1

THE COMPLEXITY OF SQUARES IN THE GROUP OF ISOMETRIES OF THE BAIRE SPACE

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Abstract. We prove that in the Polish group of isometries of the Baire space the collection of n-th powers is non-Borel. We also prove that in the Polish space of trees on $\mathbb{N}$ the collection of trees that have an automorphism under which every node has order exactly n is non-Borel.

1. Introduction

The main concern of this paper is the complexity of the set of n-th powers in a Polish group. This question arose first in the study of generic automorphisms of a measure space. Let (X, μ) be any Lebesgue space, for example the unit interval with Lebesge measure. The collection $\mathrm{Aut}(X, \mu)$ of all invertible measure-preserving transformations of (X, μ) taken modulo null sets becomes a Polish topological group when endowed with the weak topology, which comes from viewing it as a subgroup of the unitary operators on $L^2(X, \mu)$ with the strong operator topology. We say that a generic transformation satisfies a property if the collection of transformations satisfying the property is comeager in $\mathrm{Aut}(X, \mu)$.

A series of results over the past decade indicate that the centralizer of a generic transformation is quite large. In 2000, King [6] answered an old question of Halmos by showing that a generic transformation has roots of all orders. This implies that a generic transformation commutes with something other than its integral powers. The method that King devised was later used by de Sam Lazaro and de la Rue [7] to show that $\mathbb{R}$ embeds into the centralizer of a generic T (in fact, successive roots of T can be taken in a way to produce a flow, or one parameter subgroup, whose 1-value is T) and by Ageev [1] to show that every finite abelian group embeds into the centralizer of a generic T

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(essentially, that the identity has abundant roots in the centralizer of T).

In the above considerations, the collection of n-th powers plays an important role and one would like to understand this collection as completely as possible. In King's paper showing that a generic transformation has roots of all orders [6], he asked whether the collection of n-th powers (for $n > 1$ fixed) is Borel. If it were Borel, then a concrete description of it might yield additional insight into the size or structure of the centralizer of a generic transformation. On the other hand, it would also be of interest if the collection of n-th powers, an algebraicly simple set, were non-Borel, i.e., topologically complicated.

Other people have worked on this type of question in other contexts. In 1989 Humke and Laczkovich [4] showed that in the semi-group of continuous functions from an interval into itself, the collection of n-th powers is non-Borel. Gartside and Pejić [2] showed in 2009 that in the group of homeomorphisms of the circle, the collection of n-th powers is complete analytic and, in particular, non-Borel.

In both of these cases the underlying structure (the interval or the circle) carries an ordering which interacts with the functions under consideration (continuous functions or homeomorphisms). This aspect of linearity, completely absent in (X, μ), is crucial to the arguments in [4] and [2].

The main object of the present paper is to move away from this rigid linearity. We will show that in the group of isometries of the Baire space, the collection of n-th powers is non-Borel (for each $n > 1$). In the Baire space there is no ordering to interact with the functions under consideration. By restricting our attention to isometries, rather than considering all homeomorphisms, we have something that looks like a measure: each basic clopen set has an associated size, and this size must be preserved under isometries.

2. Preliminaries

Let $\mathbb{N}^\mathbb{N}$ denote the metric space whose underlying set is the collection of all sequences of natural numbers and in which the distance between two sequences α and β is given by $\frac{1}{1+l(\alpha,\beta)}$, where $l(\alpha, \beta)$ is the length of the longest common initial segment of α and β. Let $\mathrm{Iso}(\mathbb{N}^\mathbb{N})$ denote the group of isometries of $\mathbb{N}^\mathbb{N}$ equipped with the topology of pointwise convergence. It is straightforward to check that $\mathrm{Iso}(\mathbb{N}^{\mathbb{N}})$ is a Polish group. We will prove the following theorem.

Theorem 1. *For any $n > 1$, the collection of n-th powers is non-Borel in* $\mathrm{Iso}(\mathbb{N}^{\mathbb{N}})$.

We will prove the theorem using the proposition below about trees on $\mathbb{N}$. Let $\mathbb{N}^{<\mathbb{N}}$ denote the set of finite sequences of natural numbers, including the empty sequence. A *tree* (on $\mathbb{N}$) is a nonempty subset of $\mathbb{N}^{<\mathbb{N}}$ that is closed under taking initial segments. The empty sequence $\emptyset$ is an element of every tree; it is called the root. Non-root elements of a tree are called nodes. A tree is naturally composed of levels; the level k nodes are precisely those with length k. A node b is a *descendent* of $a \in T$ if a is a proper initial segment of b; the node b is a *child* of a if additionally the length of a is exactly one less than the length of b. The collection of all trees is a closed subset of $2^{(\mathbb{N}^{<\mathbb{N}})}$ and so is a Polish space. It is denoted by $\mathrm{Tr}_{\mathbb{N}}$. An *automorphism* of a tree T is a bijection $F : T \to T$ satisfying:

(1) $F(\emptyset) = \emptyset$
(2) For all $a, b \in T$, b is a child of a iff $F(b)$ is a child of $F(a)$.

It is clear that an automorphism of a tree preserves the levels of the tree. Each automorphism can be uniquely decomposed into its orbits. The restriction of an automorphism of a tree to a single orbit is called a *cycle*. If the orbit has cardinality n, it is called an n-cycle. If the orbit is infinite, it is called an infinite cycle, or ∞-cycle. We can construct an automorphism of a tree by constructing the cycles that make up the automorphism.

Let T be a tree. An *n-matching* of T is an automorphism F of T such that each node of T has order exactly n under F. In essense, a tree has an n-matching if its level 1 nodes can be partitioned into groups of size n so that two nodes in the same group have isomorphic "sets of descendents." For $n > 1$, let D_n be the the collection of trees that have an n-matching and let E_n be the collection of trees that don't have an n-matching.

Proposition 2. *For each $n > 1$, $D_n \subseteq \mathrm{Tr}_{\mathbb{N}}$ is non-Borel.*

To prove the proposition we will need the following: the collection of well-founded trees (i.e., those without an infinite branch) is a complete co-analytic subset of $\mathrm{Tr}_{\mathbb{N}}$. There is a natural co-analytic rank on well-founded trees, and any collection of well-founded trees with unbounded rank (in ω_1) is non-Borel and, moreover, non-analytic (See Theorem 35.23 in [5]).

3. Proofs

Let $T_{\mathbb{N}}$ be the tree containing all finite sequences of natural numbers. It is easy to see that the isometries of the Baire space correspond exactly to the automorphisms of $T_{\mathbb{N}}$. In the proof below we will work with $\mathrm{Aut}(T_{\mathbb{N}})$ rather than $\mathrm{Iso}(N^{\mathbb{N}})$. For $n > 1$ let P_n denote the collection of n-th powers in $\mathrm{Aut}(T_{\mathbb{N}})$.

3.1. **Proof of the theorem from the proposition.** Fix $n > 1$. We will describe a continuous function $\Phi : \mathrm{Tr}_{\mathbb{N}} \to \mathrm{Aut}(T_{\mathbb{N}})$. We will show that the preimage of P_n under Φ is D_n, which by the proposition is non-Borel. This implies that P_n is non-Borel.

First, partition $\mathbb{N}$ into infinite sets $(B_i : i \in \mathbb{N})$. Now recursively define $A_{\overline{c}}$, for $\overline{c} \in \mathbb{N}^{<\mathbb{N}}$, by $A_{\emptyset} = \{\emptyset\}$ and $A_{\langle c_0, c_1, ..., c_k\rangle} = \{\langle a_0, a_1, ..., a_k\rangle : \langle a_0, a_1, ..., a_{k-1}\rangle \in A_{\langle c_0, c_1, ..., c_{k-1}\rangle}$ and $a_k \in B_{c_k}\}$, for $k > 0$.

For $k > 0$ the sets $(A_{\overline{c}} : \overline{c} \in \mathbb{N}^k)$ form a partition of $\mathbb{N}^k$ and any element of that partition is infinite. We will now describe the function $\Phi : \mathrm{Tr}_{\mathbb{N}} \to \mathrm{Aut}(T_{\mathbb{N}})$. This will be done in such a way that each $A_{\overline{c}}$ is invariant under each $\Phi(T)$.

Define Φ in such a way that for each $T \in \mathrm{Tr}_{\mathbb{N}}$, $\Phi(T)$ is an automorphism of $T_{\mathbb{N}}$ satisfying:

(1) If $\overline{c} \in T$ then $\Phi(T) \upharpoonright A_{\overline{c}}$ consists of one n-cycle and infinitely many ∞-cycles.
(2) If $\overline{c} \notin T$, then $\Phi(T) \upharpoonright A_{\overline{c}}$ consists of infinitely many ∞-cycles.

Moreover, do this in such a way that for all $T_1, T_2 \in \mathrm{Tr}_{\mathbb{N}}$:

(1) If $\overline{c} \in T_1 \cap T_2$, then $\Phi(T_1)$ and $\Phi(T_2)$ agree on every element of $A_{\overline{c}}$.
(2) If $\langle c_0, c_1, ..., c_{k-1}\rangle \in T_1 \cap T_2$ and $\overline{c} = \langle c_0, c_1, ..., c_k\rangle \notin T_1 \cup T_2$, then $\Phi(T_1)$ and $\Phi(T_2)$ agree on every element of $A_{\overline{c}}$.
(3) If $\langle c_0, c_1, ..., c_{k-1}\rangle \in T_1 \cap T_2$ and $\overline{c} = \langle c_0, c_1, ..., c_k\rangle \in T_1 \setminus T_2$, then $\Phi(T_1)$ and $\Phi(T_2)$ disagree on every element of $A_{\overline{c}}$.

It follows from this that for each $\overline{c} = \langle c_0, c_1, ..., c_k\rangle \in \mathbb{N}^{<\mathbb{N}}$ and each $\overline{a} \in A_{\overline{c}}$, $\Phi(T_1)$ agrees with $\Phi(T_2)$ on $\overline{a}$ iff $\{\langle c_0\rangle, \langle c_0, c_1\rangle, ..., \langle c_0, c_1, ..., c_k\rangle\} \cap T_1 = \{\langle c_0\rangle, \langle c_0, c_1\rangle, ..., \langle c_0, c_1, ..., c_k\rangle\} \cap T_2$.

It remains to show that the preimage of of P_n under Φ is D_n. In other words, we need to show that $T \in \mathrm{Tr}_{\mathbb{N}}$ has an n-matching if and only if $\Phi(T)$ has an n-th root.

First, let $T \in \mathrm{Tr}_{\mathbb{N}}$, let $F = \Phi(T)$, and suppose $G^n = F$. Consider an n-cycle in F. The elements of this n-cycle must be part of an n^2-cycle in G. The orbit of size n^2 of G that consists of those n^2 elements must be the union of n orbits of size n of F. Moreover, G must permute these orbits. Since there is a 1-1 correspondence between the nodes of T and the n-cycles in F, G induces a permutation of all the nodes of T. It is easy to see that this permutation is, in fact, an n-matching.

Now, let let $T \in \mathrm{Tr}_{\mathbb{N}}$ and suppose T has an n-matching. By the definition of Φ, $\Phi(T)$ has only n-cycles and ∞-cycles. For $\Phi(T)$ to have an n-th root, there must be a way of grouping, level by level, in a coherent way, the n-cycles into groups of n and the ∞-cycles into groups of n. The construction of $\Phi(T)$ was done in such away that at each stage there are infinitely many ∞-cycles, so these can be matched into groups of n. The n-matching of T decomposes into orbits of size n, and each element of a given orbit is on the same level. Each of the nodes in an orbit corresponds to an n-cycle in $\Phi(T)$ and this provides a way of grouping the n-cycles into groups of n. It is easy to see that such a grouping gives rise to the desired n-th root of $\Phi(T)$.

We have shown that $T \in \mathrm{Tr}_{\mathbb{N}}$ has an n-matching if and only if $\Phi(T)$ has an n-th root. This concludes the proof of the theorem.

3.2. **Proof of the proposition.** We will describe a continuous function $\Psi : \mathrm{Tr}_{\mathbb{N}} \to \mathrm{Tr}_{\mathbb{N}}$. The preimage of the collection of well founded trees will be E_n. We will show that the image of E_n under Ψ has unbounded rank. This will imply that E_n, and hence D_n, is non-Borel, since the image of any Borel set under a continuous function is analytic.

We first order the elements of the $\mathbb{N}^{<\mathbb{N}}$ as follows: For each $a \in \mathbb{N}^{<\mathbb{N}}$, let s_a be the sum of the entries of a plus the length of a. If s_a is less than s_b, then we say $a < b$. If $s_a = s_b$, then we say $a < b$ if a is lexicographically less than b. It is clear this is an ω ordering of $\mathbb{N}^{<\mathbb{N}}$. Let $\{r_0, r_1, r_2, ...\}$ be the enumeration of $\mathbb{N}^{<\mathbb{N}}$ satisfying: $i < j$ iff $r_i < r_j$.

Given a tree $T \in \mathrm{Tr}_{\mathbb{N}}$ we will produce $\Psi(T)$, a tree of attempts to produce an n-matching of T. We say that a function $f : \{r_0, r_1, ..., r_k\} \to T$ is a *valid attempt* for T if for $i, j \leq k$:

(1) If $r_i \notin T$, then $f(r_i) = \emptyset$.
(2) If $r_i = \emptyset$, then $f(r_i) = \emptyset$.
(3) If r_i is a node of T and a child of r_j, then $f(r_i)$ is a child of $f(r_j)$.
(4) If r_i and r_j are distinct nodes of T, then $f(r_i) \neq f(r_j)$.
(5) If r_i is a node of T and $r_i \in dom(f^n)$, then $f^n(r_i) = r_i$.
(6) If r_i is a node of T and $r_i \in dom(f^m)$ with $0 < m < n$, then $f^m(r_i) \neq r_i$.

The sequence $\langle a_0, a_1, a_2, ..., a_k\rangle$ is a node of the tree $\Psi(T)$ if the function f defined by $f(r_i) = r_{a_i}$ is a valid attempt.

It is clear $\Psi(T)$ actually is a tree, for an initial segment of a valid attempt is a valid attempt. It is easy to see that if T has an n-matching, then $\Psi(T)$ has an infinite branch. Indeed, the initial segments of such an automorphism naturally correspond to valid attempts which in turn correspond to the nodes of an infinite branch. It is also easy to check that if $\Psi(T)$ has an infinite branch, then the sequence of valid attempts corresponding to those nodes gives rise to an n-matching of T. Thus $T \in E_n$ if and only if $\Psi(T)$ is well-founded.

It is clear that Ψ is continuous. It remains to show that $\Psi(E_n)$ is unbounded.

We will construct two sequences of trees: $(T_\alpha : \alpha < \omega_1) \subseteq E_n$ and $(T'_\alpha : \alpha < \omega_1) \subseteq D_n$. Thus T_α will not be isomorphic to T'_α. The construction will be such that the rank of $\Psi(T_\alpha)$ will be at least α. Additionally, the image under Ψ of the tree $T'_\alpha \oplus T_\alpha$ (defined below) will be well-founded and have rank greater than the rank of $\Psi(T_\alpha)$.

Let S_0 and S_1 be trees. The sequence $\langle a_0, a_1, a_2, ..., a_k\rangle$ is a node of $(S_0 \oplus S_1)$ if either $a_0 \leq (n-2)$ and $\langle a_1, a_2, ..., a_k\rangle$ is a node of S_0 or $a_0 = (n-1)$ and $\langle a_1, a_2, ..., a_k\rangle$ is a node of S_1. Thus the tree $S_0 \oplus S_1$ has exactly n level one nodes; Beneath $(n-1)$ of them is a subtree isomorphic S_0 and beneath the last of them is a subtree isomorphic to S_1. Thus $(S_0 \oplus S_1)$ has an n-matching iff S_0 and S_1 are isomorphic.

We now describe T_0 and T'_0. The tree T_0 consists of the root and the nodes $\langle 0\rangle, \langle 1\rangle, ...\langle n-2\rangle$. The tree T'_0 consists of the root and the nodes $\langle 0\rangle, \langle 1\rangle, ...\langle n-1\rangle$. It is clear T_0 does not have an n-matching and that T'_0 does have an n-matching. It is clear also that $(T'_0 \oplus T_0)$ does not have an n-matching and that its image under Ψ has rank greater than the rank of $\Psi(T_0)$.

We now describe how to produce $T_{\alpha+1}$ and $T'_{\alpha+1}$ from T_α and T'_α. The tree $T_{\alpha+1}$ is defined to be $T'_\alpha \oplus T_\alpha$ and the tree $T'_{\alpha+1}$ is defined to be $T'_\alpha \oplus T'_\alpha$. By assumption the rank of $\Psi(T'_\alpha \oplus T_\alpha)$ is at least $\alpha+1$. We need further to show that the rank of $\Psi(T'_{\alpha+1} \oplus T_{\alpha+1})$ is greater than the rank of $\Psi(T_{\alpha+1})$. Consider the structure of the two trees $T_{\alpha+1}$ and $(T'_{\alpha+1} \oplus T_{\alpha+1})$.

The structure of the tree $T_{\alpha+1}$ is this: There are exactly n level one nodes, $\langle 0\rangle, \langle 1\rangle, ...\langle n-1\rangle$. Beneath the first $n-1$ of these is a subtree isomorphic to T'_α and beneath the last is subtree isomorphic to T_α.

The structure of the tree $(T'_{\alpha+1} \oplus T_{\alpha+1})$ is this: There are exactly n level one nodes: $\langle 0\rangle, \langle 1\rangle, ..., \langle n-1\rangle$. There are exactly n^2 level two nodes: all $\langle i,j\rangle$ with i and j less than n. Beneath the node $\langle n-1, n-1\rangle$ there is a subtree isomorphic to T_α, beneath each of the other level two nodes there is a subtree isomorphic to T'_α.

Notice that in each of these two cases, the entire tree doesn't have an n-matching because T'_α and T_α are not isomorphic.

We will define an injective tree homomorphism (a map between trees which preserves the descendent relation) $H : \Psi(T_{\alpha+1}) \to \Psi(T'_{\alpha+1} \oplus T_{\alpha+1})$. The homomorphism H will be such that each level two node in $\Psi(T_{\alpha+1})$ will get sent to a node of $\Psi(T'_{\alpha+1} \oplus T_{\alpha+1})$ on a level greater than two. It is easy to see that $\Psi(T_{\alpha+1})$ has only finitely many level two nodes; it follows that the rank of $\Psi(T'_{\alpha+1} \oplus T_{\alpha+1})$ is greater than the rank of $\Psi(T_{\alpha+1})$.

We will first given an informal description of H and then give a formal description of H and show that it is a tree homomorphism. There will similar functions used later in the paper, but only the informal versions of these later functions will be given.

The informal description of H is this: H takes as an input a valid attempt for $T_{\alpha+1}$. This corresponds to an attempt to show that the sets of descendants beneath the nodes $\langle 0\rangle$, $\langle 1\rangle$, ..., $\langle n-1\rangle$ in $T_{\alpha+1}$ are pairwise isomorphic. This attempt translates directly to an attempt to show that the sets of descendants beneath the nodes $\langle 0, n-1\rangle$, $\langle 1, n-1\rangle$, ..., $\langle n-1, n-1\rangle$ in $T'_{\alpha+1} \oplus T_{\alpha+1}$ are pairwise isomorphic (the collections of "sets of descendants" are the same in the two situations).

It is easy to translate this into an attempt to show that the sets of descendants beneath the nodes $\langle 0\rangle$, $\langle 1\rangle$, ..., $\langle n-1\rangle$ in $T'_{\alpha+1} \oplus T_{\alpha+1}$ are pairwise isomorphic, because for each $i \neq n-1$, the sets of descendants beneath the nodes $\langle 0, i\rangle$, $\langle 1, i\rangle$, ..., $\langle n-1, i\rangle$ in $T'_{\alpha+1} \oplus T_{\alpha+1}$ are actually pairwise isomorphic. This in turn corresponds to a valid attempt for $T'_{\alpha+1} \oplus T_{\alpha+1}$. This is the output of our function H.

We'll now give the formal description of H. Consider a node of $\Psi(T_{\alpha+1})$. This is a sequence $\langle a_0, a_1, ..., a_k\rangle$ that is associated with a valid attempt $f : \{r_0, r_1, ..., r_k\} \to \mathbb{N}^{<\mathbb{N}}$ satisfying $f(r_i) = r_{a_i}$. For each nonempty $z = \langle z_0, z_1, ..., z_m\rangle \in \mathbb{N}^{<\mathbb{N}}$, define $\overline{z}$ to be $\langle z_0, n, z_1, ..., z_m\rangle$ (in the case $m = 0$, $\overline{z} = \langle z_0, n\rangle$). Let $\overline{\emptyset} = \emptyset$

We now define a function g as follows. For $r_i \in dom(f)$, define $g(\overline{r_i})$ to be $\overline{f(r_i)}$. This function g serves as a skeleton for a full function $g' : \{r_0, r_1, ..., \overline{r_k}\} \to \mathbb{N}^{<\mathbb{N}}$. We extend g to g' in the following way. Let $r_j \in \{r_0, r_1, ..., \overline{r_k}\} \setminus dom(g)$. If $r_j \notin T$, then define $g'(r_j) = \emptyset$. If, on the other hand, $r_j = \langle c_0, c_1, ..., c_l\rangle \in T$ (in this case $c_1 \neq n$) define $g'(r_j) = \langle d, c_1, c_2, ..., c_l\rangle$, where d is the unique entry of $f(c_0)$.

It is easy to check that g' is a valid attempt for $T'_{\alpha+1} \oplus T_{\alpha+1}$. The node of $\Psi(T'_{\alpha+1} \oplus T_{\alpha+1})$ that corresponds to g' is the definition of $H(\langle a_0, a_1, ..., a_k\rangle)$.

It is easy to see that if a and b are nodes in $\Psi(T_{\alpha+1})$ with b a descendent of a, then $F(b)$ is a descendent of $F(a)$ in $\Psi(T'_{\alpha+1} \oplus T_{\alpha+1})$. It is also straightforward to check that every level 2 node in $\Psi(T_{\alpha+1})$ gets sent to a node in $\Psi(T'_{\alpha+1} \oplus T_{\alpha+1})$ that is on a level greater than 2. Since there are only finitely many level 2 nodes in $\Psi(T_{\alpha+1})$, this implies that the rank of $\Psi(T'_{\alpha+1} \oplus T_{\alpha+1})$ is strictly greater than the rank of $\Psi(T_{\alpha+1})$.

We will next describe how to produce T_β and T'_β when β is a limit ordinal. First choose an increasing sequence of ordinals $(\alpha_i : i \in \mathbb{N})$ whose supremum is β.

We now define the auxiliary tree S_j for each $j \in \mathbb{N}$. The sequence $\langle a_0, a_1, ..., a_n\rangle$ is a node of S_0 if and only if: for some $r \in \mathbb{N}$, $a_r = 1$, $a_i = 0$ if $i < r$, and $\langle a_{r+1}, a_{r+2}, ..., a_n\rangle \in T'_{\alpha_r}$. For $j > 0$, the sequence $\langle a_0, a_1, a_2, ..., a_n\rangle$ is a node of S_j if and only if either $a_j = 1$, $a_i = 0$ for $i < j$, and $\langle a_{j+1}, a_{j+2}, ..., a_n\rangle \in T_{\alpha_j}$ or for some $r \neq j$, $a_r = 1$, $a_i = 0$ if $i < r$, and $\langle a_{r+1}, a_{r+2}, ..., a_n\rangle \in T'_{\alpha_r}$. Since T_{α_i} is not isomorphic to T'_{α_i}, S_0 is not isomorphic to any S_j with $j > 0$.

We will now describe T_β and T'_β. For each $i \in \mathbb{N}$, $\langle i\rangle$ is a level one node of T_β. Beneath each of the level one nodes $\langle 0\rangle, \langle 1\rangle, ..., \langle n-2\rangle$ there should be a subtree isomorphic to S_0. Beneath each of the other level one nodes should be a subtree isomorphic to some S_j for some $j > 0$. Furthermore, for each $j > 0$, there should be infinitely many level 1 nodes such that the nodes beneath them are isomorphic to S_j.

The tree T'_β is defined in the same way as T_β, except that the nodes beneath the node $\langle n-1\rangle$ should also be isomorphic to S_0. It is clear that T'_β has an n-matching and that T_β does not have an n-matching.

It remains to verify two things. First that the rank of $\Psi(T_\beta)$ is at least β. Second, that the rank of $\Psi(T'_\beta \oplus T_\beta)$ is greater than the rank of $\Psi(T_\beta)$.

Claim 1. The rank of $\Psi(T_\beta)$ is at least β.

Proof. It suffices to show that the rank of $\Psi(T_\beta)$ is greater than or equal to the rank of $\Psi(T'_{\alpha_r} \oplus T_{\alpha_r})$ for each $r > 0$. The structure of the tree $T'_{\alpha_r} \oplus T_{\alpha_r}$ is this: There are exactly n level one nodes, $\langle 0\rangle, \langle 1\rangle, ...\langle n-1\rangle$. Beneath the first $n-1$ of these is a subtree isomorphic to T'_{α_r} and beneath the last is subtree isomorphic to T_{α_r}.

The structure of the tree T_β is more complicated and we will describe in detail only a part of it. Beneath each of the nodes $\langle 0\rangle, \langle 1\rangle, ...\langle n-2\rangle$ is a subtree isomorphic with S_0. We can also choose some $m > n-2$ so that beneath the node $\langle m\rangle$ is a subtree isomorphic to S_r. So for each $i \leq n-2$ there is a subtree isomorphic to T'_{α_r} beneath the node $\langle i, 0, 0, ..., 0, 1\rangle$, where there are r-many zeros written. Also, there is a subtree isomorphic to T_{α_r} beneath the node $\langle m, 0, 0, ..., 0, 1\rangle$, where again there are r-many zeros written.

We will give an informal description of a function $G_r : \Psi(T'_{\alpha_r} \oplus T_{\alpha_r}) \to \Psi(T_\beta)$ that is similar to the function H described above. We leave it to the reader to completely formalize G_r and show that it is a tree homomorphism. This will imply that the rank of $\Psi(T_\beta)$ is greater than or equal to the rank of $\Psi(T'_{\alpha_r} \oplus T_{\alpha_r})$.

Here is the informal description of G_r: G_i takes as an input a valid attempt for $T'_{\alpha_r} \oplus T_{\alpha_r}$, which corresponds to an attempt to show that the sets of descendants beneath the nodes $\langle 0\rangle$, $\langle 1\rangle$, ..., $\langle n-1\rangle$ in $T'_{\alpha_r} \oplus T_{\alpha_r}$ are pairwise isomorphic. This translates directly to an attempt to show that the sets of descendants beneath the nodes $\langle 0, 0, 0, ..., 0, 1\rangle$, $\langle 1, 0, 0, ..., 0, 1\rangle$, ..., $\langle n-2, 0, 0, ..., 0, , 1\rangle$ and $\langle m, 0, 0, ..., 0, 1\rangle$ in T_β are pairwise isomorphic (the collections of "sets of descendants" are the same in the two situations). It is easy to see that there are no difficulties translating this into an attempt to show that the sets of descendants beneath the nodes $\langle 0\rangle$, $\langle 1\rangle$, ..., $\langle n-2\rangle$ and $\langle m\rangle$ in T_β are pairwise isomorphic. It is easy to see that this translates into a valid attempt for T_β. This is the output of the function G_r.

□

Claim 2. The rank of $\Psi(T'_\beta \oplus T_\beta)$ is greater than the rank of $\Psi(T_\beta)$.

Proof. We will give an informal description of a function $F : \Psi(T_\beta) \to \Psi(T'_\beta \oplus T_\beta)$ that is similar the function H described above. We leave it to the reader to completely formalize G_r and show that it is a tree homomorphism.

The function F takes as an input a valid attempt for T_β, which corresponds to an attempt to show that the sets of descendants beneath the nodes $\langle 0\rangle$, $\langle 1\rangle$, ..., $\langle n-2\rangle$ and $\langle m\rangle$ (some $m \geq n-1$) in T_β are pairwise isomorphic. This translates directly to an attempt to show that the sets of descendants beneath the nodes $\langle 0, n-1\rangle$, $\langle 1, n-1\rangle$, ..., $\langle n-2, n-1\rangle$ and $\langle n-1, m\rangle$ in $T'_\beta \oplus T_\beta$ are pairwise isomorphic (the collections of "sets of descendants" are the same in the two situations). It is easy to see that there are no difficulties translating this into an attempt to show that the sets of descendants beneath the nodes $\langle 0\rangle$, $\langle 1\rangle$, ..., $\langle n-2\rangle$ and $\langle n-1\rangle$ in $T'_\beta \oplus T_\beta$ are pairwise isomorphic, which corresponds to a valid attempt for $T'_\beta \oplus T_\beta$. This is the output of F.

It is easy to see that every level 2 node in $\Psi(T_\beta)$ gets sent to a node in $\Psi(T'_\beta \oplus T_\beta)$ on a level greater than 2. Since there are only finitely many level 2 nodes in $\Psi(T'_\beta \oplus T_\beta)$, this implies that the rank of $\Psi(T'_\beta \oplus T_\beta)$ is greater than the rank of T_β.

□

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