SOME APPLICATIONS OF MIXED SUPPORT ITERATIONS

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Abstract. We give some applications of mixed support forcing iterations to the topics of disjoint stationary sequences and internally approachable sets. In the first half of the paper we study the combinatorial content of the idea of a disjoint stationary sequence, including its relation to adding clubs by forcing, the approachability ideal, canonical structure, the proper forcing axiom, and properties related to internal approachability. In the second half of the paper we present some consistency results related to these ideas. We construct a model in which a disjoint stationary sequence exists at the successor of an arbitrary regular uncountable cardinal. We also construct models in which the properties of being internally stationary, internally club, and internally approachable are distinct.

In [16] we worked out a general schema for a mixed support forcing iteration. In this paper we present some applications of this forcing iteration to the topics of disjoint stationary sequences and internally approachable sets. Let κ be a regular uncountable cardinal. A *disjoint stationary sequence* on κ^+ is a sequence $\langle \mathcal{S}_\alpha : \alpha \in A\rangle$, where A is a stationary subset of $\kappa^+ \cap \operatorname{cof}(\kappa)$, such that for all α in A, $\mathcal{S}_\alpha$ is a stationary subset of $P_\kappa(\alpha)$, and for all $\alpha < \beta$ in A, $\mathcal{S}_\alpha \cap \mathcal{S}_\beta$ is empty. This definition is a weakened version of the idea of a disjoint club sequence which we introduced in [11].

In the first half of the paper, we study the combinatorial content of the idea of a disjoint stationary sequence. In Section 3 we study the relation of this idea to adding clubs by forcing and the approachability ideal. In Section 4 we give a canonical description of the domain of a disjoint stationary sequence on κ^+, under the assumption $\kappa^{<\kappa} \leq \kappa^+$. In Section 5 we prove that the forcing axiom PFA^+ implies the existence of a disjoint stationary sequence on ω_2. In Section 6 we relate disjoint club sequences and disjoint stationary sequences to some variations of the property of internal approachability.

In the second half of the paper, we use mixed support forcing iterations to prove some consistency results concerning the combinatorial properties studied in the first half. This requires some preparatory work. In Section 7 we refine a theorem of Gitik [12] on adding stationarily many new sets. In Section 8 we summarize the properties of the mixed support forcing iteration we will use.

In Section 9 we begin our consistency proofs, by constructing a model with a disjoint stationary sequence on the successor of an arbitrary regular uncountable cardinal. Sections 10 and 11 present very general constructions of models in which variations of the property of internal approachability are distinct, building on previous work on the topic from [14], [15], and [17]. In Section 10 we construct a model in which the properties of being internally stationary and internally club are distinct, and in Section 11 we construct a model in which the properties of being

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internally club and internally approachable are distinct. In Section 12 we present some open problems related to the material in this paper.

1. Background and notation

We assume the reader has some basic familiarity with proper forcing and iterated forcing, generic elementary embeddings, generalized stationarity, and the interaction of elementary substructures with forcing.

For a regular cardinal μ, we write $\mathrm{cof}(\mu)$ to denote the class of limit ordinals with cofinality equal to μ, and similarly for $\mathrm{cof}(<\mu)$, $\mathrm{cof}(\leq\mu)$, and so forth. We write Even and Odd to denote the classes of even and odd ordinals respectively. For ordinals $\alpha \leq \beta$, $\beta - \alpha$ is the unique ordinal γ such that $\alpha + \gamma = \beta$. If $f : X \to Y$ is a function and $A \subseteq X$, $f[A]$ denotes $\{f(x) : x \in A\}$. We say that a cardinal χ is *much larger* than a set X if $\mathcal{P}(\mathcal{P}(\mathcal{P}(X)))$ is a member of $H(\chi)$. For a regular cardinal μ, $\textsc{Add}(\mu)$ denotes the usual μ-closed Cohen forcing for adding a subset to μ. For a regular cardinal κ and a set X, $\textsc{Coll}(\kappa, X)$ denotes the Lévy collapse for adding a surjection of κ onto X.

Let κ be a regular cardinal and let X be a set. Then $P_\kappa(X)$ denotes the collection of sets $a \subseteq X$ such that $|a| < \kappa$. We will always assume in this case that $\kappa \subseteq X$, although this is not necessary in general. We assume the reader is familiar with generalized stationarity on $P_\kappa(X)$, in the sense of Jech. A characterization of stationarity we will use is that a set $S \subseteq P_\kappa(X)$ is stationary if for any function $F : [X]^{<\omega} \to X$, there is a set a in S closed under F such that $a \cap \kappa \in \kappa$. This property is equivalent to S having non-empty intersection with every club subset of $P_\kappa(X)$.

If μ is a cardinal and X is a set, $[X]^\mu$ denotes the collection of sets $a \subseteq X$ such that $|a| = \mu$. The family $[X]^\mu$ is a club subset of $P_{\mu^+}(X)$. So when we speak of stationary and club subsets of $[X]^\mu$, we mean stationary and club subsets of $P_{\mu^+}(X)$ which are contained in $[X]^\mu$. We will always assume that $\mu^+ \subseteq X$, although this is not necessary in general. A set $S \subseteq [X]^\mu$ is stationary if for any function $F : [X]^{<\omega} \to X$, there is a set N in S which is closed under F and satisfies that $\mu \subseteq N$. This is true because there is a club subset of $[X]^\mu$ such that for any N in this club, $\mu \subseteq N$ implies $N \cap \mu^+ \in \mu^+$.

We will often consider collections of the form $P_\kappa(\alpha)$, where κ is a regular uncountable cardinal and α is an ordinal with size κ. If $f : \kappa \to \alpha$ is a surjection, then $\{f[i] : i < \kappa\}$ is a club subset of $P_\kappa(\alpha)$. It follows that any stationary subset of $P_\kappa(\alpha)$ contains a stationary subset of size κ. If $\langle a_i : i < \kappa \rangle$ is an increasing and continuous sequence of sets in $P_\kappa(\alpha)$ with union equal to α, then $\{a_i : i < \kappa\}$ is a club subset of $P_\kappa(\alpha)$, and a set $S \subseteq P_\kappa(\alpha)$ is stationary iff there is a stationary set $B \subseteq \kappa$ such that $\{a_i : i \in B\} \subseteq S$.

Suppose that $j : M \to N$ is an elementary embedding between transitive models of ZFC, $\mathbb{P}$ is a forcing poset in M, G is a generic filter for $\mathbb{P}$ over M, and H is a generic filter for $j(\mathbb{P})$ over N. If $j[G] \subseteq H$, then there is a unique extension of j to $j : M[G] \to N[H]$ such that $j(G) = H$, namely, $j(\dot{a}^G) = j(\dot{a})^H$. Suppose that $M \subseteq N$ are transitive models of ZFC with the same ordinals, and λ is a regular uncountable cardinal in N such that $M^{<\lambda} \cap N \subseteq M$. If $\mathbb{P}$ is a forcing poset in M which is λ-c.c. in N, and G is a generic filter for $\mathbb{P}$ over N, then $M[G]^{<\lambda} \cap N[G] \subseteq M[G]$.

2. Internal Approachability

In this preliminary section, we review some well-known facts about internal approachability. For more information on this topic, see [8]. Let ξ be a limit ordinal. A set N is *internally approachable of length* ξ if N is the union of an increasing and continuous sequence $\langle N_i : i < \xi\rangle$ such that for all $\alpha < \xi$, $\langle N_i : i < \alpha\rangle$ is in N. We write $\mathrm{IA}(\xi)$ for the class of sets which are internally approachable of length ξ. If N is in $\mathrm{IA}(\xi)$ and N is an elementary substructure of $H(\lambda)$ for some cardinal λ, then ξ is a subset of N. Note that the property of being in $\mathrm{IA}(\xi)$ is upwards absolute.

Suppose κ is a regular uncountable cardinal and $\chi \geq \kappa$ is regular. Then for any limit ordinal ξ less than κ, the family $P_\kappa(H(\chi)) \cap \mathrm{IA}(\xi)$ is a stationary subset of $P_\kappa(H(\chi))$. Indeed, let $F : H(\chi)^{<\omega} \to H(\chi)$ be a function. Define by induction an *internally approachable chain* $\langle N_i : i < \xi\rangle$ of sets in $H(\chi)$ as follows. Let N_0 be any non-empty set in $P_\kappa(H(\chi))$. If $\delta < \xi$ is a limit ordinal and $\langle N_i : i < \delta\rangle$ is defined, let $N_\delta = \bigcup\{N_i : i < \delta\}$. Suppose $\langle N_i : i \leq \nu\rangle$ is defined for an ordinal $\nu < \xi$. Since χ is regular, the sequence $\langle N_i : i \leq \nu\rangle$ is in $H(\chi)$. Choose $N_{\nu+1}$ in $P_\kappa(H(\chi))$ such that $N_\nu \subseteq N_{\nu+1}$, $\langle N_i : i \leq \nu\rangle$ is a member of $N_{\nu+1}$, $N_{\nu+1}$ is closed under F, and $N_{\nu+1} \cap \kappa \in \kappa$. The set $N_{\nu+1}$ is in $H(\chi)$ because it is a subset of $H(\chi)$ of size less than χ. Now let $N = \bigcup\{N_i : i < \xi\}$. Then N is in $P_\kappa(H(\chi)) \cap \mathrm{IA}(\xi)$, N is closed under F, and $N \cap \kappa \in \kappa$.

Let $\kappa < \lambda$ be regular cardinals, and suppose N is a set in $[H(\lambda)]^\kappa$. In this context, we say that N is *internally approachable* if N is internally approachable of length κ. Suppose that N is an elementary substructure of $\langle H(\lambda), \in, \lhd, \kappa\rangle$ of size κ, where $\lhd$ is a well-ordering of $H(\lambda)$. We claim that N is internally approachable iff N is the union of an increasing and continuous sequence $\langle a_i : i < \kappa\rangle$ of sets *with size less than* κ such that for all $\beta < \kappa$, $\langle a_i : i < \beta\rangle$ is in N. Obviously the second condition implies N is internally approachable.

Suppose N is internally approachable, and let $\langle N_i : i < \kappa\rangle$ be an increasing and continuous sequence with union equal to N such that for all $\alpha < \kappa$, $\langle N_i : i < \alpha\rangle$ is in N. Since N has size κ, $|N_i| \leq \kappa$ for all $i < \kappa$. Without loss of generality, assume N_0 is non-empty. For all $i < \kappa$, let f_i denote the $\lhd$-least surjection of κ onto N_i. Note that for all $\alpha < \kappa$, the sequence $\langle f_i : i < \alpha\rangle$ is in N by elementarity. Fix in N a bijection $g : \kappa \to \kappa \times \kappa$. Define a function $h : \kappa \to N$ as follows. For an ordinal $\gamma < \kappa$, let $g(\gamma) = \langle i, j\rangle$, and define $h(\gamma) = f_i(j)$. Clearly h is a surjection. Note that for all $\alpha < \kappa$, $h \restriction \alpha$ is in N, since $h \restriction \alpha$ is definable in $\langle H(\lambda), \in, \lhd, \kappa\rangle$ from g and a proper initial segment of $\langle f_i : i < \kappa\rangle$. Since h is a surjection, $\langle h[i] : i < \kappa\rangle$ is an increasing and continuous sequence of sets in $P_\kappa(N)$ with union equal to N. For all $\alpha < \kappa$, the sequence $\langle h[i] : i < \alpha\rangle$ is in N, since it is definable in $\langle H(\lambda), \in, \lhd, \kappa\rangle$ from $h \restriction \alpha$.

Definition 2.1. *Let $\mu < \kappa$ be regular cardinals and suppose $\mathbb{P}$ a forcing poset. We say that $\mathbb{P}$ is* κ-proper for $\mathrm{IA}(\mu)$ *if for any regular cardinal χ much larger than κ and $\mathbb{P}$, there is a club set $C \subseteq P_\kappa(H(\chi))$ such that for all N in $C \cap IA(\mu)$, for all p in $N \cap \mathbb{P}$ there is $q \leq p$ such that q is N-generic.*

The condition q being *N-generic* in Definition 2.1 means that q forces $N[\dot{G}] \cap V = N$, or equivalently, for any dense open set $D \subseteq \mathbb{P}$ in N, q forces $\dot{G} \cap D \cap N$ is non-empty.

Lemma 2.2. *Let κ be a regular uncountable cardinal, and suppose $\mathbb{P}$ is a κ-closed forcing poset. Then for any regular cardinal $\mu < \kappa$, $\mathbb{P}$ is κ-proper for IA(μ).*

Proof. Let χ be a regular cardinal much larger than κ and $\mathbb{P}$. Let C be the club set of N in $P_\kappa(H(\chi))$ such that N is an elementary substructure of $\langle H(\chi), \in, \lhd, \mathbb{P}\rangle$, where $\lhd$ is a well-ordering of $H(\chi)$. Suppose N is in $C \cap \mathrm{IA}(\mu)$. Let $\langle N_i : i < \mu\rangle$ be an increasing and continuous sequence with union equal to N such that for all $\alpha < \mu$, $\langle N_i : i < \alpha\rangle$ is in N. Since $|N| < \kappa$, $|N_i| < \kappa$ for all $i < \kappa$.

Let p be in $N \cap \mathbb{P}$. Define by induction a descending sequence of conditions $\langle p_i : i < \mu\rangle$. Let $p_0 = p$. Suppose p_i is defined for a fixed $i < \mu$. Since $\mathbb{P}$ is κ-closed, $\mathbb{P}$ is $< \kappa$-distributive. Let p_{i+1} be the $\lhd$-least condition below p_i which is in the intersection of all dense open subsets of $\mathbb{P}$ which lie in N_i. This is possible since N_i has size less than κ and $\mathbb{P}$ is $< \kappa$-distributive. If $\delta < \mu$ is a limit ordinal and $\langle p_i : i < \delta\rangle$ is defined, let p_δ be the $\lhd$-least condition in $\mathbb{P}$ which is below p_i for all $i < \delta$. This is possible since $\mathbb{P}$ is κ-closed. This completes the definition.

Note that every proper initial segment of $\langle p_i : i < \mu\rangle$ is in N, since any such initial segment is definable in $\langle H(\chi), \in, \lhd, \mathbb{P}\rangle$ from p and a proper initial segment of $\langle N_i : i < \mu\rangle$. In particular, p_i is in N for all $i < \mu$. Let q be a lower bound to the sequence $\langle p_i : i < \mu\rangle$. We claim that q is N-generic. So let D be a dense open subset of $\mathbb{P}$ in N. Fix $i < \mu$ such that D is in N_i. Then by construction, p_{i+1} is in $D \cap N$. Since $q \le p_{i+1}$, q forces $\dot{G} \cap D \cap N$ is non-empty. □

Lemma 2.3. *Let $\mu < \kappa$ be regular cardinals, and suppose $\mathbb{P}$ is a forcing poset which is κ-proper for IA(μ). Then $\mathbb{P}$ preserves the regularity of κ.*

Proof. Suppose for a contradiction there is a condition p in $\mathbb{P}$ and an ordinal ν less than κ such that p forces $\dot{f} : \nu \to \kappa$ is unbounded in κ. Let χ be a regular cardinal much larger than κ and $\mathbb{P}$. Choose an elementary substructure N of $\langle H(\chi), \in, \mathbb{P}, p, \dot{f}\rangle$ of size less than κ such that $\nu \subseteq N$ and there is a condition $q \le p$ which is N-generic. Note that since N has size less than κ, $N \cap \kappa$ is bounded below κ.

Let G be a generic filter for $\mathbb{P}$ over V such that q is in G. Then $N[G] \cap V = N$, and in particular, $N[G] \cap \kappa = N \cap \kappa$. Moreover, $N[G]$ is an elementary substructure of $H(\chi)^{V[G]}$, and $f = \dot{f}^G$ is in $N[G]$. Since ν is a subset of $N[G]$, $f[\nu]$ is a subset of $N[G] \cap \kappa$ by elementarity. But $f[\nu]$ is a cofinal subset of κ. So $N[G] \cap \kappa = N \cap \kappa$ is unbounded in κ, which is a contradiction. □

Lemma 2.4. *Let $\mu < \kappa$ be regular cardinals, and suppose $\mathbb{P}$ is a forcing poset which is κ-proper for IA(μ). Let χ be a regular cardinal much larger than κ and $\mathbb{P}$. If S is a stationary subset of $P_\kappa(H(\chi)) \cap$ IA(μ), then $\mathbb{P}$ forces S is a stationary subset of $P_\kappa(H(\chi)^V)$.*

Proof. Let p be a condition in $\mathbb{P}$, and suppose p forces $\dot{F} : (H(\chi)^V)^{<\omega} \to H(\chi)^V$ is a function. Since $\mathbb{P}$ is κ-proper for IA(μ) and S is a stationary subset of $P_\kappa(H(\chi)) \cap$ IA(μ), we can fix a set N in S such that N is an elementary substructure of $\langle H(\chi), \in, \mathbb{P}, p, \dot{F}\rangle$, $N \cap \kappa \in \kappa$, and there is $q \le p$ in $\mathbb{P}$ which is N-generic.

Let G be a generic filter for $\mathbb{P}$ over V which contains q. Then $N[G] \cap V = N$, and in particular, $N[G] \cap H(\chi)^V = N$. Since $N[G]$ is an elementary substructure of $H(\chi)^{V[G]}$ and $F = \dot{F}^G$ is in $N[G]$, N is closed under F. Also N is in S and $N \cap \kappa \in \kappa$. □

3. Disjoint stationary sequences and adding clubs

In [11] we introduced the idea of a disjoint club sequence on ω_2 (see Definition 3.1 below). In this section we generalize this idea to the notion of a disjoint stationary sequence on κ^+, for an uncountable regular cardinal κ. Our first application of mixed support iterations, given in Section 9 below, is to construct a disjoint stationary sequence on the successor of a regular uncountable cardinal. Over the next several sections we study the combinatorial content of disjoint stationary sequences, beginning in this section with its relation to adding clubs by forcing and the approachability ideal.

First let us recall the definition of a disjoint club sequence, stated here for an arbitrary regular uncountable cardinal κ.

Definition 3.1. *Let κ be a regular uncountable cardinal. A sequence $\langle \mathcal{C}_\alpha : \alpha \in A\rangle$ is a* disjoint club sequence *on κ^+ if A is a stationary subset of $\kappa^+ \cap \mathrm{cof}(\kappa)$, for all α in A, $\mathcal{C}_\alpha$ is a club subset of $P_\kappa(\alpha)$, and for all $\alpha < \beta$ in A, $\mathcal{C}_\alpha \cap \mathcal{C}_\beta$ is empty.*

Our original motivation for this definition was to prove the consistency of the non-existence of a thin stationary subset of $P_{\omega_1}(\omega_2)$ [11], answering a question of S.D. Friedman. Thin stationary sets were used by Friedman in [10] to construct a forcing poset which adds a club subset to a stationary subset of ω_2. The impact of Definition 3.1 for adding clubs is more general.

Theorem 3.2 ([11])**.** *Suppose $V \subseteq W$ are transitive models of ZFC with the same ordinals and the same ω_1 and ω_2. Assume there exists a disjoint club sequence $\langle \mathcal{C}_\alpha : \alpha \in A\rangle$ on ω_2 in V. Then the set $A \cup \mathrm{cof}(\omega)$ does not contain a club subset in W.*

The existence of a disjoint club sequence on ω_2 is a consequence of Martin's Maximum, and is equiconsistent with the existence of a Mahlo cardinal [11]. We do not know whether it is consistent to have a disjoint club sequence on κ^+ for a regular cardinal κ larger than ω_1 (see Section 12). But if we weaken the definition so that the sets on the disjoint sequence are only stationary, instead of club, we are able to obtain the consistency of such a sequence on the successor of an arbitrary regular cardinal; this is shown in Section 9 below.

Definition 3.3. *Let κ be a regular uncountable cardinal. A sequence $\langle \mathcal{S}_\alpha : \alpha \in A\rangle$ is a* disjoint stationary sequence *on κ^+ if A is a stationary subset of $\kappa^+ \cap \mathrm{cof}(\kappa)$, for all α in A, $\mathcal{S}_\alpha$ is a stationary subset of $P_\kappa(\alpha)$, and for all $\alpha < \beta$ in A, $\mathcal{S}_\alpha \cap \mathcal{S}_\beta$ is empty.*

An important parameter related to a disjoint club sequence or disjoint stationary sequence is the domain of the sequence. Assuming $\kappa^{<\kappa} \le \kappa^+$, there is a maximum set which is the domain of such a sequence, as we show in the next section. Let us note that the domain of such a sequence cannot contain almost all ordinals of cofinality κ in κ^+.

Proposition 3.4. *Let κ be a regular uncountable cardinal, and suppose $\langle \mathcal{S}_\alpha : \alpha \in A\rangle$ is a disjoint stationary sequence on κ^+. Then $(\kappa^+ \cap \mathrm{cof}(\kappa)) \setminus A$ is stationary.*

Proof. Let $C \subseteq \kappa^+$ be a club set. We prove there is α in $C \cap \mathrm{cof}(\kappa)$ which is not in A. Choose an increasing and continuous sequence $\langle N_i : i < \kappa\rangle$ of elementary substructures of $H(\kappa^{++})$ such that for all $i < \kappa$, N_i has size less than κ, $N_i \cap \kappa \in \kappa$, the parameters C and $\langle \mathcal{S}_\alpha : \alpha \in A \rangle$ are members of N_i, and $N_i \in N_{i+1}$. Note that by elementarity, for all $i < \kappa$, the ordinal $\sup(N_i \cap \kappa^+)$ is a limit point of C, and so is in C.

Let $N = \bigcup\{N_i : i < \kappa\}$. Since $N_i \cap \kappa \in \kappa$ and $N_i \in N_{i+1}$ for all $i < \kappa$, κ is a subset of N. It follows that $N \cap \kappa^+ \in \kappa^+$. Let $\beta = N \cap \kappa^+$. Then β is the supremum of the sequence $\langle \sup(N_i \cap \kappa^+) : i < \kappa \rangle$. Since $N_i \in N_{i+1}$ for all $i < \kappa$, this sequence is increasing. So β is in $C \cap \mathrm{cof}(\kappa)$.

Suppose for a contradiction that β is in A. Then $\mathcal{S}_\beta$ is defined and is a stationary subset of $P_\kappa(\beta)$. Clearly $\{N_i \cap \kappa^+ : i < \kappa\}$ is a club subset of $P_\kappa(\beta)$. So fix an ordinal $i < \kappa$ such that $N_i \cap \kappa^+$ is in $\mathcal{S}_\beta$. Since N_i is in N, $N_i \cap \kappa^+$ is in N. Now $\langle \mathcal{S}_\alpha : \alpha \in A \rangle$ is a disjoint stationary sequence, so β is the unique ordinal in A such that $N_i \cap \kappa^+$ is in $\mathcal{S}_\beta$. By elementarity, β is in N. This contradicts that $N \cap \kappa^+$ is equal to β. □

As a corollary, we get an analogue of Theorem 3.2 for disjoint stationary sequences.

Corollary 3.5. *Suppose $V \subseteq W$ are transitive models of ZFC with the same ordinals. Let κ be a regular uncountable cardinal in W, and assume $(\kappa^+)^V = (\kappa^+)^W$. Suppose that every stationary subset of κ in V remains stationary in W. If $\langle \mathcal{S}_\alpha : \alpha \in A \rangle$ is a disjoint stationary sequence in V, then the set $A \cup \mathrm{cof}(<\kappa)$ does not contain a club subset in W.*

In particular, if κ is a regular uncountable cardinal and $\langle \mathcal{S}_\alpha : \alpha \in A \rangle$ is a disjoint stationary sequence on κ^+, then for any $<\kappa^+$-distributive forcing poset $\mathbb{P}$, $\mathbb{P}$ does not add a club subset to $A \cup \mathrm{cof}(<\kappa)$.

Proof. First note that for all α in A, $\mathcal{S}_\alpha$ is a stationary subset of $P_\kappa(\alpha)$ in W. Indeed, choose an increasing and continuous sequence $\langle a_i : i < \kappa \rangle$ in V of sets of size less than κ with union equal to α. Since $\mathcal{S}_\alpha$ is stationary in $P_\kappa(\alpha)$, there is a stationary set $B \subseteq \kappa$ in V such that $\{a_i : i \in B\} \subseteq \mathcal{S}_\alpha$. By our assumptions, B remains stationary in W. Therefore in W, $\{a_i : i \in B\}$ is a stationary subset of $P_\kappa(\alpha)$ contained in $\mathcal{S}_\alpha$, so $\mathcal{S}_\alpha$ is stationary in W.

Suppose for a contradiction there is a club $C \subseteq \kappa^+$ in W which is a subset of $A \cup \mathrm{cof}(<\kappa)$. In particular, A is still stationary in W, so $\langle \mathcal{S}_\alpha : \alpha \in A \rangle$ is still a disjoint stationary sequence. But then $(\kappa^+ \cap \mathrm{cof}(\kappa)) \setminus A$ is disjoint from C, which contradicts Proposition 3.4.

If $\mathbb{P}$ is a $<\kappa^+$-distributive forcing poset, then $\mathbb{P}$ does not add any new sets of ordinals with order type less than κ^+. So κ and κ^+ remain regular cardinals in $V^{\mathbb{P}}$. Since $\mathbb{P}$ does not add any new subsets to κ, $\mathbb{P}$ preserves the stationarity of stationary subsets of κ. Now apply the previous part of the corollary to $W = V^{\mathbb{P}}$. □

It follows in particular that $\kappa^{<\kappa} = \kappa$ implies there does not exist a disjoint stationary sequence on κ^+. For as shown in [2], $\kappa^{<\kappa} = \kappa$ implies that for every stationary set $S \subseteq \kappa^+ \cap \mathrm{cof}(\kappa)$, there is a $<\kappa^+$-distributive forcing poset which adds a club subset to $S \cup \mathrm{cof}(<\kappa)$.

We now discuss the relationship between disjoint stationary sequences and the approachability ideal. Let κ be a regular uncountable cardinal. For a sequence $\vec{a} = \langle a_i : i < \kappa^+ \rangle$ of bounded subsets of κ^+, let $S_{\vec{a}}$ denote the set of limit ordinals $\alpha < \kappa^+$ such that there is a club set $c \subseteq \alpha$ with order type equal to the cofinality of α such that for all $\beta < \alpha$, there is $i < \alpha$ such that $c \cap \beta = a_i$. The *approachability ideal* $I[\kappa^+]$ is the collection of sets $S \subseteq \kappa^+$ for which there exists a club set $C \subseteq \kappa^+$

and a sequence $\vec{a} = \langle a_i : i < \kappa^+ \rangle$ of bounded subsets of κ^+ such that $S \cap C \subseteq S_{\vec{a}}$ ([21], [22]).

We will use the following well-known result about the approachability ideal.

Theorem 3.6. *Suppose κ is a regular uncountable cardinal and S is a stationary subset of $\kappa^+ \cap \mathrm{cof}(\kappa)$ in $I[\kappa^+]$. Then there is a $<\kappa^+$-distributive forcing poset which adds a club subset to $S \cup \mathrm{cof}(<\kappa)$.*

Proof. Fix a sequence $\vec{a} = \langle a_i : i < \kappa^+ \rangle$ of bounded subsets of κ^+ and a club set $C \subseteq \kappa^+$ such that $S \cap C \subseteq S_{\vec{a}}$. Define a forcing poset $\mathbb{P}$ by letting $\mathbb{P}$ consist of conditions which are closed and bounded subsets of κ^+ contained in $S \cup \mathrm{cof}(<\kappa)$, ordered by end-extension.

We show that $\mathbb{P}$ is $< \kappa^+$-distributive. Let $\vec{D} = \langle D_i : i < \kappa \rangle$ be a sequence of dense open subsets of $\mathbb{P}$. Consider a condition p in $\mathbb{P}$. Let $\mathfrak{B}$ denote the structure $\langle H(\kappa^{++}), \in, \lhd \rangle$, where $\lhd$ is a well-ordering of $H(\kappa^{++})$.

Since S is stationary in κ^+, there exists a set $N \prec \mathfrak{B}$ with size κ such that $N \cap \kappa^+ \in S \cap C$, and the parameters $\mathbb{P}$, p, and $\vec{D}$ are in N. Let $\alpha = N \cap \kappa^+$. Then α is in $S \cap C \subseteq S_{\vec{a}}$, and since α is in S, α has cofinality κ. Fix a club set $c = \{\alpha_i : i < \kappa\} \subseteq \alpha$ with order type κ such that every proper initial segment of c is in $\{a_i : i < \alpha\}$. Since $\vec{a}$ is in N, by elementarity the set $\{a_i : i < \alpha\}$ is a subset of N. Thus every proper initial segment of c is in N.

We define by induction a descending sequence $\langle p_i : i < \kappa \rangle$ of conditions in $\mathbb{P}$. Let $p_0 = p$. Given p_i, let p_{i+1} be the $\lhd$-least member of D_i below p_i such that $\max(p_{i+1})$ is greater than both $\max(p_i)$ and α_i. Let $\delta < \kappa$ be a limit ordinal and suppose $\langle p_i : i < \delta \rangle$ is defined. Let p_δ be equal to

$$\bigcup \{p_i : i < \delta\} \cup \{\sup(\bigcup \{p_i : i < \delta\})\}.$$

For all $i < \delta$, $\max(p_i) < \max(p_{i+1})$, so $\sup(\bigcup\{p_i : i < \delta\})$ has cofinality equal to the cofinality of δ, which is smaller than κ. Therefore p_δ is a closed and bounded subset of κ^+ contained in $S \cup \mathrm{cof}(<\kappa)$. So p_δ is in $\mathbb{P}$. This completes the construction of the sequence.

Every proper initial segment of $\langle p_i : i < \kappa \rangle$ is definable in $\mathfrak{B}$ from a proper initial segment of c, together with the poset $\mathbb{P}$, p, and the sequence $\vec{D}$. But these parameters are all in N. So by elementarity, every proper initial segment of the sequence $\langle p_i : i < \kappa \rangle$ is in N.

Now for all $i < \kappa$, $\max(p_{i+1})$ is greater than α_i. On the other hand, p_{i+1} is in N, so $\max(p_{i+1})$ is in $N \cap \kappa^+ = \alpha$. So $\bigcup\{p_i : i < \kappa\}$ is unbounded in α, and therefore is a club subset of α.

Let q be equal to

$$\bigcup \{p_i : i < \kappa\} \cup \{\alpha\}.$$

Since α is in S, q is a closed bounded subset of κ^+ contained in $S \cup \mathrm{cof}(<\kappa)$. So q is in $\mathbb{P}$, $q \leq p$, and q is in D_i for all $i < \kappa$. $\square$

For an uncountable cardinal κ, the *approachability property* AP_κ is the statement that $I[\kappa^+]$ contains a club subset of κ^+.

Corollary 3.7. *Let κ be a regular uncountable cardinal and suppose $\langle \mathcal{S}_\alpha : \alpha \in A \rangle$ is a disjoint stationary sequence on κ^+. Then $I[\kappa^+]$ does not contain any stationary subset of A. In particular, AP_κ fails.*

Proof. Let S be a stationary subset of A. If S is in $I[\kappa^+]$, then by Theorem 3.6 there is a $<\kappa^+$-distributive forcing poset which adds a club subset to $S \cup \mathrm{cof}(<\kappa)$. But $S \cup \mathrm{cof}(<\kappa) \subseteq A \cup \mathrm{cof}(<\kappa)$, so $\mathbb{P}$ adds a club subset to $A \cup \mathrm{cof}(<\kappa)$, contradicting Corollary 3.5. Thus S is not in $I[\kappa^+]$. Now if C is a club subset of κ^+, $A \cap C$ is a stationary subset of A, and hence is not in $I[\kappa^+]$. But $I[\kappa^+]$ is an ideal, so C cannot be in $I[\kappa^+]$ either. So AP_κ fails. $\square$

For independent interest, we make the following observation.

Proposition 3.8. *Let κ be a regular uncountable cardinal, and assume $\kappa^{<\kappa} \leq \kappa^+$. Let $\vec{a} = \langle a_i : i < \kappa^+ \rangle$ be an enumeration of $[\kappa^+]^{<\kappa}$. Let $S \subseteq \kappa^+ \cap \mathrm{cof}(\kappa)$. Then the following are equivalent:*

(1) $S \cap S_{\vec{a}}$ is stationary.

(2) There is a $<\kappa^+$-distributive forcing poset which adds a club subset to $S \cup \mathrm{cof}(<\kappa)$.

Proof. ($1 \Rightarrow 2$) Let $T = S \cap S_{\vec{a}}$. Then T is a stationary subset of $\kappa^+ \cap \mathrm{cof}(\kappa)$ and T is in $I[\kappa^+]$. By Theorem 3.6, there is a $<\kappa^+$-distributive forcing poset which adds a club subset to $T \cup \mathrm{cof}(<\kappa)$. But $T \cup \mathrm{cof}(<\kappa) \subseteq S \cup \mathrm{cof}(<\kappa)$. So $\mathbb{P}$ adds a club subset to $S \cup \mathrm{cof}(<\kappa)$.

($2 \Rightarrow 1$) Suppose $\mathbb{P}$ is a $<\kappa^+$-distributive forcing poset which adds a club subset to $S \cup \mathrm{cof}(<\kappa)$. Let G be a generic filter for $\mathbb{P}$ over V, and let $W = V[G]$. Then in W, all cardinals and cofinalities less than or equal to κ^+ are preserved, and $\vec{a}$ still enumerates all sets in $[\kappa^+]^{<\kappa}$. Let C be a club subset of $S \cup \mathrm{cof}(<\kappa)$ in W.

We would like to show that $S \cap S_{\vec{a}}$ is stationary in V. So let D be a club subset of κ^+ in V. In W, D is still a club subset of κ^+. For the rest of the proof we work in W. Define by induction an increasing and continuous sequence $\langle N_i : i < \kappa \rangle$ of elementary substructures of $H(\kappa^{++})$ such that for all $i < \kappa$, N_i has size less than κ, $N_i \cap \kappa \in \kappa$, the parameters C, D, and $\vec{a}$ are in N_i, and $\langle N_j : j \leq i \rangle$ is in N_{i+1}. Let $N = \bigcup \{N_i : i < \kappa\}$. Let $\alpha = N \cap \kappa^+$. Then α is an ordinal in $C \cap D$ with cofinality equal to κ.

Let $c = \{\sup(N_i \cap \kappa^+) : i < \kappa\}$. Then c is a club subset of α with order type equal to κ. For all $\beta < \alpha$, $c \cap \beta$ is in N, since it is definable in $H(\kappa^{++})$ from a proper initial segment of $\langle N_i : i < \kappa \rangle$. But $\langle a_i : i < \kappa^+ \rangle$ enumerates all of $[\kappa^+]^{<\kappa}$. So by elementarity, $c \cap \beta$ is in $\{a_i : i < \alpha\}$. The poset $\mathbb{P}$ is $<\kappa^+$-distributive, so c is in V. Therefore in V, c is a witness that α is in $S_{\vec{a}}$. As α is in C, which is a subset of $S \cup \mathrm{cof}(<\kappa)$, and α has cofinality κ, α is in S. So α is in $(S \cap S_{\vec{a}}) \cap D$. $\square$

4. Disjoint Stationary Sequences and Canonical Structure

We now relate the idea of a disjoint stationary sequence to the concept of canonical structure in set theory, as discussed in [6]. Although ZFC has a wide variety of models, there are *invariants* which take different values in different models, thereby giving information about the model. Such invariants are *canonical* if, although the axiom of choice may be required to prove they exist, their definition is independent of the choices made.

An example of a canonical invariant in set theory is the set of approachable ordinals, which we describe now. Let κ be a regular uncountable cardinal, and assume $\kappa^{<\kappa} \leq \kappa^+$. Let $\vec{a} = \langle a_i : i < \kappa^+ \rangle$ be an enumeration of $[\kappa^+]^{<\kappa}$. It is not hard to show that if S is in $I[\kappa^+]$, then there is a club set $C \subseteq \kappa^+$ such that $S \cap C \subseteq S_{\vec{a}}$. Thus the set $S_{\vec{a}}$ generates $I[\kappa^+]$ modulo clubs. We refer to

$S_{\vec{a}}$ as the *maximum set* in $I[\kappa^+]$, or as the set of *approachable ordinals* in κ^+. If $\vec{b} = \langle b_i : i < \kappa^+ \rangle$ is any other enumeration of $[\kappa^+]^{<\kappa}$, then $S_{\vec{b}}$ is equal to $S_{\vec{a}}$ modulo clubs. Thus the set of approachable ordinals in κ^+ is a canonical invariant, in the sense described above, under the assumption $\kappa^{<\kappa} \leq \kappa^+$.

It turns out that a similar situation occurs with regards to the domain of a disjoint stationary sequence.

Proposition 4.1. *Let κ be a regular uncountable cardinal, and assume $\kappa^{<\kappa} \leq \kappa^+$. Let $\langle x_i : i < \kappa^+ \rangle$ be an enumeration of $[\kappa^+]^{<\kappa}$. Define S as the set of α in $\kappa^+ \cap \mathrm{cof}(\kappa)$ such that $P_\kappa(\alpha) \setminus \{x_i : i < \alpha\}$ is a stationary subset of $P_\kappa(\alpha)$.*

(1) If S is stationary, then there is a club $C \subseteq \kappa^+$ and a disjoint stationary sequence $\langle \mathcal{S}_\alpha : \alpha \in S \cap C \rangle$.

(2) If there exists a disjoint stationary sequence $\langle \mathcal{T}_\alpha : \alpha \in T \rangle$, then S is stationary and T is a subset of S modulo clubs.

Proof. (1) Suppose S is stationary. For any α in S, α has size κ, so there is a club subset of $P_\kappa(\alpha)$ of size κ. It follows that any stationary subset of $P_\kappa(\alpha)$ has a stationary subset of size κ. For each α in S, choose a stationary set $\mathcal{S}_\alpha \subseteq P_\kappa(\alpha) \setminus \{x_i : i < \alpha\}$ of size κ.

Define $F : S \to \kappa^+$ by letting $F(\alpha)$ be an ordinal less than κ^+ such that $\mathcal{S}_\alpha \subseteq \{x_i : i < F(\alpha)\}$. This is possible since $\mathcal{S}_\alpha$ has size κ. Now let C be the club set of $\nu < \kappa^+$ such that for all α in $S \cap \nu$, $F(\alpha) < \nu$. We claim that $\langle \mathcal{S}_\alpha : \alpha \in S \cap C \rangle$ is a disjoint stationary sequence. Clearly $S \cap C$ is a stationary subset of $\kappa^+ \cap \mathrm{cof}(\kappa)$, and for all α in $S \cap C$, $\mathcal{S}_\alpha$ is a stationary subset of $P_\kappa(\alpha)$. Let $\alpha < \beta$ be in $S \cap C$. Then $F(\alpha) < \beta$, so $\mathcal{S}_\alpha \subseteq \{x_i : i < F(\alpha)\} \subseteq \{x_i : i < \beta\}$. But $\mathcal{S}_\beta$ is disjoint from $\{x_i : i < \beta\}$. So $\mathcal{S}_\alpha \cap \mathcal{S}_\beta$ is empty.

(2) Suppose $\langle \mathcal{T}_\alpha : \alpha \in T \rangle$ is a disjoint stationary sequence. We claim there is a club $D \subseteq \kappa^+$ such that for all α in $T \cap D$, $\mathcal{T}_\alpha$ is disjoint from $\{x_i : i < \alpha\}$. It then follows that $T \cap D \subseteq S$. For if α is in $T \cap D$, then $\mathcal{T}_\alpha$ is a stationary subset of $P_\kappa(\alpha)$ contained in $P_\kappa(\alpha) \setminus \{x_i : i < \alpha\}$, so α is in S. Therefore S is stationary and T is a subset of S modulo clubs.

If the claim fails, then there is a stationary set $T' \subseteq T$ such that for all α in T', $\mathcal{T}_\alpha \cap \{x_i : i < \alpha\}$ is non-empty. Then there is a regressive map which sends an ordinal α in T' to some $i < \alpha$ such that x_i is in $\mathcal{T}_\alpha$. By Fodor's Lemma, there is a stationary set $T'' \subseteq T'$ and $i < \kappa^+$ such that for all α in T'', x_i is in $\mathcal{T}_\alpha$. Fix $\alpha < \beta$ in T''. Then x_i is in $\mathcal{T}_\alpha \cap \mathcal{T}_\beta$, contradicting that $\mathcal{T}_\alpha \cap \mathcal{T}_\beta$ is empty. $\square$

Corollary 4.2. *Let κ be a regular uncountable cardinal, and assume $\kappa^{<\kappa} \leq \kappa^+$. Suppose there exists a disjoint stationary sequence on κ^+. Then there exists a disjoint stationary sequence on κ^+ with a maximum domain, in the sense that its domain contains modulo clubs the domain of any other disjoint stationary sequence on κ^+.*

Proof. Let $\langle x_i : i < \kappa^+ \rangle$ be an enumeration of $[\kappa^+]^{<\kappa}$. Let S be the set of α in $\kappa^+ \cap \mathrm{cof}(\kappa)$ such that $P_\kappa(\alpha) \setminus \{x_i : i < \alpha\}$ is stationary in $P_\kappa(\alpha)$. By Proposition 4.1(2), S is stationary. By Proposition 4.1(1), there is a club $C \subseteq \kappa^+$ and a disjoint stationary sequence $\langle \mathcal{S}_\alpha : \alpha \in S \cap C \rangle$. Now suppose that $\langle \mathcal{T}_\alpha : \alpha \in T \rangle$ is a disjoint stationary sequence on κ^+. Then by Proposition 4.1(2), T is a subset of S modulo clubs. $\square$

The existence of the maximum domain S of a disjoint stationary sequence was proven using a fixed enumeration $\vec{x} = \langle x_i : i < \kappa^+ \rangle$ of $[\kappa^+]^{<\kappa}$. If S_1 is defined from another enumeration of $[\kappa^+]^{<\kappa}$ in the same manner as S is defined from $\vec{x}$, then by maximality, S and S_1 are equal modulo clubs. So assuming $\kappa^{<\kappa} \le \kappa^+$, the maximum domain of a disjoint stationary sequence is a canonical invariant.

A similar analysis holds for disjoint club sequences.

Proposition 4.3. *Let κ be a regular uncountable cardinal, and assume $\kappa^{<\kappa} \le \kappa^+$. Let $\langle x_i : i < \kappa^+ \rangle$ be an enumeration of $[\kappa^+]^{<\kappa}$. Define S as the set of α in $\kappa^+ \cap \mathrm{cof}(\kappa)$ such that $P_\kappa(\alpha) \setminus \{x_i : i < \alpha\}$ contains a club subset of $P_\kappa(\alpha)$.*

(1) If S is stationary, then there is a club $C \subseteq \kappa^+$ and a disjoint club sequence $\langle \mathcal{C}_\alpha : \alpha \in S \cap C \rangle$.

(2) If there exists a disjoint club sequence $\langle \mathcal{D}_\alpha : \alpha \in T \rangle$, then S is stationary and T is a subset of S modulo clubs.

Corollary 4.4. *Let κ be a regular uncountable cardinal, and assume $\kappa^{<\kappa} \le \kappa^+$. Suppose there exists a disjoint club sequence on κ^+. Then there exists a disjoint club sequence on κ^+ with a maximum domain, in the sense that its domain contains modulo clubs the domain of any other disjoint club sequence on κ^+.*

The proofs of Proposition 4.3 and Corollary 4.4 are minor variations of the proofs of Proposition 4.1 and Corollary 4.2.

5. Disjoint Stationary Sequences and the Proper Forcing Axiom

We proved in [11] that Martin's Maximum implies there exists a disjoint club sequence on ω_2. In this section we show that PFA^+ implies the existence of a disjoint stationary sequence on ω_2.

Definition 5.1. *The forcing axiom PFA^+ is the statement that for any proper forcing poset $\mathbb{P}$, any collection $\{D_i : i < \omega_1\}$ of dense open subsets of $\mathbb{P}$, and any $\mathbb{P}$-name $\dot{A}$ for a stationary subset of ω_1, there exists a filter $G \subseteq \mathbb{P}$ such that for all $i < \omega_1$, $G \cap D_i$ is non-empty, and moreover, the set $\dot{A}^G = \{\alpha < \omega_1 : \exists p \in G \ p \Vdash \alpha \in \dot{A}\}$ is a stationary subset of ω_1.*

Let $\chi \ge \omega_2$ be a regular cardinal, and suppose $\mathbb{P}$ is a forcing poset in $H(\chi)$. If N is an elementary substructure of $H(\chi)$ with size $\aleph_1$, $\mathbb{P} \in N$, and $G \subseteq \mathbb{P}$, we say that G is *N-generic* if for every dense open set $D \subseteq \mathbb{P}$ in N, $G \cap D \cap N$ is non-empty.

Suppose PFA^+ holds. Let $\chi \ge \omega_2$ be a regular cardinal, and let $\mathbb{P}$ be a proper forcing poset in $H(\chi)$. Then for any $\mathbb{P}$-name $\dot{A}$ for a stationary subset of ω_1, there are stationarily many $N \prec H(\chi)$ of size $\aleph_1$ such that there exists a filter $G \subseteq \mathbb{P}$ which is N-generic such that $\dot{A}^G$ is stationary in ω_1 (see Proposition 2.1 of [14]).

In the next theorem, we use the fact proven in [2] that the poset $\mathrm{Add}(\omega)$ forces that the collection $P_{\omega_1}(\omega_2) \setminus V$ is a stationary subset of $P_{\omega_1}(\omega_2)$. This fact is also a consequence of Theorem 7.1 below.

Theorem 5.2. *The forcing axiom PFA^+ implies that there exists a disjoint stationary sequence on ω_2.*

Proof. Suppose PFA^+ holds. Then $2^\omega = \omega_2$, since this follows from PFA. So $\omega_1^{<\omega_1} = \omega_2$, and thus we are in the context of Proposition 4.1 with $\kappa = \omega_1$. Let $\vec{x} = \langle x_i : i < \omega_2 \rangle$ be an enumeration of $[\omega_2]^{<\omega_1}$. Define S as the set of α in $\omega_2 \cap \mathrm{cof}(\omega_1)$ such that $P_{\omega_1}(\alpha) \setminus \{x_i : i < \alpha\}$ is a stationary subset of $P_{\omega_1}(\alpha)$. By Proposition 4.1, if

S is stationary then there exists a disjoint stationary sequence on ω_2. So it suffices to show that S is stationary. Let C be a club subset of ω_2, and we will show that $S \cap C$ is non-empty.

Let $\mathbb{P}$ be the forcing poset $\text{Add}(\omega) * \text{Coll}(\omega_1, \omega_2)$. Consider a generic filter $G * H$ for $\mathbb{P}$ over V. Note that $\omega_2^V = \omega_2^{V[G]}$, and ω_2^V has size $\aleph_1$ in $V[G*H]$. In the model $V[G]$, let $X = P_{\omega_1}(\omega_2) \setminus V$. By the comments preceding the statement of the theorem, X is a stationary subset of $P_{\omega_1}(\omega_2)$ in $V[G]$. Now the poset $\text{Coll}(\omega_1, \omega_2)$ in $V[G]$ is proper, and so preserves stationary subsets of $P_{\omega_1}(\omega_2^V)$. Therefore in $V[G*H]$, X is a stationary subset of $P_{\omega_1}(\omega_2^V)$. Fix in $V[G*H]$ an increasing and continuous sequence $\langle a_i : i < \omega_1 \rangle$ of countable sets with union equal to ω_2^V. As X is stationary in $V[G*H]$, there is a stationary set $B \subseteq \omega_1$ such that $\{a_i : i \in B\} \subseteq X$. In particular, the set $\{a_i : i \in B\}$ is disjoint from V.

Now back in V, fix a $\mathbb{P}$-name $\langle \dot{a}_i : i < \omega_1 \rangle$ for an increasing and continuous sequence of countable sets with union equal to ω_2^V, and a $\mathbb{P}$-name $\dot{B}$ for a stationary subset of ω_1, such that $\mathbb{P}$ forces $\{\dot{a}_i : i \in B\}$ is disjoint from V. Let χ be a regular cardinal larger than ω_2 such that $\mathbb{P}$, $\langle \dot{a}_i : i < \omega_1 \rangle$, and $\dot{B}$ are in $H(\chi)$. Fix an elementary substructure $N \prec H(\chi)$ of size $\aleph_1$ which contains as elements the parameters $\vec{x}$, C, $\mathbb{P}$, $\langle \dot{a}_i : i < \omega_1 \rangle$, $\dot{B}$, and all countable ordinals, and a filter $G * H \subseteq \text{Add}(\omega) * \text{Coll}(\omega_1, \omega_2)$ which is N-generic such that $B = \dot{B}^{G*H}$ is a stationary subset of ω_1. Let $\alpha = N \cap \omega_2$. By elementarity, α is a limit point of C, so α is in C. It suffices to show that α is in S.

For each $i < \omega_1$ let $a_i = \{\alpha < \omega_2 : \exists p \in G * H \ p \Vdash \alpha \in \dot{a}_i\}$. We claim that $\langle a_i : i < \omega_1 \rangle$ is an increasing and continuous sequence of countable sets with union equal to α. First let us show that $\bigcup\{a_i : i < \omega_1\} = \alpha$. Fix $i < \omega_1$. Since $\mathbb{P}$ is proper, every countable set in $V^{\mathbb{P}}$ is covered by a countable set in the ground model. So by elementarity there is a countable set $b_i \subseteq \omega_2$ in N such that $\mathbb{P}$ forces $\dot{a}_i \subseteq b_i$. Then clearly $a_i \subseteq b_i$, hence a_i is countable. Since b_i is in N, $a_i \subseteq b_i \subseteq N \cap \omega_2 = \alpha$. So $\bigcup\{a_i : i < \omega_1\} \subseteq \alpha$. On the other hand, suppose ν is in α. Let D be the dense open collection of conditions p in $\mathbb{P}$ for which there is $i < \omega_1$ such that p forces $\nu \in \dot{a}_i$. By elementarity, D is in N. As $G * H$ is N-generic, $(G * H) \cap D \cap N$ is non-empty. Let p be in this intersection, and fix $i < \omega_1$ such that p forces ν is in $\dot{a}_i$. Then ν is in a_i. This proves $\bigcup\{a_i : i < \omega_1\} = \alpha$.

To show that $\langle a_i : i < \omega_1 \rangle$ is increasing, suppose $i < j < \omega_1$ and ν is in a_i. Then there is p in $G * H$ such that p forces ν is in a_i. But $\mathbb{P}$ forces $\dot{a}_i \subseteq \dot{a}_j$. So p also forces ν is in $\dot{a}_j$, hence ν is in a_j. To prove continuity, let $\delta < \omega_1$ be a limit ordinal. By what we just proved, $\bigcup\{a_i : i < \delta\} \subseteq a_\delta$. Let ν be in a_δ. Then there is p in $G * H$ which forces that ν is in $\dot{a}_\delta$. Let E be the collection of conditions q in $\mathbb{P}$ such that either q forces ν is not in $\dot{a}_\delta$, or there exists $i < \delta$ such that q forces $\nu \in \dot{a}_i$. Since $\mathbb{P}$ forces $\bigcup\{\dot{a}_i : i < \delta\} = \dot{a}_\delta$, E is dense open, and by elementarity, E is in N. Let q be in $(G * H) \cap E \cap N$. Since p and q are compatible, there is $i < \delta$ such that q forces ν is in $\dot{a}_i$. Hence ν is in a_i. So $a_\delta = \bigcup\{a_i : i < \delta\}$.

It follows that $\{a_i : i < \omega_1\}$ is a club subset of $P_{\omega_1}(\alpha)$. Since $B \subseteq \omega_1$ is stationary, the set $\{a_i : i \in B\}$ is stationary in $P_{\omega_1}(\alpha)$. We claim that $\{a_i : i \in B\}$ is disjoint from $\{x_i : i < \alpha\}$. Then by the definition of S, α is in S, which finishes the proof.

Since $\vec{x} = \langle x_i : i < \omega_2 \rangle$ is in N, $\{x_i : i < \alpha\}$ is a subset of N. Suppose for a contradiction that for some i in B and $j < \alpha$, $a_i = x_j$. Let F be the collection of conditions p in $\mathbb{P}$ which decide for some $\nu < \omega_2$ that ν is in $\dot{a}_i \triangle x_j$. Since $\mathbb{P}$ forces $\dot{a}_i$ is not in V, whereas x_j is in V, F is a dense open set. By elementarity, F is in

N. Fix p in $(G * H) \cap F \cap N$. Then there is ν in $N \cap \omega_2 = \alpha$ such that p forces ν is in $\dot{a}_i \triangle x_j$. If ν is in x_j, then p forces ν is not in $\dot{a}_i$. Since $G * H$ is a filter, there is no condition in $G * H$ which forces that ν is in $\dot{a}_i$. So ν is not in a_i, which contradicts our assumption that $a_i = x_j$. On the other hand, if ν is not in x_j, then p forces ν is in $\dot{a}_i \setminus x_j$. So ν is in $a_i \setminus x_j$, which again contradicts that $a_i = x_j$. $\square$

6. Variations of Internal Approachability

We now discuss some variations of the notion of an internally approachable set, and relate these properties to disjoint stationary sequences and disjoint club sequences.

Definition 6.1. *Let $\kappa < \lambda$ be regular uncountable cardinals, and suppose N is a set in $[H(\lambda)]^\kappa$.*

(1) N is internally unbounded *if for any set x in $P_\kappa(N)$, there is a set a in $N \cap P_\kappa(N)$ such that $x \subseteq a$.*

(2) N is internally stationary *if $N \cap P_\kappa(N)$ is a stationary subset of $P_\kappa(N)$.*

(3) N is internally club *if $N \cap P_\kappa(N)$ contains a club subset of $P_\kappa(N)$.*

(4) N is internally approachable *if N is internally approachable of length κ.*

The properties in Definition 6.1 appear in several papers of Matt Foreman, including [6] and [9]. In [9] it is asked whether these properties are equivalent. Note that (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) holds for club many N (see Section 2). Assuming $\kappa^{<\kappa} = \kappa$, it is not hard to show that (1) implies $P_\kappa(N) \subseteq N$, from which it follows that N is internally approachable. In general, these properties are not equivalent, as we proved in [14] and [15].

Previously the properties in Definition 6.1 were studied independently of the topic of disjoint club sequences. But it turns out there is a close relation between these ideas, as we show in Theorem 6.3 below.

Lemma 6.2. *Let κ be a regular uncountable cardinal and assume $2^\kappa = \kappa^+$. Then there is a club set $C \subseteq [H(\kappa^+)]^\kappa$ such that for any N in C, if $N \cap \kappa^+$ is an ordinal with cofinality κ, then N is internally unbounded.*

Proof. Since $2^\kappa = \kappa^+$, $H(\kappa^+)$ has size κ. Fix an increasing and continuous sequence $\langle N_i : i < \kappa^+ \rangle$ of elementary substructures of $H(\kappa^+)$ with union equal to $H(\kappa^+)$ such that for all $i < \kappa^+$, $\kappa \subseteq N_i$, N_i has size κ, and $N_i \in N_{i+1}$. Let D be the club set of $\alpha < \kappa^+$ such that $N_\alpha \cap \kappa^+ = \alpha$. Let $C = \{N_i : i \in D\}$, which is a club subset of $[H(\kappa^+)]^\kappa$.

Suppose that N is in C and $N \cap \kappa^+$ is an ordinal with cofinality κ. We show that N is internally unbounded. Fix $\alpha < \kappa^+$ such that $N = N_\alpha$. Since $N_\alpha \cap \kappa^+ = \alpha$, α has cofinality κ. Fix a cofinal set $d \subseteq \alpha$ with order type κ. Then $N = \bigcup\{N_i : i \in d\}$.

Consider a set x in $P_\kappa(N)$. Since d has order type κ, there is i in d such that x is a subset of N_i. Since N_i is in N and N_i has cardinality κ, by elementarity we can fix in N an increasing sequence $\langle a_j : j < \kappa \rangle$ of sets with size less than κ such that $\bigcup\{a_j : j < \kappa\} = N_i$. Now x is a subset of N_i, so there is $j < \kappa$ such that $x \subseteq a_j$. Then a_j is in $N \cap P_\kappa(N)$ and $x \subseteq a_j$. $\square$

Theorem 6.3. *Let κ be a regular uncountable cardinal and assume $2^\kappa = \kappa^+$. Then the following statements are equivalent.*

(1) There is a disjoint club sequence on κ^+.

(2) There are stationarily many N in $[H(\kappa^+)]^\kappa$ which are internally unbounded but not internally stationary.

Proof. ($1 \Rightarrow 2$) Suppose that $\vec{C} = \langle \mathcal{C}_\alpha : \alpha \in A\rangle$ is a disjoint club sequence on κ^+. By thinning out the $\mathcal{C}_\alpha$'s if necessary, we may assume that for all α in A, $\mathcal{C}_\alpha$ is equal to $\{b^\alpha_i : i < \kappa\}$, where $\langle b^\alpha_i : i < \kappa\rangle$ is an increasing and continuous sequence of sets with union equal to α.

We would like to show that there are stationarily many N in $[H(\kappa^+)]^\kappa$ which are internally unbounded but not internally stationary. By Lemma 6.2, it suffices to show that there are stationarily many sets N in $[H(\kappa^+)]^\kappa$ such that $N \cap \kappa$ is an ordinal with cofinality κ and N is not internally stationary. So let $F : H(\kappa^+)^{<\omega} \to H(\kappa^+)$ be a function.

Since $2^\kappa = \kappa^+$, $H(\kappa^+)$ has size κ^+. Fix a bijection $g : \kappa^+ \to H(\kappa^+)$. Let $\langle N_i : i < \kappa^+\rangle$ be an increasing and continuous sequence of sets of size κ with union equal to $H(\kappa^+)$, such that for all $i < \kappa^+$, N_i is an elementary substructure of $\langle H(\kappa^+), \in, \vec{\mathcal{C}}, F, g\rangle$ and $N_i \cap \kappa^+ \in \kappa^+$. Let D be the club set of $\alpha < \kappa^+$ such that $N_\alpha \cap \kappa^+ = \alpha$.

Choose an ordinal β in $A \cap D$. Then β has cofinality equal to κ and $N_\beta \cap \kappa^+ = \beta$. Clearly N_β is closed under F and $\kappa \subseteq N_\beta$. So we need only to show that N_β is not internally stationary. Since $g : \kappa^+ \to H(\kappa^+)$ is a bijection, by elementarity $g \restriction \beta$ is a bijection from β onto N_β. Therefore the collection $\{g[b^\beta_i] : i < \kappa\}$ is a club subset of $P_\kappa(N_\beta)$. So to show that N_β is not internally stationary, it suffices to show that for all $i < \kappa$, $g[b^\beta_i]$ is not in N_β.

Let i be less than κ. Then b^β_i is the unique set in $H(\kappa^+)$ whose pointwise image under g is equal to $g[b^\beta_i]$. So b^β_i is definable in the structure $\langle H(\kappa^+), \in, \vec{\mathcal{C}}, F, g\rangle$ from the parameter $g[b^\beta_i]$. Suppose for a contradiction that $g[b^\beta_i]$ is in N_β. Then b^β_i is in N_β. Since $\langle \mathcal{C}_\alpha : \alpha \in A\rangle$ is a sequence of disjoint sets, β is the unique ordinal in A such that b^β_i is in $\mathcal{C}_\beta$. So by elementarity, β is in N_β, which contradicts that $N_\beta \cap \kappa^+ = \beta$.

($2 \Rightarrow 1$) Suppose there are stationarily many N in $[H(\kappa^+)]^\kappa$ which are internally unbounded but not internally stationary. Since $2^\kappa = \kappa^+$, $H(\kappa^+)$ has size κ^+. Fix a bijection $g : \kappa^+ \to H(\kappa^+)$. Let $\mathfrak{B}$ denote the structure $\langle H(\kappa^+), \in, \lhd, g\rangle$, where $\lhd$ is a well-ordering of $H(\kappa^+)$. Fix an increasing and continuous sequence $\langle N_i : i < \kappa^+\rangle$ of sets with union equal to $H(\kappa^+)$ such that for all $i < \kappa^+$, N_i is an elementary substructure of $\mathfrak{B}$, N_i has size κ, $\kappa \subseteq N_i$, and $N_i \in N_{i+1}$. Note that $\{N_i : i < \kappa^+\}$ is a club subset of $[H(\kappa^+)]^\kappa$. Let D be the club set of $\alpha < \kappa^+$ such that $N_\alpha \cap \kappa^+ = \alpha$.

We would like to define a disjoint club sequence $\langle \mathcal{C}_\alpha : \alpha \in A\rangle$ on κ^+. First define the domain of the sequence by letting

$$A = \{\alpha \in D \cap \mathrm{cof}(\kappa) : N_\alpha \text{ is not internally stationary}\}.$$

We claim that A is stationary. Indeed, let $C \subseteq \kappa^+$ be a club set. Since $\{N_i : i \in C \cap D\}$ is a club subset of $[H(\kappa^+)]^\kappa$, by (2) we can fix α in $C \cap D$ such that N_α is internally unbounded but not internally stationary. If α has cofinality κ, then α is in $A \cap C$ and we are done. If not, then let x be a cofinal subset of α with order type less than κ. Since N_α is internally unbounded, there is a in $N_\alpha \cap P_\kappa(N_\alpha)$ such that $x \subseteq a$. Then $x \subseteq a \cap \kappa^+ \subseteq N_\alpha \cap \kappa^+ = \alpha$, so $\sup(a \cap \kappa^+) = \alpha$. Since a is in N, α is in N_α by elementarity, which contradicts that $N_\alpha \cap \kappa^+ = \alpha$.

Let α be in A, and we define a set $\mathcal{C}_\alpha$ which is a club subset $P_\kappa(\alpha)$. Since N_α is not internally stationary, there exists an increasing and continuous sequence $\langle M_i : i < \kappa \rangle$ of sets in $P_\kappa(N_\alpha)$ with union equal to N_α such that for all $i < \kappa$, M_i is not in N_α. Let $\langle M_i^\alpha : i < \kappa \rangle$ be the $\lhd$-least such sequence.

Define b_i^α for $i < \kappa^+$ by letting $b_i^\alpha = M_i^\alpha \cap \kappa^+$. Since $N_\alpha \cap \kappa^+ = \alpha$, $\{b_i^\alpha : i < \kappa\}$ is a club subset of α. Now by elementarity, $g \restriction \alpha$ is a bijection from α onto N_α. So $\langle g[b_i^\alpha] : i < \kappa \rangle$ is an increasing and continuous sequence of sets in $P_\kappa(N_\alpha)$ with union equal to N_α. Fix a club $E_\alpha \subseteq \kappa$ such that for all i in E_α, $g[b_i^\alpha] = M_i^\alpha$. Now define $\mathcal{C}_\alpha = \{b_i^\alpha : i \in E_\alpha\}$. Clearly $\mathcal{C}_\alpha$ is a club subset of $P_\kappa(\alpha)$.

We claim that $\langle \mathcal{C}_\alpha : \alpha \in A \rangle$ is a disjoint club sequence on κ^+. We already know that A is a stationary subset of $\kappa^+ \cap \mathrm{cof}(\kappa)$, and for all α in A, $\mathcal{C}_\alpha$ is a club subset of $P_\kappa(\alpha)$. Let $\alpha < \beta$ be in A. Suppose for a contradiction that there is a set b in $\mathcal{C}_\alpha \cap \mathcal{C}_\beta$. Fix i_α in E_α and i_β in E_β such that $b = b_{i_\alpha}^\alpha = b_{i_\beta}^\beta$. Now N_α is a member of N_β, so by elementarity, the sequence $\langle M_i^\alpha : i < \kappa \rangle$ is a member of N_β. Since $\kappa \subseteq N_\beta$, for all $i < \kappa$, $M_i^\alpha \cap \kappa^+ = b_i^\alpha$ is in N_β. In particular, $b_{i_\alpha}^\alpha$ is in N_β. But $b_{i_\alpha}^\alpha = b_{i_\beta}^\beta$. So $b_{i_\beta}^\beta$ is in N_β. By elementarity, $g[b_{i_\beta}^\beta]$ is in N_β. The index i_β is in E_β, so $g[b_{i_\beta}^\beta] = M_{i_\beta}^\beta$. Therefore $M_{i_\beta}^\beta$ is in N_β, which contradicts that the set $\{M_i^\beta : i < \kappa\}$ is disjoint from N_β. □

Corollary 6.4. *The following statement is equiconsistent with the existence of a Mahlo cardinal: there are stationarily many $N \prec H(\omega_2)$ with size $\aleph_1$ such that N is internally unbounded but not internally stationary.*

Proof. By Theorem 6.3, it suffices to show that the existence of a disjoint club sequence on ω_2 together with $2^{\omega_1} = \omega_2$ is equiconsistent with the existence of a Mahlo cardinal. We proved this in [11]. □

On the other hand, the only way we know how to distinguish between internally unbounded and internally stationary for elementary substructures of $H(\omega_3)$ of size $\aleph_1$ uses a supercompact cardinal; see [14].

The next theorem is a variation of Theorem 6.3.

Theorem 6.5. *Let κ be a regular uncountable cardinal and assume $2^\kappa = \kappa^+$. Then the following statements are equivalent:*

(1) There exists a disjoint stationary sequence on κ^+.

(2) There are stationarily many N in $[H(\kappa^+)]^\kappa$ such that N is internally unbounded but not internally club.

Proof. The proof is almost the same as the proof of Theorem 6.3. We leave the minor adjustments in the proof to the interested reader. □

7. Adding Stationarily Many New IA sets

We now give a technical refinement of a theorem of Gitik [12], which we will use in the consistency proofs later in the paper. Suppose $V \subseteq W$ are models of ZFC with the same ordinals, and there is a real in $W \setminus V$. Gitik proved under these assumptions that for any regular uncountable cardinal κ in W, if $\lambda \geq (\kappa^+)^W$, then $P_\kappa(\lambda) \setminus V$ is a stationary subset of $P_\kappa(\lambda)$ in W.

Theorem 7.1. *Suppose $V \subseteq W$ are models of ZFC with the same ordinals and there is a real in $W \setminus V$. Let κ be a regular uncountable cardinal in W, and let X be a set in V such that $(\kappa^+)^W \subseteq X$. In W let $\chi \geq \kappa^+$ be a regular cardinal such* *that $X \subseteq H(\chi)$. Then in W there are stationarily many N in $P_\kappa(H(\chi)) \cap IA(\omega)$ such that $N \cap X$ is not in V.*

The proof of Theorem 7.1 which we give below is basically the same as the proof of Gitik's theorem in [12], although we provide somewhat more detail.

First let us review some notation and background material.

Definition 7.2. *Let κ be a regular uncountable cardinal. A* tree *on $\kappa^+ \cap \mathrm{cof}(\kappa)$ with* stationary splitting *is a set $T \subseteq (\kappa^+ \cap \mathrm{cof}(\kappa))^{<\omega}$ such that:*

(1) T consists of finite increasing sequences of ordinals and is closed under taking initial segments,

(2) T contains a stem *u such that for any t in T, either $t \subseteq u$ or $u \subseteq t$,*

(3) for any t in T which extends the stem u, the set $\{\alpha : t^\frown\alpha \in T\}$ is a stationary subset of κ^+.

For two such trees S and T, we say that S is a subtree *of T if $S \subseteq T$.*

Suppose T is a tree on $\kappa^+ \cap \mathrm{cof}(\kappa)$ with stationary splitting. For $n < \omega$, $T \restriction n$ denotes the set of s in T with domain less than n. The *n-th level* of T is the set of t in T with domain equal to n. If t is in T, let T_t be the subtree of T consisting of nodes s in T such that $s \subseteq t$ or $t \subseteq s$.

We write $[T]$ for the set of all functions $f : \omega \to \kappa^+$ such that $f \restriction n$ is in T for all $n < \omega$. A standard topology on $[T]$ is defined by letting the basic open sets be sets of the form $[T_t]$, where t is in T. The Borel subsets of $[T]$ are the sets in the σ-algebra generated by the open sets.

We will use the following special case of a theorem of Rubin and Shelah [20]. Let T be a tree on $\kappa^+ \cap \mathrm{cof}(\kappa)$ with stationary splitting, and suppose $\{B_i : i < \xi\}$ is a family of fewer than κ^+ many Borel subsets of $[T]$ such that $[T] = \bigcup\{B_i : i < \xi\}$. Then there is a subtree $S \subseteq T$ with stationary splitting, with the same stem as T, such that $[S] \subseteq B_i$ for some $i < \xi$. In our application of this theorem, the B_i's will be closed sets.

Lemma 7.3. *Suppose κ is a regular uncountable cardinal, and T is a tree on $\kappa^+ \cap \mathrm{cof}(\kappa)$ with stationary splitting. Let $\{N_s : s \in T\}$ be a family of sets such that for all s in T, N_s is a set of size less than κ, and for all t in T with $s \subseteq t$, $N_s \subseteq N_t$. For η in T, let $N_\eta = \bigcup\{N_{\eta\restriction n} : n < \omega\}$.*

Then there is a subtree $S \subseteq T$ with stationary splitting, with the same stem as T, and a function $F : S \to \kappa^+$ such that for all $s^\frown\alpha$ in S, $F(s) < \alpha$, and if η is in $[S]$ and $s^\frown\alpha$ is an initial segment of η, then $\sup(N_\eta \cap \alpha) \leq F(s)$.

Proof. For each α in $\kappa^+ \cap \mathrm{cof}(\kappa)$, fix an increasing sequence $\langle \xi_i^\alpha : i < \kappa\rangle$ cofinal in α. We define a sequence of trees $\langle T_n : n < \omega\rangle$, each of which has a stem equal to the stem of T. We will arrange that $T_n \restriction (n+1) = T_m \restriction (n+1)$ for all $m \geq n$. After defining the sequence of trees, we let $S = \bigcap\{T_n : n < \omega\}$. Then for all $n < \omega$, the n-th level of S is equal to the n-th level of T_n. So we can define a function $F : S \to \kappa^+$ by inductively defining $F(s)$, where s has domain n and is in T_n.

Let $T_0 = T$. Let $n < \omega$ and assume that T_n is defined. Define $T_{n+1} \restriction (n+1) = T_n \restriction (n+1)$. Consider s in T_n with domain n. For $i < \kappa$ define $B_i(s)$ as the set of η in $[(T_n)_s]$ such that $\sup(N_\eta \cap \eta(n)) \leq \xi_i^{\eta(n)}$. Clearly $[(T_n)_s] = \bigcup\{B_i(s) : i < \kappa\}$, since for any η in $[(T_n)_s]$, N_η has size less than κ, whereas $\eta(n)$ has cofinality κ.

For each $i < \kappa$, $B_i(s)$ is a closed subset of $[(T_n)_s]$. Indeed, suppose ν is in $[(T_n)_s] \setminus B_i(s)$. We find an open neighborhood of ν in $[(T_n)_s]$ disjoint from $B_i(s)$.

Since ν is not in $B_i(s)$, $\sup(N_\nu \cap \nu(n)) > \xi_i^{\nu(n)}$. But $N_\nu = \bigcup\{N_{\nu \restriction m} : m < \omega\}$. So there is $m < \omega$ larger than n and ζ in $N_{\nu \restriction m}$ such that $\xi_i^{\nu(n)} < \zeta < \nu(n)$. Then $[(T_n)_{\nu \restriction m}]$ is an open neighborhood of ν in $[(T_n)_s]$ which is disjoint from $B_i(s)$. For if η is in $[(T_n)_{\nu \restriction m}]$, then ζ is in $N_{\nu \restriction m} = N_{\eta \restriction m}$, and $\eta(n) = \nu(n)$. So $\xi_i^{\eta(n)} = \xi_i^{\nu(n)} < \zeta \le \sup(N_\eta \cap \eta(n))$.

By the theorem of Rubin and Shelah, fix a subtree $T(s) \subseteq (T_n)_s$ with stationary splitting, with the same stem as $(T_n)_s$, and an ordinal $i_s < \kappa$, such that $[T(s)] \subseteq B_{i_s}(s)$. If the stem of T_n has domain larger than n, then there is a unique α such that $s^\frown\alpha$ is in T_n. In this case, define $F(s) = \xi_{i_s}^\alpha$ and $A(s) = \{\alpha\}$. Otherwise the set $\{\alpha : s^\frown\alpha \in T(s)\}$ is a stationary subset of κ^+, and the function which maps any α in this set to $\xi_{i_s}^\alpha$ is regressive. By Fodor's Lemma, there is a stationary set $A(s) \subseteq \{\alpha : s^\frown\alpha \in T(s)\}$ and a fixed ordinal $F(s)$ such that for all α in $A(s)$, $F(s) = \xi_{i_s}^\alpha$.

Now define T_{n+1} as follows. As mentioned before, let $T_{n+1} \restriction (n+1) = T_n \restriction (n+1)$. Suppose t is in T_n and $\mathrm{dom}(t) > n$. We let t be in T_{n+1} if t is in $T(t \restriction n)$ and $t(n)$ is in $A(t \restriction n)$.

Let $S = \bigcap\{T_n : n < \omega\}$. Then S is a subtree of T with stationary splitting and with the same stem as T. Consider a node $s^\frown\alpha$ in S, and let $n = \mathrm{dom}(s)$. Since $s^\frown\alpha$ is in T_{n+1} and has domain greater than n, by definition $s^\frown\alpha$ is in $T(s)$ and α is in $A(s)$. By construction, $F(s) = \xi_{i_s}^\alpha$, which is less than α. Suppose η is in $[S]$ and $s^\frown\alpha$ is an initial segment of η. Then for all $m > n$, $\eta \restriction m$ is in T_{n+1}, so by definition, $\eta \restriction m$ is in $T(s)$. Hence η is in $[T(s)]$. But $[T(s)]$ is a subset of $B_{i_s}(s)$. So $\sup(N_\eta \cap \eta(n)) \le \xi_{i_s}^{\eta(n)} = \xi_{i_s}^\alpha = F(s)$. $\square$

Lemma 7.4. *Suppose $V \subseteq W$ are models of ZFC with the same ordinals and there is a real in $W \setminus V$. In W, let κ be a regular uncountable cardinal, and let T be a tree on $\kappa^+ \cap \mathrm{cof}(\kappa)$ with stationary splitting. Assume $\{N_s : s \in T\}$ is a family in W of sets such that for all s in T, the range of s is a subset of N_s, N_s has size less than κ, and for all t in T with $s \subseteq t$, $N_s \subseteq N_t$. For η in T, let $N_\eta = \bigcup\{N_{\eta \restriction n} : n < \omega\}$. Then there is η in $[T]$ such that $N_\eta \cap \kappa^+$ is not in V.*

Proof. By Lemma 7.3, fix a subtree $S \subseteq T$ with stationary splitting and with the same stem as T and a function $F : S \to \kappa^+$ such that for all $s^\frown\alpha$ in S, $F(s) < \alpha$, and if η is in $[S]$ and $s^\frown\alpha$ is an initial segment of η, then $\sup(N_\eta \cap \alpha) \le F(s)$. Let $h : \omega \to 2$ be a function in $W \setminus V$.

We define by induction three branches η, η_0, and η_1 in $[S]$. Suppose $m < \omega$ and $\eta \restriction m$, $\eta_0 \restriction m$, and $\eta_1 \restriction m$ are defined and are in S. Since S has stationary splitting, the set $\{\alpha : (\eta \restriction m)^\frown\alpha \in S\}$ is unbounded in κ^+. Choose $\eta(m)$ in this set which is larger than $F(\eta_0 \restriction m)$, $F(\eta_1 \restriction m)$, and any ordinal appearing in $\eta_0 \restriction m$ and $\eta_1 \restriction m$. Then choose $\eta_0(m)$ and $\eta_1(m)$ so that $\eta_0 \restriction (m+1)$ and $\eta_1 \restriction (m+1)$ are in S, $\eta_0(m)$ and $\eta_1(m)$ are distinct ordinals larger than $\eta(m)$ and $F(\eta \restriction (m+1))$, and such that $\eta_0(m) < \eta_1(m)$ iff $h(m) = 0$.

This completes the construction of η, η_0, and η_1. The following statements are easily verified from the definition.

(1) For $i < 2$ and $m < \omega$, $\min((N_{\eta_i} \cap \kappa^+) \setminus \eta(m)) = \eta_i(m)$.

(2) For $i < 2$ and $m < \omega$, $\min((N_\eta \cap \kappa^+) \setminus \eta_i(m)) = \eta(m+1)$.

(3) For $m < \omega$, $h(m) = 0$ iff $\min((N_{\eta_0} \cap \kappa^+) \setminus \eta(m)) < \min((N_{\eta_1} \cap \kappa^+) \setminus \eta(m))$.

For (1), we know $\eta_i(m)$ is in N_{η_i} by assumption, and $\eta_i(m) > \eta(m)$. So $\min((N_{\eta_i} \cap \kappa^+) \setminus \eta(m)) \leq \eta_i(m)$. On the other hand, $\sup(N_{\eta_i} \cap \eta_i(m)) \leq F(\eta_i \restriction m) < \eta(m)$, so equality holds. The proof of (2) is similar. Now (3) follows immediately from (1) and the definition of $\eta_0(m)$ and $\eta_1(m)$.

We claim that at least one of the sets $N_\eta \cap \kappa^+$, $N_{\eta_0} \cap \kappa^+$, and $N_{\eta_1} \cap \kappa^+$ is not in V. Then we are done, since η, η_0, and η_1 are in $[T]$. Suppose for a contradiction they are all in V. We show that η is in V as well. Working in V, define a function $g : \omega \to On$ by recursion as follows. Let $g(0) = \eta(0)$. Given $g(m)$, define

$$g(m+1) = \min((N_\eta \cap \kappa^+) \setminus \min((N_{\eta_0} \cap \kappa^+) \setminus g(m))).$$

We claim that $g = \eta$. Indeed, $g(0) = \eta(0)$, and if $g(m) = \eta(m)$, then by (1) and (2) and the definition of g, $g(m+1) = \eta(m+1)$. So η is in V. But now by (3), h is definable in V from η, $N_{\eta_0} \cap \kappa^+$, and $N_{\eta_1} \cap \kappa^+$, which is a contradiction. $\square$

Now we are ready to prove Theorem 7.1. Suppose $V \subseteq W$ are models of ZFC with the same ordinals and there is a real in $W \setminus V$. Let κ be a regular uncountable cardinal in W, and let X be a set in V such that $(\kappa^+)^W \subseteq X$. In W let $\chi \geq \kappa^+$ be a regular cardinal such that $X \subseteq H(\chi)$. We show there are stationarily many N in $P_\kappa(H(\chi)) \cap \mathrm{IA}(\omega)$ such that $N \cap X$ is not in V.

We work in W. Let $F : H(\chi)^{<\omega} \to H(\chi)$ be a function. We will find a set N in $P_\kappa(H(\chi)) \cap \mathrm{IA}(\omega)$ such that $N \cap \kappa \in \kappa$, N is closed under F, and $N \cap X$ is not in V. Let T be the tree of all finite increasing sequences in $(\kappa^+ \cap \mathrm{cof}(\kappa))^{<\omega}$. Let $\mathfrak{B}$ denote the structure $\langle H(\chi), \in, \lhd, F\rangle$, where $\lhd$ is a well-ordering of $H(\chi)$.

We define for each s in T a set N_s in $P_\kappa(H(\chi))$ by induction on the domain of s. Let $N_{\langle\rangle}$ be the empty set. Suppose $m < \omega$ and N_s is defined for all s in T with domain equal to m. Consider $s^\frown\alpha$ in T such that s has domain m. Define $N_{s^\frown\alpha}$ as the Skolem hull in $\mathfrak{B}$ of the set

$$N_s \cup \{N_s\} \cup \{s^\frown\alpha\} \cup \sup(N_s \cap \kappa).$$

This completes the definition of the family $\{N_s : s \in T\}$. Note that for all s in T, N_s is a set in $P_\kappa(H(\chi))$ closed under F, the range of s is a subset of N_s, and for all t in T with $s \subseteq t$, $N_s \subseteq N_t$.

For each η in $[T]$, let $N_\eta = \bigcup\{N_{\eta\restriction n} : n < \omega\}$. Then $N_\eta \prec \mathfrak{B}$, so N_η is closed under F. Also $N_\eta \cap \kappa \in \kappa$. Since $N_{\eta\restriction n} \in N_{\eta\restriction(n+1)}$ for all $n < \omega$, N_η is in $P_\kappa(H(\chi)) \cap \mathrm{IA}(\omega)$. By Lemma 7.4, choose η in $[T]$ such that $N_\eta \cap \kappa^+$ is not in V. Then $N_\eta \cap X$ is not in V either, since $N_\eta \cap \kappa^+ = (N_\eta \cap X) \cap \kappa^+$. This completes the proof.

8. Mixed Support Forcing Iterations

In the remaining sections of the paper, we present some consistency results related to the combinatorial properties studied in the first part of the paper. These consistency results will rely on a mixed support forcing iteration which we worked out in detail in our previous paper [16]. This iteration is described in the next theorem.

Theorem 8.1 ([16])**.** *Suppose $\mu^{<\mu} = \mu$, κ is a regular cardinal greater than μ, and for all $\nu < \kappa$, $\nu^{<\mu} < \kappa$. Consider a forcing iteration*

$$\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \beta, j < \beta\rangle$$

satisfying the following conditions:

(1) If $i < \beta$ is even, then $\mathbb{P}_i$ forces $\dot{\mathbb{Q}}_i = \mathrm{Add}(\mu)$,

(2) If $i < \beta$, then $\mathbb{P}_i$ forces that $\dot{\mathbb{Q}}_i$ is μ-closed,

(3) If $\delta \leq \beta$ is a limit ordinal, then $\mathbb{P}_\delta$ consists of all partial functions $p : \delta \to V$ such that $p \restriction i$ is in $\mathbb{P}_i$ for all $i < \delta$, $|\mathrm{dom}(p) \cap Even| < \mu$, and $|\mathrm{dom}(p) \cap Odd| < \kappa$. For p and q in $\mathbb{P}_\delta$, let $q \leq p$ if $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and for all i in $\mathrm{dom}(p)$, $q \restriction i$ forces $q(i) \leq p(i)$.

In addition, for all even α with $\alpha+1 < \beta$, define a weak ordering $\leq^*$ on $\mathrm{Add}(\mu) * \dot{\mathbb{Q}}_{\alpha+1} = \dot{\mathbb{Q}}_\alpha * \dot{\mathbb{Q}}_{\alpha+1}$ in $V^{\mathbb{P}_\alpha}$ by letting $p_2 * \dot{q}_2 \leq^* p_1 * \dot{q}_1$ if $p_2 * \dot{q}_2 \leq p_1 * \dot{q}_1$ and $p_2 = p_1$.

(4) If α is even and $\alpha + 1 < \beta$, then $\mathbb{P}_\alpha$ forces $\langle \mathrm{Add}(\mu) * \dot{\mathbb{Q}}_{\alpha+1}, \leq^* \rangle = \langle \dot{\mathbb{Q}}_\alpha * \dot{\mathbb{Q}}_{\alpha+1}, \leq^* \rangle$ is κ-strategically closed.

Assuming that these conditions are satisfied, then:

(a) $\mathbb{P}_\beta$ is μ-closed,

(b) $\mathbb{P}_\beta$ preserves all cardinals and cofinalities less than or equal to κ,

(c) $\mathbb{P}_\beta$ is κ-proper for IA(μ),

(d) $\mathbb{P}_\beta$ is $({<}\kappa, \infty, \mu)$-distributive; that is, for any $\xi < \kappa$, if $g : \xi \to On$ is a function in $V^{\mathbb{P}_\beta}$, then there is a function $h : \xi \to [On]^\mu$ in V such that $g(i) \in h(i)$ for all $i < \xi$,

(e) $\mathbb{P}_\beta$ forces that whenever X is a subset of V such that for all A in $([V]^{<\kappa})^V$, $X \cap A$ is in V, then X is in V.

Since $\mathbb{P}_i$ is μ-closed for all $i \leq \beta$, the poset $\mathrm{Add}(\mu)$ is the same in the ground model and in any intermediate extension. Let δ be a limit ordinal less than or equal to β. If $\mathrm{cf}(\delta) < \mu$, then (3) implies $\mathbb{P}_\delta$ is the inverse limit of $\langle \mathbb{P}_i : i < \delta \rangle$. If $\mathrm{cf}(\delta) \geq \kappa$, then (3) implies $\mathbb{P}_\delta$ is the direct limit of $\langle \mathbb{P}_i : i < \delta \rangle$. If $\mu \leq \mathrm{cf}(\delta) < \kappa$, then $\mathbb{P}_\delta$ is a mixed support limit.

Let us emphasize that in condition (4), it is not the two-step iteration itself which is κ-strategically closed, but rather the two-step iteration with the *weak ordering* $\leq^*$. A special case in which condition (4) holds is when $\mathrm{Add}(\mu)$ forces $\dot{\mathbb{Q}}_{\alpha+1}$ is κ-strategically closed. For example, in the forcing iterations defined in Sections 9 and 10, κ-closed Lévy collapses are used at odd stages.

On the other hand, (4) does *not* imply that $\mathrm{Add}(\mu)$ forces $\dot{\mathbb{Q}}_{\alpha+1}$ is κ-strategically closed. For example, in Section 11 we consider a forcing poset $\mathrm{Add}(\omega) * \dot{\mathbb{Q}}$, where $\dot{\mathbb{Q}}$ adds an increasing and continuous sequence of sets in $P_{\omega_1}(H(\omega_2)^V) \cap V$ with order type ω_1 and union $H(\omega_2)^V$. Since $\dot{\mathbb{Q}}$ destroys the stationarity of the stationary set $(P_{\omega_1}(H(\omega_2)^V) \cap V^{\mathrm{Add}(\omega)}) \setminus V$, $\dot{\mathbb{Q}}$ is not proper. In particular, $\dot{\mathbb{Q}}$ is not ω_1-strategically closed. However, the poset $\langle \mathrm{Add}(\omega) * \dot{\mathbb{Q}}, \leq^* \rangle$ is ω_1-strategically closed. For more details on this example, see Section 2 of [16].

Let us draw some additional conclusions about such an iteration.

Lemma 8.2. *Suppose $\mu^{<\mu} = \mu$, κ is a regular cardinal greater than μ, and for all $\nu < \kappa$, $\nu^{<\mu} < \kappa$. Consider a forcing iteration*

$$\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \beta, j < \beta \rangle$$

satisfying assumptions (1), (2), (3), and (4) of Theorem 8.1. Then:

(f) If δ is a limit ordinal with cofinality greater than or equal to μ, then in $V^{\mathbb{P}_\beta}$, $\mathrm{cf}(\delta) \geq \mu$.

(g) If δ is a limit ordinal with cofinality greater than or equal to κ, then in $V^{\mathbb{P}_\beta}$, $\mathrm{cf}(\delta) \geq \kappa$.

(h) $\mathbb{P}_\beta$ forces that whenever $h : \kappa \to V$ is a function, all of whose initial segments are in V, then h is in V.

Proof. Since $\mathbb{P}_\beta$ is μ-closed, $\mathbb{P}_\beta$ does not add any new sequences of ordinals of order type less than μ. Property (f) follows immediately.

To prove (g), let δ be a limit ordinal, and suppose $\mathrm{cf}(\delta) < \kappa$ in $V^{\mathbb{P}_\beta}$. We prove that $\mathrm{cf}(\delta) < \kappa$ in V. Fix a function $g : \xi \to \delta$ in $V^{\mathbb{P}_\beta}$ which is cofinal in δ, where $\xi < \kappa$. By conclusion (d) of Theorem 8.1, there is a function $h : \xi \to [On]^\mu$ in V such that $g(i) \in h(i)$ for all $i < \xi$. Define a function $f : \xi \to On$ in V by letting $f(i) = \sup(h(i) \cap \delta)$ for all $i < \xi$. If $f(i) = \delta$ for some $i < \xi$, then $\mathrm{cf}(\delta) \leq \mu < \kappa$ in V and we are done. Otherwise the codomain of f can be taken as δ, so $f : \xi \to \delta$. Since g is cofinal in δ in $V^{\mathbb{P}_\beta}$, f is cofinal in δ. Hence $\mathrm{cf}(\delta) \leq \xi < \kappa$ in V.

For (h), let $h : \kappa \to V$ be a function in $V^{\mathbb{P}_\beta}$, all of whose proper initial segments are in V. Note that h is a subset of V. Let A be in $([V]^{<\kappa})^V$. Since A has size less than κ, $h \cap A$ is a subset of $h \restriction i$ for some $i < \kappa$. Therefore $h \cap A = (h \restriction i) \cap A$, which is in V. By conclusion (e) of Theorem 8.1, it follows that h is in V.

□

In applications of this mixed support forcing iteration, oftentimes we would like to apply the properties described in the conclusions of Theorem 8.1 and Lemma 8.2 not to $\mathbb{P}_\beta$, but rather to $\mathbb{P}_{\alpha,\beta}$, where α is even, $\alpha + 1 < \beta$, and $\mathbb{P}_\beta$ is factored as $\mathbb{P}_\alpha * \mathbb{P}_{\alpha,\beta}$. This is justified by the fact that $\mathbb{P}_{\alpha,\beta}$ is equivalent in $V^{\mathbb{P}_\alpha}$ to a forcing iteration satisfying assumptions (1), (2), (3), and (4) of Theorem 8.1.

The proof of this fact is a reworking of the usual intermediate stage analysis of a forcing iteration, as given for example in Section 5 of [3], for the mixed support case. The details are trivial, but very technical, and we only give a sketch of the proof below. The proof would be better understood after having read [16]. For the reader who is not already familiar with [16], we suggest only reading the statement of the theorem and skipping the proof.

Theorem 8.3. *Suppose $\mu^{<\mu} = \mu$, κ is a regular cardinal greater than μ, and for all $\nu < \kappa$, $\nu^{<\mu} < \kappa$. Consider a forcing iteration*

$$\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \beta, j < \beta \rangle$$

satisfying assumptions (1), (2), (3), and (4) of Theorem 8.1. Then for all even α with $\alpha + 1 < \beta$, $\mathbb{P}_\beta$ factors as $\mathbb{P}_\alpha * \mathbb{P}_{\alpha,\beta}$, such that in $V^{\mathbb{P}_\alpha}$:

(I) $\mathbb{P}_{\alpha,\beta}$ preserves all cardinals and cofinalities less than or equal to κ,

(II) $\mathbb{P}_{\alpha,\beta}$ is κ-proper for $IA(\mu)$,

(III) If δ is a limit ordinal with cofinality greater than or equal to κ, then $\mathbb{P}_{\alpha,\beta}$ forces that $\mathrm{cf}(\delta) \geq \kappa$*,*

(IV) $\mathbb{P}_{\alpha,\beta}$ forces that whenever $h : \kappa \to V^{\mathbb{P}_\alpha}$ is a function, all of whose proper initial segments are in $V^{\mathbb{P}_\alpha}$, then h is in $V^{\mathbb{P}_\alpha}$.

Proof. Suppose that $\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \beta, j < \beta \rangle$ is an iterated forcing satisfying assumptions (1), (2), (3), and (4) of Theorem 8.1. By properties (b), (c), (g), and (h) of Theorem 8.1 and Lemma 8.2, to prove the theorem it will suffice to show that for all even α with $\alpha + 1 < \beta$, $\mathbb{P}_\beta$ is equivalent to $\mathbb{P}_\alpha * \mathbb{P}_{\alpha,\beta}$, where $\mathbb{P}_{\alpha,\beta}$ is equivalent in $V^{\mathbb{P}_\alpha}$ to a forcing iteration satisfying assumptions (1), (2), (3), and (4) of Theorem 8.1.

First let us describe how to factor the iteration. So let $\alpha \leq \gamma \leq \beta$ be given. Define in V a set $\mathbb{P}_{\alpha,\gamma}$ as the collection $\{p \restriction [\alpha,\gamma) : p \in \mathbb{P}_\gamma\}$. In $V^{\mathbb{P}_\alpha}$, define a partial ordering on the set $\mathbb{P}_{\alpha,\gamma}$ by letting $t \leq s$ if there is a condition p in the generic filter $\dot{G}_\alpha$ for $\mathbb{P}_\alpha$ such that $p^\frown t \leq p^\frown s$ in $\mathbb{P}_\gamma$. Then the map $\pi : \mathbb{P}_\gamma \to \mathbb{P}_\alpha * \mathbb{P}_{\alpha,\gamma}$ defined by letting $\pi(p) = (p \restriction \alpha) * (p \restriction [\alpha,\gamma))$ is an isomorphism of $\mathbb{P}_\gamma$ onto a dense subset of $\mathbb{P}_\alpha * \mathbb{P}_{\alpha,\gamma}$.

Let α be an even ordinal such that $\alpha + 1 < \beta$. We would like to see that in $V^{\mathbb{P}_\alpha}$, $\mathbb{P}_{\alpha,\beta}$ is equivalent to a forcing iteration satisfying conditions (1), (2), (3), and (4). We give only a sketch of the relevant points of the argument, and refer the reader to Section 5 of [3] for more complete details.

Fix a generic filter G_α for $\mathbb{P}_\alpha$ over V. Let $\beta^* = \beta - \alpha$, so $\alpha + \beta^* = \beta$. In the model $V[G_\alpha]$, we define a forcing iteration

$$\langle \underline{\mathbb{P}}_i, \underline{\dot{\mathbb{Q}}}_j : i \leq \beta^*, j < \beta^* \rangle,$$

which satisfies conditions (1), (2), (3), and (4). Suppose $i < \beta^*$ and $\underline{\mathbb{P}}_i$ is defined. The appropriate induction hypotheses imply $\underline{\mathbb{P}}_i$ is equivalent in $V[G_\alpha]$ to $\mathbb{P}_{\alpha,\alpha+i}$. We translate the $\mathbb{P}_{\alpha+i}$-name $\dot{\mathbb{Q}}_{\alpha+i}$ to a $\underline{\mathbb{P}}_i$ name $\underline{\dot{\mathbb{Q}}}_i$, which is interpreted as the same forcing poset as $\dot{\mathbb{Q}}_{\alpha+i}$ in any generic extension of $V[G_\alpha]$ by $\underline{\mathbb{P}}_i$. It is easily seen that $\underline{\mathbb{P}}_{i+1} = \underline{\mathbb{P}}_i * \underline{\dot{\mathbb{Q}}}_i$ is equivalent to $\mathbb{P}_{\alpha,\alpha+i+1}$.

Let δ be a limit ordinal less than or equal to β^*, and suppose $\underline{\mathbb{P}}_i$ is defined for all $i < \delta$. Let $\underline{\mathbb{P}}_\delta$ be the mixed support limit of $\langle \underline{\mathbb{P}}_i : i < \delta \rangle$ as described in condition (3). One shows that $\mathbb{P}_{\alpha,\alpha+\delta}$ is isomorphic to a dense subset of $\underline{\mathbb{P}}_\delta$, by a map which is obtained by piecing together the isomorphisms $\mathbb{P}_{\alpha,\alpha+i} \to \underline{\mathbb{P}}_i$ for $i < \delta$. The only non-trivial issue is to show that the range of the map thus obtained is dense in $\underline{\mathbb{P}}_\delta$. The problem is that there might be conditions in $\underline{\mathbb{P}}_\delta$ whose domain is not in the ground model V. To handle this, we claim that for every set b in $V[G_\alpha]$ which is the domain of a condition in $\underline{\mathbb{P}}_\delta$, there is a set $B \subseteq [\alpha, \alpha+\delta)$ in V such that $|B \cap \mathrm{Even}| < \mu$, $|B \cap \mathrm{Odd}| < \kappa$, and $b \subseteq \{\nu - \alpha : \nu \in B\}$. Then given a condition u in $\underline{\mathbb{P}}_\delta$ with domain b, we can find B as above which will serve as the domain of a condition in $\mathbb{P}_{\alpha,\alpha+\delta}$ which will map to a condition in $\underline{\mathbb{P}}_\delta$ which is below u.

So assume p is a condition in $\mathbb{P}_\alpha$ which forces that $\dot{b}$ is a subset of δ in $V^{\mathbb{P}_\alpha}$ such that $|\dot{b} \cap \mathrm{Even}| < \mu$ and $|\dot{b} \cap \mathrm{Odd}| < \kappa$. We find $s \leq p$ in $\mathbb{P}_\alpha$ and a set $B \subseteq [\alpha, \alpha+\delta)$ in V such that $|B \cap \mathrm{Even}| < \mu$, $|B \cap \mathrm{Odd}| < \kappa$, and s forces $\dot{b} \subseteq \{\nu - \alpha : \nu \in B\}$. To prove this, we will find $s \leq p$ and a set $C \subseteq \delta$ in V such that $|C \cap \mathrm{Even}| < \mu$, $|C \cap \mathrm{Odd}| < \kappa$, and s forces $b \subseteq C$. Then we let $B = \{\alpha + \gamma : \gamma \in C\}$. Since α is even, $\alpha + \gamma$ is even iff γ is even, for any γ in C. Therefore $|B \cap \mathrm{Even}| < \mu$ and $|B \cap \mathrm{Odd}| < \kappa$, and we are done.

Since $\dot{b}$ is forced to have size less than κ, we can fix $q \leq p$ and a cardinal $\xi < \kappa$ such that q forces $\dot{b}$ has size ξ. Let $\langle \dot{\nu}_i : i < \xi \rangle$ be a $\mathbb{P}_\alpha$-name which q forces is an enumeration of $\dot{b}$. Since $\mathbb{P}_\alpha$ is μ-closed and q forces that $|\dot{b} \cap \text{Even}| < \mu$, there is $r \leq q$ and a set $C_0 \subseteq \delta$ with size less than μ such that r forces $\dot{b} \cap \text{Even} = C_0$. By property (d) of $\mathbb{P}_\alpha$, there is a $s \leq r$ and a sequence $\langle a_i : i < \xi \rangle$ of subsets of δ with size less than or equal to μ such that for all $i < \xi$, s forces $\dot{\nu}_i \in a_i$. For each $i < \xi$, let c_i be the set of $\gamma < \delta$ such that there is $t \leq s$ which forces $\gamma = \dot{\nu}_i$. Clearly $c_i \subseteq a_i$, and so $|c_i| \leq \mu$. Let $C = \bigcup \{c_i : i < \xi\}$. Then s forces $\dot{b} \subseteq C$. Since s forces $\dot{b} \cap \text{Even} = C_0$, clearly $C \cap \text{Even} = C_0$, which has size less than μ. As C is the union of ξ many sets of size at most μ, $|C| \leq \xi \cdot \mu < \kappa$. So in particular, $|C \cap \text{Odd}| < \kappa$. □

9. Constructing a Disjoint Stationary Sequence

In [11] we constructed a model in which there exists a disjoint club sequence on ω_2, assuming there is a Mahlo cardinal. The next theorem can be thought of as an attempt to generalize this result to cardinals larger than ω_2. See Section 12 for some open problems related to this construction.

Theorem 9.1. *Suppose that κ is a regular uncountable cardinal, and λ is a Mahlo cardinal greater than κ. Then there exists a forcing poset $\mathbb{P}_\lambda$ satisfying:*
(1) $\mathbb{P}_\lambda$ preserves cardinals and cofinalities less than or equal to κ,
(2) $\mathbb{P}_\lambda$ collapses λ to become κ^+, and forces $2^\omega = \kappa^+$,
(3) $\mathbb{P}_\lambda$ forces there exists a disjoint stationary sequence on κ^+.

Proof. We define by recursion a forcing iteration

$$\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \lambda, j < \lambda \rangle.$$

This iteration will satisfy the assumptions of Theorem 8.1, in the case $\mu = \omega$. Clearly $\omega^{<\omega} = \omega$ and for all $\nu < \kappa$, $\nu^{<\omega} = \nu < \kappa$. In particular, the iteration will preserve cardinals and cofinalities less than or equal to κ.

Suppose $\mathbb{P}_i$ is defined for a fixed $i < \lambda$. If i is even, let $\dot{\mathbb{Q}}_i$ be a $\mathbb{P}_i$-name for $\text{Add}(\omega)$. If i is odd, let $\dot{\mathbb{Q}}_i$ be a $\mathbb{P}_i$-name for $\text{Coll}(\kappa, \kappa^+)$.

Let $\delta \leq \lambda$ be a limit ordinal, and suppose $\mathbb{P}_i$ is defined for all $i < \delta$. Let $\mathbb{P}_\delta$ consist of all partial functions $p : \delta \to V$ such that $p \restriction i$ is in $\mathbb{P}_i$ for all $i < \delta$, $|\text{dom}(p) \cap \text{Even}| < \omega$, and $|\text{dom}(p) \cap \text{Odd}| < \kappa$.

This completes the definition. The iteration clearly satisfies the hypotheses of Theorem 8.1. So we can conclude that for all even $\alpha < \lambda$, $\mathbb{P}_\lambda$ can be factored as $\mathbb{P}_\lambda = \mathbb{P}_\alpha * \mathbb{P}_{\alpha,\lambda}$, such that in $V^{\mathbb{P}_\alpha}$:

(I) $\mathbb{P}_{\alpha,\lambda}$ preserves all cardinals and cofinalities less than or equal to κ.

(II) $\mathbb{P}_{\alpha,\lambda}$ is κ-proper for $\text{IA}(\omega)$.

(III) If δ is a limit ordinal with cofinality greater than or equal to κ, then $\mathbb{P}_{\alpha,\lambda}$ forces that $\text{cf}(\delta) \geq \kappa$.

Define a set A by letting

$$A = \{\alpha < \lambda : \kappa < \alpha \ \wedge \ \alpha \text{ is strongly inaccessible}\}.$$

Since λ is a Mahlo cardinal, A is stationary.

Consider an ordinal α in $A \cup \{\lambda\}$. Then for all $i < \alpha$, $|\mathbb{P}_i| < \alpha$. Moreover, if $\delta < \alpha$ is an ordinal with cofinality greater than or equal to κ, then $\mathbb{P}_\delta$ is the direct limit of $\langle \mathbb{P}_i : i < \delta \rangle$. It follows by a standard argument that $\mathbb{P}_\alpha$ is α-c.c., and any bounded subset of α in $V^{\mathbb{P}_\alpha}$ is in $V^{\mathbb{P}_i}$ for some $i < \alpha$. (See [3] for a review of such arguments.) Since Lévy collapses are used unboundedly often below α, $\mathbb{P}_\alpha$ forces that α is equal to κ^+. Also $\mathbb{P}_\alpha$ forces that $2^\omega = \alpha = \kappa^+$. In particular, $\mathbb{P}_\lambda$ collapses λ to become κ^+, and forces that $2^\omega = \kappa^+$.

We claim that in $V^{\mathbb{P}_\lambda}$, A is a stationary subset of $\kappa^+ \cap \mathrm{cof}(\kappa)$. Since $\mathbb{P}_\lambda$ is λ-c.c., A is a stationary subset of $\lambda = \kappa^+$ in $V^{\mathbb{P}_\lambda}$. Let α be in A. Since $\mathrm{cf}(\alpha) = \alpha > \kappa$ in V, $\mathbb{P}_\lambda = \mathbb{P}_{0,\lambda}$ forces $\mathrm{cf}(\alpha) \geq \kappa$ by (III). But in $V^{\mathbb{P}_\lambda}$, α has size κ. So $\mathrm{cf}(\alpha) = \kappa$ in $V^{\mathbb{P}_\lambda}$.

Let G be a generic filter for $\mathbb{P}_\lambda$ over V. We construct in $V[G]$ a disjoint stationary sequence $\langle \mathcal{S}_\alpha : \alpha \in A \rangle$. We know that A is a stationary subset of $\kappa^+ \cap \mathrm{cof}(\kappa)$ in $V[G]$. For all $i < \lambda$, let $G_i = G \cap \mathbb{P}_i$. Fix in V a regular cardinal χ much larger than λ.

Consider an ordinal α in A, and we define $\mathcal{S}_\alpha$. We apply Theorem 7.1 in the case of the models $V[G_\alpha] \subseteq V[G_{\alpha+1}]$ and $X = \alpha$. There is a real in $V[G_{\alpha+1}] \setminus V[G_\alpha]$, and in $V[G_{\alpha+1}]$, α is equal to κ^+. Working in the model $V[G_{\alpha+1}]$, define $\mathcal{T}_\alpha$ as the set of N in $P_\kappa(H(\chi)) \cap \mathrm{IA}(\omega)$ such that $N \cap \alpha$ is not in $V[G_\alpha]$. By Theorem 7.1, $\mathcal{T}_\alpha$ is a stationary subset of $P_\kappa(H(\chi))$.

Since $\mathrm{C{\small OLL}}(\kappa, \kappa^+)$ is κ-closed, $\mathcal{T}_\alpha$ remains stationary in $V[G_{\alpha+2}]$ by Lemmas 2.2 and 2.4. By property (II), $\mathbb{P}_{\alpha+2,\lambda}$ is κ-proper for $\mathrm{IA}(\omega)$ in $V[G_{\alpha+2}]$. Therefore $\mathcal{T}_\alpha$ is a stationary subset of $P_\kappa(H(\chi)^{V[G_{\alpha+1}]})$ in $V[G]$ by Lemma 2.4. Define $\mathcal{S}_\alpha = \{N \cap \alpha : N \in \mathcal{T}_\alpha\}$. Since $\mathcal{T}_\alpha$ is stationary, $\mathcal{S}_\alpha$ is a stationary subset of $P_\kappa(\alpha)$ in $V[G]$. Note that by the definition of $\mathcal{T}_\alpha$, $\mathcal{S}_\alpha$ is a subset of $V[G_{\alpha+1}] \setminus V[G_\alpha]$.

We claim that $\langle \mathcal{S}_\alpha : \alpha \in A \rangle$ is a disjoint stationary sequence on κ^+ in $V[G]$. We know that A is a stationary subset of $\kappa^+ \cap \mathrm{cof}(\kappa)$, and for all α in A, $\mathcal{S}_\alpha$ is a stationary subset of $P_\kappa(\alpha)$. Let $\alpha < \beta$ be in A. Then $\mathcal{S}_\alpha \subseteq V[G_{\alpha+1}] \subseteq V[G_\beta]$, and $\mathcal{S}_\beta \cap V[G_\beta]$ is empty. So $\mathcal{S}_\alpha \cap \mathcal{S}_\beta$ is empty. $\square$

10. Internally Stationary and Internally Club

Let us recall the definition of internally stationary and internally club sets from Section 6. Let $\kappa < \lambda$ be regular uncountable cardinals, and let N be a set in $[H(\lambda)]^\kappa$. The set N is *internally club* if $N \cap P_\kappa(N)$ contains a club subset of $P_\kappa(N)$. This is equivalent to the existence of an increasing and continuous sequence $\langle N_i : i < \kappa \rangle$ of sets in $N \cap P_\kappa(N)$ with union equal to N. The set N is *internally stationary* if $N \cap P_\kappa(N)$ is a stationary subset of $P_\kappa(N)$. Equivalently, N is internally stationary if there exists an increasing and continuous sequence $\langle N_i : i < \kappa \rangle$ of sets in $P_\kappa(N)$ with union equal to N, and a stationary set $B \subseteq \kappa$, such that $\{N_i : i \in B\} \subseteq N$.

Foreman and Todorčević [9] asked whether the properties of being internally stationary and internally club are equivalent. We solved this problem in [14] by showing that under the forcing axiom PFA_2, for all regular $\lambda \geq \omega_2$ there are stationarily many N in $[H(\lambda)]^{\aleph_1}$ such that N is internally stationary but not internally club. We now generalize this result to sets of size larger than $\aleph_1$. See Section 12 for some open questions related to this construction.

Theorem 10.1. *Suppose that κ is a regular uncountable cardinal and λ is a supercompact cardinal larger than κ. Then there exists a forcing poset $\mathbb{P}_\lambda$ satisfying:*

(1) $\mathbb{P}_\lambda$ preserves cardinals and cofinalities less than or equal to κ,

(2) $\mathbb{P}_\lambda$ *collapses* λ *to become* κ^+*, and forces* $2^\omega = \kappa^+$*,*

(3) $\mathbb{P}_\lambda$ *forces that for all regular* $\theta \geq \kappa^+$*, there are stationarily many* N *in* $[H(\theta)]^\kappa$ *such that* N *is internally stationary but not internally club.*

Proof. We define by recursion a forcing iteration

$$\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \lambda, j < \lambda \rangle.$$

This iteration will satisfy the assumptions of Theorem 8.1, in the case $\mu = \omega$. For this purpose, fix a Laver function $f : \lambda \to V_\lambda$. Suppose $\mathbb{P}_i$ is defined for a fixed $i < \lambda$. If i is even, let $\dot{\mathbb{Q}}_i$ be a $\mathbb{P}_i$-name for $\textsc{Add}(\omega)$. Suppose i is odd. Let α be the predecessor of i, so $i = \alpha + 1$. We consider two cases.

> Case 1: α is a strongly inaccessible cardinal greater than κ, $|\mathbb{P}_j| < \alpha$ for all $j < \alpha$, and $f(\alpha)$ is a regular cardinal greater than or equal to α.
>
> Case 2: Otherwise.

If Case 2 holds, let $\dot{\mathbb{Q}}_i = \dot{\mathbb{Q}}_{\alpha+1}$ be a $\mathbb{P}_i$-name for $\textsc{Coll}(\kappa, \kappa^+)$.

Suppose Case 1 holds. As an induction hypothesis, assume that $\mathbb{P}_\alpha$ is α-c.c.; this will follow easily from the assumptions of Case 1 and the definition of the limit stages described below. Since we use the Lévy collapse unboundedly often below α, $\mathbb{P}_\alpha$ forces that α is equal to κ^+. Since $\mathbb{P}_\alpha$ is α-c.c., in $V^{\mathbb{P}_\alpha}$ the ordinal $f(\alpha)$ is a regular cardinal greater than or equal to $\alpha = \kappa^+$. Let $\dot{\mathbb{Q}}_i = \dot{\mathbb{Q}}_{\alpha+1}$ be a $\mathbb{P}_i$-name for $\textsc{Coll}(\kappa, H(f(\alpha))^{V^{\mathbb{P}_\alpha}})$.

Let δ be a limit ordinal less than or equal to λ, and suppose $\mathbb{P}_i$ is defined for all $i < \delta$. Let $\mathbb{P}_\delta$ consist of all partial functions $p : \delta \to V$ such that for all $i < \delta$, $p \restriction i$ is in $\mathbb{P}_i$, $|\mathrm{dom}(p) \cap \mathrm{Even}| < \omega$, and $|\mathrm{dom}(p) \cap \mathrm{Odd}| < \kappa$. If δ satisfies the assumptions of Case 1, standard arguments show that $\mathbb{P}_\delta$ is δ-c.c.

This completes the definition. The iteration clearly satisfies the hypotheses of Theorem 8.1. So we can conclude that for all even $\alpha < \lambda$, $\mathbb{P}_\lambda$ can be factored as $\mathbb{P}_\lambda = \mathbb{P}_\alpha * \mathbb{P}_{\alpha,\lambda}$, such that in $V^{\mathbb{P}_\alpha}$:

> (I) $\mathbb{P}_{\alpha,\lambda}$ preserves all cardinals and cofinalities less than or equal to κ.
>
> (II) $\mathbb{P}_{\alpha,\lambda}$ is κ-proper for $\mathrm{IA}(\omega)$.

Also, $\mathbb{P}_\lambda$ is λ-c.c., and $\mathbb{P}_\lambda$ forces that λ is equal to κ^+. Moreover, any bounded subset of λ in $V^{\mathbb{P}_\lambda}$ is in $V^{\mathbb{P}_i}$ for some $i < \lambda$. Hence $\mathbb{P}_\lambda$ forces that $2^\omega = \lambda$.

We claim that in $V^{\mathbb{P}_\lambda}$, for all regular $\theta \geq \lambda$, there are stationarily many N in $[H(\theta)]^\kappa$ such that N is internally stationary but not internally club.

To prove the claim, we first need to analyze an elementary embedding. Fix in V an elementary embedding $j : V \to M$ with critical point λ such that $M^{|H(\theta)|} \subseteq M$ and $j(f)(\lambda) = \theta$. This is possible since f is a Laver function. Consider in M the iteration

$$j(\langle \mathbb{P}_i, \dot{\mathbb{Q}}_i : i < \lambda \rangle) = \langle \mathbb{P}^j_i, \dot{\mathbb{Q}}^j_i : i < j(\lambda) \rangle.$$

Then for all $i < \lambda$, $\mathbb{P}^j_i = j(\mathbb{P}_i) = \mathbb{P}_i$. So in the definition of $\dot{\mathbb{Q}}_{\lambda+1}$ by cases in M, λ satisfies Case 1. Therefore $j(\mathbb{P}_\lambda)$ factors as

$$j(\mathbb{P}_\lambda) = \mathbb{P}_\lambda * \textsc{Add}(\omega) * \textsc{Coll}(\kappa, H(\theta)^{M^{\mathbb{P}_\lambda}}) * \mathbb{P}_{\lambda+2, j(\lambda)}.$$

Let $K = G * G_0 * G_1 * H$ be a generic filter for $j(\mathbb{P}_\lambda)$ over V. Since $j[G] = G \subseteq K$, in the generic extension $V[K]$ we can extend j to $j : V[G] \to M[K]$ such that $j(G) = K$.

For the rest of the proof, we write $\mathbf{H}(\theta)$ for $\mathbf{H}(\theta)^{\mathbf{V[G]}}$. As $\mathbb{P}$ is λ-c.c. and $M^\theta \cap V \subseteq M$, we have that $M[G]^\theta \cap V[G] \subseteq M[G]$ by a standard fact. So $H(\theta) = H(\theta)^{M[G]}$. Since $\mathbb{P}$ is a subset of $H(\lambda)$ and $\mathbb{P}$ is λ-c.c., $H(\theta) = H(\theta)^V[G]$. It follows that

$$j[H(\theta)] = j[H(\theta)^V][K].$$

Indeed, let $j(a)$ be in $j[H(\theta)]$, where a is in $H(\theta)$. Fix a $\mathbb{P}_\lambda$-name $\dot{a}$ in $H(\theta)^V$ such that $\dot{a}^G = a$. Then by elementarity, $j(a) = j(\dot{a}^G) = j(\dot{a})^K$, which is in $j[H(\theta)^V][K]$. On the other hand, suppose $c = j(\dot{b})^K$, where $\dot{b}$ is in $H(\theta)^V$. Then $c = j(\dot{b})^K = j(\dot{b})^{j(G)} = j(\dot{b}^G)$, which is in $j[\ H(\theta)^V[G]\ ] = j[H(\theta)]$. This proves the equality. Now by the closure of M, $j[H(\theta)^V]$ is in M, so $j[H(\theta)^V][K] = j[H(\theta)]$ is in $M[K]$. The function $j \restriction H(\theta)$ is the inverse of the transitive collapse of $j[H(\theta)]$ to $H(\theta)$, therefore $j \restriction H(\theta)$ is in $M[K]$ as well.

We would like to show that in $V[G]$ there are stationarily many N in $[H(\theta)]^\kappa$ such that N is internally stationary but not internally club. So fix a function $F : H(\theta)^{<\omega} \to H(\theta)$. In $V[K]$, $j : V[G] \to M[K]$ is an elementary embedding. So by elementarity, it suffices to show that in $M[K]$ there is a set N in $[j(H(\theta))]^{j(\kappa)} = [j(H(\theta))]^\kappa$ such that $j(\kappa) = \kappa \subseteq N$, N is closed under $j(F)$, and N is internally stationary but not internally club. So let $N = j[H(\theta)]$. We will prove that N satisfies these properties in $M[K]$.

First let us prove that N is in $[j(H(\theta))]^\kappa$. Clearly N is a subset of $j(H(\theta))$. Factor $j(\mathbb{P}_\lambda)$ as

$$j(\mathbb{P}_\lambda) = \mathbb{P}_\lambda * \textsc{Add}(\omega) * \textsc{Coll}(\kappa, H(\theta)^{M^{\mathbb{P}_\lambda}}) * \mathbb{P}_{\lambda+2,j(\lambda)}.$$

Since j is injective, N has the same size as $H(\theta)$ in $M[K]$, and $H(\theta)$ is equal to $H(\theta)^{M[G]}$. In $M[G]$, $H(\theta)$ has size at least θ, which is greater than or equal to κ^+. In $M[G * G_0 * G_1]$, $H(\theta)$ acquires size κ. Since $\mathbb{P}_{\lambda+2,j(\lambda)}$ preserves κ, $H(\theta)$ has size κ in $M[K]$. So indeed N is in $[j(H(\theta))]^\kappa$.

Now the critical point of j is λ, so $\kappa = j[\kappa] \subseteq N$. Also N is closed under $j(F)$. Indeed, let $j(a_1), \ldots, j(a_n)$ be in N, where $a_1, \ldots, a_n$ are in $H(\theta)$. Then $F(a_1, \ldots, a_n)$ is in $H(\theta)$, so $j(F(a_1, \ldots, a_n))$ is in N. But $j(F(a_1, \ldots, a_n)) = j(F)(j(a_1), \ldots, j(a_n))$.

Now we prove that N is internally stationary in $M[K]$. Fix a regular cardinal χ in V much larger than θ and $j(\lambda)$. Factor $j(\mathbb{P}_\lambda)$ as $\mathbb{P}_\lambda * \mathbb{P}_{\lambda,j(\lambda)}$. By property (II), the poset $\mathbb{P}_{\lambda,j(\lambda)}$ is κ-proper for $\mathrm{IA}(\omega)$ in $M[G]$. Define a set S in $M[G]$ by letting

$$S = P_\kappa(H(\chi)) \cap \mathrm{IA}(\omega).$$

Then S is a stationary set in $M[G]$. Since $\mathbb{P}_{\lambda,j(\lambda)}$ is κ-proper for $\mathrm{IA}(\omega)$, S remains a stationary subset of $P_\kappa(H(\chi)^{M[G]})$ in $M[K]$ by Lemma 2.4. Let $T = \{A \cap H(\theta) : A \in S\}$. Since S is stationary, T is a stationary subset of $P_\kappa(H(\theta))$ in $M[K]$. Also, since S is a subset of $M[G]$, T is a subset of $M[G]$.

Fix in $M[K]$ an increasing and continuous sequence $\langle b_i : i < \kappa \rangle$ of sets in $P_\kappa(H(\theta))$ with union equal to $H(\theta)$. This is possible since $H(\theta)$ has size κ in $M[K]$. Then $\{b_i : i < \kappa\}$ is a club subset of $P_\kappa(H(\theta))$. Since T is a stationary subset of $P_\kappa(H(\theta))$, there is a stationary set $B \subseteq \kappa$ such that $\{b_i : i \in B\}$ is a subset of T. But T is a subset of $M[G]$, so $\{b_i : i \in B\}$ is a subset of $M[G]$.

For all i in B, b_i is a subset of $H(\theta)$ in $M[G]$ with size less than θ. So b_i is in $H(\theta)$. It follows that $j(b_i) = j[b_i]$ is in $j[H(\theta)] = N$. Now $\langle j[b_i] : i < \kappa \rangle$ is an increasing and continuous sequence with union equal to N. This sequence is in $M[K]$ since it is definable in $M[K]$ from $\langle b_i : i < \kappa \rangle$ and $j \restriction H(\theta)$. So the set $\{j[b_i] : i \in B\}$ is a subset of $N \cap P_\kappa(N)$ which is stationary in $P_\kappa(N)$. Thus N is internally stationary in $M[K]$.

Suppose for a contradiction that N is internally club in $M[K]$. Fix an increasing and continuous sequence $\langle N_i : i < \kappa \rangle$ of sets with size less than κ such that for all $i < \kappa$, N_i is in $N = j[H(\theta)]$. Then $\langle j^{-1}[N_i] : i < \kappa \rangle$ is an increasing and continuous sequence of sets in $M[K]$ with union equal to $H(\theta)$. For each $i < \kappa$, fix a_i in $H(\theta)$ such that $j(a_i) = N_i$. Since $j(\kappa) = \kappa$, a_i has size less than κ. So $j(a_i) = j[a_i] = N_i$, therefore $j^{-1}[N_i] = a_i$. Hence $\langle a_i : i < \kappa \rangle$ is an increasing and continuous sequence of sets in $H(\theta) \cap P_\kappa(H(\theta))$ with union equal to $H(\theta)$.

Define a set U in $M[G * G_0]$ by letting

$$U = \{A \in P_\kappa(H(\chi)) \cap \mathrm{IA}(\omega) : A \cap H(\theta) \notin M[G]\}.$$

There is a real in $M[G*G_0] \setminus M[G]$, and $H(\theta)$ is a set in $M[G]$ such that $(\kappa^+)^{M[G*G_0]} = \lambda \subseteq H(\theta)$. Therefore by Theorem 7.1, U is a stationary subset of $P_\kappa(H(\chi))$ in $M[G * G_0]$. Since $\textsc{Coll}(\kappa, H(\theta))$ is κ-closed, U is stationary in $M[G * G_0 * G_1]$ by Lemmas 2.2 and 2.4. By property (II), $\mathbb{P}_{\lambda+2,j(\lambda)}$ is κ-proper for $\mathrm{IA}(\omega)$ in $M[G * G_0 * G_1]$. So U is a stationary subset of $P_\kappa(H(\chi)^{M[G*G_0]})$ in $M[K]$ by Lemma 2.4. Therefore in $M[K]$ the set $\{A \cap H(\theta) : A \in U\}$ is a stationary subset of $P_\kappa(H(\theta))$.

Since $\{a_i : i < \kappa\}$ is a club subset of $P_\kappa(H(\theta))$, there is $i < \kappa$ such that a_i is equal to $A \cap H(\theta)$ for some A in U. By the definition of U, $A \cap H(\theta) = a_i$ is not in $M[G]$. But a_i is in $H(\theta)$, and $H(\theta) = H(\theta)^{M[G]}$ is a subset of $M[G]$, which is a contradiction. So indeed N is not internally club. $\square$

11. Internally Club and Internally Approachable

Let $\kappa < \lambda$ be regular uncountable cardinals, and let N be a set in $[H(\lambda)]^\kappa$. The set N is internally club iff N is the union of an increasing and continuous sequence $\langle N_i : i < \kappa \rangle$ of sets in $N \cap P_\kappa(N)$. The set N is internally approachable iff N is the union of an increasing and continuous sequence $\langle N_i : i < \kappa \rangle$ of sets in $P_\kappa(N)$ such that for all $\alpha < \kappa$, $\langle N_i : i < \alpha \rangle$ is in N.

Foreman and Todorčević [9] asked whether the properties of being internally club and internally approachable are equivalent. We originally solved this problem in [15], by showing that under the Proper Forcing Axiom, for all regular $\lambda \geq \omega_2$ there are stationarily many N in $[H(\lambda)]^{\aleph_1}$ which are internally club but not internally approachable. In [17] we presented a more general argument, which distinguished internally club and internally approachable for sets of size the successor of a regular cardinal. This argument used a mixed support forcing iteration, which we later axiomatized in a general form in [16]. Using the iteration schema from [16], we are now in a position to present a very general construction of a model in which the properties of being internally club and internally approachable are distinct, which handles all possible cardinalities.

Theorem 11.1. *Suppose $\mu < \kappa < \lambda$ are regular cardinals, where λ is supercompact, $\mu^{<\mu} = \mu$, and for all $\nu < \kappa$, $\nu^{<\mu} < \kappa$. Then there exists a forcing poset $\mathbb{P}_\lambda$ satisfying:*

(1) $\mathbb{P}_\lambda$ preserves cardinals and cofinalities less than or equal to κ,

(2) $\mathbb{P}_\lambda$ collapses λ to become κ^+, and forces $2^\mu = \kappa^+$,

(3) $\mathbb{P}_\lambda$ forces that for all regular $\theta \geq \kappa^+$, there are stationarily many N in $[H(\theta)]^\kappa$ such that N is internally club but not internally approachable.

In particular, this theorem distinguishes internally club and internally approachable for sets of size an inaccessible cardinal. For sets of size the successor of a singular cardinal, Theorem 11.1 is used in combination with Prikry forcing techniques; see [13].

Before proving Theorem 11.1, let us give an overview of the forcing poset we will use. Let $\mu < \kappa$ be regular cardinals, and assume $\mu^{<\mu} = \mu$. Let $\theta \geq \kappa^+$ be regular. We define a two-step iteration $\textsc{Add}(\mu) * \dot{\mathbb{Q}}$. Let G be a generic filter for $\textsc{Add}(\mu)$ over the ground model V. In $V[G]$, define $\mathbb{Q}$ as the poset consisting of conditions which are increasing and continuous sequences $\langle a_i : i \leq \nu \rangle$ such that $\nu < \kappa$, and for all $i \leq \nu$, a_i is in $P_\kappa(H(\theta)^V) \cap V$. We let $q \leq p$ in $\mathbb{Q}$ if $q \restriction \mathrm{dom}(p) = p$.

The poset $\mathbb{Q}$ adds a generic sequence $\langle a_i : i < \kappa \rangle$ which is increasing, continuous, and cofinal in $P_\kappa(H(\theta)^V) \cap V$. In particular, $\bigcup \{a_i : i < \kappa\} = H(\theta)^V$. So $\mathbb{Q}$ collapses $H(\theta)^V$ to have size κ. In [17] it is proven that $\mathbb{Q}$ is $<\kappa$-distributive, so $\mathbb{Q}$ does not add any new sequences of ordinals with order type less than κ.

Define a weak ordering $\leq^*$ on $\textsc{Add}(\mu) * \dot{\mathbb{Q}}$ by letting $p_2 * \dot{q}_2 \leq^* p_1 * \dot{q}_1$ if $p_2 * \dot{q}_2 \leq p_1 * \dot{q}_1$ and $p_2 = p_1$. In Proposition 2.2 of [16] we proved that $\langle \textsc{Add}(\mu) * \dot{\mathbb{Q}}, \leq^* \rangle$ is κ-strategically closed. Also, $\textsc{Add}(\mu)$ forces that $\mathbb{Q}$ is μ-closed, as shown in Lemma 3.2 of [17].

Now we begin the proof of Theorem 11.1. Fix regular cardinals $\mu < \kappa < \lambda$ such that λ is supercompact, $\mu^{<\mu} = \mu$, and for all $\nu < \kappa$, $\nu^{<\mu} < \kappa$. We define by recursion an iterated forcing

$$\langle \mathbb{P}_i, \dot{\mathbb{Q}}_j : i \leq \lambda, j < \lambda \rangle.$$

For this purpose, fix a Laver function $f : \lambda \to V_\lambda$. Suppose $\mathbb{P}_i$ is defined for a fixed $i < \lambda$. If i is even, let $\dot{\mathbb{Q}}_i$ be a $\mathbb{P}_i$-name for $\textsc{Add}(\mu)$. Suppose i is odd. Let α be the predecessor of i, so $i = \alpha + 1$. We consider two cases.

Case 1: α is a strongly inaccessible cardinal greater than κ, $|\mathbb{P}_j| < \alpha$ for all $j < \alpha$, and $f(\alpha)$ is a regular cardinal greater than or equal to α.

Case 2: Otherwise.

If Case 2 holds, let $\dot{\mathbb{Q}}_i = \dot{\mathbb{Q}}_{\alpha+1}$ be a $\mathbb{P}_i$-name for $\textsc{Coll}(\kappa, \kappa^+)$.

Suppose Case 1 holds. As an induction hypothesis, assume $\mathbb{P}_\alpha$ is α-c.c.; this will follow easily from the assumptions of Case 1 and the definition of the limit stages given below. Since we use $\textsc{Coll}(\kappa, \kappa^+)$ at unboundedly many stages below α, $\mathbb{P}_\alpha$ forces that α is equal to κ^+. Since $\mathbb{P}_\alpha$ is α-c.c., $f(\alpha)$ is still a regular cardinal in $V^{\mathbb{P}_\alpha}$ and is greater than or equal to $\alpha = \kappa^+$. Working in $V^{\mathbb{P}_\alpha}$, let $\dot{\mathbb{Q}}_i = \dot{\mathbb{Q}}_{\alpha+1}$ be an $\textsc{Add}(\mu)$-name such that $\textsc{Add}(\mu) * \dot{\mathbb{Q}}_{\alpha+1}$ is the two-step iteration discussed earlier in this section for adding an increasing and continuous sequence of length κ through $P_\kappa(H(f(\alpha))^{V^{\mathbb{P}_\alpha}}) \cap V^{\mathbb{P}_\alpha}$.

Let $\delta \leq \lambda$ be a limit ordinal and suppose $\mathbb{P}_i$ is defined for all $i < \delta$. Let $\mathbb{P}_\delta$ consist of all partial functions $p : \delta \to V$ such that for all $i < \delta$, $p \restriction i$ is in $\mathbb{P}_i$, $|\mathrm{dom}(p) \cap \mathrm{Even}| < \mu$, and $|\mathrm{dom}(p) \cap \mathrm{Odd}| < \kappa$. If δ satisfies the properties listed in Case 1, then standard arguments show that $\mathbb{P}_\delta$ is δ-c.c.

This completes the definition. The iteration clearly satisfies the hypotheses of Theorem 8.1. So we can conclude that for all even $\alpha < \lambda$, $\mathbb{P}_\lambda$ factors as $\mathbb{P}_\alpha * \mathbb{P}_{\alpha,\lambda}$, where in $V^{\mathbb{P}_\alpha}$:

(I) $\mathbb{P}_{\alpha,\lambda}$ preserves all cardinals and cofinalities less than or equal to κ.

(II) $\mathbb{P}_{\alpha,\lambda}$ forces that if $h : \kappa \to V^{\mathbb{P}_\alpha}$ is a function, all of whose initial segments are in $V^{\mathbb{P}_\alpha}$, then h is in $V^{\mathbb{P}_\alpha}$.

In addition, $\mathbb{P}_\lambda$ is λ-c.c., and $\mathbb{P}_\lambda$ forces that λ is equal to κ^+. Also any bounded subset of λ in $V^{\mathbb{P}_\lambda}$ is in $V^{\mathbb{P}_i}$ for some $i < \lambda$. Therefore $\mathbb{P}_\lambda$ forces that $2^\mu = \kappa^+$.

Proposition 11.2. *In $V^{\mathbb{P}_\lambda}$, for any regular cardinal $\theta \geq \kappa^+$, there are stationarily many N in $[H(\theta)]^\kappa$ such that N is internally club but not internally approachable.*

Proof. Fix in V an elementary embedding $j : V \to M$ with critical point λ such that $M^{|H(\theta)|} \subseteq M$ and $j(f)(\lambda) = \theta$. This is possible since f is a Laver function. Consider the iteration

$$j(\langle \mathbb{P}_i, \dot{\mathbb{Q}}_i : i < \lambda \rangle) = \langle \mathbb{P}_i^j, \dot{\mathbb{Q}}_i^j : i < j(\lambda) \rangle$$

in M. For all $i < \lambda$, $\mathbb{P}_i^j = j(\mathbb{P}_i) = \mathbb{P}_i$. So in the definition of $\dot{\mathbb{Q}}_{\lambda+1}^j$ by cases in M, λ satisfies Case 1. Hence $j(\mathbb{P}_\lambda)$ factors as

$$j(\mathbb{P}_\lambda) = \mathbb{P}_\lambda * \textsc{Add}(\mu) * \dot{\mathbb{Q}} * \mathbb{P}_{\lambda+2,j(\lambda)},$$

where $\dot{\mathbb{Q}}$ is a name for the poset which adds an increasing and continuous sequence of order type κ through $P_\kappa(H(\theta)^{M^{\mathbb{P}_\lambda}}) \cap M^{\mathbb{P}_\lambda}$.

Let $K = G * G_0 * G_1 * H$ be a generic filter for $j(\mathbb{P}_\lambda)$ over V. Since $j[G] = G \subseteq K$, in the generic extension $V[K]$ we can lift j to $j : V[G] \to M[K]$ such that $j(G) = K$.

For the remainder of the proof, we write $H(\theta)$ for $H(\theta)^{V[G]}$. As $\mathbb{P}$ is λ-c.c. and $M^\theta \cap V \subseteq M$, we have that $M[G]^\theta \cap V[G] \subseteq M[G]$. So $H(\theta) = H(\theta)^{M[G]}$. Since $\mathbb{P}$ is a subset of $H(\lambda)$ and $\mathbb{P}$ is λ-c.c., $H(\theta) = H(\theta)^V[G]$. It follows that

$$j[H(\theta)] = j[H(\theta)^V][K],$$

by the same argument as given in Section 10. By the closure of M, $j[H(\theta)^V]$ is in M, so $j[H(\theta)^V][K] = j[H(\theta)]$ is in $M[K]$. Also $j \restriction H(\theta)$ is in $M[K]$, since $j \restriction H(\theta)$ is the inverse of the transitive collapse of $j[H(\theta)]$ to $H(\theta)$.

We would like to show that in $V[G]$, there are stationarily many N in $[H(\theta)]^\kappa$ such that N is internally club but not internally approachable. So fix a function $F : H(\theta)^{<\omega} \to H(\theta)$. By elementarity it suffices to show that in $M[K]$, there is a set N in $[j(H(\theta))]^{j(\kappa)} = [j(H(\theta))]^\kappa$ such that $j(\kappa) = \kappa \subseteq N$, N is closed under $j(F)$, and N is internally club but not internally approachable. Let $N = j[H(\theta)]$. We will prove that N satisfies these properties in $M[K]$.

Clearly $N \subseteq j(H(\theta))$. The critical point of j is λ, so $\kappa = j[\kappa] \subseteq N$. Also N is closed under $j(F)$, as we argued in Section 10. To show that N has size κ, let us factor $j(\mathbb{P}_\lambda)$ as

$$\mathbb{P}_\lambda * \textsc{Add}(\mu) * \dot{\mathbb{Q}} * \mathbb{P}_{\lambda+2,j(\lambda)}.$$

Clearly in $M[K]$, N has the same size as $H(\theta)$, and $H(\theta)$ is equal to $H(\theta)^{M[G]}$. In $M[G]$, $H(\theta)$ has size at least θ, which is greater than or equal to κ^+. In $M[G * G_0 * G_1]$, $H(\theta)$ acquires size κ. Since $\mathbb{P}_{\lambda+2,j(\lambda)}$ preserves κ, $H(\theta)$ has size κ in $M[K]$. So N is in $[j(H(\theta))]^\kappa$.

We will prove that N is internally club but not internally approachable in $M[K]$. First let us show that N is internally club. Let $\langle a_i : i < \kappa\rangle$ be the union of G_1, the generic filter for $\mathbb{Q}$ over $M[G * G_0]$. Then $\langle a_i : i < \kappa\rangle$ is an increasing and continuous sequence of sets in $P_\kappa(H(\theta)) \cap M[G]$ with union equal to $H(\theta)$. For all $i < \kappa$, a_i is a subset of $H(\theta)^{M[G]} = H(\theta)$ of size less than κ. Since $\theta \geq \kappa^+$ in $M[G]$, it follows that a_i is in $H(\theta)$. Since a_i has size less than λ, $j(a) = j[a_i]$. Hence $j[a_i]$ is a member of $N = j[H(\theta)]$. But clearly $\langle j[a_i] : i < \kappa\rangle$ is an increasing and continuous sequence with union equal to N, and for all $i < \kappa$, $j[a_i]$ is in N. This sequence is in $M[K]$, since $j \restriction H(\theta)$ is in $M[K]$. So N is internally club in $M[K]$.

Suppose for a contradiction that N is internally approachable in $M[K]$. Fix an increasing and continuous sequence $\langle N_i : i < \kappa\rangle$ in $M[K]$ with union equal to N, such that every proper initial segment of the sequence is in N. Then for all $\alpha < \kappa$, we can fix g_α in $H(\theta)$ such that $j(g_\alpha) = \langle N_i : i < \alpha\rangle$. Now for all $\alpha < \beta < \kappa$, $j(g_\beta) \restriction \alpha = j(g_\alpha)$. As $j(\alpha) = \alpha$, this implies $j(g_\beta \restriction \alpha) = j(g_\alpha)$. So by elementarity, $g_\beta \restriction \alpha = g_\alpha$.

Let $\langle M_i : i < \kappa\rangle = \bigcup\{g_\alpha : \alpha < \kappa\}$. The sequence $\langle N_i : i < \kappa\rangle$ and the function $j \restriction H(\theta)$ are in $M[K]$. Therefore $\langle g_\alpha : \alpha < \kappa\rangle$ is in $M[K]$ by definability, so $\langle M_i : i < \kappa\rangle$ is in $M[K]$. Note that for all $\alpha < \kappa$, $N_\alpha = j(M_\alpha)$. Indeed, M_α is the maximal element of $g_{\alpha+1}$, so $j(M_\alpha)$ is the maximal element of $j(g_{\alpha+1}) = \langle N_i : i \leq \alpha\rangle$, which is N_α. Clearly $\langle M_i : i < \kappa\rangle$ is increasing and continuous, since g_α is increasing and continuous for all $\alpha < \kappa$. Note that since g_α is in $H(\theta)$ for all $\alpha < \kappa$, M_i is in $H(\theta)$ for all $i < \kappa$.

We claim that $\bigcup\{M_i : i < \kappa\} = H(\theta)$. For all $i < \kappa$, M_i is in $H(\theta)$, so $M_i \subseteq H(\theta)$. Hence $\bigcup\{M_i : i < \kappa\} \subseteq H(\theta)$. On the other hand, let x be in $H(\theta)$. Then $j(x)$ is in N. Since $N = \bigcup\{N_i : i < \kappa\}$, there is $i < \kappa$ such that $j(x)$ is in N_i. But $N_i = j(M_i)$. So x is in M_i by elementarity. Thus $\bigcup\{M_i : i < \kappa\} = H(\theta)$.

Define $h = \langle M_i : i < \kappa\rangle$. Then for all $\alpha < \kappa$, $h \restriction \alpha = g_\alpha$. Since g_α is in $H(\theta) = H(\theta)^{M[G]}$, $h \restriction \alpha$ is in $M[G]$. So $h : \kappa \to M[G]$ is a function in $M[K]$, all of whose proper initial segments are in $M[G]$. Factor $j(\mathbb{P}_\lambda) = \mathbb{P}_\lambda * \mathbb{P}_{\lambda,j(\lambda)}$. Applying property (II) to $\mathbb{P}_{\lambda,j(\lambda)}$ and h, we can conclude that h is in $M[G]$. But if h is in $M[G]$, then $H(\theta)$ has size κ in $M[G]$. This is false, since θ is greater than or equal to κ^+ in $M[G]$. So N is not internally approachable. □

12. Open problems

There are a number of open questions remaining on the topics studied in this paper. In Section 9 we showed how to construct a model with a disjoint stationary sequence on the successor of a regular cardinal κ. But we do not know how to construct a disjoint club sequence on any cardinal larger than ω_2.

Question 12.1. *Is it consistent that there exists a disjoint club sequence on ω_3? Or on larger cardinals?*

An alternative way to weaken the idea of a disjoint club sequence is to require the stationary sets on the disjoint sequence to contain almost all sets of a particular cofinality, in the sense described in Question 12.2. This property has stronger implications than disjoint stationary sequences for the theory of adding clubs by forcing, since it is more strongly upwards absolute.

Question 12.2. *Let ν be equal to ω or ω_1. Is it consistent that there exists a sequence $\langle \mathcal{E}_\alpha : \alpha \in A\rangle$ such that A is a stationary subset of $\omega_3 \cap \mathrm{cof}(\omega_2)$, for all*

$\alpha < \beta$ in A, $\mathcal{E}_\alpha \cap \mathcal{E}_\beta$ is empty, and for all α in A, $\mathcal{E}_\alpha$ is a stationary subset of $P_{\omega_2}(\alpha)$, and there is a club $C \subseteq P_{\omega_2}(\alpha)$ such that $\{a \in C : \mathrm{cf}(a \cap \omega_2) = \nu\} \subseteq \mathcal{E}_\alpha$? What about for other cardinals and cofinalities?

The mixed support forcing iteration schema we used in our consistency results is flexible about which cardinal μ we can use when forcing with $\mathrm{ADD}(\mu)$. But in Sections 9 and 10 we were restricted to using $\mathrm{ADD}(\omega)$, since we needed to appeal to the fact that adding a real produces stationarily many new sets.

Question 12.3. *Is it consistent there is a disjoint stationary sequence on ω_3 and $2^\omega = \omega_1$? What about for other cardinals?*

Question 12.4. *Is it consistent the properties of internally stationary and internally club are distinct for sets of size $\aleph_2$, while $2^\omega = \omega_1$? What about for other sizes?*

One difficulty in answering Questions 12.3 and 12.4 is the lack of any analogue of Theorem 7.1 to higher cardinals provable in ZFC. Indeed, Magidor's Covering Lemma [18] implies, for example, that if $0^\sharp$ does not exist, then there is a club subset of $[\omega_3]^{\aleph_1}$ consisting of sets which are countable unions of sets in L. Thus if $0^\sharp$ does not exist, then $\mathrm{ADD}(\omega_1)$ does not add stationarily many new sets of size $\aleph_1$. The most we can hope for along these lines is a consistency result.

Question 12.5. *Is there a general method for constructing a model in which $\mathrm{ADD}(\mu)$ forces that $[\mu^{++}]^\mu \setminus V$ is a stationary subset of $[\mu^{++}]^\mu$, where $\mu^{<\mu} = \mu$?*

Question 12.6. *Is there a general method for constructing a model in which $\mathrm{ADD}(\mu)$ forces $([H(\chi)]^\mu \cap IA(\mu)) \setminus V$ is a stationary subset of $[H(\chi)]^\mu$, where $\mu^{<\mu} = \mu$ and $\chi \geq \mu^+$ is regular?*

Question 12.5 was answered positively in the case $\mu = \aleph_1$ by Dobrinen and Friedman [7], by adapting an argument of Baumgarter [4]. But it is not clear if this argument can apply to Question 12.6 to obtain stationarily many new internally approachable sets of size $\aleph_1$ in a generic extension by $\mathrm{ADD}(\omega_1)$. Also the arguments of [7] and [4] are very closely tied to the case $\mu = \omega_1$. The hope is, if a positive answer to Questions 12.5 and 12.6 is obtained, then the machinery developed in Sections 9 and 10 will yield positive solutions to Questions 12.3 and 12.4.

Our proofs for distinguishing variants of internal approachability used a supercompact cardinal. But in Section 6 we used a distinct club sequence to reduce the consistency strength in a special case. It is not clear whether this is possible in general.

Question 12.7. *Is it possible to reduce the large cardinal assumptions used in Theorems 10.1 and 11.1 for distinguishing the properties of internally stationary, internally club, and internally approachable?*

As we discussed in [16], some of our applications of mixed support forcing iterations bear some resemblance to Mitchell's construction of a model with no Aronszajn tree on ω_2 [19]. Abraham [1] adapted Mitchell's argument to obtain a model in which there are no Aronszajn trees on two successive cardinals. Cummings and Foreman [5] extended this result to obtain a model with no Aronszajn trees on an infinite interval of cardinals. Thus a natural question is whether the same situation can be obtained for the consistency results of Sections 9, 10, and 11.

Question 12.8. *Is it consistent to have a disjoint stationary sequence on two successive cardinals? Or on an infinite interval of cardinals?*

Question 12.9. *Is it consistent to have a distinction between variants of internal approachability simultaneously for sets of two successive cardinalities? Or for an infinite interval of cardinalities?*

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