1 Measurable Cardinals

Definition 1. A cardinal \( \kappa \) is measurable iff there is a non-principal \( \kappa \)-complete ultrafilter on \( \kappa \).

We usually characterize measurable cardinals differently.

Definition 2. Let \( M \) and \( N \) be inner models of ZFC. Then \( j : M \to N \) is an elementary embedding iff \( j \) is 1-1 and

\[
M \models \varphi (x_1, \cdots, x_n) \iff N \models \varphi (j(x_1), \cdots, j(x_n))
\]

for all sentences \( \varphi \). If \( j \) is not the identity, then there is a least cardinal \( \lambda \) so that \( j(\lambda) \neq \lambda \). We call \( \lambda \) the critical point of \( j \) and write \( \text{crit}(j) = \lambda \). Note that \( j(\lambda) > \lambda \).

Theorem 1. \( \kappa \) is measurable iff there is an elementary embedding \( j : V \to M \) so that \( \text{crit}(j) = \kappa \).

Proof. We sketch the proof. If \( \kappa \) is measurable with measure \( U \), then we can take the ultrapower of \( V \) by \( U \), call it \( \text{Ult}(V, U) \). This is a well-founded model as \( U \) is \( \kappa \)-complete and thus countably complete. Let \( M \) be the transitive collapse of \( \text{Ult}(V, U) \) and \( j : V \to M \) be the composition of the ultrapower embedding and the collapse map. This is the desired embedding.

Conversely suppose that \( j : V \to M \) is so that \( \text{crit}(j) = \kappa \). Define \( U \) by \( X \in U \) iff \( \kappa \in j(X) \). This is a non-principal, \( \kappa \)-complete, and normal ultrafilter on \( \kappa \).

Remark 1. If \( U \) on \( \kappa \) is normal, and we construct \( j : V \to M \) as in the proof above, then \( j(\langle \text{id}_\kappa \rangle) = \kappa \). This allows us to prove various things about \( \kappa \). For instance we can see that \( \{ \lambda < \kappa : \lambda \text{ is weakly compact} \} \in U \).

Remark 2. Note that in the definition of measurable above, \( U, j \in V \). This and the minimality of \( L \) show us that \( L \) cannot have any measurable cardinals. In fact measurable cardinals transcend \( L \) in a much stronger way. If there is a measurable cardinal, then \( |P(\alpha)| = |\alpha| \) for all \( \alpha \geq \omega \).

Proposition 1. Let \( \kappa \) be measurable and let \( j : V \to M \) be constructed from a \( \kappa \)-complete ultrafilter \( U \) over \( \kappa \). Then \( M^\kappa \subseteq M \), \( V_{\kappa+1} = M_{\kappa+1} \) and \( V_{\kappa+2} \not\subseteq M \).

Proof. Again we sketch the proof. That \( M^\kappa \subseteq M \) follows from the \( \kappa \)-completeness of \( U \). Let \( X \subseteq V_\kappa \). Then

\[
j(X) \cap V_\kappa = X
\]

So \( V_{\kappa+1} = M_{\kappa+1} \). Finally \( U \in V_{\kappa+2} \), but \( U \notin M \).
2 Large Large Cardinals

The last proposition showed that an embedding generated by a measurable cardinals is limited in how much of $V$ it can capture. The way to define large large cardinals is to assert the existence of embeddings which capture more and more of $V$. We state some of these axioms in increasing consistency strength.

**Definition 3.**

- Let $\alpha$ be an ordinal. $\kappa$ is $\alpha$-strong iff there is an embedding $j : V \rightarrow M$ with $\text{crit}(j) = \kappa$ so that $\alpha < j(\kappa)$ and $V_{\kappa+\alpha} \subseteq M$.
- $\kappa$ is strong iff $\kappa$ is $\alpha$-strong for all $\alpha$.
- $\kappa$ is Woodin iff for all $f : \kappa \rightarrow \kappa$ there is an $\alpha < \kappa$ so that $f''\alpha \subseteq \alpha$ and an embedding $j : V \rightarrow M$ with $\text{crit}(j) = \alpha$ and $V_{j(f(\alpha))} \subseteq M$.
- $\kappa$ is superstrong iff there is an embedding $j : V \rightarrow M$ with $\text{crit}(j) = \kappa$ so that $V_{j(\kappa)} \subseteq M$.

**Definition 4.**

- Let $\alpha$ be an ordinal. $\kappa$ is $\alpha$-supercompact iff there is an embedding $j : V \rightarrow M$ with $\text{crit}(j) = \kappa$ so that $\alpha < j(\kappa)$ and $M^\alpha \subseteq M$.
- $\kappa$ is supercompact iff $\kappa$ is $\alpha$-supercompact for all $\alpha \geq \kappa$.

**Remark 3.** $\kappa$ is measurable if $\kappa$ is 1-strong and $\kappa$ is measurable if $\kappa$ is $\kappa$-supercompact. A Woodin cardinal has a stationary set of measurable cardinals below it.

**Remark 4.** It is overkill, but the existence of a supercompact cardinal is enough to ensure that $\text{AD}$ holds in $L(\mathbb{R})$, and thus that every constructible set of reals is Lebesgue measurable, Baire measurable, and has the perfect set property.

3 Inconsistency

A natural question at this point is whether or not one can find a definable non-trivial elementary embedding $j : V \rightarrow M$ so that $V \subseteq M$. We will show that this is impossible.

**Theorem 2** (Kunen). *Suppose that $j : V \rightarrow M$ is a non-trivial elementary embedding. Then $V \neq M$.***

**Proof.** Let $\kappa = \text{crit}(j)$ and $\lambda = \sup\{j^n(\kappa) : n \in \omega\}$. Let $A = \{\xi < \lambda^+ : \text{cf}(\xi) = \omega\}$. This set is stationary. By a standard theorem, we can partition $A$ into $\kappa$ many stationary sets $S_\alpha$. Let $f : \kappa \rightarrow \mathcal{P}(\lambda^+)$ by $f(\alpha) = S_\alpha$. Note that $j(\lambda) = \lambda$ from how we have defined $\lambda$. Thus

$$\lambda^+ \leq j(\lambda^+) = (\lambda^+)^M \leq \lambda^+$$

and so $\lambda^+ = j(\lambda^+)$. Consider $j(f)$. This is creates partition of $j(A)$ into $j(\kappa)$ many stationary (in $M$) sets by elementarity. By the above argument, $j(A) = A$. Thus $j(f)(\kappa) \subseteq A$ is stationary in $M$.

By way of contradiction suppose that $V = M$. Then $j(f)(\kappa)$ is stationary in $V$. Therefore we have some $\alpha_0 < \kappa$ so that $j(f)(\kappa) \cap j(\alpha_0)$ is stationary as $j(f)(\kappa) = \bigcup_{\alpha < \kappa}[j(f)(\kappa) \cap f(\alpha)]$, $\lambda > \kappa$ and the club filter on $\lambda^+$ is $\lambda^+$-complete. Let

$$C = \{\xi < \lambda^+ : \text{cf}(\xi) = \omega \land j(\xi) = \xi\}$$

Then $C$ is unbounded in $\lambda^+$, and $C$ is closed under countable sequences. Borrowing again from infinite combinatorics, this suffices to guarantee that $C \cap j(f)(\kappa) \cap f(\alpha_0) \neq \emptyset$ as all elements of $j(f)(\kappa)$ have cofinality $\omega$. Let $\xi_0 \in C \cap j(f)(\kappa) \cap f(\alpha_0)$. Then

$$\xi_0 = j(\xi_0) \in j(f(\alpha_0)) = j(f)(j(\alpha_0))$$

So $\xi_0 \in j(f)(\kappa) \cap j(f)(j(\alpha_0))$. This contradicts the fact that $j(f)$ created a partition of $A$.  

\qed