Hausdorff Dimension of Metric Spaces and Lipschitz Maps
Onto Cubes

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1 Introduction

We follow the work of Kelete, Mathe, and Zindulka [3].

When can a metric space be mapped onto the \([0,1]^k\)? If our notion of map is too loose, the question has some bizarre answers. For instance, in 1890, Peano built a continuous function from \([0,1]\) onto \([0,1]^2\). Later work in this direction revealed that for any \(k\), there are continuous functions from \([0,1]\) onto \([0,1]^k\), and they can even be taken to be \(\frac{1}{k}\)-Holder continuous. To restrict the problem to a reasonable category, we will require that our maps be Lipschitz. The following proposition gives us one constraint.

**Proposition 1.** Let \((X,d)\) and \((Y,\rho)\) be metric spaces, \(d \geq 0\) be a real number, and \(f : X \to Y\) be Lipschitz. Then there is a constant \(C > 0\) so that

\[
\mathcal{H}^d(f[S]) \leq C\mathcal{H}^d(S)
\]

for all Borel \(S \subseteq X\).

So it must be that \(\mathcal{H}^k(X) > 0\). Does this suffice? In 1933, Kolmogorov [4] conjectured that if \(X \subseteq \mathbb{R}^n\) is so that \(\mathcal{H}^1(X) > 0\), then there is a Lipschitz map from \(X\) onto \([0,1]\). Sadly, this is false. In 1963, Vitushkin, Ivanov, and Melnikov [9] constructed a compact subset of the plane with positive 1-dimensional Hausdorff measure that cannot be mapped onto a segment by a Lipschitz map. We will show that it does suffice to assume that \(\dim_H(X) > k\).

2 Preliminaries

Frostman’s Lemma is useful tool for computing Hausdorff dimension.

**Theorem 1** (Frostman’s Lemma). Let \((X,\rho)\) be a compact metric space, \(S \subseteq X\) be Borel, and \(d > 0\). Then the following are equivalent:

- \(\mathcal{H}^d(S) > 0\), and
- there is a finite Borel measure \(\mu\) with \(\mu(S) > 0\) so that \(\mu(E) \leq |E|^d\) for any Borel \(E \subseteq S\).

Zindulka introduced the following notion in [10].

**Definition 1.** A metric space \((X,d)\) is **monotone** iff there is a linear order \(<\) and a constant \(C\) so that

\[
(\forall a,b \in X)(\text{diam}([a,b]_{<}) \leq Cd(a,b)).
\]

If this holds for a given \(C\), we say the space is \(C\)-monotone.
Nekvinda and Zindulka [8] proved that every ultrametric space is monotone. We can say even more for compact ultrametric spaces.

**Lemma 1.** Any compact ultrametric space \((X, d)\) is 1-monotone.

**Proof.** Let \(D = |X|\). If \(D = 0\), then it is a singleton, so this is trivial. Otherwise, since \(d\) is an ultrametric, the relation \(d(x, y) < D\) is an equivalence relation. The equivalence classes are open, and \(X\) is compact, so there are only finitely many equivalence classes: \(X_1, \ldots, X_k\). These then are closed and compact as well. Since \(X\) is compact and \(D \neq 0\), there are two points with distance \(D\), so \(k \geq 2\). Note also that if \(a\) and \(b\) are from distinct equivalence classes, then \(d(a, b) = D\).

Since each equivalence class is a compact metric space, we do the same for each of them. So we get a tree of clopen sets \(X_{i_1, \ldots, i_m}\) with the property that, for a fixed \(m\), these sets form a partition of \(X\), and if \(a \in X_{i_1, \ldots, i_m, j}\) and \(b \in X_{i_1, \ldots, i_m, j'}\) and \(j \neq j'\), then \(d(a, b) = \text{diam}(X_{i_1, \ldots, i_m})\). So for any \(x \in X\), there is a unique \(\bar{s} \in \omega^\omega\) so that \(x \in X_{\bar{s}|m}\) for all \(m\). Similarly, each such \(\bar{s}\) defines a unique point (This requires some argument). So define \(F : X \rightarrow \omega^\omega\) by \(F(x)\) is the unique \(\bar{s}\) with \(x \in X_{\bar{s}|m}\) for all \(m\).

We now define the order on \(X\). \(x \prec y\) iff \(F(x) <_{lex} F(y)\). We need to show that for any \(x \prec y \in X\), \(\text{diam}([x, y]_\prec) = d(x, y)\). Let \(\bar{s} \in \omega^\omega\) so that \(x, y \in X_{\bar{s}}\). Then \(d(x, y) = \text{diam}(X_{\bar{s}})\). But also, \([x, y]_\prec \subseteq X_{\bar{s}}\), so

\[
d(x, y) \leq \text{diam}([x, y]_\prec) \leq \text{diam}(X_{\bar{s}}) = d(x, y).
\]

This completes the proof. \(\square\)
3 Nice Large Metric Spaces Can be Mapped Onto Cubes

**Theorem 2.** If $(X,d)$ is a compact monotone metric space with positive $s$-dimensional Hausdorff measure, where $s > 0$, then $X$ can be mapped onto a non-degenerate interval by an $s$-Holder function.

**Proof.** By Frostman’s lemma, we can choose a non-zero finite Borel measure $\mu$ on $X$ so that $\mu(E) \leq |E|^s$ for any Borel $E \subseteq X$. Since $X$ is a monotone metric space there is a linear order $\prec$ and a constant $C$ so that

$$\forall a, b \in X(\text{diam}([a, b]_{\prec}) \leq Cd(a, b)).$$

**Claim 1.** Any open interval $(a, b)_{\prec}$ is open, and thus Borel.

**Reason.** We proceed by contradiction. So there is an $x \in (a, b)_{\prec}$ so that for all $n$, there is an $x_n \in B(x, \frac{1}{n})$ so that $x_n \leq a$ or $b \leq x_n$. Let $N$ be so that $\frac{1}{N} < \frac{1}{2} \min\{\text{diam}([a, x]_{\prec}), \text{diam}([x, b]_{\prec})\}$. First suppose that $x_N \leq a$. By assumption,

$$\text{diam}([x_N, x]_{\prec}) \leq Cd(x_N, x) \leq C \frac{1}{N} < \text{diam}([a, x]_{\prec}),$$

but this cannot be, as $[a, x]_{\prec} \subseteq [x_N, x]_{\prec}$. Now suppose that $b \leq x_N$. Again,

$$\text{diam}([x, x_N]_{\prec}) \leq Cd(x_N, x) \leq C \frac{1}{N} < \text{diam}([x, b]_{\prec}),$$

which is a contradiction as $[x, b]_{\prec} \subseteq [x, x_N]_{\prec}$. Thus the interval is open. \hfill \Box

For $x \in X$, let $(-\infty, x)_{\prec} = \{y \in X : y \prec x\}$, and let $g(x) = \mu((-\infty, x)_{\prec})$. Then $g$ is $s$-Holder, since for any $a \prec b \in X$,

$$0 \leq g(b) - g(a) = \mu([a, b]_{\prec}) \leq \text{diam}([a, b]_{\prec})^s \leq (Cd(a, b))^s.$$

Thus $g[X]$ is compact. Since $\mu$ is not the zero measure and $X$ is separable, $g[X]$ is not a singleton.

We will now show that $g[X]$ is connected. Since $g[X]$ is closed, all we need to prove is that there are no $u, v \in g[X]$ with $u \prec v$ and $(u, v) \cap g[X] = \emptyset$. Suppose otherwise, and let $u$ and $v$ witness it. Let $D \subseteq X$ be countable and dense. Let $D_1 = \{x \in D : g(x) \leq u\}$ and $D_2 = \{x \in D : g(x) \geq v\}$. Since $(u, v) \cap g[X] = \emptyset$, $D = D_1 \cup D_2$. We also get that $\mu\left(\bigcup_{x \in D_1} (-\infty, x)\right) \leq u$ and $\mu\left(\bigcap_{x \in D_2} (-\infty, x)\right) \geq v$. Let

$$A = \left(\bigcap_{x \in D_2} (-\infty, x)\right) \setminus \left(\bigcup_{x \in D_1} (-\infty, x)\right).$$

Then $\mu(A) \geq u - v > 0$. On the other hand, $A$ cannot have more than two points. For if $x \prec y \prec z \in A$, then $(x, z)_{\prec} \neq \emptyset$, but $(x, z)_{\prec} \cap D = \emptyset$, which contradicts the denseness of $D$. Since $\mu(E) \leq |E|^s$ for any Borel $E \subseteq X$, the singletons are measure zero. Hence $\mu(A) = 0$. This is a contradiction. \hfill \Box

**Corollary 1.** Let $X$ be a compact monotone metric space and let $k > 0$ be an integer. Then $X$ can be mapped onto the $k$-dimensional cube $[0, 1]^k$ by a Lipschitz map iff $X$ has positive $k$-dimensional Hausdorff measure.

**Proof.** Clearly it is necessary that $\mathcal{H}^k(X) > 0$. So suppose that $\mathcal{H}^k(X) > 0$. Then by the theorem, there is a $k$-Holder continuous map $g : X \to \mathbb{R}$ so that $g[X] = [0, 1]$. It is known that there is a $\frac{1}{k}$-Holder Peano curve $h : [0, 1] \to [0, 1]^k$, the classical construction works. Then the composition $h \circ g$ is a Lipschitz map from $X$ onto $[0, 1]^k$. \hfill \Box
In 2013, Mendel and Naor [6] proved some results about approximating sets in the context of dimension. The following is a weak version of one thing they showed.

**Theorem 3.** (Mendel and Naor) For every compact metric space \((X,d)\) and \(\varepsilon > 0\) there is a closed subset \(Y \subseteq X\) so that \(\dim_H(Y) \geq (1 - \varepsilon)\dim_H(X)\) and \((Y,d)\) is bi-Lipschitz equivalent to an ultrametric space.

**Theorem 4.** Let \(A\) be an analytic subset of a separable complete metric space \((X,d)\), and let \(k\) be an integer. If \(\dim_H(A) > k\), then \(A\) can be mapped onto the \(k\) dimensional cube \([0,1]^k\) by a Lipschitz map.

**Proof.** Let \(s \in (k, \dim_H(A)) \subseteq \mathbb{R}\). By the theorem of Howroyd, \(A\) has a compact subset \(C\) with finite and positive \(s\)-dimensional Hausdorff measure. Then by the Mendel-Naor theorem, \(C\) has a subset \(E\) with \(\dim_H(E) > k\) that is bi-Lipschitz equivalent to an ultrametric space. Say \(f : E \to (Z,\rho)\) is bi-Lipschitz. Applying the Howroyd theorem again (this time to \(k \in (k - 1, \dim_H(E))\)), we get a compact subset \(B\) of \(E\) with positive and finite \(k\)-dimensional Hausdorff measure. \(B\) is bi-Lipschitz equivalent to a compact ultrametric space \(Y\), via \(f|_B\). Now \(Y\) can be mapped onto \([0,1]^k\) by a Lipschitz map, say \(g : Y \to [0,1]^k\). Then \(h = g \circ f|_B\) is Lipschitz and \(h : B \to [0,1]^k\) is onto. Write \(h(x) = (h_1(x), \ldots, h_k(x))\). Then each of the \(h_i\) are Lipschitz and real-valued. So we can extend each \(h_i\) to a Lipschitz \(H_i : A \to [0,1]\). Then \(H(x) = (H_1(x), \cdots, H_k(x))\) is a Lipschitz function from \(A\) onto \([0,1]^k\). \(\square\)

It was conjectured by Laczkovich [5] in 1991 that if \(A \subseteq \mathbb{R}^n\) has positive Lebesgue measure, then \(A\) can be mapped onto \([0,1]^n\) with a Lipschitz map. This has been shown for \(n \leq 2\) [1], but is still open for higher \(n\).
4 Large Metric Spaces that Cannot Be Mapped Onto A Segment

Theorem 5. Assume that, in $\mathbb{R}^n$, less than continuum many closed sets of measure zero and a set of measure zero cannot cover $\mathbb{R}^n$. Then there is a non-Lebesgue-null $A \subseteq \mathbb{R}^n$ so that for any continuous function $f : \mathbb{R}^n \to \mathbb{R}$, $f[A]$ does not contain any interval.

Proof. Let $\{f_\alpha : \alpha < \chi\}$ and $\{N_\alpha : \alpha < \chi\}$ enumerate the collection of $\mathbb{R}^n \to \mathbb{R}$ continuous functions and the collection of Lebesgue null Borel subsets of $\mathbb{R}^n$, respectively.

By transfinite induction, for every $\alpha < \chi$, we construct points $x_\alpha \in \mathbb{R}^n$ and $y_\alpha \in (0, 1)$ so that

1. $x_\alpha \notin N_\alpha$,
2. $x_\alpha \notin \bigcup_{\beta < \alpha} f_\beta^{-1}(y_\beta)$,
3. $y_\alpha \notin f_\alpha([x_\beta : \beta \leq \alpha])$, and
4. $f_\alpha^{-1}(y_\alpha)$ is Lebesgue null.

$x_0$ and $y_0$ can be chosen essentially arbitrarily (up to conditions 1 and 3). Suppose we have completed all steps $\beta$ for $\beta < \alpha$. By our main assumption, and since $f_\beta^{-1}(y_\beta)$ is a closed Lebesgue null set for each $\beta < \alpha$, we can choose an $x_\alpha$ so that conditions 1 and 2 hold. Now it cannot be that more than countably many $f_\alpha^{-1}(t)$ for $t \in (0, 1)$ have positive measure, and $f_\alpha([x_\beta : \beta \leq \alpha])$ is null by our main assumption. So the set

$$\{y \in (0, 1) : f_\alpha^{-1}(y_\alpha) \text{ is Lebesgue null} \land y \notin f_\alpha([x_\beta : \beta \leq \alpha])\}$$

has full measure in $(0, 1)$. Thus we can find a $y_\alpha$ satisfying properties 3 and 4. This completes the recursive step.

Let $A = \{x_\alpha : \alpha < \chi\}$. A cannot be Lebesgue null, as property 1 shows that $A$ is not contained in any Borel Lebesgue null set. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is continuous. Let $h = g \circ f$, where $g : \mathbb{R} \to \mathbb{R}$ is a linear transformation. Then $h = f_\alpha$ for some $\alpha < \chi$. So $h(x_\beta) \neq y_\alpha$ for any $\beta \leq \alpha$ by property 3, and $h(x_\beta) \neq y_\alpha$ for any $\beta > \alpha$ by property 2. So $y_\alpha \notin h[A]$, and thus $(0, 1) \nsubseteq h[A]$. Therefore $g^{-1}((0, 1)) \nsubseteq f[A]$. Since $g$ was arbitrary, this means that no interval is contained in $f[A]$. \qed

The assumption we took for this work would be taken as $\text{cov}(\mathcal{M}) = c$. We will now suppose that $\text{cov}(\mathcal{M}) < c$.

Theorem 6. Suppose $\text{cov}(\mathcal{M}) < c$. For any gauge function $\varphi$, there is a separable metric space $(X, d)$ so that $|X| = \text{cov}(\mathcal{M})$ with $\mathcal{H}^\varphi((X, d)) > 0$.

Proof. Fremlin and Miller [7] proved that $\text{cov}(M)$ is the least cardinality of a subspace of $(\omega^\omega, d)$ that is not a strong measure zero space. So there is an $H \subseteq \omega^\omega$ so that $|H| = \text{cov}(\mathcal{M})$ and $(H, d)$ is not a strong measure zero space. Hence there is a gauge function $\varphi_0$ so that $\mathcal{H}^{\varphi_0}((H, d)) > 0$. Let $\varphi$ be a gauge function and $g : \{1, 2, \cdots\} \to (0, \infty)$ be defined by

$$g(m) = \varphi^{-1}\left(\varphi_0\left(\frac{1}{m}\right)\right).$$

Note that $\varphi\left(\frac{1}{m}\right) = \varphi(g(m))$ for all $m \in \{1, 2, \cdots\}$. Define $d_g$ on $\omega^\omega$ by

$$d_g(x, y) = g(|x \wedge y| + 1).$$

Then $(H, d_g)$ is a separable metric space and $\mathcal{H}^\varphi((H, d_g)) = \mathcal{H}^{\varphi_0}((H, d))$. Therefore $(H, d_g)$ is as desired. \qed

Theorem 7 (ZFC). There is a separable metric space with arbitrarily large Hausdorff dimension than cannot be mapped onto a segment by a uniformly continuous function.

There is a model of ZFC in which every positive Hausdorff dimensional subset of a Euclidean space can be mapped onto $[0, 1]$ by a uniformly continuous function. [2]
References


