) \Vdash_{\mathbb{P}_\nu} a_i^\xi \notin \dot{S}_\nu$, which implies $a_i^\xi \notin X$. So a_i^ξ is not in Z. Now suppose ξ is in $\mathrm{dom}(F)$. Then clearly a_i^ξ is not in X. Since $\mathrm{dom}(F) \subseteq \nu$, a_i^ξ is a subset of ν. But for all b in Y, $b \nsubseteq \nu$. So a_i^ξ is not in Y.

So indeed (Z,L) is a condition in $\mathbb{P}_\beta$. The last statement of the lemma is immediate. □

Lemma 5.8. *There is a dense set of conditions in $\mathbb{P}_\nu * \mathbb{P}(\nu,\dot{S}_\nu,\beta)$ of the form $(X,F)*(\check{Y},\check{K})$ such that $\bigcup X = (\bigcup Y) \cap \nu$. Suppose $(X,F)*(\check{Y},\check{K})$ is such a condition. Let $Z = X \cup Y$ and $L = F \cup K$. Then (Z,L) is in $\mathbb{P}_{\nu,\beta}$, and $(Z_\nu,L_\nu)*(\check{Z}^\nu,\check{L}^\nu)$ is equal to $(X,F)*(\check{Y},\check{K})$.*

Proof. Let $(X_0,F_0)*(\dot{Y}_0,\dot{K}_0)$ be in $\mathbb{P}_\nu * \mathbb{P}(\nu,\dot{S}_\nu,\beta)$. Since $\mathbb{P}_\nu$ is ω_1-distributive, we can fix $(X_1,F_1) \leq (X_0,F_0)$ and sets Y_1 and K_1 such that (X_1,F_1) forces $\dot{Y}_0 = \check{Y}_1$ and $\dot{K}_0 = \check{K}_1$.

Choose a countable set e such that $\bigcup X_1 \subsetneqq e \subseteq \nu$ and $(\bigcup Y_1) \cap \nu \subseteq e$. Let $X_2 = X_1 \cup \{e\}$. By Lemma 3.4, (X_2,F_1) is in $\mathbb{P}_\nu$ and $(X_2,F_1) \leq (X_1,F_1)$. Now choose a countable set $f \subseteq \beta$ such that $f \cap \nu = e$ and $(\bigcup Y_1) \cup \{\nu\} \subsetneqq f$. Let $Y_2 = Y_1 \cup \{f\}$. By Lemma 3.4, (X_2,F_1) forces $(\check{Y}_2,\check{K}_1)$ is in $\mathbb{P}(\nu,\dot{S}_\nu,\beta)$ and $(\check{Y}_2,\check{K}_1) \leq (\check{Y}_1,\check{K}_1)$.

It follows that $(X_2,F_1)*(\check{Y}_2,\check{K}_1) \leq (X_0,F_0)*(\dot{Y}_0,\dot{K}_0)$. Note that e is the maximum element of X_2, so $\bigcup X_2 = e$. Also f is the maximum element of Y_2, so $\bigcup Y_2 = f$. So $\bigcup X_2 = e = f \cap \nu = (\bigcup Y_2) \cap \nu$, proving the first part of the lemma.

Now suppose $(X,F)*(\check{Y},\check{K})$ is a condition in $\mathbb{P}_\nu * \mathbb{P}(\nu,\dot{S}_\nu,\beta)$ such that $\bigcup X = (\bigcup Y) \cap \nu$. Let $Z = X \cup Y$ and $L = F \cup K$. By Lemma 5.7, (Z,L) is in $\mathbb{P}_\beta$, $Z_\nu = X$, $L_\nu = F$, $Z^\nu = Y$, and $L^\nu = K$. Also $\bigcup Z_\nu = \bigcup X = (\bigcup Y) \cap \nu = (\bigcup Z^\nu) \cap \nu$, so (Z,L) is in $\mathbb{P}_{\nu,\beta}$. □

Theorem 5.9. *Define $i : \mathbb{P}_{\nu,\beta} \to \mathbb{P}_\nu * \mathbb{P}(\nu,\dot{S}_\nu,\beta)$ by letting $i(Y,K) = (Y_\nu,K_\nu)*(\check{Y}^\nu,\check{K}^\nu)$. Then i is an isomorphism of $\mathbb{P}_{\nu,\beta}$ onto a dense subset of $\mathbb{P}_\nu * \mathbb{P}(\nu,\dot{S}_\nu,\beta)$.*

Proof. By (I) and (III) of Proposition 5.5, if (Y,K) is in $\mathbb{P}_{\nu,\beta}$, then $i(Y,K)$ is in $\mathbb{P}_\nu * \mathbb{P}(\nu,\dot{S}_\nu,\beta)$. By (II) and (IV) of Proposition 5.5, if (Z,L) and (Y,K) are in $\mathbb{P}_{\nu,\beta}$ and $(Z,L) \leq (Y,K)$ in $\mathbb{P}_\beta$, then $i(Z,L) \leq i(Y,K)$ in $\mathbb{P}_\nu * \mathbb{P}(\nu,\dot{S}_\nu,\beta)$. It is easy to see that i is an injection, and the range of i is dense in $\mathbb{P}_\nu * \mathbb{P}(\nu,\dot{S}_\nu,\beta)$ by Lemma 5.8. To complete the proof, it remains to show that if (Z,L) and (Y,K) are in $\mathbb{P}_{\nu,\beta}$ and $i(Z,L) \leq i(Y,K)$ in $\mathbb{P}_\nu * \mathbb{P}(\nu,\dot{S}_\nu,\beta)$, then $(Z,L) \leq (Y,K)$ in $\mathbb{P}_\beta$.

Let (Z,L) and (Y,K) be in $\mathbb{P}_{\nu,\beta}$, and assume $i(Z,L) \leq i(Y,K)$ in $\mathbb{P}_\nu * \mathbb{P}(\nu,\dot{S}_\nu,\beta)$. This means that $(Z_\nu,L_\nu) \leq (Y_\nu,K_\nu)$ in $\mathbb{P}_\nu$, and $(Z_\nu,L_\nu) \Vdash_{\mathbb{P}_\nu} (Z^\nu,L^\nu) \leq (Y^\nu,K^\nu)$ in $\mathbb{P}(\nu,\dot{S}_\nu,\beta)$. We prove that $(Z,L) \leq (Y,K)$ in $\mathbb{P}_\beta$.

Since $(Z_\nu,L_\nu) \leq (Y_\nu,K_\nu)$ in $\mathbb{P}_\nu$, we have: (1) $Y_\nu \subseteq Z_\nu$, (2) for all b in $Z_\nu \setminus Y_\nu$, $b \nsubseteq \bigcup Y_\nu$, (3) $\mathrm{dom}(K_\nu) \subseteq \mathrm{dom}(L_\nu)$, and (4) for all ξ in $\mathrm{dom}(K_\nu)$, $K(\xi)$ is an initial segment of $L(\xi)$. Fix a generic filter H on $\mathbb{P}_\nu$ with (Z_ν,L_ν) in H. Then in $V[H]$, $(Z^\nu,L^\nu) \leq (Y^\nu,K^\nu)$ in $\mathbb{P}(\nu,S_\nu,\beta)$, which means: (5) $Y^\nu \subseteq Z^\nu$, (6) for all b in $Z^\nu \setminus Y^\nu$, $b \nsubseteq \bigcup Y^\nu$, (7) $\mathrm{dom}(K^\nu) \subseteq \mathrm{dom}(L^\nu)$, and (8) for all ξ in $\mathrm{dom}(K^\nu)$, $K(\xi)$ is an initial segment of $L(\xi)$. Note that properties (5 − 8) are absolute between V and $V[H]$, so in fact they hold in V.

To show that $(Z,L) \leq (Y,K)$ in $\mathbb{P}_\beta$, we show (a) $Y \subseteq Z$, (b) for all b in $Z \setminus Y$, $b \nsubseteq \bigcup Y$, (c) $\mathrm{dom}(K) \subseteq \mathrm{dom}(L)$, and (d) for all ξ in $\mathrm{dom}(K)$, $K(\xi)$ is an initial segment of $L(\xi)$. Since $Y = Y_\nu \cup Y^\nu$ and $Z = Z_\nu \cup Z^\nu$, (a) follows from (1) and (5). Similarly, $K = K_\nu \cup K^\nu$ and $L = L_\nu \cup L^\nu$, so (c) and (d) follow from (3), (4), (7), and (8).

It remains to prove (b). So let b be in $Z \setminus Y$, and we show $b \not\subseteq \bigcup Y$. First assume that $b \subseteq \nu$. Then b is in $Z_\nu \setminus Y_\nu$. By (2), $b \not\subseteq \bigcup Y_\nu$. Since (Y, K) is in $\mathbb{P}_{\nu,\beta}$, by Lemma 5.4(2), $(\bigcup Y) \cap \nu = \bigcup Y_\nu$. So if $b \subseteq \bigcup Y$, then $b \subseteq (\bigcup Y) \cap \nu = \bigcup Y_\nu$, which is a contradiction. Now assume that $b \not\subseteq \nu$. Then b is in $Z^\nu \setminus Y^\nu$. By (6), $b \not\subseteq \bigcup Y^\nu$. As (Y, K) is in $\mathbb{P}_{\nu,\beta}$, by Lemma 5.4(1), $\bigcup Y = \bigcup Y^\nu$. Since $b \not\subseteq \bigcup Y^\nu$, $b \not\subseteq \bigcup Y$. $\square$

As $\mathbb{P}_{\nu,\beta}$ is a dense suborder of $\mathbb{P}_\beta$, it follows that $\mathbb{P}_\beta$ is forcing equivalent to $\mathbb{P}_\nu * \mathbb{P}(\nu, \dot{S}_\nu, \beta)$.

The following is a technical lemma which we will need in the next section.

Lemma 5.10. *Let $i : \mathbb{P}_{\nu,\beta} \to \mathbb{P}_\nu * \mathbb{P}(\nu, \dot{S}_\nu, \beta)$ be the map $i(Y, K) = (Y_\nu, K_\nu) * (\check{Y}^\nu, \check{K}^\nu)$. Suppose (X, F) is a condition in $\mathbb{P}_\nu$ (and hence (X, F) is in $\mathbb{P}_\beta$ as well). Let (Y, K) be in $\mathbb{P}_{\nu,\beta}$, and assume $(Y, K) \leq (X, F)$ in $\mathbb{P}_\beta$. Then $i(Y, K) \leq (X, F) * 1$ in $\mathbb{P}_\nu * \mathbb{P}(\nu, \dot{S}_\nu, \beta)$.*

Proof. For (X, F) and (Y, K) as above, $i(Y, K) \leq (X, F) * 1$ means $(Y_\nu, K_\nu) * (\check{Y}^\nu, \check{K}^\nu) \leq (X, F) * 1$, which is equivalent to $(Y_\nu, K_\nu) \leq (X, F)$ in $\mathbb{P}_\nu$. So it suffices to show that if (X, F) is in $\mathbb{P}_\nu$, (Y, K) is in $\mathbb{P}_{\nu,\beta}$, and $(Y, K) \leq (X, F)$ in $\mathbb{P}_\beta$, then $(Y_\nu, K_\nu) \leq (X, F)$ in $\mathbb{P}_\nu$.

Since $(Y, K) \leq (X, F)$ in $\mathbb{P}_\beta$, $X \subseteq Y$. If b is in X, then b is in Y, and since (X, F) is in $\mathbb{P}_\nu$, b is a subset of ν. So b is in Y_ν. Hence $X \subseteq Y_\nu$. Suppose b is in $Y_\nu \setminus X$. Then b is in $Y \setminus X$, so $b \not\subseteq \bigcup X$. As $(Y, K) \leq (X, F)$ in $\mathbb{P}_\beta$, $\mathrm{dom}(F) \subseteq \mathrm{dom}(K)$. But also $\mathrm{dom}(F) \subseteq \nu$. So $\mathrm{dom}(F) \subseteq \mathrm{dom}(K) \cap \nu = \mathrm{dom}(K_\nu)$. If ξ is in $\mathrm{dom}(F)$, then as $(X, F) \leq (Y, K)$, $F(\xi)$ is an initial segment of $K(\xi) = K_\nu(\xi)$. $\square$

6. Extending an Elementary Embedding, Part 1

We now deal with the problem of extending an elementary embedding after Lévy collapsing a large cardinal to become ω_2, and then forcing with $\mathbb{P}_{\omega_2}$. Fix for the remainder of the section a cardinal κ which is κ^+-supercompact, and an elementary embedding $j : V \to M$ with critical point κ such that $j(\kappa) > \kappa^+$ and $M^{\kappa^+} \subseteq M$. Assume $2^\kappa = \kappa^+$.

Recall that the Lévy collapse $\textsc{Coll}(\omega_1, <\kappa)$ is ω_1-closed, κ-c.c., has size κ, and forces that κ is equal to ω_2. It is also easy to see that $\textsc{Coll}(\omega_1, <\kappa)$ forces $2^\omega = \omega_1$ and $2^{\omega_1} = \omega_2$. Since we are assuming $2^\kappa = \kappa^+$ in V, $\textsc{Coll}(\omega_1, <\kappa)$ forces $2^{\omega_2} = \omega_3$. We will be interested in the two-step iteration $\textsc{Coll}(\omega_1, <\kappa) * \dot{\mathbb{P}}_\kappa$. (Note that $\dot{\mathbb{P}}_\kappa$ is the same as $\dot{\mathbb{P}}_{\omega_2}$.) Since CH holds after forcing with the Lévy collapse, this two-step iteration is κ-c.c. and ω_1-distributive. Occasionally we will write $j(\textsc{Coll}(\omega_1, <\kappa) * \dot{\mathbb{P}}_\kappa)$ as $\textsc{Coll}(\omega_1, <j(\kappa)) * \dot{\mathbb{P}}_{j(\kappa)}$. To reduce superscripts, we will write κ^+ to denote the successor of κ in V, even when working in a generic extension in which this ordinal is no longer a cardinal.

Theorem 6.1. *Let $\overline{G} * \overline{H}$ be a generic filter on $j(\textsc{Coll}(\omega_1, <\kappa) * \dot{\mathbb{P}}_\kappa)$ over V. Let $G = \overline{G} \cap \textsc{Coll}(\omega_1, <\kappa)$ and $H = \overline{H} \cap \mathbb{P}_\kappa$. Then G is a generic filter on $\textsc{Coll}(\omega_1, <\kappa)$ over V, H is a generic filter on $\mathbb{P}_\kappa$ over $V[G]$, and j can be extended in $V[\overline{G} * \overline{H}]$ to $j : V[G * H] \to M[\overline{G} * \overline{H}]$ such that $j(G * H) = \overline{G} * \overline{H}$. Moreover:*

(1) *$M[G * H]^{\kappa^+} \cap V[G * H] \subseteq M[G * H]$,*

(2) *$M[\overline{G}{*}\overline{H}]$ is equal to $M[\overline{G}][H][H']$, where H' is a generic filter on $\mathbb{P}(\kappa, S_\kappa, j(\kappa))$ over $M[\overline{G}][H]$,*

(3) *$j(S_\kappa) = S_{j(\kappa)}$, $S_{j(\kappa)} = S_\kappa \cup S(\kappa, S_\kappa, j(\kappa))$, $S_\kappa = \{b \in S_{j(\kappa)} : b \subseteq \kappa\}$, and $S(\kappa, S_\kappa, j(\kappa)) = \{b \in S_{j(\kappa)} : b \nsubseteq \kappa\}$.*

Proof. Since the inclusion map from $\mathrm{Coll}(\omega_1, {<}\kappa)$ to $\mathrm{Coll}(\omega_1, {<}j(\kappa))$ is a complete embedding, $G = \overline{G} \cap \mathrm{Coll}(\omega_1, {<}\kappa)$ is a generic filter on $\mathrm{Coll}(\omega_1, {<}\kappa)$ over V. As $j \restriction \mathrm{Coll}(\omega_1, {<}\kappa)$ is the identity, $j[G] = G \subseteq \overline{G}$. So j can be extended to $j : V[G] \to M[\overline{G}]$ such that $j(G) = \overline{G}$.

As $\mathrm{Coll}(\omega_1, {<}\kappa)$ is κ-c.c., $M[G]^{\kappa^+} \cap V[G] \subseteq M[G]$. In particular, $\mathbb{P}_\kappa$ is computed the same in $V[G]$ and $M[G]$. By standard facts, $M[\overline{G}] = M[G][G']$, where $G' = \overline{G} \cap \mathrm{Coll}(\omega_1, [\kappa, j(\kappa)))$. Since $\mathrm{Coll}(\omega_1, [\kappa, j(\kappa)))$ is ω_1-closed, $\mathbb{P}_\kappa$ is computed the same in $M[G]$ and $M[\overline{G}]$.

In $M[\overline{G}]$ consider the map $i : \mathbb{P}_{\kappa, j(\kappa)} \to \mathbb{P}_\kappa * \mathbb{P}(\kappa, \dot{S}_\kappa, j(\kappa))$ given by $i(Z, K) = (Z_\kappa, K_\kappa) * (\check{Z}^\kappa, \check{K}^\kappa)$. By Theorem 5.9, i is an isomorphism of $\mathbb{P}_{\kappa, j(\kappa)}$ onto a dense subset of $\mathbb{P}_\kappa * \mathbb{P}(\kappa, \dot{S}_\kappa, j(\kappa))$. But being an isomorphism is absolute. It follows that $i[\overline{H} \cap \mathbb{P}_{\kappa, j(\kappa)}]$ generates a generic filter F on $\mathbb{P}_\kappa * \mathbb{P}(\kappa, \dot{S}_\kappa, j(\kappa))$ over $V[\overline{G}]$, and $M[\overline{G}][\overline{H}] = M[\overline{G}][F]$.

Let $k : \mathbb{P}_\kappa \to \mathbb{P}_\kappa * \mathbb{P}(\kappa, \dot{S}_\kappa, j(\kappa))$ be the map $k(p) = p * 1$. Then k is a complete embedding. It follows that $k^{-1}(F) = \{p \in \mathbb{P}_\kappa : p * 1 \in F\}$ is a generic filter on $\mathbb{P}_\kappa$ over $V[\overline{G}]$. Let $H = k^{-1}(F)$.

We claim that H is equal to $\overline{H} \cap \mathbb{P}_\kappa$. So let p be in H. Then p is in $\mathbb{P}_\kappa$, and $p*1$ is in F, the filter generated by $i[\overline{H} \cap \mathbb{P}_{\kappa, j(\kappa)}]$. So there is a condition (Y, K) in $\overline{H} \cap \mathbb{P}_{\kappa, j(\kappa)}$ such that $i(Y, K) = (Y_\kappa, K_\kappa) * (\check{Y}^\kappa, \check{K}^\kappa) \le p * 1$. This implies $(Y_\kappa, K_\kappa) \le p$ in $\mathbb{P}_\kappa$. By Lemma 3.3, $(Y_\kappa, K_\kappa) \le p$ in $\mathbb{P}_{j(\kappa)}$. By Lemma 5.6, $(Y, K) \le (Y_\kappa, K_\kappa)$ in $\mathbb{P}_{j(\kappa)}$. So $(Y, K) \le p$ in $\mathbb{P}_{j(\kappa)}$. Since (Y, K) is in $\overline{H}$, p is in $\overline{H}$. This shows $H \subseteq \overline{H} \cap \mathbb{P}_\kappa$.

On the other hand, let q be in $\overline{H} \cap \mathbb{P}_\kappa$, and we show q is in H. In other words, we show $q * 1$ is in F. Since $\mathbb{P}_{\kappa, j(\kappa)}$ is dense in $\mathbb{P}_{j(\kappa)}$ and q is in $\overline{H}$, choose $q' \le q$ in $\overline{H} \cap \mathbb{P}_{\kappa, j(\kappa)}$. Then $i(q') \in i[\overline{H} \cap \mathbb{P}_{\kappa, j(\kappa)}]$, so $i(q')$ is in F. Since $q' \le q$ in $\mathbb{P}_{j(\kappa)}$ and q is in $\mathbb{P}_\kappa$, by Lemma 5.10, $i(q') \le q * 1$ in $\mathbb{P}_\kappa * \mathbb{P}(\kappa, \dot{S}_\kappa, j(\kappa))$. As $i(q')$ is in F, $q * 1$ is in F. This completes the proof that $H = \overline{H} \cap \mathbb{P}_\kappa$.

Now $j \restriction \mathbb{P}_\kappa$ is the identity, as can be easily seen, so $j[H] = H = \overline{H} \cap \mathbb{P}_\kappa \subseteq \overline{H}$. Therefore we can extend j to $j : V[G][H] \to M[\overline{G}][\overline{H}]$ such that $j(H) = \overline{H}$. So $j(G * H) = j(G) * j(H) = \overline{G} * \overline{H}$. Since $\mathrm{Coll}(\omega_1, {<}\kappa) * \mathbb{P}_\kappa$ is κ-c.c., $M[G * H]^{\kappa^+} \cap V[G * H] \subseteq M[G * H]$.

Recall that $M[\overline{G}][\overline{H}] = M[\overline{G}][F]$, where F is a generic filter on $\mathbb{P}_\kappa * \mathbb{P}(\kappa, \dot{S}_\kappa, j(\kappa))$ over $M[\overline{G}]$. By standard facts about two-step iterations, if we let $H' = \{\dot{s}^H : \exists p \ p * \dot{s} \in F\}$, then H' is a generic filter on $\mathbb{P}(\kappa, S_\kappa, j(\kappa))$ over $M[\overline{G}][H]$, and $M[\overline{G}][F] = M[\overline{G}][H][H']$. So $M[\overline{G}][\overline{H}] = M[\overline{G}][H][H']$.

It is immediate by the elementarity of j and the fact that $j(H) = \overline{H}$ that $j(S_\kappa) = S_{j(\kappa)}$. If b is in S_κ, then by elementarity, $j(b) = b$ is in $j(S_\kappa) = S_{j(\kappa)}$. So $S_\kappa \subseteq S_{j(\kappa)}$.

By definition, $S(\kappa, S_\kappa, j(\kappa))$ is equal to $\bigcup\{Y : \exists K \ (Y, K) \in H'\}$. As a subclaim, let us prove that $S(\kappa, S_\kappa, j(\kappa))$ is equal to $\bigcup\{Z^\kappa : \exists L \ (Z, L) \in \overline{H} \cap \mathbb{P}_{\kappa, j(\kappa)}\}$.

On the one hand, suppose b is in Z^κ, where for some L, (Z, L) is in $\overline{H} \cap \mathbb{P}_{\kappa, j(\kappa)}$. Then $i(Z, L) = (Z_\kappa, L_\kappa) * (\check{Z}^\kappa, \check{L}^\kappa)$ is in F. By the definition of H', (Z^κ, L^κ) is in H'. So $Z^\kappa \subseteq S(\kappa, S_\kappa, j(\kappa))$. Hence b is in $S(\kappa, S_\kappa, j(\kappa))$.

On the other hand, suppose b is in $S(\kappa, S_\kappa, j(\kappa))$. Then there is (Y, K) in H' such that b is in Y. By the definition of H', fix $p * \dot{s}$ in F such that $\dot{s}^H = (Y, K)$. Since $p * \dot{s}$ is in F, there is (Z, L) in $\overline{H} \cap \mathbb{P}_{\kappa, j(\kappa)}$ such that $i(Z, L) = (Z_\kappa, L_\kappa) * (\check{Z}^\kappa, \check{L}^\kappa) \leq p * \dot{s}$. By Lemma 5.6, $(Z, L) \leq (Z_\kappa, L_\kappa)$ in $\mathbb{P}_{j(\kappa)}$, so (Z_κ, L_κ) is in $\overline{H} \cap \mathbb{P}_\kappa = H$. Since $(Z_\kappa, L_\kappa) * (\check{Z}^\kappa, \check{L}^\kappa) \leq p * \dot{s}$ and $(Z_\kappa, L_\kappa) \in H$, $(Z^\kappa, L^\kappa) \leq \dot{s}^H = (Y, K)$. In particular, $Y \subseteq Z^\kappa$. So b is in Z^κ. This proves $S(\kappa, S_\kappa, j(\kappa)) \subseteq \bigcup\{Z^\kappa : \exists L\ (Z, L) \in \overline{H} \cap \mathbb{P}_{\kappa, j(\kappa)}\}$, finishing the proof of the subclaim.

Let us prove that $S_{j(\kappa)} = S_\kappa \cup S(\kappa, S_\kappa, j(\kappa))$. We already observed $S_\kappa \subseteq S_{j(\kappa)}$. Let b be in $S(\kappa, S_\kappa, j(\kappa))$. By the subclaim, there is (Z, L) in $\overline{H} \cap \mathbb{P}_{\kappa, j(\kappa)}$ such that b is in Z^κ. Since (Z, L) is in $\overline{H}$, $Z \subseteq S_{j(\kappa)}$. But b is in $Z^\kappa \subseteq Z$, so b is in $S_{j(\kappa)}$.

Now let b be in $S_{j(\kappa)}$, and we show b is in $S_\kappa \cup S(\kappa, S_\kappa, j(\kappa))$. Fix (Y, K) in $\overline{H}$ such that b is in Y. Since $\mathbb{P}_{\kappa, j(\kappa)}$ is dense in $\mathbb{P}_{j(\kappa)}$, fix $(Z, L) \leq (Y, K)$ in $\overline{H} \cap \mathbb{P}_{\kappa, j(\kappa)}$. Then b is in Z. By the subclaim, $Z^\kappa \subseteq S(\kappa, S_\kappa, j(\kappa))$. So if b is in Z^κ, then b is in $S(\kappa, S_\kappa, j(\kappa))$. Otherwise $b \subseteq \kappa$, so b is in Z_κ. By Lemma 5.6, $(Z, L) \leq (Z_\kappa, L_\kappa)$ in $\mathbb{P}_{j(\kappa)}$, so (Z_κ, L_κ) is in $\overline{H} \cap \mathbb{P}_\kappa = H$. This implies $Z_\kappa \subseteq S_\kappa$, and hence b is in S_κ.

We have proven $S_{j(\kappa)} = S_\kappa \cup S(\kappa, S_\kappa, j(\kappa))$. Now every set in S_κ is a subset of κ, and every set in $S(\kappa, S_\kappa, j(\kappa))$ is not a subset of κ. Therefore it is immediate that $S_\kappa = \{b \in S_{j(\kappa)} : b \subseteq \kappa\}$ and $S(\kappa, S_\kappa, j(\kappa)) = \{b \in S_{j(\kappa)} : b \not\subseteq \kappa\}$. □

7. Adding a Club

We briefly describe a forcing poset due to Sakai [3] for adding a club subset of $P_{\omega_1}(\omega_2)$. Specifically, for a given set $T \subseteq P_{\omega_1}(\omega_2)$, we define a forcing poset $\mathbb{Q}(T)$ which adds a generic function $F : \omega_2 \times \omega_2 \to \omega_1$ for which there is no set in T closed under F. In the next section we describe an iteration of this forcing poset.

Definition 7.1. *Let T be a subset of $P_{\omega_1}(\omega_2)$. Define $\mathbb{Q}(T)$ as the forcing poset consisting of conditions p satisfying:*

- *p is a function of the form $p : a^p \times a^p \to \omega_1$, where a^p is a countable subset of ω_2,*
- *for all x in T, if $x \subseteq a^p$, then x is not closed under p.*

Let $q \leq p$ if q extends p as a function, that is, if $a^p \subseteq a^q$ and $q \restriction (a^p \times a^p) = p$.

Notation 7.2. *For a countable set $a \subseteq \omega_2$, we write a^2 for $a \times a$.*

Lemma 7.3. *Let p and q be conditions in $\mathbb{Q}(T)$, and assume $p \restriction (a^p \cap a^q)^2 = q \restriction (a^p \cap a^q)^2$. Then p and q are compatible.*

In particular, if $a^p \cap a^q = \emptyset$, then p and q are compatible.

Proof. We define a condition r which is below p and q. Let $a^r = a^p \cup a^q$. As a^r is countable, fix an ordinal ζ in $\omega_1 \setminus a^r$. Define $r : (a^r)^2 \to \omega_1$ as follows. Let (i, j) be in $(a^r)^2$. If i and j are both in a^p, let $r(i, j) = p(i, j)$. If i and j are both in a^q, let $r(i, j) = q(i, j)$. Note this is well-defined, since if i and j are both in $a^p \cap a^q$, then $p(i, j) = q(i, j)$ by assumption. The remaining possibility is that i is in $a^p \setminus a^q$ and j is in $a^q \setminus a^p$, or vice versa. In this case, let $r(i, j) = \zeta$.

So r is a function of the form $r : (a^r)^2 \to \omega_1$, where a^r is a countable subset of ω_2. Note that r extends both p and q. To show r is a condition, suppose x is in T and $x \subseteq a^r$. If x is a subset of a^p, then x is not closed under p, since p is a condition. So x is not closed under r. Likewise, if x is a subset of a^q, then x is not closed under q, so x is not closed under r. If neither of the above holds, then there is i in $a^r \setminus a^q$ and j in $a^r \setminus a^p$. But then i is in $a^p \setminus a^q$ and j is in $a^q \setminus a^p$. By the definition of r, $r(i,j) = \zeta$. Since ζ is not in a^r, ζ is not in x, so x is not closed under r. $\square$

Lemma 7.4. *Let b be a countable subset of ω_2. Then there are densely many conditions t in $\mathbb{Q}(T)$ such that $b \subseteq a^t$.*

Proof. Let $p : (a^p)^2 \to \omega_1$ be a given condition. If $b \subseteq a^p$, then we are done. Otherwise $b \setminus a^p$ is non-empty. As b is countable, fix an ordinal ζ in $\omega_1 \setminus b$. Define s by letting $a^s = b \setminus a^p$, and for all (i,j) in $(a^s)^2$, let $s(i,j) = \zeta$. If x is in T and is a subset of a^s, then ζ is not in x, so x is not closed under s. This shows s is in $\mathbb{Q}(T)$. Now $a^s \cap a^p = \emptyset$, so by Lemma 7.3, s and p are compatible. Let $t \leq s, p$. Then $b \subseteq a^s \cup a^p \subseteq a^t$. $\square$

Suppose I is a generic filter on $\mathbb{Q}(T)$ over V. Then by the last lemma, it is easy to see that $F = \bigcup I$ is a total function $F : \omega_2^V \times \omega_2^V \to \omega_1^V$, and if b is in T, then b is not closed under F. So if ω_1 and ω_2 are preserved, then $\mathbb{Q}(T)$ destroys the stationarity of T.

Depending on the assumptions of the ground model and on the choice of T, $\mathbb{Q}(T)$ may or may not preserve cardinals. In the sections which follow, we will define and analyze an iteration of forcing posets of the form $\mathbb{Q}(T)$. In our context, the forcing posets $\mathbb{Q}(T)$ which we use, together with the iteration itself, are ω_1-distributive and ω_2-c.c.

8. Iteration of Club Adding

In this section we define an iteration of forcing posets of the form $\mathbb{Q}(T)$ from the previous section. We work in a model $V[H]$, where H is a generic filter on $\mathbb{P}_{\omega_2}$ over V. As usual, let $S_{\omega_2} = \bigcup\{X : \exists F\ (X,F) \in H\}$. So S_{ω_2} is a stationary subset of $P_{\omega_1}(\omega_2)$ in $V[H]$ which does not reflect to any ordinal in ω_2 with cofinality ω_1.

Our goal is to define a forcing poset $\mathbb{Q}$ which forces that every stationary subset of S_{ω_2} reflects to an uncountable ordinal in ω_2 with cofinality ω. The forcing poset $\mathbb{Q}$ will be an iteration of forcing posets of the form $\mathbb{Q}(T)$, where T is a subset of S_{ω_2} which does not reflect to any uncountable ordinal in ω_2 of cofinality ω.

The iteration we define is an adaptation of an iteration developed by Sakai, which he used to prove the following theorem:

Theorem 8.1 (Sakai [3])**.** *Assume GCH and $\square_{\omega_1}$. Then there exists a forcing poset which forces that there is a stationary set $S \subseteq P_{\omega_1}(\omega_2)$ such that every stationary subset of S reflects to an uncountable ordinal in ω_2 with cofinality ω.*

Let us describe in more detail the context in which we will define $\mathbb{Q}$. We start with a ground model V which satisfies $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, and $2^{\omega_2} = \omega_3$. Let H be a generic filter on $\mathbb{P}_{\omega_2}$ over V. Let $S_{\omega_2} = \bigcup\{X : \exists F\ (X,F) \in H\}$.

We verify that in $V[H]$, it is still the case that $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, and $2^{\omega_2} = \omega_3$. Since $\mathbb{P}_{\omega_2}$ is ω_1-distributive, $\mathbb{P}_{\omega_2}$ does not add any new subsets of ω. So $2^\omega = \omega_1$ in $V[H]$. As CH holds in V, it is easy to see that $\mathbb{P}_{\omega_2}$ has cardinality ω_2. Since $\mathbb{P}_{\omega_2}$ is ω_2-c.c., it has at most $\omega_2^{\omega_1} = 2^{\omega_1} = \omega_2$ many antichains. So the number of nice $\mathbb{P}_{\omega_2}$-names in V for a subset of ω_1 is bounded by $\omega_2^{\omega_1} = 2^{\omega_1} = \omega_2$. It follows that $2^{\omega_1} = \omega_2$ in $V[H]$. Similarly, the number of nice $\mathbb{P}_{\omega_2}$-names in V for a subset of ω_2 is $\omega_2^{\omega_2} = 2^{\omega_2} = \omega_3$. So $2^{\omega_2} = \omega_3$ in $V[H]$.

We will work in $V[H]$ to define a forcing iteration $\mathbb{Q}$ which destroys the stationarity of any subset of S_{ω_2} which does not reflect to an uncountable ordinal in ω_2 with cofinality ω. Actually, we will find it convenient to define $\mathbb{Q}$ in a way that is not literally a forcing iteration, although it is forcing equivalent to one.

Before beginning the definition of $\mathbb{Q}$, we fix several auxiliary objects which will be needed in the definition. Since we will remind the reader of these objects later in the paper, we provide them with numbers for reference.

Notation 8.2. *For the purposes of bookkeeping, fix a surjective function $f : \omega_3 \to \omega_3 \times \omega_3$ satisfying that whenever $f(\alpha) = (i, j)$, then $i \leq \alpha$.*

It will be convenient to have at hand an auxiliary poset $\mathbb{R}$, which will be useful for notational purposes. However, we will not be interested in forcing with $\mathbb{R}$.

Notation 8.3. *Let $\mathbb{R}$ denote the set of all non-empty partial functions s of the form $s : (a^s)^2 \to \omega_1$, where a^s is a countable subset of ω_2. Let $t \leq_{\mathbb{R}} s$ if t extends s as a function, that is, $a^s \subseteq a^t$ and $t \restriction (a^s)^2 = s$.*

Note that by CH, $\mathbb{R}$ has cardinality ω_2.

Notation 8.4. *For each ordinal α in ω_2, fix a surjective function $g_\alpha : \omega_1 \to \alpha$. Let E be the set of b in $P_{\omega_1}(\omega_2)$ such that $\mathrm{ot}(b)$ is a limit ordinal and for all α in b, b is closed under g_α. Note that E is a club subset of $P_{\omega_1}(\omega_2)$.*

We are now ready to define by recursion the forcing poset $\mathbb{Q}$ in the model $V[H]$. Specifically, we define by recursion sequences $\langle \mathbb{Q}_\alpha : \alpha \leq \omega_3 \rangle$ and $\langle \dot{T}^i_j : i, j \leq \omega_3 \rangle$. For each $\alpha < \omega_3$, we let $\dot{T}(\alpha)$ be $\dot{T}^i_j$, where $f(\alpha) = (i, j)$. Finally, the forcing poset $\mathbb{Q}$ is defined as $\mathbb{Q}_{\omega_3}$.

We will maintain several recursion hypotheses.

Recursion Hypotheses 8.5. *For all $\alpha \leq \omega_3$:*

(1) *If p is in $\mathbb{Q}_\alpha$, then p is a partial function $p : \alpha \to \mathbb{R}$ with a countable domain, and for all p and q in $\mathbb{Q}_\alpha$, $q \leq p$ iff $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and for all ν in $\mathrm{dom}(p)$, $q(\nu) \leq_{\mathbb{R}} p(\nu)$.*
(2) *Let $\beta < \alpha$. Then: (a) for all q in $\mathbb{Q}_\alpha$, $q \restriction \beta$ is in $\mathbb{Q}_\beta$, (b) $\mathbb{Q}_\beta \subseteq \mathbb{Q}_\alpha$, (c) if q is in $\mathbb{Q}_\alpha$ and $s \leq q \restriction \beta$ in $\mathbb{Q}_\beta$, then letting $t = s \cup (q \restriction [\beta, \alpha))$, t is in $\mathbb{Q}_\alpha$ and $t \leq s, q$ in $\mathbb{Q}_\alpha$, and (d) the inclusion map $\mathbb{Q}_\beta \to \mathbb{Q}_\alpha$ is a complete embedding.*
(3) *$\mathbb{Q}_\alpha$ is ω_1-distributive and ω_2-c.c.*
(4) *If $\alpha < \omega_3$, then the sequence $\langle \dot{T}^\alpha_i : i < \omega_3 \rangle$ is an enumeration of all nice $\mathbb{Q}_\alpha$-names $\dot{T}$ for a subset of $P_{\omega_1}(\omega_2)$ satisfying the property that $\mathbb{Q}_\alpha$ forces $\dot{T} \subseteq S_{\omega_2} \cap E$ and $\dot{T}$ does not reflect to any uncountable ordinal in ω_2 with cofinality ω.*

Let $\alpha \leq \omega_3$ be given, and assume the sequences $\langle \mathbb{Q}_i : i < \alpha \rangle$ and $\langle \dot{T}^\beta_i : \beta < \alpha, i < \omega_3 \rangle$ are defined and satisfy the recursion hypotheses.

If $\alpha < \omega_3$ and we have defined $\mathbb{Q}_\alpha$ which satisfies recursion hypotheses (1), (2), and (3), then recursion hypothesis (4) is straightforward. This is clear if $\alpha = 0$. Suppose $\alpha > 0$. By CH the poset $\mathbb{R}$ has cardinality ω_2. So the collection of all partial functions $p : \alpha \to \mathbb{R}$ with a countable domain has cardinality ω_2. Since

$\mathbb{Q}_\alpha$ is a subset of this collection by recursion hypothesis (1), $\mathbb{Q}_\alpha$ has cardinality at most ω_2. By recursion hypothesis (3), $\mathbb{Q}_\alpha$ is ω_2-c.c., so the number of antichains of $\mathbb{Q}_\alpha$ is at most $\omega_2^{\omega_1} = \omega_2$. Hence the number of nice $\mathbb{Q}_\alpha$-names for a subset of $P_{\omega_1}(\omega_2)$ is at most $\omega_2^{\omega_2} = \omega_3$. Consequently, it is possible to choose an enumeration $\langle \dot{T}_i^\alpha : i < \omega_3 \rangle$ satisfying recursion hypothesis (4).

Now we define $\mathbb{Q}_\alpha$. The definition splits into three cases.

α is equal to 0

Let $\mathbb{Q}_0$ be the trivial forcing poset consisting of just the empty function. Note that recursion hypotheses (1), (2), and (3) are trivially true.

α is a limit ordinal

Let $\mathbb{Q}_\alpha$ consist of all partial functions $p : \alpha \to \mathbb{R}$ with a countable domain which satisfy the property that for all $\beta < \alpha$, $p \restriction \beta$ is in $\mathbb{Q}_\beta$. Let $q \le p$ in $\mathbb{Q}_\alpha$ if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and for all ν in $\mathrm{dom}(p)$, $q(\nu) \le_{\mathbb{R}} p(\nu)$.

Note that recursion hypothesis (1) is immediate.

α is a successor ordinal

Let $\alpha = \alpha_0 + 1$. Recall our bookkeeping function f from Notation 8.2. Let $f(\alpha_0) = (i, j)$. Then $i \le \alpha_0$ and $j < \omega_3$, so $\dot{T}_j^i$ is defined and is equal to $\dot{T}(\alpha_0)$.

By recursion hypothesis (4), $\dot{T}(\alpha_0)$ is a nice $\mathbb{Q}_i$-name for a subset of $P_{\omega_1}(\omega_2)$. Since the inclusion map $\mathbb{Q}_i \to \mathbb{Q}_{\alpha_0}$ is a complete embedding by recursion hypothesis (2), $\dot{T}(\alpha_0)$ is a nice $\mathbb{Q}_{\alpha_0}$-name for a subset of $P_{\omega_1}(\omega_2)$. Also by (4), $\mathbb{Q}_i$ forces that $\dot{T}(\alpha_0)$ is a subset of $S_{\omega_2} \cap E$ which does not reflect to any uncountable ordinal in ω_2 with cofinality ω. But this property of $\dot{T}(\alpha_0)$ is upwards absolute between a generic extension by $\mathbb{Q}_i$ and a generic extension by $\mathbb{Q}_{\alpha_0}$. Therefore $\mathbb{Q}_{\alpha_0}$ forces that $\dot{T}(\alpha_0)$ is a subset of $S_{\omega_2} \cap E$ which does not reflect to any uncountable ordinal in ω_2 with cofinality ω.

Let $\mathbb{Q}_\alpha$ be the set of all partial functions $p : \alpha \to \mathbb{R}$ with a countable domain which satisfy that $p \restriction \alpha_0$ is in $\mathbb{Q}_{\alpha_0}$, and if α_0 is in $\mathrm{dom}(p)$, then $p \restriction \alpha_0 \Vdash_{\mathbb{Q}_{\alpha_0}} p(\alpha_0) \in \mathbb{Q}(\dot{T}(\alpha_0))$. Let $q \le p$ if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and for all ν in $\mathrm{dom}(p)$, $q(\nu) \le_{\mathbb{R}} p(\nu)$.

Note that recursion hypothesis (1) is immediate.

Lemma 8.6. *Let p and q be in $\mathbb{Q}_{\alpha_0+1}$. Then $q \le p$ iff $q \restriction \alpha_0 \le p \restriction \alpha_0$ in $\mathbb{Q}_{\alpha_0}$ and if α_0 is in $\mathrm{dom}(p)$, then α_0 is in $\mathrm{dom}(q)$ and $q(\alpha_0) \le_{\mathbb{R}} p(\alpha_0)$.*

The proof is straightforward, using recursion hypothesis (1) applied to $\mathbb{Q}_{\alpha_0}$.

Lemma 8.7. *The forcing poset $\mathbb{Q}_{\alpha_0+1}$ is isomorphic to a dense subset of $\mathbb{Q}_{\alpha_0} * \mathbb{Q}(\dot{T}(\alpha_0))$. So $\mathbb{Q}_{\alpha_0+1}$ and $\mathbb{Q}_{\alpha_0} * \mathbb{Q}(\dot{T}(\alpha_0))$ are forcing equivalent.*

Proof. Define $i : \mathbb{Q}_{\alpha_0+1} \to \mathbb{Q}_{\alpha_0} * \mathbb{Q}(\dot{T}(\alpha_0))$ by letting $i(p) = (p \restriction \alpha_0) * \dot{1}$ if α_0 is not in $\mathrm{dom}(p)$, and $i(p) = (p \restriction \alpha_0) * p(\alpha_0)$ if α_0 is in $\mathrm{dom}(p)$. By the definition of $\mathbb{Q}_{\alpha_0+1}$, it is immediate that $i(p)$ is in $\mathbb{Q}_{\alpha_0} * \mathbb{Q}(\dot{T}(\alpha_0))$.

Clearly i is an injection. Note that $\mathbb{Q}_{\alpha_0}$ forces that for all a and b in $\mathbb{Q}(\dot{T}(\alpha_0))$, a and b are in $\mathbb{R}$, and $b \le a$ in $\mathbb{Q}(\dot{T}(\alpha_0))$ iff $b \le_{\mathbb{R}} a$. Using this observation and Lemma 8.6, it is routine to verify that for all p and q in $\mathbb{Q}_{\alpha_0+1}$, $q \le p$ iff $i(q) \le i(p)$.

It remains to show that the range of i is dense. So let $p * \dot{s}$ be in $\mathbb{Q}_{\alpha_0} * \mathbb{Q}(\dot{T}(\alpha_0))$. By recursion hypothesis (3), $\mathbb{Q}_{\alpha_0}$ is ω_1-distributive, so $\mathbb{Q}_{\alpha_0}$ forces $\dot{s}$ is in the ground model. Hence we can find a set s and a condition $q \leq p$ in $\mathbb{Q}_{\alpha_0}$ such that $q \Vdash \check{s} = \dot{s}$. Let $r = q \cup \{(\alpha_0, s)\}$ if s is non-empty, and $r = q$ if s is empty. Then r is in $\mathbb{Q}_{\alpha_0+1}$ and $i(r) = q * \check{s} \leq p * \dot{s}$. $\square$

This completes the recursive definition.

We have taken care of recursion hypotheses (1) and (4). Proving that $\mathbb{Q}_\alpha$ is ω_1-distributive and ω_2-c.c. will take some work, and we postpone it until the next section. The proof of recursion hypothesis (2) is routine, but we include it for completeness.

Lemma 8.8. *Let $\beta < \alpha$. Then:*

(a) *for all q in $\mathbb{Q}_\alpha$, $q \restriction \beta \in \mathbb{Q}_\beta$,*
(b) *$\mathbb{Q}_\beta \subseteq \mathbb{Q}_\alpha$,*
(c) *if q is in $\mathbb{Q}_\alpha$ and $s \leq q \restriction \beta$ in $\mathbb{Q}_\beta$, then letting $t = s \cup (q \restriction [\beta, \alpha))$, t is in $\mathbb{Q}_\alpha$ and $t \leq s, q$ in $\mathbb{Q}_\alpha$,*
(d) *the inclusion map $\mathbb{Q}_\beta \to \mathbb{Q}_\alpha$ is a complete embedding.*

Proof. (a) Let q be in $\mathbb{Q}_\alpha$. If α is a limit ordinal, then $q \restriction \beta$ is in $\mathbb{Q}_\beta$ by the definition of $\mathbb{Q}_\alpha$. Otherwise α is a successor ordinal. Let $\alpha = \alpha_0 + 1$. If $\beta = \alpha_0$, then $q \restriction \alpha_0$ is in $\mathbb{Q}_{\alpha_0}$ by the definition of $\mathbb{Q}_\alpha$. If $\beta < \alpha_0$, then $q \restriction \beta$ is equal to $(q \restriction \alpha_0) \restriction \beta$, which is in $\mathbb{Q}_\beta$ by recursion hypothesis (2a) applied to $\mathbb{Q}_{\alpha_0}$.

(b) First suppose α is a successor ordinal. Let $\alpha = \alpha_0 + 1$. Note that by the definition of $\mathbb{Q}_\alpha$, $\mathbb{Q}_{\alpha_0} \subseteq \mathbb{Q}_\alpha$. By recursion hypothesis (2b) applied to $\mathbb{Q}_{\alpha_0}$, $\mathbb{Q}_\beta \subseteq \mathbb{Q}_{\alpha_0}$. So $\mathbb{Q}_\beta \subseteq \mathbb{Q}_\alpha$. Now assume α is a limit ordinal, and let p be in $\mathbb{Q}_\beta$. To show p is in $\mathbb{Q}_\alpha$, we show that p is a partial function $p : \alpha \to \mathbb{R}$ with a countable domain, and for all $\gamma < \alpha$, $p \restriction \gamma$ is in $\mathbb{Q}_\gamma$. The first statement follows from recursion hypothesis (1) applied to $\mathbb{Q}_\beta$. Now let $\gamma < \alpha$ and we show $p \restriction \gamma$ is in $\mathbb{Q}_\gamma$. If $\gamma < \beta$, then $p \restriction \gamma$ is in $\mathbb{Q}_\gamma$ by recursion hypothesis (2a) applied to $\mathbb{Q}_\beta$. If $\beta \leq \gamma$, then $p \restriction \gamma = p$, and by recursion hypothesis (2b) applied to $\mathbb{Q}_\gamma$, p is in $\mathbb{Q}_\gamma$.

(c) Let q be in $\mathbb{Q}_\alpha$, and suppose $s \leq q \restriction \beta$ in $\mathbb{Q}_\beta$. Let $t = s \cup (q \restriction [\beta, \alpha))$. We will prove t is in $\mathbb{Q}_\alpha$. It is then easy to verify that $t \leq s, q$ in $\mathbb{Q}_\alpha$. First assume α is a limit ordinal. Clearly t is a partial function $t : \alpha \to \mathbb{R}$ with a countable domain. Let $\gamma < \alpha$ and we show $t \restriction \gamma$ is in $\mathbb{Q}_\gamma$. If $\gamma \leq \beta$, then $t \restriction \gamma = s \restriction \gamma$, which is in $\mathbb{Q}_\gamma$ by recursion hypothesis (2a) applied to $\mathbb{Q}_\beta$. Suppose $\beta < \gamma$. Let $\overline{q} = q \restriction \gamma$. Then by (a), $\overline{q}$ is in $\mathbb{Q}_\gamma$, and $s \leq q \restriction \beta = \overline{q} \restriction \beta$. By recursion hypothesis (2c) applied to $\mathbb{Q}_\gamma$, $s \cup (\overline{q} \restriction [\beta, \gamma))$ is in $\mathbb{Q}_\gamma$. But $s \cup (\overline{q} \restriction [\beta, \gamma))$ is equal to $t \restriction \gamma$.

Now assume α is a successor ordinal. Let $\alpha = \alpha_0 + 1$. First consider the case when $\beta = \alpha_0$. If α_0 is not in $\mathrm{dom}(q)$, then $t = s$, which is in $\mathbb{Q}_\alpha$ by (b). Suppose α_0 is in $\mathrm{dom}(q)$. Then $t = s \cup \{(\alpha_0, q(\alpha_0))\}$. So $t \restriction \alpha_0 = s$, which is in $\mathbb{Q}_{\alpha_0}$. By the definition of $\mathbb{Q}_\alpha$, $q \restriction \alpha_0 \Vdash q(\alpha_0) \in \mathbb{Q}(\dot{T}(\alpha_0))$. As $s \leq q \restriction \alpha_0$, $s = t \restriction \alpha_0$, and $t(\alpha_0) = q(\alpha_0)$, $t \restriction \alpha_0 \Vdash t(\alpha_0) \in \mathbb{Q}(\dot{T}(\alpha_0))$. So t is in $\mathbb{Q}_\alpha$.

Now suppose $\beta < \alpha_0$. Then $s \leq q \restriction \beta = (q \restriction \alpha_0) \restriction \beta$. By recursion hypothesis (2c) applied to $\mathbb{Q}_{\alpha_0}$, letting $t' = s \cup (q \restriction [\beta, \alpha_0))$, t' is in $\mathbb{Q}_{\alpha_0}$ and $t' \leq s, q \restriction \alpha_0$ in $\mathbb{Q}_{\alpha_0}$. But $t' = t \restriction \alpha_0$, so $t \restriction \alpha_0$ is in $\mathbb{Q}_{\alpha_0}$. If α_0 is not in $\mathrm{dom}(q)$, then $t' = t$ and we are done by (b). Otherwise since $t' \leq q \restriction \alpha_0$, t' forces $q(\alpha_0) = t(\alpha_0)$ is in $\mathbb{Q}(\dot{T}(\alpha_0))$. So t is in $\mathbb{Q}_\alpha$.

(d) We show that the inclusion map from $\mathbb{Q}_\beta$ to $\mathbb{Q}_\alpha$ is a complete embedding. Let $q \leq p$ in $\mathbb{Q}_\beta$. Then $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, and for all ν in $\mathrm{dom}(p)$, $q(\nu) \leq_{\mathbb{R}} p(\nu)$.

But this is also the definition of the ordering on $\mathbb{Q}_\alpha$. So $q \leq p$ in $\mathbb{Q}_\alpha$. Suppose p_1 and p_2 are in $\mathbb{Q}_\beta$ and $r \leq p_1, p_2$ in $\mathbb{Q}_\alpha$. By (a), $r \restriction \beta$ is in $\mathbb{Q}_\beta$, and clearly $r \restriction \beta \leq p_1, p_2$ in $\mathbb{Q}_\beta$. Finally, let q be in $\mathbb{Q}_\alpha$. By (c), if $p \leq q \restriction \beta$ in $\mathbb{Q}_\beta$, then $p \cup (q \restriction [\beta, \alpha))$ is in $\mathbb{Q}_\alpha$ and is below p and q in $\mathbb{Q}_\alpha$. Thus p and q are compatible in $\mathbb{Q}_\alpha$. $\square$

9. Distributivity and the Chain Condition

Our next goal is to prove that $\mathbb{Q}_\alpha$ is ω_1-distributive and ω_2-c.c., which will complete the verification of the recursion hypotheses. For this purpose, it will be useful to first analyze the iteration in more detail. The proofs in this section are independent of whether α is a successor ordinal or a limit ordinal.

Notation 9.1. *Suppose that p is a partial function $p : \alpha \to \mathbb{R}$ with a countable domain (for example, if p is in $\mathbb{Q}_\alpha$). For each ξ in* $\mathrm{dom}(p)$, *let a^p_ξ be $a^{p(\xi)}$, that is, a^p_ξ is the non-empty countable set a such that* $\mathrm{dom}(p(\xi)) = a^2$.

Occasionally we will consider a partial function $p : \alpha \to \mathbb{R}$, and write a^p_ξ and $p(\xi)$, even though we do not know whether ξ is in $\mathrm{dom}(p)$. In this case, a^p_ξ and $p(\xi)$ both denote the empty set if ξ is not in $\mathrm{dom}(p)$.

The next observation is obvious, but we record it for later reference.

Lemma 9.2. *Let p be in $\mathbb{Q}_\alpha$. Suppose β is in* $\mathrm{dom}(p)$ *and x is a countable subset of a^p_β which is closed under $p(\beta)$. Then $p \restriction \beta \Vdash_{\mathbb{Q}_\beta} x \notin \dot{T}(\beta)$.*

Proof. By Lemma 8.8(a), $p \restriction (\beta + 1)$ is in $\mathbb{Q}_{\beta+1}$. By the definition of $\mathbb{Q}_{\beta+1}$, $p \restriction \beta \Vdash_{\mathbb{Q}_\beta} p(\beta) \in \mathbb{Q}(\dot{T}(\beta))$. The conclusion of the lemma now follows by the definition of $\mathbb{Q}(\dot{T}(\beta))$ as given in Definition 7.1. $\square$

The next lemma will be useful throughout the remainder of the paper. The proof is straightforward, but we include it for completeness.

Lemma 9.3. *Suppose p is a partial function $p : \alpha \to \mathbb{R}$ with a countable domain. If p is not in $\mathbb{Q}_\alpha$, then there exist β, t, and x such that:*

- *β is in* $\mathrm{dom}(p)$,
- *$p \restriction \beta$ is in $\mathbb{Q}_\beta$,*
- *$t \leq p \restriction \beta$ in $\mathbb{Q}_\beta$,*
- *x is a countable subset of a^p_β closed under $p(\beta)$,*
- *$t \Vdash_{\mathbb{Q}_\beta} x \in \dot{T}(\beta)$.*

Proof. We assume the statement holds for $\mathbb{Q}_\nu$, for all $\nu < \alpha$, and verify that it also holds for $\mathbb{Q}_\alpha$. It is clear when $\alpha = 0$.

Assume α is a successor ordinal $\alpha_0 + 1$. Let p be a partial function $p : \alpha \to \mathbb{R}$ with a countable domain, and suppose p is not in $\mathbb{Q}_\alpha$. There are two ways this could happen. The first possibility is that $p \restriction \alpha_0$ is not in $\mathbb{Q}_{\alpha_0}$. In this case, the conclusion of the lemma follows from the induction hypothesis applied to $p \restriction \alpha_0$ in $\mathbb{Q}_{\alpha_0}$.

The second possibility is that $p \restriction \alpha_0$ is in $\mathbb{Q}_{\alpha_0}$, α_0 is in $\mathrm{dom}(p)$, and $p \restriction \alpha_0$ does not force that $p(\alpha_0)$ is in $\mathbb{Q}(\dot{T}(\alpha_0))$. So there is $t_0 \leq p \restriction \alpha_0$ such that t_0 forces $p(\alpha_0)$ is not in $\mathbb{Q}(\dot{T}(\alpha_0))$. Since $p(\alpha_0)$ is in $\mathbb{R}$, t_0 forces there exists a countable set $x \subseteq a^{p(\alpha_0)}$ such that x is closed under $p(\alpha_0)$ and x is in $\dot{T}(\alpha_0)$. By recursion hypothesis (3) applied to $\mathbb{P}_{\alpha_0}$, $\mathbb{P}_{\alpha_0}$ is ω_1-distributive. So t_0 forces there is such a set x in the ground model. Hence we can choose $t \leq t_0$ and x such that t forces x satisfies the above properties. Then x is a countable subset of $a^p_{\alpha_0}$ closed under $p(\alpha_0)$ and t forces x is in $\dot{T}(\alpha_0)$.

Now assume α is a limit ordinal. If p is not in $\mathbb{Q}_\alpha$, then by the definition of $\mathbb{Q}_\alpha$, there is $\gamma < \alpha$ such that $p \restriction \gamma$ is not in $\mathbb{Q}_\gamma$. The conclusion of the lemma now follows by applying the induction hypothesis to $p \restriction \gamma$ in $\mathbb{Q}_\gamma$. $\square$

Recall that $\Vdash_{\mathbb{Q}_\beta} \dot{T}(\beta) \subseteq S_{\omega_2} \cap E$. So if $t \Vdash_{\mathbb{Q}_\beta} x \in \dot{T}(\beta)$, as in the conclusion of Lemma 9.3, then x is in $S_{\omega_2} \cap E$.

Lemma 9.4. *Suppose p and q are conditions in $\mathbb{Q}_\alpha$ such that for all β in $\mathrm{dom}(p) \cap \mathrm{dom}(q)$, $p(\beta) \restriction (a^p_\beta \cap a^q_\beta)^2 = q(\beta) \restriction (a^p_\beta \cap a^q_\beta)^2$. Then p and q are compatible. Moreover, there is $r \leq p, q$ such that $\mathrm{dom}(r) = \mathrm{dom}(p) \cup \mathrm{dom}(q)$, for all β in $\mathrm{dom}(p) \setminus \mathrm{dom}(q)$, $r(\beta) = p(\beta)$, and for all β in $\mathrm{dom}(q) \setminus \mathrm{dom}(p)$, $r(\beta) = q(\beta)$.*

In particular, if p and q are conditions in $\mathbb{Q}_\alpha$ such that for all β in $\mathrm{dom}(p) \cap \mathrm{dom}(q)$, $a^p_\beta \cap a^q_\beta$ is empty, then p and q are compatible.

Proof. Choose an ordinal ζ in ω_1 which is not in $a^p_\beta \cup a^q_\beta$ for any β in $\mathrm{dom}(p) \cap \mathrm{dom}(q)$. Define r as follows. The domain of r is equal to $\mathrm{dom}(p) \cup \mathrm{dom}(q)$. Let β be in $\mathrm{dom}(r)$, and we define $r(\beta)$. If $\beta \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$, let $r(\beta) = p(\beta)$. If $\beta \in \mathrm{dom}(q) \setminus \mathrm{dom}(p)$, let $r(\beta) = q(\beta)$. Suppose β is in $\mathrm{dom}(p) \cap \mathrm{dom}(q)$. Define $a^r_\beta = a^p_\beta \cup a^q_\beta$. Let (i, j) be in $(a^r_\beta)^2$, and we define $r(\beta)(i, j)$. If (i, j) is in $(a^p_\beta)^2$, let $r(\beta)(i, j) = p(\beta)(i, j)$. If (i, j) is in $(a^q_\beta)^2$, let $r(\beta)(i, j) = q(\beta)(i, j)$. Note this is well-defined, since if i and j are both in $a^p_\beta \cap a^q_\beta$, then $p(\beta)(i, j) = q(\beta)(i, j)$. In any other case, let $r(\beta)(i, j) = \zeta$.

It is easy to see that if r is a condition in $\mathbb{Q}_\alpha$, then $r \leq p, q$. Suppose for a contradiction r is not in $\mathbb{Q}_\alpha$. By Lemma 9.3, fix β, t, and x such that β is in $\mathrm{dom}(r)$, $r \restriction \beta$ is in $\mathbb{Q}_\beta$, $t \leq r \restriction \beta$ in $\mathbb{Q}_\beta$, x is a countable subset of a^r_β closed under $r(\beta)$, and $t \Vdash_{\mathbb{Q}_\beta} x \in \dot{T}(\beta)$. Note that $r \restriction \beta$ is below $p \restriction \beta$ and $q \restriction \beta$ in $\mathbb{Q}_\beta$. Therefore $t \leq p \restriction \beta, q \restriction \beta$.

We consider different cases regarding β, and derive a contradiction in each case. First consider the case that β is in $\mathrm{dom}(p) \setminus \mathrm{dom}(q)$. Then by the definition of r, $r(\beta) = p(\beta)$. Therefore x is a countable subset of a^p_β closed under $p(\beta)$. By Lemma 9.2, $p \restriction \beta \Vdash_{\mathbb{Q}_\beta} x \notin \dot{T}(\beta)$. But this contradicts that $t \leq p \restriction \beta$. A similar argument handles the case when β is in $\mathrm{dom}(q) \setminus \mathrm{dom}(p)$.

Now consider the case when β is in $\mathrm{dom}(p) \cap \mathrm{dom}(q)$. If x is a subset of a^p_β or x is a subset of a^q_β, then the same argument as in the previous paragraph yields a contradiction. Otherwise there is i in $x \setminus a^p_\beta$ and j in $x \setminus a^q_\beta$. Then by the definition of r, $r(\beta)(i, j) = \zeta$, which is not in x. So x is not closed under $r(\beta)$, contradicting the choice of x. $\square$

Lemma 9.5. *Let z be a countable subset of α and let b be a countable subset of ω_2. Then there are densely many conditions t in $\mathbb{Q}_\alpha$ such that $z \subseteq \mathrm{dom}(t)$ and for all β in z, $b \subseteq a^t_\beta$.*

Proof. Let p be a condition in $\mathbb{Q}_\alpha$. We will find a condition $t \leq p$ such that $z \subseteq \mathrm{dom}(t)$ and for all β in z, $b \subseteq a^t_\beta$. Fix ζ in $\omega_1 \setminus b$. Let X be the set of β in $\mathrm{dom}(p) \cap z$ such that $b \subseteq a^p_\beta$.

Define s in $\mathbb{Q}_\alpha$ as follows. The domain of s is equal to $z \setminus X$. Consider β in $\mathrm{dom}(s)$. If β is in $z \setminus \mathrm{dom}(p)$, let $s(\beta) : b^2 \to \omega_1$ satisfy $s(\beta)(i, j) = \zeta$ for all

$(i,j) \in b^2$. Suppose β is in $z \cap \mathrm{dom}(p)$. Since β is not in X, $b \setminus a^p_\beta$ is non-empty. Define $s(\beta) : (b \setminus a^p_\beta)^2 \to \omega_1$ by letting $s(\beta)(i,j) = \zeta$ for all (i,j) in $(b \setminus a^p_\beta)^2$.

The set s is a partial function $s : \alpha \to \mathbb{R}$ with a countable domain. Note that s satisfies the property that for all β in $\mathrm{dom}(s)$, there are no subsets of a^s_β which are closed under $s(\beta)$. It follows easily from Lemma 9.3 that s is in $\mathbb{Q}_\alpha$. Also note that if β is in $\mathrm{dom}(s) \cap \mathrm{dom}(p)$, then $a^s_\beta \cap a^p_\beta$ is empty. By Lemma 9.4, s and p are compatible. Fix $t \leq s, p$ in $\mathbb{Q}_\alpha$. Then clearly t is as desired. $\square$

It will be useful to consider a certain kind of condition in $\mathbb{Q}_\alpha$, which we call a square condition.

Notation 9.6. *A condition p in $\mathbb{Q}_\alpha$ is* square *if there is a set a such that for all β in $\mathrm{dom}(p)$, $a^p_\beta = a$. In this case, we let a^p denote the set a as above.*

Proposition 9.7. *The forcing poset $\mathbb{Q}_\alpha$ is ω_1-distributive. The set of square conditions is dense in $\mathbb{Q}_\alpha$.*

Proof. We prove both statements simultaneously. So let $\{D_n : n < \omega\}$ be a family of dense open subsets of $\mathbb{Q}_\alpha$, and let p be a condition in $\mathbb{Q}_\alpha$. We prove there exists a square condition $q \leq p$ which is in $\bigcap\{D_n : n < \omega\}$.

Fix a regular cardinal θ much larger than $\mathbb{Q}_\alpha$. Recall that we are working in the model $V[H]$, where H is a generic filter on $\mathbb{P}_{\omega_2}$ over V. We will apply Lemma 4.7 to the forcing poset $\mathbb{P}_{\omega_2} = \mathbb{P}(\emptyset, 0, \omega_2)$. By Lemma 4.7, in $V[H]$ there are stationarily many N in $P_{\omega_1}(H(\theta))$ such that $S_{\omega_2} \cap P(N \cap \omega_2) \subseteq N$. Choose N in $P_{\omega_1}(H(\theta))$ such that $N \prec H(\theta)$, the objects $\mathbb{Q}_\alpha$, p, and $\{D_n : n < \omega\}$ are in N, and $S_{\omega_2} \cap P(N \cap \omega_2) \subseteq N$.

Choose an N-generic sequence $\langle p_n : n < \omega\rangle$ for $\mathbb{Q}_\alpha$, with $p_0 = p$. By Lemma 9.5 and N-genericity, it is easy to see that $\bigcup\{\mathrm{dom}(p_n) : n < \omega\} = N \cap \alpha$, and for each β in $N \cap \alpha$, $\bigcup\{a^{p_n}_\beta : n < \omega\} = N \cap \omega_2$.

Define q by letting $\mathrm{dom}(q) = \bigcup\{\mathrm{dom}(p_n) : n < \omega\}$, and for each β in $\mathrm{dom}(q)$, letting $q(\beta)$ be the union of the set of functions $\{p_n(\beta) : n < \omega\}$. Thus q is a partial function $q : \alpha \to \mathbb{R}$ with a countable domain. We will prove that q is in $\mathbb{Q}_\alpha$. It is then easy to see that $q \leq p_n$ for all $n < \omega$, and hence q is in $\bigcap\{D_n : n < \omega\}$. Also, for all β in $\mathrm{dom}(q)$, $a^q_\beta = N \cap \omega_2$, so q is a square condition.

Suppose for a contradiction q is not in $\mathbb{Q}_\alpha$. By Lemma 9.3, fix β, t, and x such that β is in $\mathrm{dom}(q) = N \cap \alpha$, $q \restriction \beta$ is in $\mathbb{Q}_\beta$, $t \leq q \restriction \beta$ in $\mathbb{Q}_\beta$, x is a countable subset of $a^q_\beta = N \cap \omega_2$ closed under $q(\beta)$, and $t \Vdash_{\mathbb{Q}_\beta} x \in \dot{T}(\beta)$. Note that $q \restriction \beta \leq p_n \restriction \beta$ for all $n < \omega$. Since $t \Vdash_{\mathbb{Q}_\beta} x \in \dot{T}(\beta)$, x is in S_{ω_2}. Also $x \subseteq N \cap \omega_2$. So x is in $S_{\omega_2} \cap P(N \cap \omega_2)$. By the choice of N, x is in N.

Let D be the set of conditions s in $\mathbb{Q}_\alpha$ such that β is in $\mathrm{dom}(s)$ and $x \subseteq a^s_\beta$. By Lemma 9.5, D is dense open, and by elementarity, D is in N. So fix $n < \omega$ such that p_n is in D. Since x is closed under $q(\beta)$ and $q(\beta) \restriction (a^{p_n}_\beta)^2 = p_n(\beta)$, x is closed under $p_n(\beta)$. By Lemma 9.2, $p_n \restriction \beta \Vdash_{\mathbb{Q}_\beta} x \notin \dot{T}(\beta)$. As $t \Vdash_{\mathbb{Q}_\beta} x \in \dot{T}(\beta)$, $p_n \restriction \beta$ and t are incompatible. But $t \leq q \restriction \beta \leq p_n \restriction \beta$, so we have a contradiction. $\square$

Proposition 9.8. *The forcing poset $\mathbb{Q}_\alpha$ is ω_2-c.c.*

Proof. Recall that CH holds in $V[H]$. We will show that $\mathbb{Q}_\alpha$ is ω_2-Knaster. Let $\langle p_i : i < \omega_2\rangle$ be a sequence of conditions in $\mathbb{Q}_\alpha$. Without loss of generality we may assume each p_i is a square condition.

Using CH, apply the Δ-System Lemma to the sequence of countable sets $\langle \mathrm{dom}(p_i) : i < \omega_2 \rangle$ to get an unbounded set $Z_0 \subseteq \omega_2$ and a countable set b such that for all $i < j$ in Z_0, $\mathrm{dom}(p_i) \cap \mathrm{dom}(p_j) = b$. Now apply the Δ-System Lemma again to the sequence of countable sets $\langle a^{p_i} : i \in Z_0 \rangle$ to get an unbounded set $Z_1 \subseteq Z_0$ and a countable set c such that for all $i < j$ in Z_1, $a^{p_i} \cap a^{p_j} = c$.

An easy counting argument using CH shows there are at most ω_1 many possibilities for a sequence $\langle p_i(\beta) \restriction c^2 : \beta \in b \rangle$, for i in Z_1. So we can fix an unbounded set $Z_2 \subseteq Z_1$ such that for all $i < j$ in Z_2, for all β in b, $p_i(\beta) \restriction c^2 = p_j(\beta) \restriction c^2$.

Consider $i < j$ in Z_2. Then for all β in $\mathrm{dom}(p_i) \cap \mathrm{dom}(p_j) = b$, $p_i(\beta) \restriction (a^{p_i} \cap a^{p_j})^2 = p_i(\beta) \restriction c^2 = p_j(\beta) \restriction c^2 = p_j(\beta) \restriction (a^{p_i} \cap a^{p_j})^2$. By Lemma 9.4, p_i and p_j are compatible. □

This completes the verification of the recursion hypotheses.

We conclude this section with several technical lemmas which will be needed later in the paper. Recall that $\mathbb{Q}$ is $\mathbb{Q}_{\omega_3}$.

Lemma 9.9. *Let p and q be conditions in $\mathbb{Q}$, and assume $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$, and for all β in $\mathrm{dom}(p)$, $a^p_\beta \subseteq a^q_\beta$. If p and q are compatible, then $q \leq p$.*

Proof. To show $q \leq p$, it suffices to show that whenever β is in $\mathrm{dom}(p)$, $q(\beta) \restriction (a^p_\beta)^2 = p(\beta)$. Fix $r \leq p, q$. Then $q(\beta) \restriction (a^p_\beta)^2 = r(\beta) \restriction (a^p_\beta)^2 = p(\beta)$. □

Lemma 9.10. *Suppose I is a generic filter on $\mathbb{Q}$ over $V[H]$. Let $\{p_n : n < \omega\}$ be a countable subset of I. Then there is a condition r in I such that $r \leq p_n$ for all $n < \omega$.*

Proof. Since $\mathbb{Q}$ is ω_1-distributive, the set $\{p_n : n < \omega\}$ is in the ground model $V[H]$. Let $z = \bigcup\{\mathrm{dom}(p_n) : n < \omega\}$ and let $b = \bigcup\{a^{p_n}_\beta : n < \omega, \beta \in \mathrm{dom}(p_n)\}$. By Lemma 9.5, there are densely many conditions t in $\mathbb{Q}$ such that $z \subseteq \mathrm{dom}(t)$, and for all β in z, $b \subseteq a^t_\beta$. Fix such a condition t in I. Let $n < \omega$, and we show $t \leq p_n$. By the choice of t, $\mathrm{dom}(p_n) \subseteq z \subseteq \mathrm{dom}(t)$, and for all β in $\mathrm{dom}(p_n)$, $a^{p_n}_\beta \subseteq b \subseteq a^t_\beta$. Also p_n and t are both in I, so they are compatible. By Lemma 9.9, $t \leq p_n$. □

The next easy lemma is technical and unmotivated, but we will need it in a construction later in the paper.

Lemma 9.11. *Suppose p is a square condition in $\mathbb{Q}$. Let ν be an ordinal in $\omega_2 \setminus a^p$ and let ζ be an ordinal in ω_1 which is different from ν. Let b be a subset of $\mathrm{dom}(p)$. Then there is a condition $t \leq p$ in $\mathbb{Q}$ such that $\mathrm{dom}(t) = \mathrm{dom}(p)$, for all β in b, $t(\beta) = p(\beta)$, and for all β in $\mathrm{dom}(t) \setminus b$, $\nu \in a^t_\beta$ and $t(\beta)(\nu, \nu) = \zeta$.*

Proof. Define s with domain equal to $\mathrm{dom}(p) \setminus b$ so that for all β in $\mathrm{dom}(s)$, $a^s_\beta = \{\nu\}$ and $s(\beta)(\nu, \nu) = \zeta$. Since for each β in $\mathrm{dom}(s)$, there do not exist any subsets of a^s_β which are closed under $s(\beta)$, it follows easily from Lemma 9.3 that s is a condition in $\mathbb{Q}$. Now for all β in $\mathrm{dom}(p) \cap \mathrm{dom}(s)$, $a^p_\beta \cap a^s_\beta$ is empty. So by Lemma 9.4, we can choose $t \leq s, p$ such that $\mathrm{dom}(t) = \mathrm{dom}(s) \cup \mathrm{dom}(p)$ (which is equal to $\mathrm{dom}(p)$), and for all β in $\mathrm{dom}(p) \setminus \mathrm{dom}(s)$, $t(\beta) = p(\beta)$.

Let us verify that t is as desired. We already noted that $\mathrm{dom}(t) = \mathrm{dom}(p)$. If β is in b, then by the choice of s, β is in $\mathrm{dom}(p) \setminus \mathrm{dom}(s)$, so $t(\beta) = p(\beta)$. If β is in $\mathrm{dom}(t) \setminus b$, then β is in $\mathrm{dom}(p) \setminus b = \mathrm{dom}(s)$. By the choice of s, ν is in a^s_β. Since $t \leq s$, ν is in a^t_β, and $t(\beta)(\nu, \nu) = s(\beta)(\nu, \nu) = \zeta$. □

10. Preserving the Stationarity of S_{ω_2}

In this section we will prove that $\mathbb{Q}$ preserves the stationarity of S_{ω_2}, and forces that every stationary subset of S_{ω_2} reflects to an uncountable ordinal in $\omega_2 \cap \mathrm{cof}(\omega)$.

Proposition 10.1. *The forcing poset $\mathbb{Q}$ forces that S_{ω_2} is stationary in $P_{\omega_1}(\omega_2)$.*

Proof. Recall that $\mathbb{Q}$ is defined in the model $V[H]$, where H is a generic filter on $\mathbb{P}_{\omega_2}$ over V. So it suffices to show that in V, the forcing poset $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}$ forces $\dot{S}_{\omega_2}$ is stationary.

Let $(X, F) * \dot{p}$ be a condition in $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}$, and suppose $(X, F) * \dot{p}$ forces $\dot{h} : [\omega_2]^{<\omega} \to \omega_2$ is a function. We will find $(Y, K) * \dot{q} \le (X, F) * \dot{p}$ and a set z such that $(Y, K) * \dot{q}$ forces z is in $\dot{S}_{\omega_2}$ and z is closed under $\dot{h}$. Note that $(X, F) * \dot{p}$ forces there are club many ordinals $\nu < \omega_2$ which are closed under $\dot{h}$. Choose $(X', F') * \dot{p}' \le (X, F) * \dot{p}$ and an uncountable ordinal ν in $\omega_2 \cap \mathrm{cof}(\omega)$ such that $(X', F') * \dot{p}'$ forces ν is closed under $\dot{h}$.

For each $\alpha < \omega_3$, $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}_\alpha$ forces that $\dot{T}(\alpha)$ is a subset of $P_{\omega_1}(\omega_2)$ which does not reflect to any uncountable ordinal in ω_2 with cofinality ω. In particular, $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}_\alpha$ forces that $\dot{T}(\alpha)$ does not reflect to ν. So choose a $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}_\alpha$-name $\dot{f}_\alpha$ for a function $\dot{f}_\alpha : [\nu]^{<\omega} \to \nu$ such that $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}_\alpha$ forces there is no set in $\dot{T}(\alpha) \cap P_{\omega_1}(\nu)$ which is closed under $\dot{f}_\alpha$.

Fix a regular cardinal θ much larger than $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}$, $\dot{h}$, and $\langle \dot{f}_\alpha : \alpha < \omega_3 \rangle$. Let N be a countable elementary substructure of $H(\theta)$ which contains as elements the objects $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}$, $(X', F') * \dot{p}'$, $\dot{h}$, ν, and $\langle \dot{f}_\alpha : \alpha < \omega_3 \rangle$. We attempt to define a condition $(Y, K) * \dot{q}$ below $(X', F') * \dot{p}'$ which is N-generic for $\mathbb{P}_{\omega_2} * \mathbb{Q}$.

Choose an N-generic sequence $\langle (X_n, F_n) * \dot{p}_n : n < \omega \rangle$ such that $(X_0, F_0) * \dot{p}_0 = (X', F') * \dot{p}'$. Let K be the infinum of the sequence $\langle F_n : n < \omega \rangle$. Let $Y = \bigcup \{X_n : n < \omega\} \cup \{N \cap \nu\}$. By Proposition 3.8, (Y, K) is a condition in $\mathbb{P}_{\omega_2}$ and $(Y, K) \le (X_n, F_n)$ for all $n < \omega$. In particular, (Y, K) is N-generic.

Since $N \cap \nu$ is in Y, $(Y, K) \Vdash N \cap \nu \in \dot{S}_{\omega_2}$. Using Lemma 3.7, it is easy to see that $\bigcup Y = N \cap \omega_2$. So by Lemma 4.5, $(Y, K) \Vdash \dot{S}_{\omega_2} \cap P(N \cap \omega_2) = Y$.

Let H be a generic filter on $\mathbb{P}_{\omega_2}$ over V which contains the condition (Y, K). Since (Y, K) is N-generic, $N[H] \cap V = N$. For each $n < \omega$, let $p_n = \dot{p}_n^H$. Then $\langle p_n : n < \omega \rangle$ is an $N[H]$-generic sequence for $\mathbb{Q}$. By Lemma 9.5 and $N[H]$-genericity, $\bigcup \{\mathrm{dom}(p_n) : n < \omega\} = N[H] \cap \omega_3 = N \cap \omega_3$, and for each β in $\bigcup \{\mathrm{dom}(p_n) : n < \omega\}$, $\bigcup \{a_\beta^{p_n} : n < \omega\} = N[H] \cap \omega_2 = N \cap \omega_2$.

We define a lower bound q of the sequence $\langle p_n : n < \omega \rangle$. Let the domain of q be equal to $\bigcup \{\mathrm{dom}(p_n) : n < \omega\}$, and for each β in $\mathrm{dom}(q)$, let $q(\beta)$ be the union of the set of functions $\{p_n(\beta) : n < \omega\}$. Clearly q is a partial function $q : \omega_3 \to \mathbb{R}$ with a countable domain. Also, $\mathrm{dom}(q) = N \cap \omega_3$, and for each β in $\mathrm{dom}(q)$, $a_\beta^q = N \cap \omega_2$.

We show that q is a condition in $\mathbb{Q}$. It is then clear that $q \le p_n$ for all $n < \omega$. Suppose for a contradiction q is not a condition. By Lemma 9.3, fix β, t, and x such that β is in $\mathrm{dom}(q)$, $q \restriction \beta$ is in $\mathbb{Q}_\beta$, $t \le q \restriction \beta$ in $\mathbb{Q}_\beta$, x is a countable subset of a_β^q which is closed under $q(\beta)$, and $t \Vdash_{\mathbb{Q}_\beta} x \in \dot{T}(\beta)$. Note that β is in $N \cap \omega_3$ and x is in $S_{\omega_2} \cap P(N \cap \omega_2) = Y$. Clearly $q \restriction \beta \le p_n \restriction \beta$ for all $n < \omega$, so in particular, $q \restriction \beta$ is $N[H]$-generic for $\mathbb{Q}_\beta$.

Now $Y = \bigcup \{X_n : n < \omega\} \cup \{N \cap \nu\}$, and $\bigcup \{X_n : n < \omega\} \subseteq N$. So either x is in N, or $x = N \cap \nu$. First suppose x is in N. Let D be the dense set of conditions s in $\mathbb{Q}$ such that $\beta \in \mathrm{dom}(s)$ and $x \subseteq a_\beta^s$. By elementarity, D is in N. Fix $n < \omega$ such that

$p_n \in D$. Then $x \subseteq a_\beta^{p_n}$. Since x is closed under $q(\beta)$ and $q(\beta) \restriction (a_\beta^{p_n})^2 = p_n(\beta)$, x is closed under $p_n(\beta)$. By Lemma 9.2, $p_n \restriction \beta \Vdash_{\mathbb{Q}_\beta} x \notin \dot{T}(\beta)$. Since $t \Vdash_{\mathbb{Q}_\beta} x \in \dot{T}(\beta)$, $p_n \restriction \beta$ and t are incompatible. But $t \leq q \restriction \beta \leq p_n \restriction \beta$, and we have a contradiction.

Now assume $x = N \cap \nu$. Recall that $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}_\beta$ forces that $\dot{f}_\beta : [\nu]^{<\omega} \to \nu$ is a function such that there is no set in $\dot{T}(\beta) \cap P_{\omega_1}(\nu)$ which is closed under $\dot{f}_\beta$. Since β is in N, $\dot{f}_\beta$ is in N. Let I_β be a generic filter on $\mathbb{Q}_\beta$ over $V[H]$ such that t is in I_β. Since $t \leq q \restriction \beta$ and $q \restriction \beta$ is $N[H]$-generic, $N[H][I_\beta] \cap V[H] = N[H]$. In particular, $N[H][I_\beta] \cap \nu = N[H] \cap \nu = N \cap \nu$. Let $f_\beta = \dot{f}_\beta^{H * I_\beta}$. Since $\dot{f}_\beta$ is in N, f_β is in $N[H][I_\beta]$. By the elementarity of $N[H][I_\beta]$, $N[H][I_\beta] \cap \nu = N \cap \nu$ is closed under f_β. On the other hand, as t is in I_β, $x = N \cap \nu$ is in $T(\beta) = \dot{T}(\beta)^{I_\beta}$. So x is a member of $T(\beta) \cap P_{\omega_1}(\nu)$ which is closed under f_β, contradicting the choice of f_β. This completes the proof that q is in $\mathbb{Q}$.

Let $\dot{q}$ be a $\mathbb{P}_{\omega_2}$-name such that (Y, K) forces $\dot{q}$ is in $\dot{\mathbb{Q}}$ and $\dot{q} \leq \dot{p}_n$ for all $n < \omega$. Then $(Y, K) * \dot{q}$ is N-generic for $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}$, and $(Y, K) * \dot{q} \leq (X, F) * \dot{p}$. We claim $(Y, K) * \dot{q}$ forces that $N \cap \nu$ is in $\dot{S}_{\omega_2}$ and is closed under $\dot{h}$, which completes the proof. Since $(Y, K) * \dot{q} \leq (X', F') * \dot{p}'$, $(Y, K) * \dot{q}$ forces ν is closed under $\dot{h}$. As $\dot{h}$ is in N, $(Y, K) * \dot{q}$ forces $N[\dot{H} * \dot{I}] \cap \omega_2$ is closed under $\dot{h}$, where $\dot{H} * \dot{I}$ is the canonical $\mathbb{P}_{\omega_2} * \dot{\mathbb{Q}}$-name for the generic filter. So $(Y, K) * \dot{q}$ forces $N[\dot{H} * \dot{I}] \cap \nu$ is closed under $\dot{h}$. As $(Y, K) * \dot{q}$ is N-generic, $(Y, K) * \dot{q} \Vdash N[\dot{H} * \dot{I}] \cap \nu = N \cap \nu$. But $N \cap \nu$ is in Y, so $(Y, K) * \dot{q}$ forces $N \cap \nu$ is in $\dot{S}_{\omega_2}$ and is closed under $\dot{h}$. $\square$

Theorem 10.2. *The forcing poset $\mathbb{Q}$ forces that S_{ω_2} is a stationary subset of $P_{\omega_1}(\omega_2)$ which does not reflect to any ordinal in ω_2 with cofinality ω_1, and for every stationary set $T \subseteq S_{\omega_2}$, there is an uncountable ordinal ν in ω_2 with cofinality ω such that T reflects to ν.*

Proof. Let p be a condition in $\mathbb{Q}$ and suppose p forces $\dot{U}$ is a subset of S_{ω_2} which does not reflect to any uncountable ordinal in ω_2 with cofinality ω. We will prove that p forces $\dot{U}$ is non-stationary. Since $\mathbb{Q}$ is ω_1-distributive, it is easy to see that the set E from Notation 8.4 is still club in any generic extension by $\mathbb{Q}$. So it suffices to show p forces $\dot{U} \cap E$ is non-stationary.

Fix a nice $\mathbb{Q}$-name $\dot{T}$ for a subset of $P_{\omega_1}(\omega_2)$ such that $p \Vdash_{\mathbb{Q}} \dot{T} = \dot{U} \cap E$, and $\mathbb{Q}$ forces if p is not in the generic filter, then $\dot{T}$ is the empty set. For some sequence of antichains $\langle A_x : x \in P_{\omega_1}(\omega_2) \rangle$ of $\mathbb{Q}$, $\dot{T}$ is equal to $\bigcup \{A_x \times \{\check{x}\} : x \in P_{\omega_1}(\omega_2)\}$. As $\mathbb{Q}$ is ω_2-c.c., each A_x has cardinality at most ω_1. Since $P_{\omega_1}(\omega_2)$ has cardinality ω_2 and $\mathbb{Q} = \bigcup \{\mathbb{Q}_\alpha : \alpha < \omega_3\}$, we can fix $\beta < \omega_3$ such that for all x in $P_{\omega_1}(\omega_2)$, $A_x \subseteq \mathbb{Q}_\beta$. Then $\dot{T}$ is a nice $\mathbb{Q}_\beta$-name.

For each ν in ω_2 with cofinality ω, $\Vdash_{\mathbb{Q}} \dot{T}$ does not reflect to ν. So we can choose a nice $\mathbb{Q}$-name $\dot{C}_\nu$ for a club subset of $P_{\omega_1}(\nu)$ which is disjoint from $\dot{T}$. As above, choose $\xi < \omega_3$ larger than β such that for all such ν, $\dot{C}_\nu$ is a $\mathbb{Q}_\xi$-name. Now it is easy to see that $\mathbb{Q}_\xi$ forces that $\dot{C}_\nu \cap \dot{T}$ is empty, for all uncountable ν in ω_2 with cofinality ω.

In summary, $\dot{T}$ is a nice $\mathbb{Q}_\xi$-name for a subset of $P_{\omega_1}(\omega_2)$ such that $\mathbb{Q}_\xi$ forces that $\dot{T} \subseteq S_{\omega_2} \cap E$, and for all uncountable ν in ω_2 with cofinality ω, $\dot{T}$ does not reflect to ν. By recursion hypothesis (4), there is $i < \omega_3$ such that $\dot{T} = \dot{T}_i^\xi$.

Let $\gamma < \omega_3$ be an ordinal such that $f(\gamma) = (\xi, i)$. Then $\dot{T}(\gamma) = \dot{T}_i^\xi = \dot{T}$. By Lemma 8.7, $\mathbb{Q}_{\gamma+1}$ adds a club subset of $P_{\omega_1}(\omega_2)$ which is disjoint from $\dot{T}$. Since the inclusion map $\mathbb{Q}_{\gamma+1} \to \mathbb{Q}$ is a complete embedding, $\mathbb{Q}$ adds a club which is disjoint from $\dot{T}$. As $p \Vdash \dot{T} = \dot{U} \cap E$, p forces $\dot{U} \cap E$ is non-stationary. $\square$

11. Extending an Elementary Embedding, Part 2

Fix for the remainder of the section a cardinal κ which is κ^+-supercompact, and assume $2^\kappa = \kappa^+$. Let $j : V \to M$ be an elementary embedding with critical point κ such that $j(\kappa) > \kappa^+$ and $M^{\kappa^+} \subseteq M$. In Section 6 we showed how to extend the elementary embedding j after forcing with $\textsc{Coll}(\omega_1, {<}\kappa) * \dot{\mathbb{P}}_\kappa$. Specifically, let $\overline{G} * \overline{H}$ be a generic filter on $j(\textsc{Coll}(\omega_1, {<}\kappa) * \dot{\mathbb{P}}_\kappa)$ over V. Let $G = \overline{G} \cap \textsc{Coll}(\omega_1, {<}\kappa)$ and $H = \overline{H} \cap \mathbb{P}_\kappa$. Then $G * H$ is a generic filter on $\textsc{Coll}(\omega_1, {<}\kappa) * \dot{\mathbb{P}}_\kappa$ over V, and in $V[\overline{G} * \overline{H}]$ we can extend j to $j : V[G * H] \to M[\overline{G} * \overline{H}]$ such that $j(G * H) = \overline{G} * \overline{H}$.

In this section we will prove that in $M[\overline{G} * \overline{H}]$, there is a projection mapping $\pi : j(\mathbb{Q}) \to \mathbb{Q}$. As a consequence, if $\overline{I}$ is a generic filter for $j(\mathbb{Q})$ over $V[\overline{G} * \overline{H}]$, then in $V[\overline{G} * \overline{H} * \overline{I}]$ we can extend j to $j : V[G * H * I] \to M[\overline{G} * \overline{H} * \overline{I}]$ such that $j(G * H * I) = \overline{G} * \overline{H} * \overline{I}$, where I is the filter on $\mathbb{Q}$ generated by $\pi[\overline{I}]$.

As in Section 6, we write κ^+ to indicate the ordinal which is the successor of κ in V, even when working in a model in which this ordinal is no longer a cardinal. In the models $M[G * H]$ and $V[G * H]$, κ is equal to ω_2 and κ^+ is equal to ω_3. However, in the model $M[\overline{G} * \overline{H}]$, $j(\kappa)$ is equal to ω_2, and κ and κ^+ are both ordinals of cofinality ω_1 strictly between ω_1 and ω_2.

By Theorem 6.1(1), $M[G * H]^{\kappa^+} \cap V[G * H] \subseteq M[G * H]$. It easily follows from this fact and from Recursion Hypotheses 8.5 (1) and (4) that the sequences $\langle \mathbb{Q}_\alpha : \alpha \le \kappa^+ \rangle$ and $\langle \dot{T}(\alpha) : \alpha < \kappa^+ \rangle$ are in $M[G * H]$.

Recall that in $V[G * H]$, $\mathbb{R}$ is the poset consisting of all non-empty functions $s : (a^s)^2 \to \omega_1$, where a^s is a countable subset of ω_2, ordered by extension of functions.

In the next lemma and in what follows, $\mathbb{R}$ means $(\mathbb{R})^{V[G*H]}$.

Lemma 11.1. *In $M[\overline{G} * \overline{H}]$, $\mathbb{R} = \{s \in j(\mathbb{R}) : a^s \subseteq \kappa\}$. Hence if t is in $j(\mathbb{R})$, then $t \restriction \kappa^2$ is in $\mathbb{R}$.*

Proof. Let s be in $\mathbb{R}$. Since s is a countable subset of $\kappa^2 \times \omega_1$, $j(s) = s$. So by the elementarity of j, s is in $j(\mathbb{R})$, and obviously $a^s \subseteq \kappa$. On the other hand, suppose s is in $j(\mathbb{R})$ and $a^s \subseteq \kappa$. Since s is a countable subset of $\kappa^2 \times \omega_1$ and $M[\overline{G} * \overline{H}]$ is a generic extension of M by an ω_1-distributive forcing poset, s is in M. So s is in V, and thus s is in $V[G * H]$. As above, $j(s) = s$. Since $j(s)$ is in $j(\mathbb{R})$, s is in $\mathbb{R}$. $\square$

We will define for each $\alpha \le \kappa^+$ a map $\pi_\alpha : j(\mathbb{Q}_\alpha) \to \mathbb{Q}_\alpha$ in $M[\overline{G} * \overline{H}]$. So fix $\alpha \le \kappa^+$, and consider a condition q in $j(\mathbb{Q}_\alpha)$. Then q is a partial function $q : j(\alpha) \to j(\mathbb{R})$ with a countable domain. So $\mathrm{dom}(q) \cap j[\alpha]$ is a countable subset of $j[\alpha]$, and the set $j^{-1}(\mathrm{dom}(q) \cap j[\alpha])$ is a countable subset of α. Since $M[\overline{G} * \overline{H}]$ is a generic extension of M by an ω_1-distributive forcing poset, $j^{-1}(\mathrm{dom}(q) \cap j[\alpha])$ is in M, and hence is in $V[G * H]$.

For each ordinal γ in $j^{-1}(\mathrm{dom}(q) \cap j[\alpha])$, $q(j(\gamma))$ is in $j(\mathbb{R})$. So $q(j(\gamma)) \restriction \kappa^2$ is in $\mathbb{R}$ by Lemma 11.1. Define $\pi_\alpha(q)$ as the function with domain equal to $j^{-1}(\mathrm{dom}(q) \cap j[\alpha])$ such that for all γ in this domain, $\pi_\alpha(q)(\gamma) = q(j(\gamma)) \restriction \kappa^2$. Clearly $\pi_\alpha(q)$ is a partial function $\pi_\alpha(q) : \alpha \to \mathbb{R}$ with a countable domain, and again by distributivity, $\pi_\alpha(q)$ is in $V[G * H]$. It will take some work to prove that $\pi_\alpha(q)$ is in $\mathbb{Q}_\alpha$.

We record our definition of π_α for easy reference.

Definition 11.2. *Let $\alpha \leq \kappa^+$. In $M[\overline{G}*\overline{H}]$ define a map π_α with domain $j(\mathbb{Q}_\alpha)$ as follows. Given q in $j(\mathbb{Q}_\alpha)$, let the domain of $\pi_\alpha(q)$ be equal to $j^{-1}(\mathrm{dom}(q) \cap j[\alpha])$, and for each γ in this domain, let $\pi_\alpha(q)(\gamma) = q(j(\gamma)) \restriction \kappa^2$.*

The next lemma is immediate from the definition.

Lemma 11.3. *Let $\alpha \leq \kappa^+$. For all q in $j(\mathbb{Q}_\alpha)$, $j[\mathrm{dom}(\pi_\alpha(q))] = \mathrm{dom}(q) \cap j[\alpha]$, and for all γ in the domain of $\pi_\alpha(q)$, $a_\gamma^{\pi_\alpha(q)} = a_{j(\gamma)}^q \cap \kappa$.*

Lemma 11.4. *Let $\alpha_1 \leq \alpha_2 \leq \kappa^+$ and let q be in $j(\mathbb{Q}_{\alpha_2})$. Then $\pi_{\alpha_1}(q \restriction j(\alpha_1)) = \pi_{\alpha_2}(q) \restriction \alpha_1$.*

Proof. The domain of $\pi_{\alpha_1}(q \restriction j(\alpha_1))$ is equal to $j^{-1}(\mathrm{dom}(q \restriction j(\alpha_1)) \cap j[\alpha_1])$, whereas the domain of $\pi_{\alpha_2}(q) \restriction \alpha_1$ is $j^{-1}(\mathrm{dom}(q) \cap j[\alpha_2]) \cap \alpha_1$, and it is easy to see these sets are equal. Let γ be in the domain of $\pi_{\alpha_1}(q \restriction j(\alpha_1))$. Then $\pi_{\alpha_1}(q \restriction j(\alpha_1))(\gamma)$ is equal to $(q \restriction j(\alpha_1))(j(\gamma)) \restriction \kappa^2 = q(j(\gamma)) \restriction \kappa^2 = \pi_{\alpha_2}(q)(\gamma) = (\pi_{\alpha_2}(q) \restriction \alpha_1)(\gamma)$. $\square$

Lemma 11.5. *Let $\alpha \leq \kappa^+$ and let q be in $j(\mathbb{Q}_\alpha)$. Suppose t is in $\mathbb{Q}_\alpha$, $\mathrm{dom}(\pi_\alpha(q)) \subseteq \mathrm{dom}(t)$, and for all γ in $\mathrm{dom}(\pi_\alpha(q))$, $t(\gamma) \leq_{\mathbb{R}} \pi_\alpha(q)(\gamma)$. Then $j(t)$ and q are compatible in $j(\mathbb{Q}_\alpha)$. In particular, if $\pi_\alpha(q)$ is in $\mathbb{Q}_\alpha$ and $t \leq \pi_\alpha(q)$ in $\mathbb{Q}_\alpha$, then $j(t)$ and q are compatible in $j(\mathbb{Q}_\alpha)$.*

Proof. To show that $j(t)$ and q are compatible in $j(\mathbb{Q}_\alpha)$, by Lemma 9.4 it suffices to show that for all β in $\mathrm{dom}(j(t)) \cap \mathrm{dom}(q)$,

$$j(t)(\beta) \restriction (a_\beta^{j(t)} \cap a_\beta^q)^2 = q(\beta) \restriction (a_\beta^{j(t)} \cap a_\beta^q)^2.$$

First we claim that

$$\mathrm{dom}(j(t)) \cap \mathrm{dom}(q) = \mathrm{dom}(q) \cap j[\alpha].$$

The domain of t is a countable subset of α, so $\mathrm{dom}(j(t)) = j(\mathrm{dom}(t)) = j[\mathrm{dom}(t)]$, which is a subset of $j[\alpha]$. So $\mathrm{dom}(j(t)) \cap \mathrm{dom}(q) \subseteq \mathrm{dom}(q) \cap j[\alpha]$. Suppose on the other hand β is in $\mathrm{dom}(q) \cap j[\alpha]$. Recall that $\mathrm{dom}(\pi_\alpha(q)) = j^{-1}(\mathrm{dom}(q) \cap j[\alpha])$. So letting $\gamma = j^{-1}(\beta)$, γ is in the domain of $\pi_\alpha(q)$. But $\mathrm{dom}(\pi_\alpha(q)) \subseteq \mathrm{dom}(t)$ by assumption, so γ is in the domain of t. Hence $j(\gamma) = \beta$ is in $\mathrm{dom}(j(t)) \cap \mathrm{dom}(q)$.

Next we claim that for all β in the domain of $j(t)$, letting $\gamma = j^{-1}(\beta)$,

$$a_\beta^{j(t)} = a_\gamma^t$$

and

$$j(t)(\beta) = t(\gamma).$$

So let $\beta = j(\gamma)$ be in the domain of $j(t)$. Then $t(\gamma) : (a_\gamma^t)^2 \to \omega_1$, where a_γ^t is a countable subset of κ. So $a_\beta^{j(t)} = j(a_\gamma^t) = a_\gamma^t$, proving the first claim. Now for any pair (i, k) in $(a_\beta^{j(t)})^2$, $j(t)(\beta)(i, k) = j(t)(j(\gamma))(j(i), j(k)) = j(t(\gamma)(i, k)) = t(\gamma)(i, k)$. So $j(t)(\beta) = t(\gamma)$.

Now we prove that for all β in $\mathrm{dom}(j(t)) \cap \mathrm{dom}(q)$, letting $\gamma = j^{-1}(\beta)$,

$$a_\beta^{j(t)} \cap a_\beta^q = a_\gamma^{\pi_\alpha(q)}.$$

This makes sense since if β is in $\mathrm{dom}(j(t)) \cap \mathrm{dom}(q)$, then β is in $\mathrm{dom}(q) \cap j[\alpha]$, so γ is in $j^{-1}(\mathrm{dom}(q) \cap j[\alpha]) = \mathrm{dom}(\pi_\alpha(q))$. Since $a_\beta^{j(t)} = a_\gamma^t$, which is a subset of κ, $a_\beta^{j(t)} \cap a_\beta^q \subseteq a_\beta^q \cap \kappa = a_\gamma^{\pi_\alpha(q)}$. On the other hand, $t(\gamma) \leq_{\mathbb{R}} \pi_\alpha(q)(\gamma)$ by assumption. In particular, $a_\gamma^{\pi_\alpha(q)} \subseteq a_\gamma^t = a_\beta^{j(t)}$. Also $a_\gamma^{\pi_\alpha(q)} = a_\beta^q \cap \kappa \subseteq a_\beta^q$. So $a_\gamma^{\pi_\alpha(q)} \subseteq a_\beta^{j(t)} \cap a_\beta^q$.

We finish the proof. Let β be in $\mathrm{dom}(j(t)) \cap \mathrm{dom}(q)$. Then β is in $\mathrm{dom}(q) \cap j[\alpha]$. Let $\gamma = j^{-1}(\beta)$. Then $a^{j(t)}_\beta \cap a^q_\beta = a^{\pi_\alpha(q)}_\gamma$ and $j(t)(\beta) = t(\gamma)$. Since $t(\gamma) \leq_{\mathbb{R}} \pi_\alpha(q)(\gamma)$, $j(t)(\beta) \restriction (a^{j(t)}_\beta \cap a^q_\beta)^2 = t(\gamma) \restriction (a^{\pi_\alpha(q)}_\gamma)^2 = \pi_\alpha(q)(\gamma) = q(\beta) \restriction (a^{\pi_\alpha(q)}_\gamma)^2 = q(\beta) \restriction (a^{j(t)}_\beta \cap a^q_\beta)^2$. $\square$

Lemma 11.6. *Let $\alpha \leq \kappa^+$ and let t be in $\mathbb{Q}_\alpha$. Suppose $u \leq j(t)$ in $j(\mathbb{Q}_\alpha)$. Then $\mathrm{dom}(t) \subseteq \mathrm{dom}(\pi_\alpha(u))$, and for all γ in $\mathrm{dom}(t)$, $\pi_\alpha(u)(\gamma) \leq_{\mathbb{R}} t(\gamma)$. In particular, if $\pi_\alpha(u)$ is in $\mathbb{Q}_\alpha$, then $\pi_\alpha(u) \leq t$ in $\mathbb{Q}_\alpha$.*

Proof. Since $\mathrm{dom}(t)$ is a countable subset of α, $\mathrm{dom}(j(t)) = j(\mathrm{dom}(t)) = j[\mathrm{dom}(t)]$. As $u \leq j(t)$ in $j(\mathbb{Q}_\alpha)$, $j[\mathrm{dom}(t)] \subseteq \mathrm{dom}(u)$. So if γ is in $\mathrm{dom}(t)$, $j(\gamma)$ is in $\mathrm{dom}(u) \cap j[\alpha]$, so γ is in $j^{-1}(\mathrm{dom}(u) \cap j[\alpha]) = \mathrm{dom}(\pi_\alpha(u))$. This shows $\mathrm{dom}(t) \subseteq \mathrm{dom}(\pi_\alpha(u))$.

Now let γ be in $\mathrm{dom}(t)$, and we show $\pi_\alpha(u)(\gamma) \leq_{\mathbb{R}} t(\gamma)$. Note that $j(t)(j(\gamma)) = j(t(\gamma)) = t(\gamma)$, because $t(\gamma)$ is a countable subset of $\kappa^2 \times \omega_1$. Since $u \leq j(t)$ in $j(\mathbb{Q}_\alpha)$, $u(j(\gamma)) \leq_{j(\mathbb{R})} j(t)(j(\gamma)) = t(\gamma)$. In other words, $u(j(\gamma))$ extends $t(\gamma)$ as a function. But a^t_γ is a subset of κ, so $u(j(\gamma)) \restriction \kappa^2 = \pi_\alpha(u)(\gamma)$ extends $t(\gamma)$ as a function. So $\pi_\alpha(u)(\gamma) \leq_{\mathbb{R}} t(\gamma)$. $\square$

Now we are ready to show $\pi_\alpha : j(\mathbb{Q}_\alpha) \to \mathbb{Q}_\alpha$ is a projection mapping.

Proposition 11.7. *For all $\alpha \leq \kappa^+$, π_α is a projection mapping $\pi_\alpha : j(\mathbb{Q}_\alpha) \to \mathbb{Q}_\alpha$. Moreover, if $t \leq \pi_\alpha(q)$ in $\mathbb{Q}_\alpha$, then $j(t)$ and q are compatible in $j(\mathbb{Q}_\alpha)$.*

Proof. The proof is by induction on $\alpha \leq \kappa^+$. So let $\alpha \leq \kappa^+$ be given, and assume for all $\beta < \alpha$, π_β satisfies the statement of the proposition.

First we prove that if q is in $j(\mathbb{Q}_\alpha)$, then $\pi_\alpha(q)$ is in $\mathbb{Q}_\alpha$. We observed above that $\pi_\alpha(q)$ is in $V[G * H]$, and $\pi_\alpha(q)$ is a partial function $\pi_\alpha(q) : \alpha \to \mathbb{R}$ with a countable domain. So if $\pi_\alpha(q)$ is not in $\mathbb{Q}_\alpha$, then by Lemma 9.3, we can fix β, u, and x such that β is in $\mathrm{dom}(\pi_\alpha(q))$, $\pi_\alpha(q) \restriction \beta$ is in $\mathbb{Q}_\beta$, $u \leq \pi_\alpha(q) \restriction \beta$ in $\mathbb{Q}_\beta$, x is a countable subset of $a^{\pi_\alpha(q)}_\beta$ which is closed under $\pi_\alpha(q)(\beta)$, and $u \Vdash_{\mathbb{Q}_\beta} x \in \dot{T}(\beta)$.

The set x is a countable subset of κ, so $j(x) = x$. By Lemma 11.4, $\pi_\beta(q \restriction j(\beta)) = \pi_\alpha(q) \restriction \beta$. So $u \leq \pi_\beta(q \restriction j(\beta))$ in $\mathbb{Q}_\beta$. Applying Lemma 11.5 to u and $q \restriction j(\beta)$, we get that $j(u)$ and $q \restriction j(\beta)$ are compatible in $j(\mathbb{Q}_\beta)$. Fix $s \leq j(u), q \restriction j(\beta)$ in $j(\mathbb{Q}_\beta)$.

Recall that $u \Vdash_{\mathbb{Q}_\beta} x \in \dot{T}(\beta)$. Since $j(x) = x$ and $s \leq j(u)$, by the elementarity of j, $s \Vdash_{j(\mathbb{Q}_\beta)} x \in j(\dot{T}(\beta))$. The set x is a subset of $a^{\pi_\alpha(q)}_\beta = a^q_{j(\beta)} \cap \kappa$, and x is closed under $\pi_\alpha(q)(\beta) = q(j(\beta)) \restriction \kappa^2$. Therefore x is a countable subset of $a^q_{j(\beta)}$ which is closed under $q(j(\beta))$. By Lemma 9.2, $q \restriction j(\beta) \Vdash_{j(\mathbb{Q}_\beta)} x \notin j(\dot{T}(\beta))$. But $s \leq q \restriction j(\beta)$, and $s \Vdash_{j(\mathbb{Q}_\beta)} x \in j(\dot{T}(\beta))$. This is a contradiction. So indeed $\pi_\alpha(q)$ is in $\mathbb{Q}_\alpha$.

Now we prove that $\pi_\alpha : j(\mathbb{Q}_\alpha) \to \mathbb{Q}_\alpha$ is a projection mapping. Clearly π_α sends the empty condition in $j(\mathbb{Q}_\alpha)$ to the empty condition in $\mathbb{Q}_\alpha$. Suppose $r \leq q$ in $j(\mathbb{Q}_\alpha)$, and we show $\pi_\alpha(r) \leq \pi_\alpha(q)$ in $\mathbb{Q}_\alpha$. Since $\mathrm{dom}(q) \subseteq \mathrm{dom}(r)$, $\mathrm{dom}(\pi_\alpha(q)) = j^{-1}(\mathrm{dom}(q) \cap j[\alpha]) \subseteq j^{-1}(\mathrm{dom}(r) \cap j[\alpha]) = \mathrm{dom}(\pi_\alpha(r))$. Let γ be in $\mathrm{dom}(\pi_\alpha(q))$. Then $j(\gamma)$ is in $\mathrm{dom}(q)$, and since $r \leq q$, $r(j(\gamma))$ extends $q(j(\gamma))$ as a function. It follows that $r(j(\gamma)) \restriction \kappa^2$ extends $q(j(\gamma)) \restriction \kappa^2$ as a function, that is, $\pi_\alpha(r)(\gamma) \leq_{\mathbb{R}} \pi_\alpha(q)(\gamma)$. Finally, let q be in $j(\mathbb{Q}_\alpha)$, and suppose $v \leq \pi_\alpha(q)$ in $\mathbb{Q}_\alpha$. We show there is $u \leq q$ in $j(\mathbb{Q}_\alpha)$ such that $\pi_\alpha(u) \leq v$ in $\mathbb{Q}_\alpha$. By Lemma 11.5, $j(v)$ and q are compatible in $j(\mathbb{Q}_\alpha)$. So fix $u \leq j(v), q$. Then by Lemma 11.6, $\pi_\alpha(u) \leq v$ in $\mathbb{Q}_\alpha$.

We have established that $\pi_\alpha : j(\mathbb{Q}_\alpha) \to \mathbb{Q}_\alpha$ is a projection mapping. If $t \leq \pi_\alpha(q)$ in $\mathbb{Q}_\alpha$, then $j(t)$ and q are compatible in $j(\mathbb{Q}_\alpha)$ by Lemma 11.5. $\square$

Let $\pi : j(\mathbb{Q}) \to \mathbb{Q}$ be the map π_{κ^+}.

We now turn to showing that we can extend the elementary embedding j after forcing with $\mathbb{Q}$. First we need several easy technical lemmas.

Lemma 11.8. *Let $\alpha \leq \kappa^+$. If q is in $j(\mathbb{Q}_\alpha)$ and $\pi_\alpha(q) \leq p$ in $\mathbb{Q}_\alpha$, then $q \leq j(p)$ in $j(\mathbb{Q}_\alpha)$.*

Proof. The domain of $j(p)$ is equal to $j(\mathrm{dom}(p)) = j[\mathrm{dom}(p)]$, and for all $j(\gamma)$ in $\mathrm{dom}(j(p))$, $j(p)(j(\gamma)) = j(p(\gamma)) = p(\gamma)$, since $p(\gamma)$ is a countable subset of $\kappa^2 \times \omega_1$.

First we show $\mathrm{dom}(j(p)) \subseteq \mathrm{dom}(q)$. So let β be in $\mathrm{dom}(j(p))$. Then $\beta = j(\gamma)$ for some γ in $\mathrm{dom}(p)$. Since $\pi_\alpha(q) \leq p$, γ is in $\mathrm{dom}(\pi_\alpha(q)) = j^{-1}(\mathrm{dom}(q) \cap j[\alpha])$. So $j(\gamma) = \beta$ is in $\mathrm{dom}(q)$.

Now let $\beta = j(\gamma)$ be in $\mathrm{dom}(j(p))$. Then $q(\beta) \restriction \kappa^2 = \pi_\alpha(q)(\gamma) \leq_{\mathbb{R}} p(\gamma) = j(p)(\beta)$, that is, $q(\beta) \restriction \kappa^2$ extends $j(p)(\beta)$ as a function. Therefore $q(\beta)$ extends $j(p)(\beta)$ as a function, that is, $q(\beta) \leq_{j(\mathbb{R})} j(p)(\beta)$. $\square$

Lemma 11.9. *Let $\alpha \leq \kappa^+$. If p is in $\mathbb{Q}_\alpha$, then $\pi_\alpha(j(p)) = p$.*

Proof. The domain of $\pi_\alpha(j(p))$ is equal to $j^{-1}(\mathrm{dom}(j(p)) \cap j[\alpha])$. But $\mathrm{dom}(j(p)) = j(\mathrm{dom}(p)) = j[\mathrm{dom}(p)]$, which is a subset of $j[\alpha]$. So the domain of $\pi_\alpha(j(p))$ is equal to $j^{-1}(j[\mathrm{dom}(p)]) = \mathrm{dom}(p)$. For each γ in the domain of p, $\pi_\alpha(j(p))(\gamma) = j(p)(j(\gamma)) \restriction \kappa^2 = j(p(\gamma)) \restriction \kappa^2$. But $p(\gamma)$ is a countable subset of $\kappa^2 \times \omega_1$, so $j(p(\gamma)) = p(\gamma)$. Since the domain of $p(\gamma)$ is a subset of κ^2, $\pi_\alpha(j(p))(\gamma) = j(p(\gamma)) \restriction \kappa^2 = p(\gamma) \restriction \kappa^2 = p(\gamma)$. So $\pi_\alpha(j(p)) = p$. $\square$

Since $\pi : j(\mathbb{Q}) \to \mathbb{Q}$ is a projection mapping, if $\overline{I}$ is a generic filter on $j(\mathbb{Q})$ over $V[\overline{G} * \overline{H}]$, then $\pi[\overline{I}]$ generates a generic filter I on $\mathbb{Q}$ over $V[\overline{G} * \overline{H}]$.

Lemma 11.10. $\pi[\overline{I}] = I$.

Proof. It suffices to show $\pi[\overline{I}]$ is upwards closed. So let q be in $\overline{I}$ and suppose $\pi(q) \leq p$ in $\mathbb{Q}$. By Lemma 11.8, $q \leq j(p)$ in $j(\mathbb{Q})$. Therefore $j(p)$ is in $\overline{I}$. By Lemma 11.9, $\pi(j(p)) = p$, so p is in $\pi[\overline{I}]$. $\square$

The next theorem is an extension of Theorem 6.1.

Theorem 11.11. *Let $\overline{I}$ be a generic filter on $j(\mathbb{Q})$ over $V[\overline{G} * \overline{H}]$. Let $I = \pi[\overline{I}]$. Then j can be lifted to $j : V[G*H*I] \to M[\overline{G}*\overline{H}*\overline{I}]$ such that $j(G*H*I) = \overline{G}*\overline{H}*\overline{I}$.*

Proof. It suffices to show $j[I] \subseteq \overline{I}$. So let p be in I, and we show $j(p)$ is in $\overline{I}$. Since p is in $I = \pi[\overline{I}]$, fix q in $\overline{I}$ such that $\pi(q) = p$. In particular, $\pi(q) \leq p$. By Lemma 11.8, $q \leq j(p)$ in $j(\mathbb{Q})$. As q is in $\overline{I}$, $j(p)$ is in $\overline{I}$. $\square$

12. Preserving Stationary Sets

As usual, fix a cardinal κ which is κ^+-supercompact, and assume $2^\kappa = \kappa^+$. Let $j : V \to M$ be an elementary embedding with critical point κ such that $j(\kappa) > \kappa^+$ and $M^{\kappa^+} \subseteq M$. Let $\overline{G} * \overline{H} * \overline{I}$ be a generic filter on $j(\mathrm{Coll}(\omega_1, <\kappa) * \dot{\mathbb{P}}_\kappa * \dot{\mathbb{Q}})$ over V. Let $G = \overline{G} \cap \mathrm{Coll}(\omega_1, <\kappa)$, $H = \overline{H} \cap \mathbb{P}_\kappa$, and $I = \pi[\overline{I}]$. Extend j to $j : V[G * H * I] \to M[\overline{G} * \overline{H} * \overline{I}]$.

Our goal in this section is to prove the following theorem. After we complete the proof of this result, we will have all the tools necessary to prove the main theorem of the paper.

Theorem 12.1. *In $V[G * H * I]$, let T be a stationary subset of $P_{\omega_1}(\kappa)$ which is disjoint from S_κ. Then T is a stationary subset of $P_{\omega_1}(\kappa)$ in the model $M[\overline{G} * \overline{H} * \overline{I}]$.*

By Theorem 6.1, $M[\overline{G} * \overline{H}]$ is equal to $M[\overline{G}][H][H']$, where H' is a generic filter on $\mathbb{P}(\kappa, S_\kappa, j(\kappa))$ over $M[\overline{G}][H]$. Let $G' = G \cap \textsc{Coll}(\omega_1, [\kappa, j(\kappa)))$. Then $M[\overline{G}] = M[G][G']$. So $M[\overline{G} * \overline{H}] = M[G][G'][H][H']$.

Lemma 12.2. $M[\overline{G} * \overline{H} * \overline{I}] = M[G * H * I][G'][H'][\overline{I}]$.

Proof. We already know

$$M[\overline{G} * \overline{H} * \overline{I}] = M[G][G'][H][H'][\overline{I}].$$

Since $\pi : j(\mathbb{Q}) \to \mathbb{Q}$ is a projection mapping and $I = \pi[\overline{I}]$, we have

$$M[G][G'][H][H'][\overline{I}] = M[G][G'][H][H'][I][\overline{I}].$$

So

$$M[\overline{G} * \overline{H} * \overline{I}] = M[G][G'][H][H'][I][\overline{I}].$$

Now we repeatedly apply the Product Lemma. Since $\mathbb{P}_\kappa$ is in $M[G]$,

$$M[G][G'][H] = M[G][H][G'].$$

So

$$M[\overline{G} * \overline{H} * \overline{I}] = M[G][H][G'][H'][I][\overline{I}].$$

Since $\mathbb{Q}$ is in $M[G][H]$,

$$M[G][H][G'][H'][I] = M[G][H][I][G'][H'].$$

Therefore

$$M[\overline{G} * \overline{H} * \overline{I}] = M[G * H * I][G'][H'][\overline{I}].$$

□

Now we begin the (long) proof of Theorem 12.1.

Proof. Let T be a stationary subset of $P_{\omega_1}(\kappa)$ in $V[G*H*I]$ which is disjoint from S_κ. Since $\textsc{Coll}(\omega_1, {<}\kappa) * \dot{\mathbb{P}}_\kappa * \dot{\mathbb{Q}}$ is κ-c.c., $M[G*H*I]^{\kappa^+} \cap V[G*H*I] \subseteq M[G*H*I]$. So T is in $M[G * H * I]$.

By the preceding lemma, $M[\overline{G} * \overline{H} * \overline{I}]$ is equal to $M[G * H * I][G'][H'][\overline{I}]$. Since $\textsc{Coll}(\omega_1, [\kappa, j(\kappa)))$ is ω_1-closed, and hence proper, T remains a stationary subset of $P_{\omega_1}(\kappa)$ in the model $M[G * H * I][G']$.

By Corollary 4.8, letting $\alpha = \kappa$, $A = S_\kappa$, and $\beta = j(\kappa)$, we get that $\mathbb{P}(\kappa, S_\kappa, j(\kappa))$ forces over $M[G * H * I][G']$ that T is stationary in $P_{\omega_1}(\kappa)$. Thus T is stationary in the model $M[G * H * I][G'][H']$. So to show that T is stationary in $M[G * H * I][G'][H'][\overline{I}]$, it suffices to show that in the model $M[G * H * I][G'][H']$, the forcing poset $j(\mathbb{Q})/\mathbb{Q} = \{q \in j(\mathbb{Q}) : \pi(q) \in I\}$ forces that T is stationary in $P_{\omega_1}(\kappa)$.

Goal: To prove that in the model $M[G * H * I][G'][H'] = M[\overline{G} * \overline{H}][I]$, the forcing poset $j(\mathbb{Q})/\mathbb{Q}$ forces that T is a stationary subset of $P_{\omega_1}(\kappa)$.

Before beginning the proof, let us recall some of the notation used in the definition of $\mathbb{Q}$, and fix some additional parameters.

The forcing poset $\mathbb{Q}$ is defined in the model $V[G*H]$, and in this model, $\kappa=\omega_2$ and $\kappa^+=\omega_3$. (As before, we will always write κ^+ to denote the ordinal which is the successor of κ in the ground model V.) In $M[\overline{G}*\overline{H}]$, both κ and κ^+ are ordinals with cofinality ω_1 strictly between ω_1 and $j(\kappa)=\omega_2$.

Recall Notation 8.4, where we fixed, in $V[G*H]$, a surjective function $g_\alpha:\omega_1\to\alpha$ for each α in $\omega_2=\kappa$. We let E be the set of b in $P_{\omega_1}(\kappa)$ such that $\mathrm{ot}(b)$ is a limit ordinal and for all α in b, b is closed under g_α. Note that the sequence $\langle g_\alpha:\alpha\in\kappa\rangle$ is in $M[G*H]$.

The definition of $\mathbb{Q}$ involved the sequence of nice names $\langle\dot{T}(\alpha):\alpha<\kappa^+\rangle$. Recall that for all $\alpha<\kappa^+$, $\mathbb{Q}_\alpha$ forces that $\dot{T}(\alpha)$ is a subset of $S_\kappa\cap E$ which does not reflect to any ordinal in κ with cofinality ω. By elementarity, $j(\mathbb{Q})$ is defined in $M[\overline{G}*\overline{H}]$ by the sequence of nice names $j(\langle\dot{T}(\alpha):\alpha<\kappa^+\rangle)$, which we denote by $\langle\dot{T}^j(\beta):\beta<j(\kappa^+)\rangle$.

By upwards absoluteness, in the model $M[\overline{G}*\overline{H}][I]$, S_κ is a subset of $P_{\omega_1}(\kappa)$ which does not reflect to any ordinal in κ with cofinality ω_1. Applying Lemma 4.10, fix a club set $D\subseteq P_{\omega_1}(\kappa)$ as follows.

Notation 12.3. *Let D be a club subset of $P_{\omega_1}(\kappa)$ in $M[\overline{G}*\overline{H}][I]$ such that for all b in D, for any ν in $b\cap\mathrm{cof}(\omega_1)$, $b\cap\nu$ is not in S_κ.*

For the remainder of the proof, we work in the model $M[\overline{G}*\overline{H}][I]$, unless specified otherwise. We will prove that in this model, the forcing poset $j(\mathbb{Q})/\mathbb{Q}$ forces that T is a stationary subset of $P_{\omega_1}(\kappa)$. To prove this, it suffices to show that whenever p is a condition in $j(\mathbb{Q})/\mathbb{Q}$ and $p\Vdash\dot{C}\subseteq P_{\omega_1}(\kappa)$ is club, then there is $q\le p$ in $j(\mathbb{Q})/\mathbb{Q}$ and a set z in T such that $q\Vdash z\in\dot{C}$. So let p and $\dot{C}$ be given, and assume $p\Vdash\dot{C}\subseteq P_{\omega_1}(\kappa)$ is club.

Fix a regular cardinal θ much larger than $j(\mathbb{Q})/\mathbb{Q}$ and $\dot{C}$. We use Lemma 4.7 to choose an appropriate elementary substructure of $H(\theta)$. The model we are working in, namely, $M[\overline{G}*\overline{H}][I]$, is equal to $M[G*H*I][G'][H']$, which is a generic extension of $M[G*H*I][G']$ by the forcing poset $\mathbb{P}(\kappa,S_\kappa,j(\kappa))$. In $M[G*H*I][G']$, θ is a regular cardinal larger than $\omega_2=j(\kappa)$, κ has cofinality ω_1, and T is a stationary subset of $P_{\omega_1}(\kappa)$ which is disjoint from S_κ. By Lemma 4.7, in $M[G*H*I][G'][H']$ there are stationarily many N in $P_{\omega_1}(H(\theta))$ such that $S(\kappa,S_\kappa,j(\kappa))\cap P(N\cap j(\kappa))\subseteq N$ and $N\cap\kappa\in T$.

Notation 12.4. *Let N be a countable subset of $H(\theta)$ in $M[\overline{G}*\overline{H}][I]$ satisfying:*
- *$N\prec H(\theta)$,*
- *$N\cap\kappa\in T\cap D$,*
- *$S(\kappa,S_\kappa,j(\kappa))\cap P(N\cap j(\kappa))\subseteq N$,*
- *N contains as members the objects $j(\mathbb{Q})$, $\langle\dot{T}^j(\beta):\beta<j(\kappa^+)\rangle$, I, $j(\mathbb{Q})/\mathbb{Q}$, p, $\dot{C}$, $j\restriction\kappa^+$, and $\langle g_\alpha:\alpha\in\kappa\rangle$.*

Since $N\cap\kappa$ is in T, it suffices to find $q\le p$ in $j(\mathbb{Q})/\mathbb{Q}$ such that $q\Vdash N\cap\kappa\in\dot{C}$. If $q\le p$ is N-generic, then since $\dot{C}$ is in N, $q\Vdash N[\dot{K}]\cap\kappa=N\cap\kappa\in\dot{C}$, where $\dot{K}$ is the canonical $j(\mathbb{Q})/\mathbb{Q}$-name for the generic filter. So it suffices to define an N-generic sequence $\langle p_n:n<\omega\rangle$ for $j(\mathbb{Q})/\mathbb{Q}$, with $p_0=p$, and find a lower bound q of this sequence.

We will choose the N-generic sequence carefully. Enumerate all dense open subsets of $j(\mathbb{Q})/\mathbb{Q}$ in N as $\langle D_n:n<\omega\rangle$. By the ω_1-distributivity of the forcing posets, the set $N \cap \kappa$ is in $M[G * H]$. So by Lemma 4.6, $S_\kappa \cap P(N \cap \kappa)$ is countable in $M[G * H]$, and hence is countable in $M[\overline{G} * \overline{H}][I]$. Let $\langle x_n : n < \omega \rangle$ be an enumeration of $S_\kappa \cap P(N \cap \kappa)$ in which each set appears infinitely often.

Now we define the N-generic sequence $\langle p_n : n < \omega \rangle$. Let $p_0 = p$. Suppose $n < \omega$ and p_n is defined. Choose a condition $p_n^* \leq p_n$ which is in $D_n \cap N$. Since the set of square conditions in $j(\mathbb{Q})$ is dense in $j(\mathbb{Q})$, the set of square conditions in $j(\mathbb{Q})/\mathbb{Q}$ is dense in $j(\mathbb{Q})/\mathbb{Q}$ by Lemma 1.1. Choose a square condition $p_n' \leq p_n^*$ in $(j(\mathbb{Q})/\mathbb{Q}) \cap N$. We split the definition of p_{n+1} into three cases.

Case 1: $x_n \subseteq a^{p_n'}$.

In this case, let $p_{n+1} = p_n'$.

Case 2: Case 1 fails and $N \cap \omega_1 \subseteq x_n$.

In this case, again let $p_{n+1} = p_n'$.

Case 3: Case 1 and Case 2 fail.

By the failure of Case 1, x_n is not a subset of $a^{p_n'}$. Fix an ordinal ν_n which is in $x_n \setminus a^{p_n'}$. By the failure of Case 2, $N \cap \omega_1$ is not a subset of x_n. Fix an ordinal ζ_n in $(N \cap \omega_1) \setminus x_n$. Since ν_n is in x_n and ζ_n is not in x_n, ν_n is not equal to ζ_n.

To define p_{n+1}, we apply Lemma 9.11. The condition p_n' is a square condition in $j(\mathbb{Q})$. The ordinal ν_n is in $j(\kappa) \setminus a^{p_n'}$ and ζ_n is an ordinal in ω_1 different from ν_n. Let $b = \mathrm{dom}(p_n') \cap j[\kappa^+]$. By Lemma 9.11, there is a condition $p_{n+1} \leq p_n'$ in $j(\mathbb{Q})$ such that $\mathrm{dom}(p_{n+1}) = \mathrm{dom}(p_n')$, for all β in b, $p_{n+1}(\beta) = p_n'(\beta)$, and for all β in $\mathrm{dom}(p_{n+1}) \setminus b$, ν_n is in $a_\beta^{p_{n+1}}$ and $p_{n+1}(\beta)(\nu_n, \nu_n) = \zeta_n$. Since $j(\mathbb{Q})$, p_n', b, ν_n, and ζ_n are all in N, by elementarity we can choose p_{n+1} in N.

We claim that p_{n+1} is in $j(\mathbb{Q})/\mathbb{Q}$, that is, $\pi(p_{n+1}) \in I$. Now $\mathrm{dom}(p_{n+1}) = \mathrm{dom}(p_n')$ and $p_{n+1}(\beta) = p_n'(\beta)$ for all β in $\mathrm{dom}(p_n') \cap j[\kappa^+]$. It follows that $\pi(p_{n+1}) = \pi(p_n')$. For $\mathrm{dom}(\pi(p_{n+1})) = j^{-1}(\mathrm{dom}(p_{n+1}) \cap j[\kappa^+]) = j^{-1}(\mathrm{dom}(p_n') \cap j[\kappa^+]) = \mathrm{dom}(\pi(p_n'))$, and for all γ in $\mathrm{dom}(\pi(p_{n+1}))$, $\pi(p_{n+1})(\gamma) = p_{n+1}(j(\gamma)) \restriction \kappa^+ = p_n'(j(\gamma)) \restriction \kappa^2 = \pi(p_n')(\gamma)$. Since $\pi(p_n')$ is in I, $\pi(p_{n+1})$ is in I. Therefore p_{n+1} is in $j(\mathbb{Q})/\mathbb{Q}$.

This completes the definition of the N-generic sequence $\langle p_n : n < \omega \rangle$.

We define q, a lower bound of the sequence $\langle p_n : n < \omega \rangle$, as follows. Let the domain of q be equal to $\bigcup\{\mathrm{dom}(p_n) : n < \omega\}$. If β is in the domain of q, define $q(\beta)$ as the union of the set of functions $\{p_n(\beta) : n < \omega\}$. Clearly q is a partial function $q : j(\kappa^+) \to j(\mathbb{R})$ with a countable domain. Furthermore, by the N-genericity of the sequence $\langle p_n : n < \omega \rangle$, $\mathrm{dom}(q) = N \cap j(\kappa^+)$, and for each β in $\mathrm{dom}(q)$, $a_\beta^q = N \cap j(\kappa)$.

We will prove that q is in $j(\mathbb{Q})/\mathbb{Q}$. It then follows easily that $q \leq p_n$ for all $n < \omega$, which completes the proof. The proof that q is in $j(\mathbb{Q})/\mathbb{Q}$ splits into two parts: (1) proving that q is in $j(\mathbb{Q})$, and (2) proving that $\pi(q)$ is in I. The second part can be handled quickly, so we take care of it first.

Claim 12.5. *Suppose q is in $j(\mathbb{Q})$. Then $\pi(q) \in I$, so q is in $j(\mathbb{Q})/\mathbb{Q}$.*

Proof. Assume q is in $j(\mathbb{Q})$. Then $\pi(q)$ is in $\mathbb{Q}$. As a subclaim, we prove

$$\mathrm{dom}(\pi(q)) = \bigcup\{\mathrm{dom}(\pi(p_n)) : n < \omega\},$$

and for all γ in $\mathrm{dom}(\pi(q))$,

$$\pi(q)(\gamma) = \bigcup\{\pi(p_n)(\gamma) : n < \omega\}.$$

We have that $\mathrm{dom}(\pi(q)) = j^{-1}(\mathrm{dom}(q) \cap j[\kappa^+]) = j^{-1}((\bigcup\{\mathrm{dom}(p_n) : n < \omega\}) \cap j[\kappa^+]) = j^{-1}(\bigcup\{\mathrm{dom}(p_n) \cap j[\kappa^+] : n < \omega\}) = \bigcup\{j^{-1}(\mathrm{dom}(p_n) \cap j[\kappa^+]) : n < \omega\} = \bigcup\{\mathrm{dom}(\pi(p_n)) : n < \omega\}$. Now let γ be in $\mathrm{dom}(\pi(q))$. Then $\pi(q)(\gamma) = q(j(\gamma)) \restriction \kappa^2 = (\bigcup\{p_n(j(\gamma)) : n < \omega\}) \restriction \kappa^2 = \bigcup\{p_n(j(\gamma)) \restriction \kappa^2 : n < \omega\} = \bigcup\{\pi(p_n)(\gamma) : n < \omega\}$.

For each $n < \omega$, p_n is in $j(\mathbb{Q})/\mathbb{Q}$, and so $\pi(p_n)$ is in I. Hence $\langle \pi(p_n) : n < \omega\rangle$ is a descending sequence of conditions in I. By Lemma 9.10, there is r in I such that $r \le \pi(p_n)$ for all $n < \omega$. It follows from the subclaim that $\mathrm{dom}(\pi(q)) \subseteq \mathrm{dom}(r)$, and for all γ in $\mathrm{dom}(\pi(q))$, $r(\gamma) \restriction a_\gamma^{\pi(q)} = \pi(q)(\gamma)$. Thus $r \le \pi(q)$ in $\mathbb{Q}$. Since r is in I and I is a filter, $\pi(q)$ is in I. □

It remains to show that q is in $j(\mathbb{Q})$. We defined the condition q in the model $M[\overline{G} * \overline{H}][I]$, whereas the forcing poset $j(\mathbb{Q})$ is defined in $M[\overline{G} * \overline{H}]$. But $\mathbb{Q}$ is ω_1-distributive in $M[\overline{G} * \overline{H}]$. This is true since $j(\mathbb{Q})$ is ω_1-distributive in $M[\overline{G} * \overline{H}]$, and $\pi : j(\mathbb{Q}) \to \mathbb{Q}$ is a projection mapping. It follows that q is in $M[\overline{G} * \overline{H}]$. We will prove in the model $M[\overline{G} * \overline{H}]$ that q is in $j(\mathbb{Q})$.

Suppose for a contradiction q is not in $j(\mathbb{Q})$. Applying Lemma 9.3 in the model $M[\overline{G} * \overline{H}]$, we can fix β, t, and x such that:

- β is in $\mathrm{dom}(q)$,
- $q \restriction \beta$ is in $j(\mathbb{Q})_\beta$,
- $t \le q \restriction \beta$ in $j(\mathbb{Q})_\beta$,
- x is a countable subset of a_β^q closed under $q(\beta)$,
- $t \Vdash_{j(\mathbb{Q})_\beta} x \in \dot{T}^j(\beta)$.

Recall that $\Vdash_{j(\mathbb{Q})_\beta} \dot{T}^j(\beta) \subseteq j(S_\kappa) \cap j(E)$. Therefore, since $t \Vdash_{j(\mathbb{Q})_\beta} x \in \dot{T}^j(\beta)$, x is in $j(S_\kappa) \cap j(E)$. By Theorem 6.1, $j(S_\kappa) = S_{j(\kappa)} = S_\kappa \cup S(\kappa, S_\kappa, j(\kappa))$. So there are two possibilities concerning x: either $x \subseteq \kappa$ and $x \in S_\kappa$, or $x \nsubseteq \kappa$ and $x \in S(\kappa, S_\kappa, j(\kappa))$.

Note that β is in N, since $\mathrm{dom}(q) = N \cap j(\kappa^+)$. Since $q \restriction \beta$ is in $j(\mathbb{Q})_\beta$, it is easy to see that $q \restriction \beta \le p_n \restriction \beta$ for all $n < \omega$. In particular, $q \restriction \beta$ is N-generic for $j(\mathbb{Q})_\beta$.

Claim 12.6. *For all $n < \omega$, if $\beta \in \mathrm{dom}(p_n)$, then x is not a subset of $a_\beta^{p_n}$.*

Proof. Suppose x is a subset of $a_\beta^{p_n}$. Since $q(\beta)$ extends $p_n(\beta)$ as a function, x is closed under $p_n(\beta)$. By Lemma 9.2, $p_n \restriction \beta \Vdash_{j(\mathbb{Q})_\beta} x \notin \dot{T}^j(\beta)$. But $t \le q \restriction \beta \le p_n \restriction \beta$, so $t \Vdash_{j(\mathbb{Q})_\beta} x \notin \dot{T}^j(\beta)$, contradicting the choice of t. □

Claim 12.7. *The set x is not in N.*

Proof. Let D be the dense open set of conditions s in $j(\mathbb{Q})/\mathbb{Q}$ such that β is in $\mathrm{dom}(s)$ and $x \subseteq a_\beta^s$. If x is in N, then by elementarity, D is in N. By N-genericity, fix $n < \omega$ such that p_n is in D. Then $x \subseteq a_\beta^{p_n}$, contradicting the last claim. □

Claim 12.8. *The set x is in S_κ.*

Proof. If x is not in S_κ, then x is in $S(\kappa, S_\kappa, j(\kappa))$. Since $q(\beta) = \bigcup\{p_n(\beta) : n < \omega\}$, $a^q_\beta = \bigcup\{a^{p_n}_\beta : n < \omega\}$, which by N-genericity is equal to $N \cap j(\kappa)$. Since $x \subseteq a^q_\beta$, x is a subset of $N \cap j(\kappa)$. Therefore x is in $S(\kappa, S_\kappa, j(\kappa)) \cap P(N \cap j(\kappa))$. By the choice of N, as described in Notation 12.4, $S(\kappa, S_\kappa, j(\kappa)) \cap P(N \cap j(\kappa)) \subseteq N$. Thus x is in N. But this contradicts Claim 12.7. □

So x is a subset of κ. Also x is a subset of a^q_β, which is equal to $N \cap j(\kappa)$. So x is a subset of $N \cap \kappa$.

Let $j(\langle g_\alpha : \alpha < \kappa\rangle) = \langle g^j_\delta : \delta < j(\kappa)\rangle$. For all $\alpha < \kappa$, $g^j_\alpha = g^j_{j(\alpha)} = j(g_\alpha)$, which is equal to g_α, since g_α is a subset of $\omega_1 \times \alpha$. Since x is in $j(E)$, it follows that for all α in x, x is closed under g_α.

Claim 12.9. *The ordinal β is not in $j[\kappa^+]$.*

Proof. Suppose for a contradiction β is in $j[\kappa^+]$. Let $\gamma = j^{-1}(\beta)$, so $j(\gamma) = \beta$. Then γ is in $j^{-1}(\mathrm{dom}(q) \cap j[\kappa^+]) = \mathrm{dom}(\pi(q))$.

By Claim 12.8, $x \subseteq \kappa$. Since $q(\beta) \restriction \kappa^2 = \pi(q)(\gamma)$, x is a countable subset of $a^{\pi(q)}_\gamma$ which is closed under $\pi(q)(\gamma)$. Since $M[\overline{G} * \overline{H}]$ is a generic extension of M by an ω_1-distributive forcing poset, x is in M, and hence is in $V[G * H]$. Applying Lemma 9.2 in $V[G * H]$, $\pi(q) \restriction \gamma \Vdash_{\mathbb{Q}_\gamma} x \notin \dot{T}(\gamma)$.

By Lemma 11.4, $\pi_\gamma(q \restriction \beta) = \pi(q) \restriction \gamma$. So $\pi_\gamma(q \restriction \beta) \Vdash_{\mathbb{Q}_\gamma} x \notin \dot{T}(\gamma)$. Now $t \le q \restriction \beta$, so $\pi_\gamma(t) \le \pi_\gamma(q \restriction \beta)$. Hence $\pi_\gamma(t) \Vdash_{\mathbb{Q}_\gamma} x \notin \dot{T}(\gamma)$. Since $j(x) = x$, $j(\mathbb{Q}_\gamma) = j(\mathbb{Q})_\beta$, and $j(\dot{T}(\gamma)) = \dot{T}^j(\beta)$, by the elementarity of j we have $j(\pi_\gamma(t)) \Vdash_{j(\mathbb{Q})_\beta} x \notin \dot{T}^j(\beta)$. On the other hand, $t \Vdash_{j(\mathbb{Q})_\beta} x \in \dot{T}^j(\beta)$. But by Proposition 11.7, $j(\pi_\gamma(t))$ and t are compatible in $j(\mathbb{Q})_\beta$, which is a contradiction. □

We split the remainder of the proof into two cases. The first case is that x is an initial segment of $N \cap \kappa$, and the second case is that x is not an initial segment of $N \cap \kappa$. We will derive a contradiction in both cases. This shows that q is indeed a condition in $j(\mathbb{Q})$, which completes the proof.

Case a: x is an initial segment of $N \cap \kappa$.

Note that x cannot be equal to $N \cap \kappa$ itself. For by the choice of N, as described in Notation 12.4, $N \cap \kappa \in T$. But x is in S_κ, and T is disjoint from S_κ.

Let ν be the least ordinal in $N \cap \kappa$ which is not in x. Then $x = N \cap \nu$. Recall that since $t \Vdash_{j(\mathbb{Q})_\beta} x \in \dot{T}^j(\beta)$, x is in $j(S_\kappa) \cap j(E)$. By the definition of E, this implies $\mathrm{ot}(x)$ is a limit ordinal. It follows that ν is a limit ordinal.

Recall by Notation 12.3, D is a club subset of $P_{\omega_1}(\kappa)$ in $M[\overline{G} * \overline{H}][I]$ such that for all b in D, if γ is in $b \cap \mathrm{cof}(\omega_1)$, then $b \cap \gamma$ is not in S_κ. By the choice of N, as described in Notation 12.4, $N \cap \kappa$ is in D. Since $N \cap \nu = x$ is in S_κ, ν is not in $(N \cap \kappa) \cap \mathrm{cof}(\omega_1)$. So $\mathrm{cf}(\nu) = \omega$.

In the model $M[\overline{G} * \overline{H}]$, the forcing poset $j(\mathbb{Q})_\beta$ forces that $\dot{T}^j(\beta)$ does not reflect to any ordinal in $j(\kappa)$ with cofinality ω. As ν has cofinality ω, we can fix a $j(\mathbb{Q})_\beta$-name $\dot{F}_\nu$ for a function $\dot{F}_\nu : [\nu]^{<\omega} \to \nu$ such that any b in $\dot{T}^j(\beta)$ which is a subset of ν is not closed under $\dot{F}_\nu$. Since $j(\mathbb{Q})$, β, $\dot{T}^j(\beta)$, and ν are in N, by elementarity we can choose such a name $\dot{F}_\nu$ in N.

As $\dot{F}_\nu$ is in N, $j(\mathbb{Q})_\beta$ forces that $N[\dot{K}] \cap \nu$ is closed under $\dot{F}_\nu$, where $\dot{K}$ is the canonical $j(\mathbb{Q})_\beta$-name for a generic filter. Recall that $q \restriction \beta \le p_n \restriction \beta$ for all $n < \omega$, and so $q \restriction \beta$ is N-generic for $j(\mathbb{Q})_\beta$. Since $t \le q \restriction \beta$, t is also N-generic. So t forces

$N[\dot{K}] \cap \nu = N \cap \nu = x$ is closed under $\dot{F}_\nu$. But $t \Vdash_{j(\mathbb{Q})_\beta} x \in \dot{T}^j(\beta)$, contradicting the choice of $\dot{F}_\nu$.

Case b: x is not an initial segment of $N \cap \kappa$.

Fix ordinals $\gamma < \alpha < \kappa$ such that γ is in $N \setminus x$ and α is in x. Recall that g_α is a surjection of ω_1 onto α. Since α is in N, g_α is in N. As γ is in $N \cap \alpha$, by elementarity fix ξ in $N \cap \omega_1$ such that $g_\alpha(\xi) = \gamma$. Since α is in x, x is closed under g_α. Therefore ξ is not in x, since otherwise $g_\alpha(\xi) = \gamma$ would be in x.

Recall that $\langle x_n : n < \omega \rangle$ is an enumeration of $S_\kappa \cap P(N \cap \kappa)$ in which each set appears infinitely often. But x is in $S_\kappa \cap P(N \cap \kappa)$. Since β is in $\bigcup\{\mathrm{dom}(p_n) : n < \omega\}$, choose n large enough so that β is in $\mathrm{dom}(p_n)$ and $x_n = x$.

Now recall the three cases in the definition of p_{n+1}. The first case is that $x_n \subseteq a_\beta^{p'_n}$. This is false by Claim 12.6. The second case is that $N \cap \omega_1 \subseteq x_n$. This is false, since ξ is in $N \cap \omega_1$ but not in x. So we are in the third case in the definition of p_{n+1}. In that case we fixed an ordinal ν_n which is in $x_n \setminus a^{p'_n}$, and an ordinal ζ_n which is in $(N \cap \omega_1) \setminus x_n$. Then we chose p_{n+1} so that for all δ in $\mathrm{dom}(p_{n+1}) \setminus j[\kappa^+]$, ν_n is in $a_\delta^{p_{n+1}}$ and $p_{n+1}(\delta)(\nu_n, \nu_n) = \zeta_n$. By Claim 12.9, β is in $\mathrm{dom}(p_{n+1}) \setminus j[\kappa^+]$. Therefore ν_n is in $a_\beta^{p_{n+1}}$ and $p_{n+1}(\beta)(\nu_n, \nu_n) = \zeta_n$. It follows that $q(\beta)(\nu_n, \nu_n) = \zeta_n$. Since ν_n is in x_n and ζ_n is not in x_n, x_n is not closed under $q(\beta)$. But $x = x_n$ and x was assumed to be closed under $q(\beta)$. Thus we have reached our final contradiction. □

13. The Main Theorem

We now have all the tools necessary to complete the proof of the main theorem of the paper.

Theorem 13.1. *Suppose κ is a κ^+-supercompact cardinal and $2^\kappa = \kappa^+$. Then there is a forcing poset $\mathbb{P}$, namely,*

$$\mathrm{Coll}(\omega_1, {<}\kappa) * \dot{\mathbb{P}}_\kappa * \dot{\mathbb{Q}},$$

such that

$$\Vdash_{\mathbb{P}} WRP(\omega_2) \wedge \neg RP(\omega_2).$$

Proof. Let $\mathbb{P}$ denote $\mathrm{Coll}(\omega_1, {<}\kappa) * \dot{\mathbb{P}}_\kappa * \dot{\mathbb{Q}}$. Then $\mathbb{P}$ is ω_1-distributive and ω_2-c.c. Let $j : V \to M$ be an elementary embedding with critical point κ such that $j(\kappa) > \kappa^+$ and $M^{\kappa^+} \subseteq M$.

Let φ be the statement $\mathrm{WRP}(\omega_2) \wedge \neg\mathrm{RP}(\omega_2)$. To show $\Vdash_{\mathbb{P}} \varphi$, it suffices to show whenever $\overline{G} * \overline{H} * \overline{I}$ is a generic filter on $j(\mathbb{P})$ over V, then letting $G = \overline{G} \cap \mathrm{Coll}(\omega_1, {<}\kappa)$, $H = \overline{H} \cap \mathbb{P}_\kappa$, and $I = \pi[\overline{I}]$, $V[G * H * I] \models \varphi$. For if $\Vdash_{\mathbb{P}} \varphi$ is false, then there is a condition $p * \dot{q} * \dot{r}$ in $\mathbb{P}$ such that $p * \dot{q} * \dot{r} \Vdash_{\mathbb{P}} \neg\varphi$. Choose $\overline{G} * \overline{H} * \overline{I}$ which contains the condition $j(p * \dot{q} * \dot{r})$. By Theorems 6.1 and 11.11, extend j to $j : V[G * H * I] \to M[\overline{G} * \overline{H} * \overline{I}]$ such that $j(G * H * I) = \overline{G} * \overline{H} * \overline{I}$. Since $j(p) = p$ is in $\overline{G} = j(G)$, p is in G. As $j(q) = q$ is in $\overline{H} = j(H)$, q is in H. By Lemma 11.9, $\pi(j(r)) = r$ is in $\pi[\overline{I}] = I$. So $p * \dot{q} * \dot{r}$ is in $G * H * I$, and thus $V[G * H * I] \models \neg\varphi$.

So let $\overline{G} * \overline{H} * \overline{I}$ be a generic filter on $j(\mathbb{P})$ over V, and let G, H, and I be as above. We prove $V[G * H * I] \models \varphi$. By Theorems 6.1 and 11.11, we can extend j to $j : V[G * H * I] \to M[\overline{G} * \overline{H} * \overline{I}]$ such that $j(G * H * I) = \overline{G} * \overline{H} * \overline{I}$. By Theorem

10.2, S_κ is a stationary subset of $P_{\omega_1}(\omega_2)$ in $V[G * H * I]$ which does not reflect to any ordinal in ω_2 with cofinality ω_1. Therefore $V[G * H * I] \models \neg\mathrm{RP}(\omega_2)$.

It remains to show that $V[G * H * I] \models \mathrm{WRP}(\omega_2)$. So let U be a stationary subset of $P_{\omega_1}(\omega_2)$ in $V[G * H * I]$. First assume $U \cap S_\kappa$ is stationary. Then by Theorem 10.2, there is an uncountable ordinal ν in ω_2 with cofinality ω such that $(U \cap S_\kappa) \cap P_{\omega_1}(\nu)$ is stationary in $P_{\omega_1}(\nu)$. So U reflects to ν.

Now assume $U \cap S_\kappa$ is non-stationary. Let $T = U \setminus S_\kappa$. Then T is a stationary subset of $P_{\omega_1}(\omega_2)$ in $V[G*H*I]$ which is disjoint from S_κ. By Theorem 12.1, T is a stationary subset of $P_{\omega_1}(\kappa)$ in $M[\overline{G} * \overline{H} * \overline{I}]$. For any b in T, $j(b) = b$ is in $j(T)$ by the elementarity of j. So $T \subseteq j(T) \cap P_{\omega_1}(\kappa)$ in $M[\overline{G} * \overline{H} * \overline{I}]$ (in fact, these sets are equal). It follows that $j(T) \cap P_{\omega_1}(\kappa)$ is stationary in $P_{\omega_1}(\kappa)$ in $M[\overline{G} * \overline{H} * \overline{I}]$. So $M[\overline{G} * \overline{H} * \overline{I}]$ models the statement that $j(T)$ reflects to an ordinal in ω_2 with cofinality ω_1. By the elementarity of j, $V[G * H * I]$ models the statement that T reflects to an ordinal in ω_2 with cofinality ω_1. Since $T \subseteq U$, U reflects to an uncountable ordinal in ω_2. $\square$

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