

Geometric Realizations of Representations of Finite Length

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Abstract

Let $G = H \times \mathbb{R}^n$ be a semidirect product Lie group. We reduce the problem of deciding which indecomposable representations of G may be realized in subquotients of spaces of sections of vector bundles over infinitesimal neighborhoods of orbits of H in the dual of \mathbb{R}^n , to a problem involving only representations of the H -stabilizers of the orbits.

0 Introduction

Let \mathcal{P} be the Poincare group, $\mathrm{SL}_2\mathbb{C} \times_s \mathbb{R}^{1,3}$, and let X_0^+ be the forward light cone, a massless orbit of $\mathrm{SL}_2\mathbb{C}$ in the dual of $\mathbb{R}^{1,3}$. The Gupta-Bleuler theory of quantum electrodynamics is based on the natural representation, \mathcal{W}_0 , of \mathcal{P} in the space of $\mathbb{R}^{1,3}$ -valued functions on X_0^+ . It is neither unitary nor irreducible; it is an indecomposable representation that is induced from the inhomogeneous stabilizer, or little group, of X_0^+ . In [9], G. Rideau considered the problem of constructing alternate theories of quantum electrodynamics based on other indecomposable representations \mathcal{W} having the same composition series as \mathcal{W}_0 . It is a well known consequence of the Mackey machine that all irreducible representations of semidirect product groups such as \mathcal{P} are induced, but this project led Rideau to discover that there are non-induced indecomposable representations of \mathcal{P} . He found that \mathcal{W}_0 admits deformations whose moduli space is $\mathbb{C}\mathbb{P}^1$, none of which is induced, and at the end of [9] he proposed a general study of non-induced indecomposable representations of semidirect products, which is the framework into which this paper fits.

Let $G = H \times_s A$ be the semidirect product Lie group formed by a real Lie group H acting linearly on a real vector group A , and assume that the orbits of the dual action of H on A^* are locally closed. Fix an orbit \mathcal{O} of H in A^* and a point p_0 in \mathcal{O} , and let S be the subgroup of H stabilizing p_0 . Given any representation σ of S in a finite dimensional complex vector space V , let $\mathrm{Ind}_S^H \sigma$ be the representation of H induced by σ in the space $C_c^\infty(\mathcal{O} : H \times_S V)$ of smooth compactly supported sections of the H -vector bundle over \mathcal{O} associated to V .

For any element p of A^* , let e^{ip} denote the character $a \mapsto \exp i\langle a, p \rangle$ of A , and note that $\sigma \otimes e^{ip_0}$ is a representation of the inhomogeneous stabilizer $S \times_s A$ in the space V . Here we consider smooth representations of finite length of G , having a topologically split composition series of representations of the form $\text{Ind}_{SA}^G(\sigma \otimes e^{ip_0})$. We will view such composition series representations as acting in the space $C_c^\infty(\mathcal{O} : H \times_S V)$, where the action of H is the induced one and the action of $a \in A$ is multiplication by the character function $\xi_a : p \mapsto \exp i\langle a, p \rangle$.

The first general results on this category of representations were obtained by A. Guichardet. In [6], he proved (among many other things) that it contains no indecomposable representations having in their composition series representations associated to more than one orbit \mathcal{O} , and so in this paper we restrict our study to the category $\text{Ext}_G \mathcal{O}$ of smooth topologically split representations of G whose composition series elements are all induced from the stabilizer of \mathcal{O} . This category is defined more precisely in Section 2. In [7], he proved that when the tangent bundle $T\mathcal{O}$ admits an H -covariant complementary bundle in the trivial bundle $\mathcal{O} \times A^*$ over \mathcal{O} , all representations in $\text{Ext}_G \mathcal{O}$ are induced and $\text{Ext}_G \mathcal{O}$ is categorically isomorphic to the category of representations of finite length of $S \times_s (T_{p_0} \mathcal{O})^\perp$, such that $T_{p_0} \mathcal{O}^\perp$ acts by the character e^{ip_0} on each element of the composition series. This is a very strong result which completes the study proposed by Rideau for such orbits, which are generic for most G . For example, if $G = \mathcal{P}$ then Guichardet's theorem applies to all orbits except the two light cones.

Our approach stems from the work of Cassinelli, Truini, and Varadarajan, who discovered that the non-induced deformations of \mathcal{W}_0 found by Rideau may be realized in subquotients of the space of $\mathbb{R}^{1,3}$ -valued functions *on the first-order infinitesimal neighborhood* of the light cone in its ambient space $(\mathbb{R}^{1,3})^*$ [2]. This suggests that there is a generalization of the Mackey machine which realizes the representations from $\text{Ext}_G \mathcal{O}$ in H -vector bundles over infinitesimal neighborhoods of \mathcal{O} in its ambient space A^* . In order to decide this question, it will be necessary to define the appropriate notion of homogeneous vector bundles over infinitesimal neighborhoods. Using the results of the present paper, we have proven in [5] the following preliminary version. Let $C_c^\infty \mathcal{O}^{(n)}$ be the functions on the n^{th} -order infinitesimal neighborhood of \mathcal{O} , which carry a natural representation $\mathcal{V}^{(n)}$ of G . Under the assumption that H is an algebraic group whose finite dimensional representations are all rational, any element of $\text{Ext}_G \mathcal{O}$ of length $n + 1$ may be realized as a subquotient of the representation $\mathcal{V}^{(n)} \otimes V$, where V is some finite dimensional representation of H , extended trivially to G .

The proof of this depends on the results of [3], [4], and as mentioned, this paper. The main result of [4] is the construction of a category equivalence \mathcal{F} from $\text{Ext}_G \mathcal{O}$ to a certain subcategory \mathcal{C} of the finite dimensional representations of SA . The definition of \mathcal{C} is complicated: if \mathcal{U} is an object of $\text{Ext}_G \mathcal{O}$, then as an object of \mathcal{C} , the representation $\mathcal{F}\mathcal{U}$ of SA is given with several additional structures. Our results here are the proof that the representation $\mathcal{V}^{(n)} \otimes V$ above

lies in $\text{Ext}_G \mathcal{O}$, the calculation of $\mathcal{F}(\mathcal{V}^{(n)} \otimes V)$, and our main theorem, which gives a condition for \mathcal{U} to be a subquotient of $\mathcal{V}^{(n)} \otimes V$ in terms of $\mathcal{F}\mathcal{U}$ and $\mathcal{F}(\mathcal{V}^{(n)} \otimes V)$.

In order to handle the extra structures attached to $\mathcal{F}\mathcal{U}$, we have found it convenient to define a notion of “representations of the infinitesimal neighborhood S^∞ of a Lie subgroup S in its ambient group H ” (we remark that we do not feel that these objects are worth study for their own sake). It turns out that $\mathcal{F}\mathcal{U}$ is a representation of $(SA)^\infty$, where the ambient group is G , and that it can be restricted to a representation of S^∞ , where the ambient group is H . Any representation V of H can also be restricted to a representation of S^∞ , and our main theorem is that \mathcal{U} is a subquotient of $\mathcal{V}^{(n)} \otimes V$ if as a representation of S^∞ , $\mathcal{F}\mathcal{U}$ is a subquotient of V .

It is well known that if S is an algebraic subgroup of an algebraic group H , then any rational representation of S is a subquotient of the restriction of a rational representation of H to S ; see for example [1]. In [5] we prove the analog of this with S^∞ replacing S . Coupled with the main theorem of this paper, this gives the subquotient theorem of [5] stated above.

Our results are organized as follows. In Section 1 we define representations of S^∞ , and in Section 2 we review our prior results in [3] and [4]. In Section 3 we link subquotients in $\text{Ext}_G \mathcal{O}$ to subquotients in \mathcal{C} , and in Section 4 we associate a representation of S^∞ to each object in \mathcal{C} . In Section 5 we give a precise definition of $\mathcal{C}_c^\infty \mathcal{O}^{(n)}$, and Section 6 we show that $\mathcal{V}^{(n)}$ is in $\text{Ext}_G \mathcal{O}$. In Section 7 we give explicit formulae for $\mathcal{F}(\mathcal{V}^{(n)} \otimes V)$, in Section 8 we prove the main theorem described above, and in Section 9 we apply it to Rideau’s original example.

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1 Representations of Neighborhoods of Subgroups

We begin with some definitions which are motivated by our main results. Throughout this paper, we will use gothic letters for complexified Lie algebras of real Lie groups, and we will write $\mathfrak{U}(\mathfrak{h})$ and $\mathfrak{U}_r(\mathfrak{h})$ for the universal enveloping algebra of a Lie algebra \mathfrak{h} and its standard filtration, respectively. Wherever it is convenient, we will write $\mathbf{1}$ for the identity map of any set with itself, and if π is a representation of some group $K \ni k$ we will write π_k for $\pi(k)$ whenever it abbreviates the notation.

Definition. Let H be a real Lie group, and let S be a Lie subgroup. A representation of the infinitesimal neighborhood of S in H , or more briefly a representation of S^∞ , is a complex finite dimensional vector space W_0 , a representation σ of S on W_0 , an S -invariant subspace W_1 of W_0 , and for each $X \in \mathfrak{h}$ a linear map $\sigma(X) : W_1 \rightarrow W_0$ such that

1. Let $d\sigma : \mathfrak{s} \rightarrow \text{End}(W_0)$ be the differential of $\sigma|_S$. Then for all $X \in \mathfrak{s}$, $d\sigma(X)|_{W_1} = \sigma(X)$.
2. For all $s \in S$ and $X \in \mathfrak{h}$, $\sigma(s) \circ \sigma(X) \circ \sigma(s^{-1}) = \sigma(\text{Ad}_s X)$ on W_1 .
3. For all $X, Y \in \mathfrak{h}$, $[\sigma(X), \sigma(Y)] = \sigma([X, Y])$ wherever the left hand side is defined, i.e., on the intersection of the inverse images of W_1 under the maps $\sigma(X)$ and $\sigma(Y)$.

We will often refer to such representations simply as σ, W , where it is understood that W denotes the flag $W_0 \supset W_1$ and σ denotes the actions of both S and \mathfrak{h} . Associated to any representation σ, W of S^∞ is a flag $W_0 \supset W_1 \supset W_2 \supset \dots$ which extends the given flag $W_0 \supset W_1$, defined inductively by

$$W_k = \bigcap_{X \in \mathfrak{h}} \sigma(X)^{-1}(W_{k-1}),$$

the intersection of the inverse images of W_{k-1} under $\sigma(X)$ as X runs over \mathfrak{h} . Note that the inverse image of W_0 under $\sigma(X)$ is W_1 for all $X \in \mathfrak{h}$ by definition, so this definition is consistent when $k = 1$. It follows from an inductive argument that W_k is indeed a subspace of W_{k-1} for all k , and another inductive argument using condition (2) in the definition of representations of S^∞ shows that W_k is S -invariant for all $k \geq 0$.

The flag $\{W_k\}$ has the property that $\sigma(X)W_k \subset W_{k-1}$ for all $X \in \mathfrak{h}$, and in fact its definition is the same as defining it to be the finest such flag. It follows from conditions (1) and (3) above that for each Z in $\mathfrak{U}(\mathfrak{s})\mathfrak{U}_r(\mathfrak{h})$, there is a map $\sigma(Z) : W_r \rightarrow W_0$ such that $\sigma(Z)(W_k) \subset W_{k-r}$ for all k , and $\sigma(Z)\sigma(Z') = \sigma(ZZ')$ wherever both sides are defined for all Z and Z' in $\mathfrak{U}(\mathfrak{h})$. The point is that for X_1, \dots, X_p in \mathfrak{h} , we may define $\sigma(X_1 \cdots X_p) : W_p \rightarrow W_0$ to be $\sigma(X_1) \cdots \sigma(X_p)$; this definition is unambiguous by condition (3) and the Poincare-Birkhoff-Witt (PBW) theorem. Furthermore, if all but r of the X_i 's lie in \mathfrak{s} , $\sigma(X_1 \cdots X_p)$ is defined from W_r to W_0 .

The representations of S^∞ form a category, for which we write $\mathcal{R}S^\infty$. A morphism T from an object σ, W to an object σ', W' is a linear map $T : W_0 \rightarrow W'_0$ such that $T(W_1) \subset W'_1$ and T intertwines both the S and \mathfrak{h} actions. An inductive argument shows that T respects the flags associated to W and W' , i.e., $T(W_k) \subset W'_k$ for all $k \geq 0$. We now generalize some of the standard constructions for representations of S to $\mathcal{R}S^\infty$.

There is an obvious notion of subobjects in $\mathcal{R}S^\infty$. If σ, W is as above and $V = \{V_0 \supset V_1\}$ is a subflag of $W_0 \supset W_1$, such that V_k is an S -invariant subspace of W_k for $k = 0, 1$ and $\sigma(X)(V_1) \subset V_0$ for all $X \in \mathfrak{h}$, then V is a subobject of σ, W . In this case one checks that for all $k \geq 0$ the associated flag space V_k is an S -invariant subspace of W_k .

There is also a notion of quotient objects, but only certain subobjects are permitted as factors. Specifically, if σ, W and V are as above and $V_1 = V_0 \cap W_1$,

then we define the quotient representation $\bar{\sigma}, W/V$ of S^∞ by $(W/V)_k = W_k/V_k$ for $k = 0, 1$ and $\bar{\sigma}$ the obvious quotient map. The property $V_1 = V_0 \cap W_1$ is necessary and sufficient to insure that $(W/V)_1$ is canonically a subspace of $(W/V)_0$. We will refer to this property as “the quotient property.” An inductive argument shows that if V has this property then $V_k = W_k \cap V_0$ for all k , and also that W_k/V_k is canonically a subspace of $(W/V)_k$ for all k . Note that in general $(W/V)_k$ may not be equal to W_k/V_k . Our first lemma shows that with these definitions morphisms have the usual property. The proof is left as an exercise.

Lemma 1.1 *Let σ, W and σ', W' be representations of S^∞ , and let $T : W \rightarrow W'$ be a morphism. Then the subflag $T(W)$ of W' is σ' -invariant, and so is a subobject of σ', W' . The kernel subflag $K_k = T^{-1}(0) \cap W_k$ ($k = 0, 1$) of W is a subobject of σ, W having the quotient property, and T factors through to an isomorphism $\bar{T} : W/K \rightarrow T(W)$. \square*

Retain σ, W and σ', W' as representations of S^∞ . We define the direct sum object $\sigma \oplus \sigma', W \oplus W'$ to be the obvious direct sum action on the flag $(W \oplus W')_k = W_k \oplus W'_k$. We also define the tensor product object $\sigma \otimes \sigma'$ acting in the flag $(W \otimes W')_k = W_k \otimes W'_k$, $k = 0, 1$, by $(\sigma \otimes \sigma')(s) = \sigma(s) \otimes \sigma'(s)$ for $s \in S$, and $(\sigma \otimes \sigma')(X) = \sigma(X) \otimes \mathbf{1} \oplus \mathbf{1} \otimes \sigma'(X)$ for $X \in \mathfrak{h}$. It is easy to check that this definition makes $\sigma \otimes \sigma'$ a representation of S^∞ . Here $(W \otimes W')_k \supset W_k \otimes W'_k$ for all $k \geq 0$, but in general the containment may be proper.

This tensor product has the usual properties of associativity and commutativity, and the trivial object $W_0^{triv} = W_1^{triv} = \mathbb{C}$, $\sigma^{triv}|_S = \mathbf{1}$, $\sigma^{triv}|_{\mathfrak{h}} = 0$ is a unit for it. For us, its most important property is that $W \otimes W'$ contains a subobject V defined by

$$V_k = \sum_{i+j=n-k} W_i \otimes W'_j, \quad k = 0, 1.$$

If $k \geq 2$, the associated flag space V_k is easily seen to contain the space $\sum_{i+j=n-k} W_i \otimes W'_j$, although the containment may be proper.

A few comments are in order regarding the category $\mathcal{R}S^\infty$. First, we do not have dual objects, because for an object σ, W of $\mathcal{R}S^\infty$, W_1^* is not a subspace of W_0^* . Also, it is not possible to divide by subrepresentations unless they have the quotient property, and so $\mathcal{R}S^\infty$ is not an abelian category. Finally, it is not clear which objects should be thought of as irreducible. Objects with no subrepresentations of any kind take the form that W_0 is an irreducible representation of S and $W_1 = 0$, but objects with no subrepresentations that have the quotient property are more difficult to describe. Thus we suspect that the definition of $\mathcal{R}S^\infty$ can be improved upon, but it will do for our applications. We will conclude this section with some examples of objects of S^∞ that will be useful later on.

Examples. The (complex finite dimensional) representations of S sit inside $\mathcal{R}S^\infty$ as the full subcategory of objects σ, W such that $W_1 = 0$. At the opposite extreme, the ordinary (S, \mathfrak{h}) -representations sit inside $\mathcal{R}S^\infty$ as the subcategory of objects σ, W such that $W_1 = W_0$; these representations of S^∞ are called restrictions of representations of S, \mathfrak{h} . We will need the following lemma in Section 9.

Lemma 1.2 *If $\tilde{\sigma}$ is a representation of S on a space W_0 and $W_1 = W_0^S$ is the subspace of invariant vectors, then there is a representation σ of S^∞ on $W_0 \supset W_1$ defined by $\sigma|_S = \tilde{\sigma}$ and $\sigma|_{\mathfrak{h}} = 0$.*

If π is a representation of H on a space V_0 , let $\tilde{\sigma} = \pi|_S$, let $V_1 = V_0^S$, and let σ be the representation of S^∞ on $V_0 \supset V_1$ defined as in the last paragraph. Let \tilde{V}_1 be a linearly isomorphic copy of V_1 , and define a representation $\tilde{\pi}$ of H in $V_0 \oplus \tilde{V}_1$ by $\tilde{\pi} = \pi \oplus \mathbf{1}$. Then σ is isomorphic to an S^∞ -subquotient of the restriction of $\tilde{\pi}$ to the S^∞ representation $\tilde{\pi}|_{S^\infty}$.

Proof. The first sentence is easy. For the second, let $j : V_1 \rightarrow \tilde{V}_1$ be an isomorphism, and let $Y_0 = \{(v, jv) : v \in V_1\} \subset V_0 \oplus \tilde{V}_1$. By definition, $\tilde{\pi}|_{S^\infty}$ acts in the flag $V_0 \oplus \tilde{V}_1 \supset V_0 \oplus \tilde{V}_1$, and the flag $V_0 \oplus \tilde{V}_1 \supset \tilde{V}_1$ is an S^∞ -subrepresentation of $\tilde{\pi}|_{S^\infty}$ on which \mathfrak{h} acts trivially (and which does not have the quotient property). Furthermore, the flag $Y_0 \supset 0$ is an S^∞ -subrepresentation of $V_0 \oplus \tilde{V}_1 \supset \tilde{V}_1$ which does have the quotient property, and one checks that the quotient is isomorphic to σ . Note that σ is usually not a subrepresentation of $\pi|_{S^\infty}$, because V_1 is usually not V_0^H . \square

If N, π is any representation of S and n is any non-negative integer, then we may define a representation W, σ of S^∞ by $W_0 = \mathfrak{U}_n(\mathfrak{h}) \otimes_s N$, $W_1 = \mathfrak{U}_{n-1}(\mathfrak{h}) \otimes_s N$, $\sigma|_S = \text{Ad} \otimes \pi$, and $\sigma|_{\mathfrak{h}}$ the usual left action on the first factor. In this case it is an exercise to prove that $W_k = \mathfrak{U}_{n-k}(\mathfrak{h}) \otimes_s N$ for $k \leq n$.

An interesting example for us is the following. Let A be a real representation of H , let \mathcal{O} be the orbit under H of a point p_0 in the dual A^* of A , and let S be the stabilizer of p_0 . Assume that \mathcal{O} is locally closed in A . Let $\Delta_{p_0}^n A^*$ be the space of differential operators of order $\leq n$ on A^* supported at p_0 , so that elements of $\Delta_{p_0}^n A^*$ are linear maps $\delta : C_c^\infty(A^*) \rightarrow \mathbb{C}$ such that δf depends only on the n -jet of f at p_0 . Then there is a natural (S, \mathfrak{h}) -action on $\Delta_{p_0}^\infty A^*$ such that S leaves $\Delta_{p_0}^n A^*$ invariant and \mathfrak{h} maps it into $\Delta_{p_0}^{n+1} A^*$, and so we get for each n a representation W of S^∞ such that $W_k = \Delta_{p_0}^{n-k} A^*$ for $k = 0, 1$. Later we will develop a more explicit description of W , and prove that in fact $W_k = \Delta_{p_0}^{n-k} A^*$ for all k .

2 Review of Prior Results

Throughout this paper fix $G = H \times_s A$ and \mathcal{O}, p_0 , and S as in the introduction. In this section we define the category of smooth representations of G of finite

length with a topologically split composition series as described in the introduction, and we recall the main results of [3] and [4]. We will use Schwartz' notations \mathcal{D} and \mathcal{E} in place of C_c^∞ and C^∞ , and if B is a vector bundle over a space \mathcal{O} then $\mathcal{D}(B)$ and $\mathcal{E}(B)$ will always denote the sections of B , equipped with their usual topologies of uniform convergence of all derivatives on all compact sets [10]. We will write $\mathcal{D}B$ instead of $\mathcal{D}(B)$ whenever it simplifies the notation, and similarly for \mathcal{E} . We will write $B(p)$ or B_p for the fiber of B at $p \in \mathcal{O}$.

Definition. Let $\text{Ext}_G \mathcal{O}$ be the category of smooth representations of G in topological vector spaces V that admit a G -invariant flag

$$V = V_0 \supset \cdots \supset V_n \supset V_{n+1} = 0$$

with the following properties. First, for each i there is a closed subspace of V_i complementary to V_{i+1} , and second, each subquotient V_i/V_{i+1} is topologically equivalent to the representation $\text{Ind}_{SA}^G(\sigma \otimes e^{ip_0})$ for some finite dimensional complex representation σ of S . Morphisms are continuous linear intertwining maps.

Henceforth fix a real subbundle C of the vector bundle $\mathcal{O} \times A^*$ over \mathcal{O} that is complementary to the tangent bundle $T\mathcal{O}$. As we said in the introduction, when C can be chosen to be an H -bundle Guichardet has completely described $\text{Ext}_G \mathcal{O}$ [7], and so our previous work was concerned mainly with the exceptional cases where this cannot be done. However, it applies in all cases, and so what follows is general. Associated to C there is a full subcategory $\text{Ext}_G^C \mathcal{O}$ of $\text{Ext}_G \mathcal{O}$ such that the inclusion functor is an equivalence of categories, and which is much easier to study.

Definition. An object of the category $\text{Ext}_G^C \mathcal{O}$ is a representation \mathcal{U} in $\text{Ext}_G \mathcal{O}$, together with an integer n and an ordered set $\sigma^0, \dots, \sigma^n$ of finite dimensional representations of S in spaces V_i , which are related to \mathcal{U} as follows. Let F_i be the H -vector bundle $H \times_S V_i$ over \mathcal{O} , and let \mathcal{U}^i be the representation $\text{Ind}_{SA}^G(\sigma^i \otimes e^{ip_0})$ acting in $\mathcal{D}F_i$ as in the introduction.

1. The representation \mathcal{U} acts in the topological vector space $\bigoplus_0^n \mathcal{D}F_i$, and so for all $g \in G$ we may view \mathcal{U}_g as a matrix with entries

$$\mathcal{U}_g^{ij} : \mathcal{D}F_j \rightarrow \mathcal{D}F_i.$$

2. The flag $\bigoplus_j^n \mathcal{D}F_i$, $0 \leq j \leq n$, is G -invariant, and \mathcal{U} defines the representations \mathcal{U}^j in the subquotients $\mathcal{D}F_j$. In other words, for all $g \in G$ the matrix \mathcal{U}_g^{ij} is lower triangular, with diagonal entries $\mathcal{U}_g^{ii} = \mathcal{U}_g^i$.
3. For all $a \in A$, \mathcal{U}_a^{ij} is a smooth section of $\text{Hom}(F_j, F_i)$, i.e., \mathcal{U}_a acts fiberwise. It has the form $\mathcal{U}_a = \xi_a \exp l_a$, where ξ_a is defined in the introduction and

l_a is a continuous endomorphism of $\bigoplus_0^n \mathcal{D}F_i$ depending linearly on a such that l_a^{ij} is 0 for $i \leq j$ and is a smooth section of $\text{Hom}(F_j, F_i)$ for $i > j$. Furthermore, for all $p \in \mathcal{O}$, $l_a(p) = 0$ for all a in the subspace C_p^\perp of A that annihilates C_p . Note that $\exp l_a$ is polynomial in l_a , as l_a is nilpotent.

Let $\mathcal{U}, F_0, \dots, F_n$ and $\mathcal{U}', F'_0, \dots, F'_{n'}$ be two objects of $\text{Ext}_G^C \mathcal{O}$. A morphism between them is a continuous linear map

$$T : \bigoplus_0^n \mathcal{D}F_i \rightarrow \bigoplus_0^{n'} \mathcal{D}F'_i$$

intertwining the actions of \mathcal{U} and \mathcal{U}' , such that the matrix entries $T^{ij} : \mathcal{D}F_j \rightarrow \mathcal{D}F'_i$ are smooth sections of $\text{Hom}(F_j, F'_i)$, i.e., such that T acts fiberwise.

Theorem 2.1 *The forgetful functor from $\text{Ext}_G^C \mathcal{O}$ to $\text{Ext}_G \mathcal{O}$ which maps an object $\mathcal{U}, F_0, \dots, F_n$ to \mathcal{U} is an equivalence of categories. If $\mathcal{U}, F_0, \dots, F_n$ is any object of $\text{Ext}_G^C \mathcal{O}$, then for all $h \in H$ and $0 \leq j \leq i \leq n$ the matrix entry \mathcal{U}_h^{ij} is an order $\leq i - j$ differential operator from F_j to F_i above the diffeomorphism $h : \mathcal{O} \rightarrow \mathcal{O}$, and for all $X \in \mathfrak{h}$ the matrix entry \mathcal{U}_X^{ij} is an order ≤ 1 differential operator above the identity map from \mathcal{O} to \mathcal{O} .*

Proof. A proof of this theorem is given mainly in [3], but first see Section 2 of [4]. By a differential operator of order $\leq r$ from F_j to F_i above the diffeomorphism $h : \mathcal{O} \rightarrow \mathcal{O}$, we mean a continuous linear map $D : \mathcal{D}F_j \rightarrow \mathcal{D}F_i$ such that for any $s \in \mathcal{D}F_j$ and $p \in \mathcal{O}$, $Ds(p)$ depends only on the r^{th} jet of s at $h^{-1}p$. For a more detailed definition, see Section 1 of [4]. \square

This theorem is the starting point of [4]. It shows that any object $\mathcal{U}, F_0, \dots, F_n$ of $\text{Ext}_G^C \mathcal{O}$ acts locally, and this may be used to associate to \mathcal{U} a representation of SA . This is complicated because H can act by differential operators, and so \mathcal{U} may not be induced, but Theorem 2.1 allows us to proceed as in Sections 3 and 4 of [4], which we will now review.

For any vector bundle B over \mathcal{O} , let $\Delta^r B$ be the vector bundle over \mathcal{O} whose sections are differential operators of order $\leq r$ from $\mathcal{D}B$ to $\mathcal{D}\mathcal{O}$. Let λ be the quasiregular representation of H on $\mathcal{D}\mathcal{O}$, and extend it to G by making A act so that λ_a is multiplication by the character function ξ_a . Fix an object $\mathcal{U}, F_0, \dots, F_n$ of $\text{Ext}_G^C \mathcal{O}$, and for $0 \leq l \leq k < \infty$ define vector bundles

$$\tilde{E}_{kl}(\mathcal{U}) = \bigoplus_{i=0}^l \Delta^{k-i} F_i.$$

We will usually write simply \tilde{E}_{kl} for $\tilde{E}_{kl}(\mathcal{U})$, and similarly for all the other maps and spaces we will associate to \mathcal{U} . Note that $\tilde{E}_{kl} \subset \tilde{E}_{k'l'}$ if $k \leq k'$ and $l \leq l'$, so that $\{\tilde{E}_{kl}\}$ is a biflag. A section D of \tilde{E}_{kl} is a differential operator

from $\oplus_{i=0}^n \mathcal{D}F_i$ to $\mathcal{D}\mathcal{O}$ in an obvious way, and its order is graded in a way compatible with the grading of the order of the differential operator \mathcal{U}_h given by Theorem 2.1. Consequently, $\lambda_h \circ D \circ \mathcal{U}_h^{-1}$ is again a section of \tilde{E}_{kl} , and we define a representation $\Delta\mathcal{U}$ of G on the biflag $\{\mathcal{D}\tilde{E}_{kl}\}$ by

$$\Delta\mathcal{U}_g(D) = \lambda_g \circ D \circ \mathcal{U}_g^{-1}.$$

It is easy to see that $\Delta\mathcal{U}$ is induced from SA , and we will write $\sigma(\mathcal{U})$ or simply σ for the inducing representation, which acts in the biflag of fibers $\{\tilde{E}_{kl}(p_0)\}$. This is the representation of SA which is associated to \mathcal{U} in [4].

There is a similar construction for morphisms in $\text{Ext}_G^C \mathcal{O}$. Let $\mathcal{U}', F'_0, \dots, F'_{n'}$ be a second object of $\text{Ext}_G^C \mathcal{O}$, and write \tilde{E}'_{kl} for $\tilde{E}_{kl}(\mathcal{U}')$, and so on. If $T : \mathcal{U} \rightarrow \mathcal{U}'$ is a morphism, then there is an induced morphism $\Delta T : \Delta\mathcal{U}' \rightarrow \Delta\mathcal{U}$ defined by $\Delta T(D) = D \circ T$, i.e., ΔT is the transpose T^T of T . Since morphisms in $\text{Ext}_G^C \mathcal{O}$ act fiberwise, we find that ΔT maps $\mathcal{D}\tilde{E}'_{kl}$ into $\mathcal{D}\tilde{E}_{k+n, n}$ for all $0 \leq l \leq k$. We will write $\mu(T)$ or simply μ for the inducing morphism, and we will see later in this section that it is closely related to $T(p_0)^T$. When $n = n'$ and T maps the invariant subspace $\oplus_{i=j}^n \mathcal{D}F_i$ of \mathcal{U} into the invariant subspace $\oplus_{i=j}^{n'} \mathcal{D}F'_i$ of \mathcal{U}' for $0 \leq j \leq n$, we say that T is a *flag morphism*. In this case we find that $\Delta T(\mathcal{D}\tilde{E}'_{kl}) \subset \mathcal{D}\tilde{E}_{kl}$.

The maps $\mathcal{U} \mapsto \sigma(\mathcal{U})$ and $T \mapsto \mu(T)$ make up a functor from $\text{Ext}_G^C \mathcal{O}$ to the finite dimensional representations of SA , which is easily seen to be faithful. However, in order to have a useful “little group method” for $\text{Ext}_G^C \mathcal{O}$ it is necessary to determine the range of this functor. This is done in [4]; the essential idea is that the representation σ of SA on $\tilde{E}_{nn}(p_0) = \oplus_{i=0}^n \Delta^{n-i} F_i(p_0)$ is determined by its restriction to the *non-invariant* subspace $\oplus_{i=0}^n F_i^*(p_0)$. To be more precise, we need several definitions.

Let $S' = SA$, $\mathfrak{s}' = \mathfrak{s} \oplus \mathfrak{a}$, and $\bar{\mathfrak{U}}_r = \mathfrak{U}_r(\mathfrak{g})\mathfrak{U}(\mathfrak{s}')$ for $0 \leq r$. Choose a subspace \mathfrak{t} of \mathfrak{h} complementary to \mathfrak{s} and an ordered basis r_1, \dots, r_q of \mathfrak{t} , and let R_r be the the subspace of $\mathfrak{U}_r(\mathfrak{g})$ spanned by the monomials $r^I = r_1^{I_1} \dots r_q^{I_q}$ such that $|I| = \sum_1^q I_i \leq r$. Note that $\bar{\mathfrak{U}}_r = \mathfrak{U}(\mathfrak{s}')\mathfrak{U}_r(\mathfrak{g}) = \mathfrak{U}(\mathfrak{s}')R_r$.

Let $L_i(\mathcal{U}) = F_i^*(p_0)$, let $E_{kl}(\mathcal{U}) = \tilde{E}_{kl}(p_0)$, and let $N_k(\mathcal{U}) = \oplus_0^k L_i$. Define biflags

$$J_{kl}(\mathcal{U}) = \bigoplus_0^l L_i \otimes \bar{\mathfrak{U}}_{k-i}, \quad M_{kl}(\mathcal{U}) = \bigoplus_0^l L_i \otimes R_{k-i}.$$

An element $\omega \in N_n$ may be viewed as an order 0 differential operator at p_0 on $\oplus_0^n \mathcal{D}F_i$, and one checks that if $\omega \in N_l$ and $Z \in \bar{\mathfrak{U}}_k$ are such that $\omega \otimes Z$ is in J_{kl} , then $\omega \circ \mathcal{U}_Z$ is in E_{kl} . Let $\rho(\mathcal{U}) : J_{kl} \rightarrow E_{kl}$ be the resulting map of biflags $\omega \otimes Z \mapsto \omega \circ \mathcal{U}_Z$. It turns out that $\rho(J_{kl}) = E_{kl}$ and that ρ maps M_{kl} isomorphically onto E_{kl} , and so we may let $q(\mathcal{U}) : E_{kl} \rightarrow M_{kl}$ be the right inverse of ρ .

In order to understand the way in which σ is determined by its restriction

to N_n , we will use some rather cumbersome constructions. Define a map

$$\eta(\mathcal{U}) : S' \rightarrow \text{Hom}_{\mathbb{C}}(N_n, M_{nn})$$

by $\eta_s = q \circ \sigma_s|_{N_n}$ for $s \in S'$. Note that N_n is a subspace of both E_{nn} and J_{nn} , and that $\rho|_{N_n} = \mathbf{1}$, so η_s may also be written as $q \circ \sigma_s \circ \rho|_{N_n}$. Let $d\eta : \mathfrak{s}' \rightarrow \text{Hom}_{\mathbb{C}}(N_n, M_{nn})$ be the differential, so that for $X \in \mathfrak{s}'$, $d\eta_X = q \circ \sigma_X|_{N_n} = q \circ \sigma_X \circ \rho|_{N_n}$. We will usually write simply η for $d\eta$.

For $\delta \in J_{kl}$ and $Z \in \overline{\mathfrak{U}}_r$, let $\delta \cdot Z \in J_{k+r,l}$ be the obvious product: when $\delta = \omega \otimes Z'$, $\delta \cdot Z = \omega \otimes Z'Z$. Define a map $\mathcal{N}(\mathcal{U}) : S' \rightarrow \text{End}_{\mathbb{C}}(\sum_{kl} J_{kl})$ by

$$\mathcal{N}_s(\omega \otimes Z) = (\eta_s \omega) \cdot \text{Ad}_s Z.$$

We will write \mathcal{N} for the differential $d\mathcal{N}$, so that if $X \in \mathfrak{s}'$ then $\mathcal{N}_X(\omega \otimes Z) = \eta_X \omega \cdot Z + \omega \otimes \text{ad}_X Z$. Roughly, we may think of \mathcal{N} as $\eta \otimes \text{Ad}$, but this is imprecise because $\eta \otimes \mathbf{1}$ and $\mathbf{1} \otimes \text{Ad}$ do not commute.

Let r be the right regular representation of $\mathfrak{U}(\mathfrak{s}')$ on the biflag $\{J_{kl}\}$, so that if $Z \in \mathfrak{U}(\mathfrak{s}')$ and $\delta \in J_{kl}$, then $r_Z \delta = \delta \cdot Z^T$, where $Z \mapsto Z^T$ is the usual antiautomorphism of \mathfrak{U} . It is easy to check that η maps N_k into M_{kk} for $0 \leq k \leq n$, that \mathcal{N} and r both leave J_{kl} and also the space K_{kl} defined below invariant for $0 \leq l \leq k$, and that for $s \in S'$ and $X \in \mathfrak{s}'$ we have $\rho \circ \mathcal{N}_s = \sigma_s \circ \rho$ and $\rho \circ r_X = \sigma_X \circ \rho$. However, \mathcal{N} is not usually a representation of S' on J_{kl} .

Let K_{kl} be the subspace of J_{kl} defined by

$$K_{kl} = \{\eta_X \omega \cdot Z + \omega \otimes XZ : \omega \in N_i, Z \in \overline{\mathfrak{U}}_{k-i}, X \in \mathfrak{s}', 0 \leq i \leq l\}.$$

It is not too hard to prove that $K_{kl} = \text{kernel } \rho|_{J_{kl}}$, and hence that $J_{kl} = M_{kl} \oplus K_{kl}$. It follows from this that for any $r, s \in S'$ and $X, Y \in \mathfrak{s}'$, $\mathcal{N}_r \mathcal{N}_s - \mathcal{N}_{rs}$ and $[N_X, N_Y] - \mathcal{N}([X, Y])$ both map N_k into $K_{k-1, k-1}$ for $0 \leq k \leq n$.

We have seen that starting out knowing only the spaces L_i and the map η , we can construct the biflags J_{kl} , M_{kl} , and K_{kl} , and the map \mathcal{N} . Since ρ is essentially the quotient map which divides by K_{kl} , and ρ intertwines \mathcal{N} and σ , we can reconstruct the representation σ from η and the L_i . This will serve as motivation for the definition of the category \mathcal{C} of representations of SA to which $\text{Ext}_{\mathcal{C}}^0 \mathcal{O}$ is isomorphic.

Definition An object σ of the category \mathcal{C} is a representation of $S' = SA$, along with the following additional structures. First, we are given an integer $n(\sigma) \geq 0$ and $n(\sigma)+1$ complex finite dimensional vector spaces $L_0(\sigma), \dots, L_{n(\sigma)}(\sigma)$. From these we form biflags $J_{kl}(\sigma)$ and $M_{kl}(\sigma)$ and a flag $N_k(\sigma)$, exactly as above. We will usually suppress the argument σ on all the maps and spaces associated to σ . Second, we are given a smooth map $\eta(\sigma) : S' \rightarrow \text{Hom}_{\mathbb{C}}(N_n, M_{nn})$, such that $\eta_s(N_k) \subset M_{kk}$ for all $s \in S'$. From η we form a map $\mathcal{N}(\sigma) : S' \rightarrow \text{End}_{\mathbb{C}} \sum_{kl} J_{kl}$ and another biflag $K_{kl}(\sigma)$, again exactly as above, and we also write η and \mathcal{N} for the differentials of η and \mathcal{N} at $e \in S'$. It is easy to check that \mathcal{N} leaves both J_{kl} and K_{kl} invariant. The map η is required to have the following properties.

1. For all r, t in SA , $\mathcal{N}_r \circ \mathcal{N}_t - \mathcal{N}_{rt}$ maps N_k into $K_{k-1, k-1}$ for $1 \leq k \leq n$, and it maps N_0 to 0.
2. There is a linear map $L(\sigma) : A \rightarrow \text{End}(N_n)$ such that for all $a \in A$, $\eta_a = e^{-i\langle a, p_0 \rangle} \exp L_{-a}$. Furthermore, for all $a \in A$, $L_a(N_k) \subset N_{k-1}$ for $1 \leq k \leq n$, $L_a(N_0) = 0$, and if $\langle a, C_{p_0} \rangle = 0$, $L_a = 0$. Note that L_a is nilpotent, and that η_a leaves N_k invariant.

Property 2 derives from Property 3 in the definition of $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$. In [4] we prove (with considerable difficulty) that Property 1 is equivalent to requiring that $J_{kl} = M_{kl} \oplus K_{kl}$. It follows that J_{kl}/K_{kl} is naturally a subspace of $J_{k'l'}/K_{k'l'}$ for $k \leq k'$ and $l \leq l'$, and so we may define a biflag $E_{kl}(\sigma) = J_{kl}/K_{kl}$. As before, let $\rho(\sigma) : J_{kl} \rightarrow E_{kl}$ be the quotient map, and let $q(\sigma) : E_{kl} \rightarrow M_{kl}$ be its right inverse. Since \mathcal{N} leaves K_{kl} invariant, it factors through ρ to a map from S' to $\text{End} \sum E_{kl}$. It is immediate from Property 1 that this map is a representation leaving E_{kl} invariant, and we define σ itself by making it equal to this representation. Note that if we are given only η and the L_i , we can reconstruct σ .

In order to define morphisms in \mathcal{C} , let σ' be another object and denote all the associated maps and spaces by primes. A morphism $\mu : \sigma \rightarrow \sigma'$ is a linear map $\mu : \sum E_{kl} \rightarrow \sum E'_{kl}$ intertwining σ and σ' , given along with a linear map $\tau(\mu) : N_n \rightarrow N'_{n'}$ such that $\rho \circ (\tau \otimes \mathbf{1}) = \mu \circ \rho$. Here $\tau \otimes \mathbf{1}$ is the obvious endomorphism of $\sum J_{kl}$, and we have suppressed the argument μ of τ as usual.

The condition that μ be an S' -map is equivalent to $(\tau \otimes \mathbf{1}) \circ \eta_s = \eta'_s \circ \tau$ for all $s \in S'$, and it is a consequence of the relation between μ and τ that $\mu(E_{kl}) \subset E'_{k+n', n'}$. If $n = n'$ and $\tau(N_k) \subset N'_k$ for $0 \leq k \leq n$, then one checks that $\mu(E_{kl}) \subset E'_{kl}$ and we say that μ is a *flag morphism*.

That concludes the definition of \mathcal{C} . It is clumsy, but it is useful for computations. The reader may ask why we do not replace the strange Property 1 with the requirement that $J_{kl} = M_{kl} \oplus K_{kl}$; the reason is that Property 1 is more useful for treating the types of examples in [4]. We remark that Properties 1 and 2 together lead to the fact that for any $X \in \mathfrak{s}'$, η_X maps N_k into $N_k + (N_{k-1} \otimes R_1)$, which is expected from the form of \mathcal{U}_X given in Theorem 2.1. The following theorem is the main result of [4].

Theorem 2.2 *There is a functor $\mathcal{F} : \text{Ext}_G^{\mathcal{C}} \mathcal{O} \rightarrow \mathcal{C}$ defined by*

$$\mathcal{F}(\mathcal{U}) = \{\sigma(\mathcal{U}), \eta(\mathcal{U}), L_0(\mathcal{U}), \dots, L_n(\mathcal{U})\}$$

and $\mathcal{F}(T) = \{\mu(T), \tau(T)\}$, where $\tau(T)$ is the transpose $T(p_0)^T$ of $T(p_0)$. This functor is a contravariant equivalence of categories.

Definition. We conclude our review by defining an extension of any object σ of \mathcal{C} to a representation of S'^{∞} (see Section 1), where S' is viewed as a subgroup of G . We will write also σ for this extension. The flag for σ as a

representation of S'^∞ is $E_{nn} \supset E_{n-1,n-1}$, and for $X \in \mathfrak{g}$, σ_X is obtained by factoring the right regular representation r of \mathfrak{g} on $\sum J_{kl} = N_n \otimes \mathfrak{U}(\mathfrak{g})$ through ρ . The point is that $r_X(J_{kl}) \subset J_{k+1,l}$ and $r_X(K_{kl}) \subset K_{k+1,l}$, so we may define $\sigma_X : E_{kl} \rightarrow E_{k+1,l}$ by $\sigma_X \circ \rho = \rho \circ r_X$. The compatibility conditions σ must meet to be a representation of S'^∞ follow from $\rho \circ r_X = \rho \circ \mathcal{N}_X = \sigma_X \circ \rho$ for all $X \in \mathfrak{s}'$. It is not hard to use the PBW Theorem to check that the flag associated to σ is $E_{nn} \supset E_{n-1,n-1} \supset \cdots \supset E_{1,1} \supset E_{0,0} \supset 0$. If $\mu : \sigma \rightarrow \sigma'$ is a flag morphism between two objects of \mathcal{C} , then it is also an S'^∞ -morphism. In Section 4 we will describe those representations of S'^∞ which arise from objects of \mathcal{C} in this way.

In Sections 7 and 8 we will need the following facts. First, $(\sigma_\alpha + i\langle \alpha, p_0 \rangle)(E_{kk}) \subset E_{k-1,k-1}$ for any $\sigma \in \mathcal{C}$ and $\alpha \in \mathfrak{a}$. This will be clear if we show that $\mathcal{N}_\alpha + i\langle \alpha, p_0 \rangle$ maps J_{kk} into $J_{k-1,k-1}$. Since $\mathcal{N}_\alpha = \eta_\alpha \otimes \mathbf{1} + \mathbf{1} \otimes \text{ad}_\alpha$, and $\eta_\alpha + i\langle \alpha, p_0 \rangle = -L_\alpha$ maps N_k into N_{k-1} by the definition of \mathcal{C} while ad_α maps $\underline{\mathfrak{M}}_r$ into $\underline{\mathfrak{M}}_{r-1}$ because \mathfrak{a} is an ideal in \mathfrak{g} , the result follows. Second, if $\sigma = \mathcal{F}(\mathcal{U})$ for some \mathcal{U} in $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$, then for $X \in \mathfrak{g}$ the action of σ_X on $E_{n-1,n-1}$ is $\sigma_X \delta = -\delta \circ \mathcal{U}_X$, as here ρ is defined by $\rho(\delta \otimes Z) = \delta \circ \mathcal{U}_Z$. Third, if $\sigma \in \mathcal{C}$ and we write $\bar{\sigma}$ for the restriction of the representation σ of S'^∞ to a representation of S^∞ with the same flag $E_{nn} \supset E_{n-1,n-1}$, then it follows again from the PBW theorem that the flag associated to $\bar{\sigma}$ is $E_{nn} \supset E_{n-1,n-1} \supset \cdots \supset E_{0,0} \supset 0$, just as it is for σ .

3 Subquotients

Recall from the introduction that this paper is a step towards deciding which objects of $\text{Ext}_G \mathcal{O}$ are subquotients of certain geometrically natural representations of G . By Theorem 2.1 it is enough to do this for objects of $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$. To do so we will use the functor \mathcal{F} to move the problem to the category \mathcal{C} , and so we must study the relation between subquotients in $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$ and in \mathcal{C} . Throughout this section fix an object \mathcal{U} , F_0, \dots, F_n of $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$, let $\sigma, \eta, L_0, \dots, L_n$ be the object $\mathcal{F}(\mathcal{U})$ of \mathcal{C} , and let $B_j = \bigoplus_j^n F_i$ so that \mathcal{U} acts in the flag $\mathcal{D}B_0 \supset \cdots \supset \mathcal{D}B_n$. By refining the flag associated to \mathcal{U} if necessary, we may and do assume in this section that the H -bundles F_i are irreducible. Our first lemma shows that $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$ is closed under taking the natural type of subquotient.

Lemma 3.1 *Suppose that B'_0 is a subbundle of B_0 such that $\mathcal{D}B'_0$ is a G -invariant subspace of $\mathcal{D}B_0$, and let $\mathcal{U}' = \mathcal{U}|_{\mathcal{D}B'_0}$. Let $B'_j = B'_0 \cap B_j$, and choose subbundles F'_i of B'_0 such that $B'_j = \bigoplus_j^n F'_i$ for $0 \leq j \leq n$. Then F'_i is either 0 or isomorphic to F_i , and $\mathcal{U}', F'_0, \dots, F'_n$ is an object of $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$.*

Define a flag of bundles $B''_j = B_j/B'_j$, let \mathcal{U}'' be the quotient representation on the flag $\mathcal{D}B''_0 \supset \cdots \supset \mathcal{D}B''_n$, and choose subbundles F''_i of B''_0 such that $B''_j = \bigoplus_j^n F''_i$ for $0 \leq j \leq n$. Then F''_i is either 0 or isomorphic to F_i , and $\mathcal{U}'', F''_0, \dots, F''_n$ is an object of $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$.

Proof. First note that it is indeed possible to choose the F'_i as required, for example by putting a Hermitian metric on B_0 . Since $\mathcal{D}(B'_j/B'_{j+1})$ is a closed G -subspace of the irreducible induced representation $\mathcal{D}(B_j/B_{j+1})$, it is 0 or isomorphic to $\mathcal{D}F_j$, and so F'_j is 0 or isomorphic to F_j . Both \mathcal{U}' and \mathcal{U}'' will inherit property 3 in the definition of $\text{Ext}_G^{\mathcal{C}}\mathcal{O}$ from \mathcal{U} ; the rest is easy. \square

The reader should keep in mind that B_0 is not naturally an H -bundle unless \mathcal{U} is induced. Recall the elementary fact that if F is any H -bundle and \mathcal{V} is the induced representation of G on $\mathcal{D}F$, then any closed G -invariant subspace of $\mathcal{D}F$ is of the form $\mathcal{D}F'$ for some H -subbundle F' of F . Lemma 3.1 does *not* generalize this fact to say that any closed G -invariant subspace of $\mathcal{D}B_0$ is of the form $\mathcal{D}B'_0$ for some B'_0 as in the lemma. Such a generalization would be of interest to the author.

We remark that F'_j need not be a subbundle of F_j . This points out a flaw in our exposition: in the definition of objects of $\text{Ext}_G^{\mathcal{C}}\mathcal{O}$ we should specify only the flag $\{B_j\}$ and not the splitting $\{F_i\}$. We gave the definition as it stands in [4] so that we could write \mathcal{U} as a matrix, and we will not change it here so that it will be easier to refer to that paper. Similarly, in the definition of objects of \mathcal{C} it would be better to specify only the flag $\{N_j\}$ rather than the splitting $\{L_i\}$.

We now define subquotients in \mathcal{C} . Suppose that N'_n is a subspace of N_n such that $\eta_s(N'_n) \subset N'_n \otimes \mathfrak{U}(\mathfrak{g})$ for all $s \in S'$. Let $N'_j = N'_n \cap N_j$, and choose subspaces L'_i of N'_n such that $N'_j = \bigoplus_{i=0}^j L'_i$ for $0 \leq j \leq n$ (some L'_i may be 0). Let $\eta' = \eta|_{N'_n}$ and define M'_{kl} , J'_{kl} , and K'_{kl} as in the definition of \mathcal{C} , but using η' and the L'_i in place of η and the L_i .

Lemma 3.2 *For all $s \in S'$ and $0 \leq j \leq n$, $\eta'_s(N'_j) \subset M'_{jj}$ and η' has properties 1 and 2 in the definition of \mathcal{C} , so there is an object σ' of \mathcal{C} such that $\eta(\sigma') = \eta'$ and $L_i(\sigma') = L'_i$. Such objects will be called subobjects of σ . The associated spaces E'_{kl} are naturally subspaces of E_{kl} , and the inclusion $\mu : E'_{kl} \rightarrow E_{kl}$ is a \mathcal{C} -morphism such that $\tau(\mu)$ is the inclusion $N'_n \rightarrow N_n$.*

Proof. Since $M'_{jj} = \sum_0^j N'_i \otimes R_{j-i}$, we have

$$\eta_s(N'_j) \subset \eta_s(N'_n) \cap \eta_s(N_j) \subset (N'_n \otimes \mathfrak{U}(\mathfrak{g})) \cap \sum_0^j N_i \otimes R_{j-i} = M'_{jj}.$$

Clearly η' inherits property 2 from η . Now by Lemma 4.2 of [4], $J'_{kl} = K'_{kl} + M'_{kl}$. On the other hand, $K'_{kl} \subset K_{kl}$, $M'_{kl} \subset M_{kl}$, and $K_{kl} \cap M_{kl} = 0$, so $J'_{kl} = K'_{kl} \oplus M'_{kl}$. We have mentioned that this is equivalent to η' having property 1; the proof is essentially that of Lemma 3.8 in [4]. Last, $E'_{kl} = J'_{kl}/K'_{kl}$ is a subspace of E_{kl} because $J'_{kl} \cap K_{kl} = (K'_{kl} \oplus M'_{kl}) \cap K_{kl} = K'_{kl}$. \square

Let σ' be a subobject of σ as above, and define spaces $N''_j = N_j/N'_j$. Then $N''_0 \subset \dots \subset N''_n$ is a flag, although some of these inclusions may not be proper, and we may define J''_{kl} and M''_{kl} as usual: $J''_{kl} = \sum_0^l N''_i \otimes \bar{\mathfrak{U}}_{k-i}$ and $M''_{kl} =$

$\sum_0^l N_i'' \otimes R_{k-i}$. Let $\pi : N_n \otimes \mathfrak{U}(\mathfrak{g}) \rightarrow N_n'' \otimes \mathfrak{U}(\mathfrak{g})$ be the canonical projection, and note that $\pi(M_{kl}) = M_{kl}'', \pi(J_{kl}) = J_{kl}'', \text{kernel}(\pi) = \sum J_{kl}'$, and $J_{kl}' \cap M_{kl} = M_{kl}'$. Since $\eta_s(N_k') \subset M_{kk}'$ for all $s \in S'$, η_s factors through π to a map $\eta_s'' : N_n'' \rightarrow M_{kk}''$ such that $\eta_s''(N_k'') \subset M_{kk}''$ for $0 \leq k \leq n$.

Lemma 3.3 *The map η'' has properties 1 and 2 in the definition of \mathcal{C} , and so by choosing spaces L_i'' such that $N_k'' = \oplus_0^k L_i''$ for $0 \leq k \leq n$ we get an object σ'' of \mathcal{C} such that $\eta(\sigma'') = \eta''$ and $L_i(\sigma'') = L_i''$ (some of the L_i'' may be 0). The biflag E_{kl}'' is the quotient of E_{kl} by E_{kl}' , and the canonical projection $\mu' : E_{kl} \rightarrow E_{kl}''$ is a \mathcal{C} -morphism from σ to σ'' such that $\tau(\mu')$ is the projection $\pi : N_n \rightarrow N_n''$. The object σ'' will be called the quotient of σ by σ' , written σ/σ' .*

Proof. Clearly η'' inherits property 2 from η . To check property 1, note that $\pi \circ \mathcal{N}_s = \mathcal{N}_s'' \circ \pi$ and $N_k'' = \pi(N_k)$, so

$$(\mathcal{N}_s'' \circ \mathcal{N}_t'' - \mathcal{N}_{st}'')(N_k'') = \pi \circ (\mathcal{N}_s \circ \mathcal{N}_t - \mathcal{N}_{st})(N_k) \subset \pi(K_{k-1, k-1}).$$

Everything else follows from the easy fact that $\pi(K_{kl}) = K_{kl}''$. \square

The next lemma says that the definitions of subobjects and quotient objects in \mathcal{C} give the usual homomorphism theorem. Its proof follows from the fact that if $\mu : \sigma \rightarrow \tilde{\sigma}$ is a \mathcal{C} -morphism, then $(\tau \otimes \mathbf{1}) \circ \eta_s = \tilde{\eta}_s \circ \tau$, where we write $\tilde{\eta} = \eta(\tilde{\sigma})$, $\tau = \tau(\mu)$, and so on.

Lemma 3.4 *If $\mu : \sigma \rightarrow \tilde{\sigma}$ is a \mathcal{C} -morphism, then the subspace $N_n' = \text{kernel}(\tau)$ of N_n defines a subobject σ' of σ . The subspace $\tilde{N}_n' = \text{image}(\tau)$ of \tilde{N}_n defines a subobject $\tilde{\sigma}'$ of $\tilde{\sigma}$, and μ factors through to an isomorphism from σ/σ' to $\tilde{\sigma}'$. \square*

Here by an isomorphism $\alpha : \nu \rightarrow \nu'$ between two objects of \mathcal{C} we mean a morphism such that $\tau(\alpha)$ is a linear isomorphism; the two objects may be of different lengths. We will leave it to the reader to check that if α is an arbitrary morphism, then $\alpha : \sum_{kl} E_{kl}(\nu) \rightarrow \sum_{kl} E_{kl}(\nu')$ is injective (respectively, surjective) if and only if $\tau(\alpha)$ is injective (respectively, surjective). Next we must check that the functor \mathcal{F} defines a contravariant bijection from the subquotients of \mathcal{U} of the type discussed at the beginning of this section to the subquotients of $\sigma = \mathcal{F}(\mathcal{U})$.

Lemma 3.5 *Let all notation be as in Lemma 3.1, and let $\sigma' = \mathcal{F}(\mathcal{U}')$ and $\sigma'' = \mathcal{F}(\mathcal{U}'')$. Then σ'' is a subobject of σ and $\sigma' = \sigma/\sigma''$. Conversely, suppose that σ'' is any subobject of σ and let $\sigma' = \sigma/\sigma''$. Then there is a subbundle B_0' of B_0 such that $\mathcal{D}B_0'$ is G -invariant and $\mathcal{F}(\mathcal{U}') = \sigma'$, where $\mathcal{U}' = \mathcal{U}|_{\mathcal{D}B_0'}$. Furthermore, if \mathcal{U}'' is the quotient representation on $\mathcal{D}(B_0/B_0')$ then $\mathcal{F}(\mathcal{U}'') = \sigma''$.*

Proof. We will leave the first statement to the reader and prove only its converse. Let $\pi : \sigma \rightarrow \sigma'$ be the projection morphism. Since \mathcal{F} is essentially surjective there is an object \mathcal{V} of $\text{Ext}_G^C \mathcal{O}$ acting in a flag of bundles $\mathcal{D}C_0 \supset \cdots \supset \mathcal{D}C_m$

such that $\mathcal{F}(\mathcal{V}) = \sigma'$, and since \mathcal{F} is fully faithful there is a morphism $T : \mathcal{V} \rightarrow \mathcal{U}$ such that $\mathcal{F}(T) = \pi$. The map $T : C_0 \rightarrow B_0$ acts fiberwise, and so if we can show that it is injective on every fiber then its image $T(C_0)$ will be the required subbundle B'_0 of B_0 .

Recall the bundles of graded differential operators $\tilde{E}_{kl}(\mathcal{U})$, and write $\tilde{E}(\mathcal{U})$ for $\sum_{kl} \tilde{E}_{kl}(\mathcal{U})$ and similarly $E(\mathcal{U})$ for the fiber $\sum_{kl} E_{kl}(\mathcal{U})$ of $\tilde{E}(\mathcal{U})$ at p_0 . In Section 2 we defined a fiberwise map $\Delta T : \tilde{E}(\mathcal{U}) \rightarrow \tilde{E}(\mathcal{V})$ by $\Delta T(D) = D \circ T$, such that $\Delta T(p_0) = \mathcal{F}(T) = \pi : E(\mathcal{U}) \rightarrow E(\mathcal{V})$. Now $\tilde{E}(\mathcal{U})$ and $\tilde{E}(\mathcal{V})$ are G -bundles and ΔT is a G -map, so since π is surjective, ΔT is surjective on all fibers. It follows that T is injective; we leave the rest to the reader. \square

4 Relations Between \mathcal{C} , $\mathcal{R}S'^{\infty}$, and $\mathcal{R}S^{\infty}$.

Recall that at the end of Section 2 we defined a map from objects of \mathcal{C} to objects of $\mathcal{R}S'^{\infty}$, and from flag morphisms in \mathcal{C} to $\mathcal{R}S'^{\infty}$ -morphisms. Here we will describe the image of this map. Whenever we refer to S'^{∞} , S' is being regarded as a subgroup of G , while whenever we refer to S^{∞} , S is being regarded as a subgroup of H . Let π be a representation of S'^{∞} acting in $V_0 \supset V_1$, and let $\{V_i\}$ be the associated flag defined in Section 1. Assume that we are given an integer n and subspaces $N_n \subset V_0, N_{n-1} \subset V_1, \dots, N_0 \subset V_n$, such that $N_0 \subset \dots \subset N_n$ is a flag with the following properties. First, if we let $J_{kl} = \sum_0^l N_i \otimes \bar{\mathfrak{U}}_{k-i}$ and $M_{kl} = \sum_0^l N_i \otimes R_{k-i}$ and define $\rho : J_{nn} \rightarrow V_0$ by $\rho(\omega \otimes Z) = \pi(Z^T)\omega$, then ρ restricts to an isomorphism from M_{nn} to V_0 . Second, $\pi|_A$ leaves N_n invariant, and $\pi(a) - \exp i\langle a, p_0 \rangle$ maps N_k to N_{k-1} for $0 \leq k \leq n$ and is 0 for $a \in C_{p_0}^{\perp}$.

We will see that such π comprise the image of our object map from \mathcal{C} to $\mathcal{R}S'^{\infty}$. Define E_{kl} to be $\rho(J_{kl})$, and note that $E_{kl} \subset V_{n-k}$ because $\rho(N_i \otimes \bar{\mathfrak{U}}_{k-i}) = \pi(\bar{\mathfrak{U}}_{k-i})N_i \subset \pi(\bar{\mathfrak{U}}_{k-i})V_{n-i} \subset V_{n-k}$. Let $q : V_0 \rightarrow M_{nn}$ be the right inverse of ρ , and for $s \in S'$ define $\eta_s : N_n \rightarrow M_{nn}$ to be $q \circ \pi_s$. As usual, define

$$K_{kl} = \{ \eta_X \omega \cdot Z + \omega \otimes XZ : \omega \in N_i, Z \in \bar{\mathfrak{U}}_{k-i}, X \in \mathfrak{s}', 0 \leq i \leq l \}.$$

Lemma 4.1 *In the setting just described, let L_0, \dots, L_n be any subspaces of N_n such that $N_k = \bigoplus_0^k L_i$. Then η, L_0, \dots, L_n define an object σ of \mathcal{C} such that $\eta(\sigma) = \eta, L_i(\sigma) = L_i, \rho(\sigma) = \rho, q(\sigma) = q, E_{kl}(\sigma) = E_{kl}$, and the extension of σ to a representation of S'^{∞} on $E_{nn} \supset E_{n-1, n-1}$ is equal to π . In particular, $\rho(M_{kk}) = E_{kk} = V_{n-k}$ for $0 \leq k \leq n$ and $V_{n+1} = 0$. Conversely, the extension of any object σ of \mathcal{C} to an object of $\mathcal{R}S'^{\infty}$ has all of the above additional structures and properties in the way indicated by the notation.*

Proof. We start by checking that $\rho(M_{kk}) = E_{kk} = V_{n-k}$ for $0 \leq k \leq n$; the same argument also shows that $V_{n+1} = 0$. This is true for $k = n$, and so we may induct downward on k . We know that $\rho(M_{kk}) \subset E_{kk} \subset V_{n-k}$, and so it will do to suppose that there exists $v \in V_{n-k} - \rho(M_{kk})$ and derive a contradiction.

Recall that $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{r}$, r_1, \dots, r_q is an ordered basis of \mathfrak{r} , and the subspace R_r of \overline{M}_r used to define M_{kl} has the monomials r^I with $|I| \leq r$ as a basis. Therefore any element of M_{nn} may be written uniquely as $\sum_I \omega_I \otimes r^I$, where $\omega_I \in N_{n-|I|}$. In particular, since $q(v) \in M_{nn} - M_{kk}$ we have $q(v) = \sum_I \omega_I \otimes r^I$, where $\omega_I \notin N_i$ and $|I| \geq k - i$ for some I . It follows that $q(v) \cdot r_q \in M_{nn} - M_{k+1, k+1}$, but $-\rho(q(v) \cdot r_q) = \pi(r_q)v$ is in V_{n-k-1} because $v \in V_{n-k}$. By induction, $V_{n-k-1} = \rho(M_{k+1, k+1})$, and so $q(\pi(r_q)v) \in M_{k+1, k+1}$ is not equal to $-q(v) \cdot r_q$. This is a contradiction, because both of them lie in M_{nn} and have the same image under ρ .

Next we will verify that η defines an object of \mathcal{C} . To check that $\eta_s(N_k) \subset M_{kk}$, note that $\eta_s(N_k) \subset q \circ \pi_s(V_{n-k}) = q(V_{n-k}) = M_{kk}$. It follows that $K_{kl} \subset J_{kl}$, and so as we saw in the proof of Lemma 3.2, $J_{kl} = M_{kl} + K_{kl}$. By construction $\rho(K_{kl}) = 0$, and so in fact $J_{kl} = M_{kl} \oplus K_{kl}$. We also saw in the proof of Lemma 3.2 that this means that η has property 1. The condition on $\pi|_A$ implies that $\eta|_A = \pi|_A$ and hence that η has property 2, so the object σ of \mathcal{C} exists. We find from $J_{kl} = M_{kl} \oplus K_{kl}$ that $K_{nn} = \text{kernel}(\rho)$ and $K_{kl} = K_{nn} \cap J_{kl}$, from which it follows that $E_{kl}(\sigma) = E_{kl}$. We leave the rest to the reader. \square

Lemma 4.2 *Let σ and σ' be two objects of \mathcal{C} such that $n = n'$. If $\mu : E_n \rightarrow E'_n$ is an S'^∞ -morphism between the S'^∞ -actions of σ and σ' such that $\mu(N_k) \subset N'_k$ for all k , then μ is a flag morphism in the category \mathcal{C} . In particular, $\mu(E_{kl}) \subset E'_{kl}$.*

Proof. Let $\tau : N_n \rightarrow N'_n$ be $\mu|_{N_n}$. Then for $\omega \otimes Z \in J_{kl}$ we have

$$\begin{aligned} \rho' \circ (\tau \otimes \mathbf{1})(\omega \otimes Z) &= \rho' \circ r_{Z^T} \circ \tau(\omega) = \sigma'_{Z^T} \circ \mu(\omega) \\ &= \mu \circ \sigma_{Z^T}(\omega) = \mu \circ \rho(\omega \otimes Z). \end{aligned}$$

Hence $\rho' \circ \tau \otimes \mathbf{1} = \mu \circ \rho$, the condition for μ to be a \mathcal{C} -morphism. It is a flag morphism by definition, and as we remarked in the definition of \mathcal{C} the last sentence is easy. \square

We will finish this section by developing a technique for combining an object $\sigma, \eta, N_0 \subset \dots \subset N_n$ of \mathcal{C} with an object π of $\mathcal{R}S^\infty$ (not $\mathcal{R}S'^\infty$) to form a new object of \mathcal{C} , which we refer to as $\sigma \otimes \pi$. Write E_{kl} for $E_{kl}(\sigma)$ and so on, as usual, and let π act in $V_0 \supset V_1$ with associated flag $\{V_i\}$. For convenience we will write E_k, M_k , and J_k for E_{kk}, M_{kk} , and J_{kk} , respectively, and we write W_k for V_{n-k} . Define spaces

$$\tilde{E}_k = \sum_0^k E_i \otimes W_{k-i} \quad \text{and} \quad \tilde{N}_k = \sum_0^k N_i \otimes W_{k-i}$$

for $0 \leq k \leq n$, view σ as a representation of S'^∞ as in Lemma 4.1, extend π to a representation of S'^∞ by making A act trivially, and let $\tilde{\sigma}$ be the tensor product representation $\sigma \otimes \pi$ of S'^∞ acting in the flag $\tilde{E}_n \supset \tilde{E}_{n-1}$, as defined in Section 1.

Lemma 4.3 *The flag associated to $\tilde{\sigma}$ is $\tilde{E}_n \supset \tilde{E}_{n-1} \supset \cdots \supset \tilde{E}_0 \supset 0$, and the subflag $\tilde{N}_n \supset \cdots \supset \tilde{N}_0$ of this flag makes $\tilde{\sigma}$ into an object of \mathcal{C} as in Lemma 4.1. We define the object $\sigma \otimes \pi$ of \mathcal{C} to be $\tilde{\sigma}$.*

Suppose that $\pi', V'_0 \supset V'_1$ is another representation of S^∞ and that $T : \pi \rightarrow \pi'$ is a S^∞ -morphism. Then $\mu = \mathbf{1} \otimes T$ is a \mathcal{C} -morphism from $\sigma \otimes \pi$ to $\sigma \otimes \pi'$, and in fact it is a flag morphism. The map $\tau(\mu)$ is $\mu|_{\tilde{N}_n}$.

Proof. Define \tilde{J}_{kl} and \tilde{M}_{kl} from the flag $\{\tilde{N}_k\}$ as usual, define $\tilde{\rho} : \tilde{J}_{nn} \rightarrow \tilde{E}_n$ by $\tilde{\rho}(\omega \otimes Z) = \tilde{\sigma}(Z^T)\omega$, and for convenience define $\tilde{J}_k = \tilde{J}_{kk}$ and $\tilde{M}_k = \tilde{M}_{kk}$. It is easy to check that $\tilde{\rho}(\tilde{J}_k) \subset \tilde{E}_k$, and so by Lemma 4.1 we need only prove that $\tilde{\rho}$ maps \tilde{M}_k isomorphically onto E_k for $0 \leq k \leq n$, and that $\tilde{\sigma}|_A$ has the special form given in that lemma. This last statement is easy, because $\sigma|_A$ has this special form and $\pi|_A = \mathbf{1}$.

To prove that $\tilde{\rho} : \tilde{M}_k \rightarrow \tilde{E}_k$ is an isomorphism, we first check that \tilde{M}_k and \tilde{E}_k have the same dimension. Since $\dim(M_i) = \dim(E_i)$, $\dim(\tilde{E}_k) = \dim(\sum_0^k M_i \otimes W_{k-i})$. But both $\sum_0^k M_i \otimes W_{k-i}$ and \tilde{M}_k are naturally isomorphic to $\sum_{i+j+l=k} N_i \otimes R_j \otimes W_l$, and so we are done. Next we will check that $\tilde{\rho}(\tilde{M}_k) = \tilde{E}_k$ by induction on k , which will complete the proof of the first paragraph of the lemma. Since $\tilde{M}_0 = \tilde{E}_0 = \tilde{N}_0$, the induction starts. We will proceed in two steps, where the first step is a proof that $\tilde{\sigma}(R_1)\tilde{E}_{k-1} + \tilde{N}_k = \tilde{\rho}(\tilde{M}_k)$. To show the forward containment we only need $\tilde{\sigma}(R_1)\tilde{\rho}(\tilde{M}_{k-1}) \subset \tilde{\rho}(\tilde{M}_k)$, as $\tilde{\rho}(\tilde{M}_{k-1}) = \tilde{E}_{k-1}$ by induction. For this, note that

$$\tilde{\sigma}(R_1)\tilde{\rho}(\tilde{M}_{k-1}) = \sum_0^{k-1} \tilde{\sigma}(R_1)\tilde{\sigma}(R_{k-i-1})\tilde{N}_i \subset \sum_0^{k-1} \tilde{\sigma}(R_{k-i} + \bar{\mathbf{M}}_{k-i-1})\tilde{N}_i$$

because $R_1 R_{k-i-1} \subset R_{k-i} + \bar{\mathbf{M}}_{k-i-1}$. But this last sum is $\tilde{\rho}(\tilde{M}_k + \tilde{J}_{k-1}) = \tilde{\rho}(\tilde{M}_k) + \tilde{E}_{k-1} \subset \tilde{\rho}(\tilde{M}_k)$, as needed. The reverse containment is easy.

For the second step we must prove that $\tilde{\sigma}(R_1)\tilde{E}_{k-1} + \tilde{N}_k = \tilde{E}_k$. The forward containment is clear; for the reverse containment, recall that $\tilde{E}_k = \sum_0^k E_i \otimes W_{k-i}$. We will show that $E_i \otimes W_{k-i} \subset \tilde{\sigma}(R_1)\tilde{E}_{k-1} + \tilde{N}_k$ by induction on i . Since $E_0 \otimes W_k = N_0 \otimes W_k \subset \tilde{N}_k$, the induction begins and we may assume $E_{i-1} \otimes W_{k-i+1} \subset \tilde{\sigma}(R_1)\tilde{E}_{k-1} + \tilde{N}_k$. Then we have that $E_{i-1} \otimes W_{k-i+1} + \tilde{\sigma}(R_1)(E_{i-1} \otimes W_{k-i}) + N_i \otimes W_{k-i}$ is in $\tilde{\sigma}(R_1)\tilde{E}_{k-1} + \tilde{N}_k$, and so since $\tilde{\sigma} = \sigma \otimes \pi$ as a representation of S'^∞ , $\sigma(R_1)E_{i-1} + N_i = E_i$, and $\pi(R_1)W_{k-i} \subset W_{k-i+1}$, the result follows. We leave the second paragraph to the reader. \square

5 Functions on Neighborhoods of Submanifolds

In Section 6 we will define certain objects of $\text{Ext}_G \mathcal{O}$ which arise geometrically as spaces of vector-valued functions on infinitesimal neighborhoods of the orbit \mathcal{O} in its ambient space A^* , and so in this section we must define functions

on neighborhoods of \mathcal{O} . Heuristically we think of the n^{th} order infinitesimal neighborhood of \mathcal{O} in A^* as a space, $\mathcal{O}^{(n)}$, but in fact we will not define $\mathcal{O}^{(n)}$ but only the functions on it. Roughly, a function on $\mathcal{O}^{(n)}$ assigns smoothly to each point of \mathcal{O} the n -jet of a function on A^* . Similar constructions may be found in the beginning of Griffiths' paper [8]. We remark that the functional analysis we will use in this section is all easy; most of the work is in setting up notation.

Suppose that M is any smooth manifold and N is any regularly imbedded submanifold. Let ΔM be the vector bundle of differential operators on M . This is a complex vector bundle of countably infinite rank with the usual filtration of bundles $\Delta^r M$ of differential operators of order $\leq r$, which are of finite rank. A section of $\mathcal{E}\Delta M$ or $\mathcal{D}\Delta M$ is a smooth or smooth compactly supported differential operator mapping $\mathcal{D}M$ to itself, respectively. An element of the fiber $\Delta M(p)$ at a point $p \in M$ is a differential operator supported at p mapping $\mathcal{D}M$ to \mathbb{C} .

For each integer $r \geq 0$, we define the r^{th} jet bundle $J^r M$ above M to be the vector bundle dual to $\Delta^r M$. If F and D are sections of $J^r M$ and $\Delta^r M$, respectively, we have the function $\langle D, F \rangle$ on M whose value $\langle D(p), F(p) \rangle$ at p is given by the duality. Let $j_r : \mathcal{D}M \rightarrow \mathcal{D}J^r M$ be the map assigning to a function f its r^{th} jet. In other words, $j_r f$ is defined by the equation $\langle D, j_r f \rangle = Df$ for all $D \in \mathcal{D}\Delta^r M$. The map j_r is not a vector bundle map; it is easily seen to be a differential operator of order r . The fibers of $J^r M$ have an algebra structure induced by multiplication of functions, and so $\mathcal{D}J^r M$ is an algebra under fiberwise multiplication and j_r is an algebra homomorphism. Since $\Delta^r M$ is a subbundle of $\Delta^s M$ for $r \leq s$, there is a quotient map of bundles $\pi_{rs} : J^s M \rightarrow J^r M$. One checks that π_{rs} is an algebra map, that $\pi_{qr} \circ \pi_{rs} = \pi_{qs}$ for $q \leq r$, and that j_r and $\pi_{rs} \circ j_s$ are equal as maps from $\mathcal{D}M$ to $\mathcal{D}J^r M$. Note that $\mathcal{D}(J^r M|_N)$ is an algebra under fiberwise multiplication, and that we may regard $\Delta^r N$ as a subbundle of $\Delta^r M|_N$ by restriction in the obvious way.

Definition. We define the space $\mathcal{D}N^{(n)}$ of smooth compactly supported functions on $N^{(n)}$ to be the subspace of $\mathcal{D}(J^n M|_N)$ consisting of sections F such that for each $p \in N$, there is an open neighborhood V of p in M (not N) and a smooth function f_V on V such that for all $q \in V \cap N$ we have

$$j_n f_V(q) = F(q).$$

So locally, functions on $N^{(n)}$ are restrictions to N of n -jets of functions on M . We give $\mathcal{D}N^{(n)}$ the subspace topology it inherits from $\mathcal{D}(J^n M|_N)$, and we define the support $\text{supp}(F) \subset N$ of $F \in \mathcal{D}(N^{(n)})$ to be its support as a section of $J^n M|_N$.

Lemma 5.1 *The space $\mathcal{D}N^{(n)}$ is a closed subalgebra of $\mathcal{D}(J^n M|_N)$, and so it is an inductive limit of Frechet spaces and is complete.*

Proof. It is clear that $\mathcal{DN}^{(n)}$ is a subalgebra from the definitions. To prove that it is closed, let F be any section in $\mathcal{D}(J^n M|_N)$, and choose D and E in $\mathcal{D}\Delta^n M$ such that the composition $E \circ D$ is still of order $\leq n$, and $E|_N$ lies in the subspace $\mathcal{D}\Delta^n N$ of $\mathcal{D}(\Delta^n M|_N)$. Since $\Delta^n M|_N$ and $J^n M|_N$ are dual to each other, there is a fiberwise pairing which gives a function $\langle D|_N, F \rangle$ in \mathcal{DN} . A local calculation shows that if $F \in \mathcal{DN}^{(n)}$, then the function obtained by applying the differential operator $E|_N \in \mathcal{D}\Delta^n N$ to $\langle D|_N, F \rangle$ is equal to $\langle (E \circ D)|_N, F \rangle$, and furthermore that F is in $\mathcal{DN}^{(n)}$ if and only if this equality holds for all E and D as above. For each choice of D and E this is a closed condition, and so $\mathcal{DN}^{(n)}$ is closed.

Recall the following standard notation: if L is any manifold, B is any vector bundle over L , and K is any subset of L , let $\mathcal{D}_K B = \{s \in \mathcal{D}B : \text{supp}(s) \subset K\}$. When K is compact, $\mathcal{D}_K B$ is a closed Frechet subspace of $\mathcal{D}B$, and $\mathcal{D}B$ is the inductive limit of the spaces $\mathcal{D}_K B$ as K runs over a compact exhaustion of L . Similarly, for any subset K of N let $\mathcal{D}_K N^{(n)} = \mathcal{D}_K(J^n M|_N) \cap \mathcal{DN}^{(n)}$. Then $\mathcal{DN}^{(n)}$ is the inductive limit of the closed Frechet subspaces $\mathcal{D}_K N^{(n)}$ as K runs over a compact exhaustion of N . \square

Remark. Perhaps the first definition of $\mathcal{DN}^{(n)}$ one thinks of is the following. Since N is regularly imbedded in M it is locally closed in M , and so we may choose V open in M such that N is a closed subset of V . Let $I_N(V)$ be the ideal of functions in $\mathcal{D}V$ that vanish on N . It would be natural to define $\mathcal{DN}^{(n)}$ to be the space $\mathcal{D}V/I_N(V)^{n+1}$ with the quotient topology, but this adds unnecessary difficulty to our results. Note that for the definition of $\mathcal{DN}^{(n)}$ we are using, there is a natural continuous linear map $\pi_n : \mathcal{D}V \rightarrow \mathcal{DN}^{(n)}$ defined by $\pi_n f = (j_n f)|_N$, and it is clear that π_n is an algebra homomorphism whose kernel is $I_N(V)^{n+1}$. By Lemma 5.2, π_n is surjective, and we expect that it factors through to a linear homeomorphism from $\mathcal{D}V/I_N(V)^{n+1}$ with the quotient topology to $\mathcal{DN}^{(n)}$, but we do not have and will not need a proof of this.

The next two lemmas give some basic properties of $\mathcal{DN}^{(n)}$. We will use the obvious fact that for any V as above, $\mathcal{DN}^{(n)} \ni F$ is a module for the algebra $\mathcal{E}V \ni g$, where $gF = (j_n g)|_N F$.

Lemma 5.2 *First, if $\{U_\alpha\}$ is any open cover of N and $F \in \mathcal{DN}^{(n)}$, then F is a finite sum of elements of $\mathcal{DN}^{(n)}$ each having support in one of the U_α . Second, the map π_n defined above is surjective.*

Proof. First, let us replace the given open cover by a locally finite refinement $\{U_1, U_2, \dots\}$ such that for each open set U_i in N there is an open set V_i in M such that $U_i = V_i \cap N$ and U_i is closed in V_i . Let $V' = \cup_i V_i$, and let $\{\theta_i\}$ be a partition of unity for the open cover $\{V_i\}$ of V' . One checks that $F = \sum_i \theta_i F$ is the desired finite sum.

Second, choose a locally finite open cover U_1, U_2, \dots of N such that for each i , there is an open set V_i in V with $U_i = V_i \cap N$ and U_i closed in V_i , and a function f_i in $\mathcal{E}V_i$ such that $(j_n f_i)|_{U_i} = F|_{U_i}$. Let $V' = \cup_i V_i$, and let $\{\theta_i\}$ again be a

partition of unity for V' subordinate to $\{V_i\}$. One checks that $F = \pi_n(\sum_i \theta_i f_i)$.
 \square

Recall from above the surjective bundle maps $\pi_{rs} : J^s M \rightarrow J^r M$ for $r \leq s$, and write also π_{rs} for the associated surjective fiberwise maps from $\mathcal{D}(J^s M|_N)$ to $\mathcal{D}(J^r M|_N)$. Let $I_{rs} = \{F \in \mathcal{D}N^{(s)} : \pi_{rs}F = 0\}$; then I_{rs} is the closed subspace of $\mathcal{D}N^{(s)}$ of functions that vanish to order r .

Let $\Omega^1 M$ be the 1-form bundle over M . The tangent bundle TN is canonically a subbundle of $TM|_N$, and we define $T^\perp N \subset \Omega^1 M|_N$ to be the annihilating bundle of TN . For any vector bundle B , let $\mathcal{S}^n B$ be its n^{th} symmetric power bundle. When B is a real bundle, we will adopt the convention that $\mathcal{S}^n B$ is complexified.

Lemma 5.3 *First, the maps $\pi_{rs} \circ \pi_s$ and π_r defined above are equal, $\pi_{rs}(\mathcal{D}N^{(s)}) = \mathcal{D}N^{(r)}$, and I_{rs} is an ideal in the algebra $\mathcal{D}N^{(s)}$. Second, π_{rs} factors through to a linear homeomorphism from $\mathcal{D}N^{(s)}/I_{rs}$ with the quotient topology to $\mathcal{D}N^{(r)}$. Third, I_{rs} admits a closed complementary subspace in $\mathcal{D}N^{(s)}$. Fourth, the product $I_{rs}I_{qs}$ is contained in $I_{r+q+1,s}$, and so the action of $\mathcal{D}N^{(s)}$ on I_{rs} factors through to an action of $\mathcal{D}N^{(s-r-1)}$. Fifth, the $\mathcal{D}N = \mathcal{D}N^{(0)}$ -module $I_{r-1,r}$ is $\mathcal{D}N$ -isomorphic and linearly homeomorphic to $\mathcal{D}(\mathcal{S}^r T^\perp N)$.*

Proof. Let V be as above Lemma 5.2. The first statement follows from the facts that $\pi_{rs} \circ j_s$ and j_r are equal as maps from $\mathcal{D}V$ to $\mathcal{D}J^r V$, that $\pi_r(\mathcal{D}V) = \mathcal{D}N^{(r)}$, and that π_{rs} is an algebra map. Second, to see that π_{rs} factors through to a linear homeomorphism it is enough to prove that π_{rs} is an open map. For any compact subset K of N it is clear that $\pi_{rs}(\mathcal{D}_K N^{(s)}) \subset \mathcal{D}_K N^{(r)}$. If we can prove that this containment is an equality, then since $\mathcal{D}_K N^{(s)}$ is Frechet the open mapping theorem implies that the restriction of π_{rs} to it is open, and so π_{rs} is open because $\mathcal{D}N^{(r)}$ is the inductive limit of spaces $\mathcal{D}_K N^{(r)}$.

Thus we come down to proving that given $F \in \mathcal{D}N^{(r)}$, there exists $F' \in \mathcal{D}N^{(s)}$ such that $\pi_{rs}F' = F$ and $\text{supp}F' = \text{supp}F$. Choose a locally finite open cover $\{U_\alpha\}$ of N with the following properties. For each U_α there is an open set V_α in M such that $U_\alpha = V_\alpha \cap N$, U_α is closed in V_α , and V_α is a coordinate chart on M with coordinates $(x, y) = (x_1, \dots, x_p, y_1, \dots, y_q)$ such that V_α is the cube $\{(x, y) : |x_i| < 1, |y_j| < 1\}$, and U_α is the subset of V_α cut out by $y = 0$. Note that x gives coordinates for U_α . Let $V' = \cup_\alpha V_\alpha$, and let $\{\theta_\alpha\}$ be a partition of unity for V' subordinate to $\{V_\alpha\}$ (the V_α may not be locally finite, so to construct the θ_α , choose a locally finite refinement). A local calculation shows that we may choose F'_α in $\mathcal{D}N^{(s)}$ such that $\text{supp}F'_\alpha = \text{supp}(\theta_\alpha F)$ and $\pi_{rs}F'_\alpha = \theta_\alpha F$, and so $F' = \sum_\alpha F'_\alpha$ is the required element of $\mathcal{D}N^{(s)}$.

For the third statement, retain the open cover U_α . We will need products of the coordinates x_i and y_j , and we will use the multinomial notations $y^J = y_1^{J_1} \dots y_q^{J_q}$ and $|J| = \sum_j J_j$. Given $F \in \mathcal{D}_{U_\alpha} N^{(s)}$, a local calculation shows that there are unique functions $c_J(x) \in \mathcal{D}_{U_\alpha} N$ for all $|J| \leq s$ such that $F = j_s(\sum_{|J| \leq s} c_J y^J)$ on U_α . Since $\text{supp}F \subset U_\alpha$, F lies in I_{rs} if and only if $c_J \equiv 0$

whenever $|J| \leq r$. We define a continuous linear projection $p_\alpha : \mathcal{D}_{U_\alpha} N^{(s)} \rightarrow I_{rs}$ by

$$p_\alpha(F) = j_s \left(\sum_{r < |J| \leq s} c_J y^J \right)$$

on U_α and $p_\alpha(F) \equiv 0$ off U_α . Define $p : \mathcal{D}N^{(s)} \rightarrow I_{rs}$ by $p(F) = \sum_\alpha p_\alpha(\theta_\alpha F)$; then since I_{rs} is an ideal and the p_α are projections, p is a continuous linear projection. Therefore $\mathcal{D}N^{(s)} = \text{kernel}(p) \oplus I_{rs}$, which proves the third statement.

Fourth, recall that $\pi_{rs} : J^s M \rightarrow J^r M$ is a bundle map and an algebra map on fibers. One checks locally that the kernels of π_{rs} on the fibers form a subbundle K_{rs} of $J^s M$ whose fibers are ideals, and that the product of the fibers $K_{rs}(p)K_{qs}(p)$ is contained in $K_{r+q+1,s}(p)$ for all $p \in M$. Since $J^r M = J^s M / K_{rs}$ and $K_{r+q+1,s} = 0$ if $q \geq s - r - 1$, the $J^s M$ action on K_{rs} factors through to an action of $J^{s-r-1} M$. The statement now follows from $I_{rs} = \mathcal{D}(K_{rs}|_N) \cap \mathcal{D}N^{(s)}$.

For the fifth statement we must define the transversal order of elements of the fiber $\Delta^r M(p)$ for $p \in N$. One of the sets U_α from above contains p , and the coordinates (x, y) on V_α give us the basis $\partial_{x_i}|_p$ and $\partial_{y_j}|_p$ of $TM(p)$. If we write ∂_y^J for $\partial_{y_1}^{J_1} \cdots \partial_{y_q}^{J_q}$, then $\Delta^r M(p)$ has the basis $\{\partial_x^I \partial_y^J|_p : |I + J| \leq r\}$. For $k \leq r$, we define the subbundle $\Delta^r(M, N, k)$ of differential operators of order $\leq r$ and transversal order $\leq k$ by letting the fiber $\Delta^r(M, N, k)(p)$ have the basis $\{\partial_x^I \partial_y^J|_p : |J| \leq k, |I + J| \leq r\}$. This definition is coordinate free, and one sees that the quotient bundle $\Delta^r M|_N / \Delta^r(M, N, r-1)$ is naturally isomorphic to $\mathcal{S}^r(T^\perp N)^*$ in much the same way that one sees that $\Delta^r M / \Delta^{r-1} M \cong \mathcal{S}^r TM$.

Recall that for $D \in \mathcal{D}(\Delta^r M|_N)$ and $F \in \mathcal{D}N^{(r)}$ we have the pairing $\langle D, F \rangle \in \mathcal{D}N$. One checks that if $F \in I_{r-1,r}$ and $f \in \mathcal{D}N = \mathcal{D}N^{(0)}$, then $\langle D, fF \rangle = \langle fD, F \rangle = f \langle D, F \rangle$, and that $\langle D, F \rangle \equiv 0$ for all $F \in I_{r-1,r}$ if and only if $D \in \mathcal{D}\Delta^r(M, N, r-1)$. Since $(\Delta^r M|_N / \Delta^r(M, N, r-1))^*$ and $\mathcal{S}^r T^\perp N$ are isomorphic, it follows that there is a $\mathcal{D}N$ -isomorphism from $I_{r-1,r}$ to $\mathcal{D}(\mathcal{S}^r T^\perp N)$, and using the open mapping theorem as above shows that it is a homeomorphism. \square

6 Representations in Functions on Neighborhoods of Orbits

Recall from the introduction that \mathcal{O} is a locally closed orbit of H in A^* , and so \mathcal{O} is regularly imbedded in A^* . In this section we will prove that there is a canonical representation of G in $\mathcal{D}\mathcal{O}^{(n)}$ which is an object of $\text{Ext}_G \mathcal{O}$, and exhibit an equivalence from this object to an object of $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$ which will be useful for computations. Note that $\mathcal{D}\mathcal{O}^{(n)}$ is not canonically a vector bundle over \mathcal{O} , while objects in $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$ act in vector bundles.

The H -action on A^* defines an H -bundle structure on $J^n A^*$ such that j_n intertwines the representations of H on $\mathcal{D}A^*$ and $\mathcal{D}J^n A^*$. Since \mathcal{O} is H -invariant,

there is a smooth representation of H on $\mathcal{D}(J^n A^*|_{\mathcal{O}})$, and one finds that $\mathcal{DO}^{(n)}$ is an invariant subspace. We write $\mathcal{V}^{(n)}$ for the subrepresentation on $\mathcal{DO}^{(n)}$, and we extend $\mathcal{V}^{(n)}$ to a representation of G by letting $\mathcal{V}_a^{(n)}$ be multiplication by $(j_n \xi_a)|_{\mathcal{O}}$ for all $a \in A$, where ξ_a is the character function from the introduction. It may be of interest to note that if we let $V = A^* \setminus \partial \mathcal{O}$, where $\partial \mathcal{O}$ is the boundary of \mathcal{O} , then V is a H -invariant open set in A^* containing \mathcal{O} as a closed subset, and $\pi_n : \mathcal{DV} \rightarrow \mathcal{DO}^{(n)}$ is a G -map.

Clearly the maps $\pi_{r,s}$ are G -maps, and so the ideals I_{kn} of $\mathcal{DO}^{(n)}$ are G -invariant. Thus $\mathcal{V}^{(n)}$ admits the G -flag

$$\mathcal{DO}^{(n)} \supset I_{0,n} \supset \cdots \supset I_{n-1,n}.$$

Using Lemma 5.3 one checks that the isomorphism from $\mathcal{DO}^{(n)}/I_{kn}$ to $\mathcal{DO}^{(k)}$ restricts to an isomorphism from $I_{k-1,n}/I_{k,n}$ to $I_{k-1,k}$, and also that the isomorphism from $I_{n-1,n}$ to $\mathcal{D}(\mathcal{S}^n T^\perp \mathcal{O})$ is a G -map, where $\mathcal{D}(\mathcal{S}^n T^\perp \mathcal{O})$ is a G -space via the H -bundle structure on $\mathcal{S}^n T^\perp \mathcal{O}$ coming from the natural H -bundle structure on $T^\perp \mathcal{O}$, and the character action of A . Lemma 5.3 now yields the following proposition.

Proposition 6.1 *The representation $\mathcal{V}^{(n)}$ of G on $\mathcal{DO}^{(n)}$ is an object of $\text{Ext}_G \mathcal{O}$ whose composition series is isomorphic to $\mathcal{D}(\mathcal{S}^k T^\perp \mathcal{O})$, $0 \leq k \leq n$. \square*

Our plan is to apply the results of [4] to reduce the study of $\mathcal{DO}^{(n)}$ to the study of certain representations of SA , and to use [4] we must find an isomorphism of $\mathcal{DO}^{(n)}$ with an object of $\text{Ext}_G^C \mathcal{O}$. To this aim let e be the endpoint map from the total space of the real subbundle C of $\mathcal{O} \times A^*$ to A^* : for $p \in \mathcal{O}$ and $c \in C_p \subset A^*$, $e(p, c) = p + c$.

We will write C_t for the total space of the bundle C , so that for example $\mathcal{E}C_t$ denotes the smooth functions from C_t to \mathbb{C} , while $\mathcal{E}C$ denotes the smooth sections of C . Given any vector space V , recall that $\mathcal{S}^k V$ is its k^{th} symmetric tensor power (complexified if V is real), and let $\mathcal{S}_n V = \bigoplus_0^n \mathcal{S}^i V$. We will view $\mathcal{S}_n V$ as an algebra via the projection from the algebra $\mathcal{S}V = \mathcal{S}_\infty V$ to $\mathcal{S}_n V$ whose kernel is the ideal $\langle V \rangle^{n+1} = \bigoplus_{i > n} \mathcal{S}^i V$. Similarly, for any vector bundle B we have the vector bundle $\mathcal{S}_n B$, whose fibers $\mathcal{S}_n(B_p)$ are algebras.

We will use the endpoint map e to define an isomorphism $\epsilon_n : \mathcal{DS}_n C^* \rightarrow \mathcal{DO}^{(n)}$. Elements of the fiber $\mathcal{S}_n C_p^*$ are polynomials of degree $\leq n$ on C_p , and so a section $s \in \mathcal{DS}_n C^*$ defines a function $\tilde{s} \in \mathcal{E}C_t$ that is polynomial on fibers: for $(p, c) \in C_p$, $\tilde{s}(p, c)$ is $s(p)(c)$, the value of the polynomial $s(p)$ at c . For future reference, for $k \leq n$ let $\tilde{\pi}_{kn} : \mathcal{S}_n C^* \rightarrow \mathcal{S}_k C^*$ be the projection along the bundle of ideals $\tilde{K}_{kn} = \bigoplus_{k < i \leq n} \mathcal{S}^i C^*$, and write also $\tilde{\pi}_{kn}$ for the corresponding map $\mathcal{DS}_n C^* \rightarrow \mathcal{DS}_k C^*$ whose kernel is $\mathcal{D}\tilde{K}_{kn}$. Note that the equality $\mathcal{O} \times A^* = T\mathcal{O} \oplus C$ defines an isomorphism $C^* \cong T^\perp \mathcal{O}$, which gives rise to an isomorphism $\mathcal{S}^n C^* \cong \mathcal{S}^n T^\perp \mathcal{O}$.

Since C is complementary to $T\mathcal{O}$ in $\mathcal{O} \times A^*$, the differential of $e : C_t \rightarrow A^*$ is bijective everywhere. Therefore given $p \in \mathcal{O}$, there is an open neighborhood \tilde{V}

of the point $(p, 0)$ in C_t such that $V = e(\tilde{V})$ is an open neighborhood of p in A^* and $e : \tilde{V} \rightarrow V$ is a diffeomorphism. Given $s \in \mathcal{DS}_n C^*$, we define $\epsilon_n(s) \in \mathcal{DO}^{(n)}$ by $(\epsilon_n s)(p) = j_n(\tilde{s} \circ (e|_{\tilde{V}})^{-1})(p)$.

It is clear that $\epsilon_n s(p)$ is a well-defined element of $J^n A^*(p)$, independent of the choice of \tilde{V} . To see that $\epsilon_n s \in \mathcal{DO}^{(n)}$, let $\tilde{U} = \tilde{V} \cap (\mathcal{O} \times 0)$ and let U be the open neighborhood $e(\tilde{U})$ of p in \mathcal{O} . In order to exclude the possibility that $V \cap \mathcal{O}$ is larger than U , shrink V so that $V \cap \mathcal{O} = U$ and shrink \tilde{V} accordingly. Define $f_V \in \mathcal{EV}$ to be $\tilde{s} \circ e|_{\tilde{V}}^{-1}$; then $\epsilon_n s|_U = j_n f_V|_U$ and so $\epsilon_n s \in \mathcal{DO}^{(n)}$.

Lemma 6.2 *First, for $k \leq n$ we have $\epsilon_k \circ \tilde{\pi}_{kn} = \pi_{kn} \circ \epsilon_n$, $\epsilon_n(\mathcal{DK}_{kn}) = I_{kn}$, and $\text{supp} \epsilon_n s = \text{supps}$ for all $s \in \mathcal{DS}_n C^*$. Second, $\epsilon_n : \mathcal{DS}_n C^* \rightarrow \mathcal{DO}^{(n)}$ is a homeomorphism and an isomorphism of algebras. Third, the isomorphism from $\mathcal{DK}_{n-1, n} = \mathcal{DS}^n C^*$ to $\mathcal{DS}^n T^\perp \mathcal{O}$ obtained by following ϵ_n with the isomorphism from $I_{n-1, n}$ defined in Lemma 5.3 is same as the one coming from the isomorphism $C^* \cong T^\perp \mathcal{O}$.*

Proof. First, retain the neighborhoods \tilde{V} , V , V' , and U' from above, write \tilde{e} for the diffeomorphism $e|_{\tilde{V}} : \tilde{V} \rightarrow V$, and shrink U' and V' so that U' is a coordinate chart on \mathcal{O} with coordinates $x = (x_1, \dots, x_p)$ and $C|_{U'}$ is trivial. Then we may extend x to coordinates (x, y) on V' such that the equations $y = 0$ define U' , and for any $q \in U'$, $(x(q), y)$ are linear coordinates on the plane $q + C_q = e(C_q) \subset A^*$ defined near $(q, 0)$. In other words, for $q \in U'$ fixed and $c \in C_q$ varying $c \mapsto y(q + c)$ are linear coordinates on C_q defined near 0 (of course, linear coordinates are defined everywhere, but $V' \cap (q + C_q)$ is a neighborhood of $(q, 0)$), and $x(q + c) = x(q)$ is independent of c .

Given $F \in \mathcal{DO}^{(n)}$ such that $\text{supp} F \subset U'$, there are unique functions $c_J \in \mathcal{D}_{U'} \mathcal{O}$ such that $F = j_n(\sum_{|J| \leq n} c_J y^J)$ on U' . For x fixed at the value $x(q)$ for any $q \in U'$, y^J is a polynomial on C_q , and so $\sum_{|J| \leq n} c_J y^J$ may be viewed as a section s in $\mathcal{D}_{U'} \mathcal{S}_n C^*$ such that $\epsilon_n s = F$. This shows that at least when s and F have support in U' , $\text{supp} \epsilon_n s = \text{supps}$ and F is in the image of ϵ_n . It also shows that F is in $\epsilon_n(\mathcal{DK}_{kn})$ if and only if $c_J \equiv 0$ for $|J| \leq k$, which is the same as $F \in I_{kn}$. Now the usual partition of unity arguments imply that $\text{supp} \epsilon_n s = \text{supps}$ in general, ϵ_n is surjective, and $\epsilon_n(\mathcal{DK}_{kn}) = I_{kn}$.

To prove that $\epsilon_k \tilde{\pi}_{kn} s = \pi_{kn} \epsilon_n s$, it is enough to prove it on U' . Since $\mathcal{S}_n C^* = \mathcal{S}_k C^* \oplus \tilde{K}_{kn}$, there are unique sections s' of $\mathcal{S}_k C^*$ and s'' of \tilde{K}_{kn} such that $s = s' + s''$. Now using $\epsilon_n s = j_n(\tilde{s} \circ \tilde{e}^{-1})$ on U' , we find that we come down to proving $\pi_{kn} \epsilon_n s'' = 0$, and this follows from our choice of coordinates.

Second, ϵ_n is injective because it preserves supports, and we have already seen that it is surjective. It is a homeomorphism by the same type of open mapping theorem argument used in the proof of Lemma 5.3. To check that it is an algebra map, it will do to prove $\epsilon_n(ss') = \epsilon_n(s)\epsilon_n(s')$ for both s and s' having support in U' . This comes down to proving that the difference $t = \tilde{s}\tilde{s}' - (ss')^\sim$ is a function in $\mathcal{E}_t C$ such that $j_n(t \circ \tilde{e}^{-1})$ is 0 on U' . Now \tilde{s} and \tilde{s}' are polynomial of degree $\leq n$ on the fibers C_p , and a review of the definition of multiplication in

$\mathcal{S}_n C^*$ shows that $(ss')^\sim$ is the part of $\tilde{s}\tilde{s}'$ of degree $\leq n$. A coordinate calculation finishes the proof.

We will leave the third statement to the reader; it may be proven by using the differential operators ∂_{y_j} , which are sections of $C|_{U'} \subset TA^*|_{U'}$, and the definition of the isomorphism in Lemma 5.3. \square

The isomorphism ϵ_n allows us to transport $\mathcal{V}^{(n)}$ to an equivalent representation $\tilde{\mathcal{V}}^{(n)}$ on $\mathcal{DS}_n C^*$, and we will prove that $\tilde{\mathcal{V}}^{(n)}$ is an object of $\text{Ext}_G^C \mathcal{O}$. To simplify notation let us write ϵ , \mathcal{V} , and $\tilde{\mathcal{V}}$ in place of ϵ_n , $\mathcal{V}^{(n)}$, and $\tilde{\mathcal{V}}^{(n)}$ when n is specified by the context. We begin with some isomorphisms, which will be viewed as identifications when it is convenient. Any $\delta \in \mathcal{S}_n A^*$ defines a constant coefficient differential operator D_δ on A^* , and there is an isomorphism between the bundles $A^* \times \mathcal{S}_n A^*$ and $\Delta^n A^*$ defined by $(p, \delta) \mapsto D_\delta(p)$. Any $\Omega \in \mathcal{S}_n A$ defines a polynomial P_Ω on A^* , and this defines a duality $\langle \cdot, \cdot \rangle_A$ between $\mathcal{S}_n A^*$ and $\mathcal{S}_n A$: $\langle \delta, \Omega \rangle_A = D_\delta(0)P_\Omega$. Note that this procedure defines a duality $\langle \cdot, \cdot \rangle_V$ between $\mathcal{S}_n V$ and $\mathcal{S}_n V^*$ for any vector space V , such that $\langle \cdot, \cdot \rangle_V = \langle \cdot, \cdot \rangle_{V^*}$.

For $p \in A^*$, let $P_{\Omega,p}$ be the shifted polynomial $P_{\Omega,p}(q) = P_\Omega(q - p)$. Since $J^n A^*$ is dual to $\Delta^n A^*$ and $\mathcal{S}_n A$ is dual to $\mathcal{S}_n A^*$, we get an isomorphism between $J^n A^*$ and $A^* \times \mathcal{S}_n A$. One checks that it maps the n -jet of $P_{\Omega,p}$ at p to (p, Ω) , and hence that it is an algebra isomorphism on each fiber which identifies the ideals $K_{kn}(p)$ and $\langle A \rangle^{k+1}$.

Under these identifications, $\mathcal{DO}^{(n)}$ sits inside $\mathcal{D}(J^n A^*|_{\mathcal{O}}) = \mathcal{D}(\mathcal{O} : \mathcal{S}_n A)$, the functions on \mathcal{O} with values in $\mathcal{S}_n A$, and the ideal I_{kn} of $\mathcal{DO}^{(n)}$ sits inside the ideal $\mathcal{D}(\mathcal{O} : \langle A \rangle^{k+1})$. Let $r : \mathcal{O} \times \mathcal{S}_n A \rightarrow \mathcal{S}_n C^*$ be the fiberwise map $r(p, \Omega) = P_\Omega|_{C_p}$, and write also r for the resulting algebra map from $\mathcal{D}(\mathcal{O} : \mathcal{S}_n A)$ to $\mathcal{DS}_n C^*$. An easy calculation using the definition of ϵ gives the following lemma.

Lemma 6.3 *The isomorphism $\epsilon^{-1} : \mathcal{DO}^{(n)} \rightarrow \mathcal{DS}_n C^*$ is the restriction of r to $\mathcal{DO}^{(n)}$. Consequently, for $p \in \mathcal{O}$, $s \in \mathcal{DS}_n C^*$, and $\delta \in \mathcal{S}_n C_p \subset \mathcal{S}_n A^*$ we have $\langle \delta, s(p) \rangle_{C_p} = \langle \delta, \epsilon s(p) \rangle_{A^*}$. \square*

We remark that as a map from $\mathcal{DS}_n C^*$ to $\mathcal{D}(\mathcal{O} : \mathcal{S}_n A)$, ϵ is not a fiberwise map but a differential operator of order n . We are now ready to compute the action of the Lie algebra \mathfrak{a} on $\mathcal{DS}_n C^*$ under the representation $\tilde{\mathcal{V}}$. Let $\zeta = d\xi$, so that for $\alpha \in \mathfrak{a}$, $\zeta_\alpha : A^* \rightarrow \mathbb{C}$ is the function $\zeta_\alpha(p) = i\langle \alpha, p \rangle$.

Proposition 6.4 *The representation $\tilde{\mathcal{V}}$ together with the splitting $\mathcal{S}_n C^* = \bigoplus_0^n \mathcal{S}^i C^*$ is an object of $\text{Ext}_G^C \mathcal{O}$. If $\alpha \in \mathfrak{a}$, $s \in \mathcal{DS}_n C^*$, and $p \in \mathcal{O}$, then $(\tilde{\mathcal{V}}_\alpha s)(p) = r(p, \zeta_\alpha(p) + i\alpha)s(p)$. Here $\zeta_\alpha(p) + i\alpha \in \mathcal{S}_1 A$ is a polynomial on A^* with constant term $\zeta_\alpha(p)$, $r(p, \zeta_\alpha(p) + i\alpha) \in \mathcal{S}_1 C_p^*$ is its restriction to C_p , and the product is in the algebra $\mathcal{S}_n C_p^*$.*

Proof. Let $F = \epsilon(s)$. We must prove that $\epsilon^{-1}(\mathcal{V}_\alpha F)(p) = r(p, \zeta_\alpha(p) + i\alpha)s(p)$. Since $\epsilon^{-1}(\mathcal{V}_\alpha F) = r(\mathcal{V}_\alpha F)$ and r is an algebra map, it is enough to prove

$(\mathcal{V}_\alpha F)(p) = (\zeta_\alpha(p) + i\alpha)F(p)$, where the product is in the algebra $\mathcal{S}_n A$. But $(\mathcal{V}_\alpha F)(p) = j_n \zeta_\alpha(p) F(p)$, and since ζ_α is just the linear polynomial $i\alpha$ on A^* , our identifications give $j_n \zeta_\alpha(p) = \zeta_\alpha(p) + i\alpha$.

In light of Proposition 6.1 and the fact that $\epsilon(\oplus_{k+1}^n \mathcal{D}\mathcal{S}^i C^*) = I_{kn}$, $\tilde{\mathcal{V}}$ will be an object of $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$ provided it satisfies the third property in the definition of that category. We have just seen that $\tilde{\mathcal{V}}_\alpha$ acts fiberwise, and if α lies in the subspace $C^\perp \otimes \mathbb{C}$ of \mathfrak{a} then $r(p, \alpha) = 0$, so $\tilde{\mathcal{V}}_\alpha F(p) = \zeta_\alpha(p) F(p)$. Lifting these statements from \mathfrak{a} to A completes the proof. \square

Note that now Theorem 2.1 applies to describe the actions of H and \mathfrak{h} under $\tilde{\mathcal{V}}$, although in fact it is easy to prove this description directly for this particular representation.

7 Computation of $\mathcal{F}\tilde{\mathcal{V}}^{(n)}$

Let π be any representation of H in a finite dimensional complex space V , and extend it to a representation of G by making A act trivially. Then it follows from Proposition 6.1 that the representation $\mathcal{V}^{(n)} \otimes \pi$ of G is in $\text{Ext}_G \mathcal{O}$, and our goal is to reduce the problem of deciding which objects \mathcal{U} of $\text{Ext}_G \mathcal{O}$ are subquotients of $\mathcal{V}^{(n)} \otimes \pi$, for some n and π , to a problem involving only the representation of \mathcal{S}^∞ associated to \mathcal{U} at the end of Section 2.

The map $\epsilon \otimes \mathbf{1}$ is an equivalence from $\mathcal{V}^{(n)} \otimes \pi$ to $\tilde{\mathcal{V}}^{(n)} \otimes \pi$, and it follows from Proposition 6.4 that $\tilde{\mathcal{V}}^{(n)} \otimes \pi$ together with the splitting $\mathcal{S}_n C^* \otimes V = \oplus_0^n \mathcal{S}^i C^* \otimes V$ is in $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$. In this section we will compute the action of the functor $\mathcal{F} : \text{Ext}_G^{\mathcal{C}} \mathcal{O} \rightarrow \mathcal{C}$ from Section 2 on $\tilde{\mathcal{V}}^{(n)} \otimes \pi$, which will allow us to reduce our subquotient question to a problem in the category \mathcal{C} .

As before we write $\tilde{\mathcal{V}}$ and \mathcal{V} for $\tilde{\mathcal{V}}^{(n)}$ and $\mathcal{V}^{(n)}$. We begin by proving that for $0 \leq k \leq n$ the bundles $\tilde{E}_{kk}(\tilde{\mathcal{V}})$ of graded differential operators on $\mathcal{S}_n C^*$ are isomorphic to the bundles $\Delta^k A^*|_{\mathcal{O}}$ of differential operators of order $\leq k$ on A^* , restricted to \mathcal{O} . This isomorphism is the transpose ϵ^T of the map ϵ of Section 6, and it will allow us to move $\mathcal{F}\tilde{\mathcal{V}}$ to an equivalent object of \mathcal{C} acting in the flag $\Delta^n A^*(p_0) = \mathcal{S}_n A^*$. This object may be thought of roughly as $\mathcal{F}\mathcal{V}$, although \mathcal{V} is not in $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$.

We saw in the discussion around Lemma 6.2 that for any $p \in \mathcal{O}$, we may choose neighborhoods $V \supset U \ni p$ as follows. First, U is open in \mathcal{O} and closed in V , V is open in A^* , and $U = V \cap \mathcal{O}$. Second, U is a coordinate chart on \mathcal{O} with coordinates x_i and $C|_U$ is trivial, and so we may define coordinates \tilde{x}_i, \tilde{y}_j on the total space of $C|_U$ such that for any $(q, c) \in C|_U$, $\tilde{x}_i(q, c) = x_i(q)$ and $c \mapsto \tilde{y}_j(q, c)$ are linear coordinates on C_q . Third, there is an open set $\tilde{V} \subset C|_U$ such that the endpoint map e is a diffeomorphism from \tilde{V} to V . We will write $\tilde{e} : \tilde{V} \rightarrow V$ for this diffeomorphism, which we use to define coordinates x_i, y_j on V by $x_i \circ \tilde{e} = \tilde{x}_i$ and $y_j \circ \tilde{e} = \tilde{y}_j$.

Recall that $\epsilon : \mathcal{D}\mathcal{S}_n C^* \rightarrow \mathcal{D}\mathcal{O}^{(n)}$ is defined on U as follows. Given $s \in$

$\mathcal{DS}_n C^*$, we view s as a function $\tilde{s} : C_t \rightarrow \mathbb{C}$ that is polynomial on fibers, and we let $\epsilon s|_U = j_n(\tilde{s} \circ \tilde{e}^{-1})|_U$. So roughly speaking, ϵ is the transpose of e^{-1} . Now recall from the beginning of Section 5 that elements $D \in \mathcal{D}(\Delta^n A^*|_{\mathcal{O}})$ act on elements $F \in \mathcal{DO}^{(n)}$ to give functions $DF \in \mathcal{DO}$, and so we have the map $\epsilon^T D = D \circ \epsilon : \mathcal{DS}_n C^* \rightarrow \mathcal{DO}$. Note that with the identifications at the end of Section 6, $D(p) \in \mathcal{S}_n A^*$, $F(p) \in \mathcal{S}_n A$, and $DF(p) = \langle D(p), F(p) \rangle_A$.

Lemma 7.1 *For any D in $\mathcal{D}(\Delta^n A^*|_{\mathcal{O}})$, $\epsilon^T D$ is in $\mathcal{D}\tilde{E}_{nn}(\tilde{\mathcal{V}})$, and ϵ^T is a fiberwise isomorphism from $\mathcal{D}(\Delta^n A^*|_{\mathcal{O}})$ to $\mathcal{D}\tilde{E}_{nn}(\tilde{\mathcal{V}})$. Therefore it arises from a vector bundle isomorphism $\epsilon^T : \Delta^n A^*|_{\mathcal{O}} \rightarrow \tilde{E}_{nn}(\tilde{\mathcal{V}})$, and ϵ^T maps $\Delta^k A^*$ isomorphically to $\tilde{E}_{kk}(\tilde{\mathcal{V}})$ for $0 \leq k \leq n$.*

Proof. Let us write \tilde{E}_{kl} for $\tilde{E}_{kl}(\tilde{\mathcal{V}})$. First we check that if D is in $\mathcal{D}(\Delta^k A^*|_{\mathcal{O}})$ then $\epsilon^T D$ is in $\mathcal{D}\tilde{E}_{kk}$. It follows from our choice of coordinates that there are unique functions $d_{IJ}(x)$ on U such that $D|_U = \sum_{|I+J| \leq k} d_{IJ} \partial_x^I \partial_y^J$, where we use the multinomial notation from the proof of Lemma 5.3. Now the \tilde{y}_j are nothing but a section basis of the bundle $C^*|_U$, and so for any s in $\mathcal{DS}_n C^*$ there are unique functions $s_J(x)$ on U such that $s|_U = \sum_{|J| \leq n} s_J \tilde{y}^J$. Since $\tilde{x} = x \circ \tilde{e}$, $\tilde{y} = y \circ \tilde{e}$, and $\partial_y^J \tilde{y}^{j'}$ is $J! \delta_{JJ'}$, we find that $\epsilon^T D(s)|_U = \sum J! d_{IJ} \partial_x^I s_J(x)$. Since $d_{IJ} = 0$ if $|I+J| > k$, we know at least that $\epsilon^T D$ is a differential operator of order $\leq k$. To check that it is graded so as to be in $\mathcal{D}\tilde{E}_{kk}$, recall the flag of vector bundles $\mathcal{S}_n C^* \supset \tilde{K}_{0n} \supset \cdots \supset \mathcal{D}\tilde{K}_{n-1,n} \supset 0$ defined below Proposition 6.1. This is the flag left invariant by $\tilde{\mathcal{V}}$, where $s \in \tilde{K}_{kn}$ if and only if $s_J = 0$ for $|J| \leq k$ for all choices of U as above. A review of the definition of $\mathcal{D}\tilde{E}_{kl}$ shows that it consists of differential operators of order $\leq k$ from $\mathcal{DS}_n C^* \rightarrow \mathcal{DO}$ that restrict to 0 on $\mathcal{D}\tilde{K}_{ln}$, and to differential operators of order $\leq k-i-1$ on $\mathcal{D}\tilde{K}_{in}$ for $0 \leq i \leq l$. In particular, $\epsilon^T D$ is in $\mathcal{D}\tilde{E}_{kk}$ if and only if $d_{IJ} = 0$ whenever $|I+J| > k$, and so ϵ^T maps $\mathcal{D}(\Delta^k A^*|_{\mathcal{O}})$ surjectively to $\mathcal{D}\tilde{E}_{kk}$.

For any $f \in \mathcal{DO}$, $\epsilon^T(fD) = f\epsilon^T D$, and so ϵ^T is a fiberwise map. We leave it to the reader to check that it is injective. \square

Recall the fiberwise restriction map $r : \mathcal{DO} \otimes \mathcal{S}_n A \rightarrow \mathcal{DS}_n C^*$, which restricts to the inverse of ϵ on $\mathcal{DO}^{(n)}$. For any $D \in \mathcal{D}\tilde{E}_{nn}$, let $r^T(D)$ be $D \circ r : \mathcal{DO}^{(n)} \rightarrow \mathcal{DO}$. It follows from Lemma 7.1 that r^T is a bundle isomorphism from \tilde{E}_{nn} to $\Delta^n A^*|_{\mathcal{O}}$ which takes \tilde{E}_{kk} to $\Delta^k A^*|_{\mathcal{O}}$, and that it is the inverse of ϵ^T .

Definition. Let $\tilde{\phi}^{(n)} = \mathcal{F}\tilde{\mathcal{V}}^{(n)}$, the object of \mathcal{C} acting in \tilde{E}_{nn} . Let $\mu : \tilde{E}_{nn}(p_0) \rightarrow \Delta^n A^*(p_0) = \mathcal{S}_n A^*$ be the isomorphism $r^T(p_0)$, and use μ to transport $\tilde{\phi}^{(n)}$ to an object $\phi^{(n)}$ of \mathcal{C} acting in $\mathcal{S}_n A^*$. We will usually write simply $\tilde{\phi}$ and ϕ for these objects.

Instead of finding the action of $\tilde{\phi}$ we will find the action of ϕ , which should be thought of as $\mathcal{F}\mathcal{V}$. It may seem that we could have avoided $\tilde{\mathcal{V}}$ altogether, but since \mathcal{V} is not canonically an object of $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$ we defined $\tilde{\mathcal{V}}$ in order to use the machinery of [4]. To understand the sense in which $\phi = \mathcal{F}\mathcal{V}$, let us

define a representation $\Delta\mathcal{V}$ of G on $\mathcal{D}(\Delta^n A^*|_{\mathcal{O}})$ by $\Delta\mathcal{V}_g(D) = \lambda_g \circ D \circ \mathcal{V}_g^{-1}$. Here $\Delta\mathcal{V}_g D$ is well defined because sections of $\Delta^n A^*|_{\mathcal{O}}$ are determined by their action on $\mathcal{D}\mathcal{O}^{(n)}$. By the above remarks r^T intertwines $\Delta\tilde{\mathcal{V}}$ and $\Delta\mathcal{V}$, and so since $\mu = r^T(p_0)$ is a \mathcal{C} -isomorphism by definition, we see that the S' -action of ϕ is simply $\Delta\mathcal{V}|_{S'}(p_0)$, the little group representation of $\Delta\mathcal{V}$ at p_0 . Using the last paragraph of Section 2, we see that the \mathfrak{g} -action $\phi_X : E_{n-1, n-1}(\phi) \rightarrow E_{nn}(\phi)$ is $\phi_X(\delta) = -\delta \circ \mathcal{V}_X$.

We will now compute $E_{kk}(\phi)$ and $N_k(\phi)$, and from now on we will write E_k instead of E_{kk} . By definition $E_k(\phi) = \mu(E_k(\tilde{\phi})) = \Delta^k A^*(p_0) = \mathcal{S}_k A^*$. In computing $N_k(\phi) = \mu(N_k(\tilde{\phi}))$ we will use the fact that if $Y \subset Z$ is any subspace of any vector space and $r : SZ^* \rightarrow SY^*$ is restriction of polynomials, then $r^T : SY \rightarrow SZ$ is inclusion. Since $N_k(\tilde{\phi})$ is the space of order 0 differential operators at p_0 on $\mathcal{D}\mathcal{S}_n C^*$ that annihilate the ideal $\mathcal{D}\tilde{K}_{kn}(p_0)$, we find that $N_k(\tilde{\phi}) = \tilde{K}_{kn}(p_0)^\perp \subset (\mathcal{S}_n C_{p_0}^*)^*$. Henceforth we will write C_0 for C_{p_0} and identify the dual of $\mathcal{S}_n C_0^*$ with $\mathcal{S}_n C_0$, which gives $N_k(\tilde{\phi}) = \mathcal{S}_k C_0$. Now on order 0 differential operators $\mu = r^T(p_0)$ is $r(p_0)^T$, and so the fact noted above gives $N_k(\phi) = \mathcal{S}_k C_0$.

Next we will use the fact that $\phi = \mathcal{F}\mathcal{V}$ to compute the S'^∞ -action of ϕ . Let us write λ for the canonical action of H on $\mathcal{D}A^*$, extended to an action of G by the character action of A . We will also continue to use λ for the same action on $\mathcal{D}\mathcal{O}$, as we did in Section 2. Let us write ℓ for the graded action of H on $\mathcal{S}A$, and ℓ^* for the graded action on $\mathcal{S}A^*$. Then ℓ and ℓ^* are dual actions, and we make $A^* \times \mathcal{S}A^*$ and $A^* \times \mathcal{S}A$ into H -bundles by the product action. Now λ makes $\Delta^n A^*$ and $J^n A^*$ into H -bundles, and we leave it to the reader to check that our identification of $\Delta^n A^*$ with $A^* \times \mathcal{S}_n A^*$ is an H -isomorphism, and so the identification of $J^n A^*$ with $A^* \times \mathcal{S}_n A$ is also, by duality.

The bundle $\mathcal{O} \times \mathcal{S}_n A^* = \Delta^n A^*|_{\mathcal{O}}$ is also an H -bundle by the product action, and a review of the definition of \mathcal{V} shows that the restriction $\Delta\mathcal{V}|_H$ of the representation $\Delta\mathcal{V}$ defined above is the representation $\lambda \otimes \ell$ on $\mathcal{D}\mathcal{O} \otimes \mathcal{S}_n A^*$ associated to this H -bundle structure.

If Z is any vector space and $z \in Z$, let $m_z : SZ \rightarrow SZ$ be multiplication by z , and for $\alpha \in Z^*$ let $\partial_\alpha : SZ \rightarrow SZ$ be the derivative along α of elements of SZ , viewed as polynomials on Z^* . Then ∂_α is a derivation of the algebra SZ such that $\partial_\alpha(\mathcal{S}_n Z) \subset \mathcal{S}_{n-1} Z$, and on Z we have $\partial_\alpha z = \langle \alpha, z \rangle$. It is elementary that the transpose $\partial_\alpha^T : SZ^* \rightarrow SZ^*$ is m_α . We are now in a position to compute ϕ .

Proposition 7.2 *The object ϕ of \mathcal{C} acts in $\mathcal{S}_n A^*$, with $E_k(\phi) = \mathcal{S}_k A^*$ and $N_k(\phi) = \mathcal{S}_k C_0$. For $\alpha \in \mathfrak{a}$, $s \in S$, and $X \in \mathfrak{h}$, we have $\phi_\alpha = -\zeta_\alpha(p_0) - i\partial_\alpha$, $\phi_s = \ell_s$, and $\phi_X = \ell_X + m_{Xp_0} : \mathcal{S}_{n-1} A^* \rightarrow \mathcal{S}_n A^*$.*

Proof. We have already seen the first sentence. The formula for ϕ_α follows from the proof of Proposition 6.4 and the fact that $m_\alpha^T = \partial_\alpha$, and $\phi_s = \ell_s$ follows from $\Delta\mathcal{V} = \lambda \otimes \ell$, since $\phi|_S$ is the little group representation of $\Delta\mathcal{V}|_H$ at p_0 . For $\delta \in \mathcal{S}_n A^*$, we compute $\phi_X \delta = -\delta \circ \mathcal{V}_X$ using the identifications from the end

of Section 6. As an element of $\Delta^n A^*(p_0)$, δ is $D_\delta|_{p_0}$, and if $\Omega \in \mathcal{S}_n A$ then as an element of $J^n A^*(p_0)$, Ω is the n -jet $j_n P_{\Omega, p_0}(p_0)$ of the polynomial P_{Ω, p_0} at p_0 . Now $\mathcal{V}_X j_n P_{\Omega, p_0} = j_n \lambda_X P_{\Omega, p_0}$, and one finds that $\lambda_X P_{\Omega, p_0} = P_{\Omega', p_0}$, where $\Omega' = \ell_X^* \Omega - \partial_{X p_0} \Omega$. Thus $\langle \phi_X \delta, \Omega \rangle = -\langle \delta, (\ell_X^* - \partial_{X p_0}) \Omega \rangle = \langle (\ell_X + m_{X p_0}) \delta, \Omega \rangle$, completing the proof. \square .

We will now consider the representation $\tilde{\mathcal{V}} \otimes \pi$ defined at the beginning of this section. In fact, it will be more convenient to study $\tilde{\mathcal{V}} \otimes \pi^*$, and we will write \mathcal{V}^π and $\tilde{\mathcal{V}}^\pi$ for $\mathcal{V} \otimes \pi^*$ and $\tilde{\mathcal{V}} \otimes \pi^*$. Then $r \otimes \mathbf{1} : \mathcal{V}^\pi \rightarrow \tilde{\mathcal{V}}^\pi$ is a G -isomorphism, and clearly $\Delta \tilde{\mathcal{V}}^\pi = (\Delta \tilde{\mathcal{V}}) \otimes \pi$, so the object $\tilde{\phi}^\pi$ of \mathcal{C} acts in $E_k(\tilde{\phi}^\pi) = E_k(\tilde{\phi}) \otimes V$ with $N_k(\tilde{\phi}^\pi) = N_k(\tilde{\phi}) \otimes V$. Define an object ϕ^π of \mathcal{C} with $E_k(\phi^\pi) = \mathcal{S}_k A^* \otimes V$ and $N_k(\phi^\pi) = \mathcal{S}_k C_0 \otimes V$ by making $\mu \otimes \mathbf{1} : \tilde{\phi}^\pi \rightarrow \phi^\pi$ an equivalence, where $\mu : \tilde{\phi} \rightarrow \phi$ is the equivalence from above.

Proposition 7.3 *The object ϕ^π of \mathcal{C} is $\phi \otimes \pi$, where the representation π of H is restricted to a representation of S^∞ with flag $V = V_0 = V_1$, and the tensor product is the one between \mathcal{C} and $\mathcal{R}S^\infty$ defined in Section 4.*

Proof. If we define a representation $\Delta \mathcal{V}^\pi$ of G on $\mathcal{D}(\Delta^n A^*|_{\mathcal{O}}) \otimes V$ by $\Delta \mathcal{V}_g^\pi(D) = \lambda_g \circ D \circ \mathcal{V}_{g^{-1}}^\pi$, then $r^T \otimes \mathbf{1}$ is a G -isomorphism and so $\Delta \mathcal{V}^\pi$ is actually $(\Delta \mathcal{V}) \otimes \pi$. Furthermore, $\phi^\pi = \mathcal{F} \mathcal{V}^\pi$ in the same sense as before, and so $\phi^\pi|_{S'}$ is the inhomogeneous little group representation of $\Delta \mathcal{V} \otimes \pi$ at p_0 , which is $\phi|_{S'} \otimes \pi$. To check that $\phi_X^\pi = \phi_X \otimes \mathbf{1} + \mathbf{1} \otimes \pi_X$, choose $\delta \in \mathcal{S}_n A^*$ and $v \in V$ and compute $\phi_X^\pi(\delta \otimes v) = -(\delta \otimes v) \circ (\mathcal{V} \otimes \pi^*)_X$, using the fact that $\mathcal{V} \otimes \pi^*$ is an ordinary representation of H . \square

In Section 8 we will need the fact that the restriction of ϕ to a representation of S^∞ is in fact the restriction of a representation $\tilde{\phi}$ of H on $\mathcal{S}_n A^*$, i.e., $\phi|_{S^\infty} = \tilde{\phi}|_{S^\infty}$. To understand why this is the case, note that ϕ is an S'^∞ -subrepresentation of an ordinary S', \mathfrak{g} -representation on the entire symmetric algebra $\mathcal{S} A^*$, and the restriction of this representation to S, \mathfrak{h} leaves the ideal $\langle A^* \rangle^{n+1}$ invariant. Hence it defines a quotient representation on $\mathcal{S}_n A^*$, and this representation extends $\phi|_{S^\infty}$ to an ordinary representation of S, \mathfrak{h} .

To extend $\phi|_{S^\infty}$ further to a representation of H , recall that ϕ is defined using the identification $\delta \mapsto D_\delta|_{p_0}$ of $\mathcal{S}_n A^*$ with $\Delta^n A^*(p_0)$. The action of ϕ on $\Delta^n A^*(p_0)$ is dual to the natural action on $J^n A^*(p_0)$, and this space is naturally isomorphic to the polynomials $\mathcal{P}_n A^*$ on A^* of degree $\leq n$. Now $\mathcal{P}_n A^*$ carries a natural action of H , the dual of which extends $\phi|_{S^\infty}$ to H . Furthermore, translation by p_0 in $\mathcal{P}_n A^*$ intertwines the H action on $J^n A^*(p_0)$ with that on $J^n A^*(0)$, which is just ℓ^* . Working this out yields the following results.

For any $p \in A^*$, define an endomorphism $m_p : \delta \mapsto p\delta$ of $\mathcal{S}^n A^*$ such that m_p maps the ideal $\langle A^* \rangle^n$ to 0. Let $\tilde{\phi}$ be the representation of H on $\mathcal{S}_n A^*$ defined by $\tilde{\phi}_h = \exp(m_{h p_0} - m_{p_0}) \circ \ell_h$; since $\ell_h \circ m_p \circ \ell_h^{-1} = m_{h p}$, $\tilde{\phi}$ is indeed a representation. Differentiating at e yields $\tilde{\phi}_X = \ell_X + m_{X p_0}$, which gives the next proposition.

Proposition 7.4 *The restriction of $\tilde{\phi}$ to a representation of S^∞ on the flag $\mathcal{S}_n A^* \supset \mathcal{S}_{n-1} A^*$ is $\phi|_{S^\infty}$. Furthermore, $\exp(m_{p_0})$ intertwines the representations $\tilde{\phi}$ and ℓ of H . \square*

8 The Main Theorem

We may now prove our main theorem, which gives a condition for an object \mathcal{U} of $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$ to be isomorphic to a subquotient of \mathcal{V}^π for some π , in terms of the restriction of the representation of S'^∞ associated to $\mathcal{F}\mathcal{U}$ to S^∞ . The main strength of the theorem is that in deciding whether or not \mathcal{U} is a subquotient of \mathcal{V}^π , it is not necessary to consider the A -action of $\mathcal{F}\mathcal{U}$.

Theorem 8.1 *Let \mathcal{U} be an object of $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$, and write σ for $\mathcal{F}\mathcal{U}$. Then \mathcal{U} is isomorphic to a subquotient of \mathcal{V}^π for some representation π of H on a finite dimensional complex vector space V if the representation $\tilde{\sigma} = \sigma|_{S^\infty}$ of S^∞ on the flag $E_n \supset E_{n-1}$ is a subquotient of the restriction of π to a representation of S^∞ on the flag $V = V_0 = V_1$. Conversely, if \mathcal{U} is isomorphic to a subquotient of \mathcal{V}^π for some π , then $\tilde{\sigma}$ is a subquotient of the restriction of a representation of H to S^∞ (this representation of H is not necessarily π).*

Remark. It is well known that if H is a real linear algebraic group and $S \subset H$ is an algebraic subgroup, then all rational representations of S are subquotients of restrictions of rational representations of H , see e.g. [1]. If there is an analog of this fact showing that for some wide class of subgroups S of algebraic groups H , a wide class of representations of S^∞ are subquotients of restrictions of representations of H , we will be able to say that “most” of the representations in $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$ are isomorphic to subquotients of representations of the form \mathcal{V}^π . In light of Theorem 2.1, this would mean that generically, representations in $\text{Ext}_G^{\mathcal{C}} \mathcal{O}$ are isomorphic to subquotients of representations of the form \mathcal{V}^π .

Proof. If \mathcal{U} is isomorphic to a subquotient of \mathcal{V}^π , then σ is a subquotient of $\mathcal{F}\mathcal{V}^\pi = \phi \otimes \pi$ in both the categories \mathcal{C} and $\mathcal{R}S'^\infty$ by Lemma 3.5. By Proposition 7.4, $(\phi \otimes \pi)|_{S^\infty}$ is itself the restriction of the representation $\tilde{\phi} \otimes \pi$ of H , and so the converse statement of the theorem is proven.

Suppose now that $\tilde{\sigma}$ is isomorphic to a subquotient of $\pi|_{S^\infty}$ in the category $\mathcal{R}S^\infty$. Then Lemma 4.3 shows that $\phi \otimes \tilde{\sigma}$ is isomorphic to a subquotient of $\phi \otimes \pi$ in the category \mathcal{C} , where the tensor product is the one defined in Section 4 mapping $\mathcal{C} \times \mathcal{R}S^\infty$ to \mathcal{C} . Therefore if we can show that σ is isomorphic to a subquotient of $\phi \otimes \tilde{\sigma}$ in \mathcal{C} , we will have that σ is isomorphic to a subquotient of $\phi \otimes \pi$ in \mathcal{C} . Lemma 3.5 will then show that \mathcal{U} is isomorphic to a subquotient of \mathcal{V}^π , completing the proof of the theorem.

We will in fact show that σ is isomorphic to a subobject of $\phi \otimes \tilde{\sigma}$ in the category \mathcal{C} , by defining a \mathcal{C} -monomorphism $z : \sigma \rightarrow \phi \otimes \tilde{\sigma}$. Write E_k and N_k for $E_k(\sigma)$ and $N_k(\sigma)$, and recall from the end of Section 2 that the flag associated

to the representation $\tilde{\sigma}$ of S^∞ on $E_n \supset E_{n-1}$ is $E_n \supset \cdots \supset E_0 \supset 0$. Let $\dot{E}_k = E_k(\phi \otimes \tilde{\sigma})$ and $\dot{N}_k = N_k(\phi \otimes \tilde{\sigma})$; by definition

$$\dot{E}_k = \bigoplus_{p+q=k} \mathcal{S}_p A^* \otimes E_q \quad \text{and} \quad \dot{N}_k = \bigoplus_{p+q=k} \mathcal{S}_p C_0 \otimes E_q.$$

In order to define z , we must define an algebra representation Σ of $\mathcal{S}_n A$ on E_n . For $\alpha \in \mathfrak{a} = A_{\mathbb{C}}$, let $\Sigma_\alpha = i(\sigma_\alpha + \zeta_\alpha(p_0))$. Since σ is a representation of the abelian Lie algebra \mathfrak{a} , this defines a representation Σ of the $\mathfrak{U}(\mathfrak{a}) = \mathcal{S}A$. Using the last paragraph of Section 2 we find that $\Sigma_\alpha(E_k) \subset E_{k-1}$, so the ideal $\langle A \rangle^{n+1}$ acts as 0 under Σ and hence Σ factors through to a representation of $\mathcal{S}_n A$.

For any $\Omega \in \mathcal{S}_n A$, let $\Sigma_\Omega^T : E_n^* \rightarrow E_n^*$ be the transpose of Σ_Ω . Then Σ^T is an algebra representation of $\mathcal{S}_n A$ on E_n^* , and we define the map $z : E_n \rightarrow \mathcal{S}_n A^* \otimes E_n$ by defining its transpose $z^T : \mathcal{S}_n A \otimes E_n^* \rightarrow E_n^*$ to be $z^T(\Omega \otimes \epsilon) = \Sigma_\Omega^T \epsilon$. We remark that if $\sigma = \phi$, $\Sigma_\alpha^T = m_\alpha$ and $z^T : \mathcal{S}_n A \otimes \mathcal{S}_n A \rightarrow \mathcal{S}_n A$ is just multiplication; this case led us to discover z . Since z^T is surjective, z is injective, and we will show that $z(N_k) \subset \dot{N}_k$ and that z is a S'^∞ -map. Then it will follow from Lemma 4.2 that $z(E_k) \subset \dot{E}_k$ and that z is a flag monomorphism in \mathcal{C} , which means that σ is isomorphic to a subobject of $\phi \otimes \tilde{\sigma}$ and Theorem 8.1 is proven. In order to proceed we need a lemma, whose proof will be omitted.

Lemma 8.2 *Let A, B, Y , and Z be finite dimensional vector spaces, $\alpha : A \rightarrow Z$ and $\beta : B \rightarrow Y$ linear maps, Z_0 and Z_1 subspaces of Z , $A_0 \subset \cdots \subset A_n \subset A$ and $B_0 \subset \cdots \subset B_n \subset B$ flags, and denote transposes with a superscript T . First, $\alpha(A_0) \subset Z_0$ if and only if $\alpha^T(Z_0^\perp) \subset A_0^\perp$. Second, if $j : Z_0 \rightarrow Z$ is inclusion, j^T is projection. Third, $(A_0 \otimes B_0)^\perp = A_0^\perp \otimes B^* + A^* \otimes B_0^\perp$. Fourth, $(Z_0 + Z_1)^\perp = Z_0^\perp \cap Z_1^\perp$. Fifth, $(\alpha \otimes \beta)^T = \alpha^T \otimes \beta^T$. Last,*

$$\left(\sum_{p+q=k} A_p \otimes B_q \right)^\perp = \sum_{r+s=k-1} A_r^\perp \otimes B_s^\perp.$$

In proving that $z(N_k) \subset \dot{N}_k$, we will actually prove that $z(N_k) \subset \sum_{p+q=k} \mathcal{S}_p C_0 \otimes N_q$. Writing $\langle C_0^\perp \rangle$ for the ideal in $\mathcal{S}_n A$ generated by C_0^\perp , we have $(\mathcal{S}_p C_0)^\perp = \langle C_0^\perp \rangle + \langle A \rangle^{p+1}$, and so by Lemma 8.2 it will do to prove that

$$\left\langle N_k, \sum_{r+s=k-1} \Sigma^T(\langle C_0^\perp \rangle + \langle A^{r+1} \rangle) N_s^\perp \right\rangle = 0.$$

Since $\Sigma \langle C_0^\perp \rangle N_k = 0$, the terms containing C_0^\perp are 0, and since $\Sigma^T \langle A^{r+1} \rangle N_s^\perp \subset N_{r+s+1} = N_k$, the equation holds.

We will check that z is an S'^∞ -map by checking separately that it is an S -map, an \mathfrak{a} -map, and a \mathfrak{h} -map. For $s \in S$ we need $z \circ \sigma_s = (\phi_s \otimes \tilde{\sigma}_s) \circ z$, or

$\sigma_s^T \circ z^T = z^T \circ (\phi_s^T \otimes \sigma_s^T)$. Applying both sides to some $\Omega \otimes \epsilon \in \mathcal{S}_n A \otimes E_n^*$, we see that we need $\sigma_s^T \circ \Sigma^T(\Omega)\epsilon = \Sigma^T(\phi_s^T \Omega) \circ \sigma_s^T \epsilon$. This comes down to $\Sigma(\Omega) = \sigma_s \circ \Sigma(\phi_s^T \Omega) \circ \sigma_s^{-1}$, and it is enough to check this for $\Omega = \alpha \in \mathfrak{a}$. By Proposition 7.2, $\phi_s = \ell_s$, and so $\phi_s^T = \ell^*(s^{-1})$ acts on α to give $\text{Ad}_s^{-1} \alpha$. Now the definition of Σ and the fact that $sp_0 = p_0$ give the desired equality.

For $\alpha \in \mathfrak{a}$, $(\phi \otimes \tilde{\sigma})_\alpha = \phi_\alpha \otimes \mathbf{1}$ because $\tilde{\sigma}$ is trivial on A , and so we need to check that $z \circ \sigma_\alpha = (\phi_\alpha \otimes \mathbf{1}) \circ z$. Taking the transpose and applying both sides to $\Omega \otimes \epsilon$, we need $\sigma_\alpha^T \circ \Sigma^T(\Omega)\epsilon = \Sigma^T(\phi_\alpha^T \Omega)\epsilon$, or $\Sigma(\Omega) \circ \sigma_\alpha = \Sigma(\phi_\alpha^T \Omega)$. Now $\phi_\alpha^T = -\zeta_\alpha(p_0) - im_\alpha$, so

$$\Sigma(\phi_\alpha^T \Omega) = \Sigma(-\zeta_\alpha(p_0) - i\alpha) \circ \Sigma(\Omega) = \sigma_\alpha \circ \Sigma(\Omega) = \Sigma(\Omega) \circ \sigma_\alpha.$$

It is a little tricky to check that $z \circ \sigma_X = (\phi \otimes \tilde{\sigma})_X \circ z$ for $X \in \mathfrak{h}$, because σ_X and ϕ_X are not endomorphisms of E_n and $\mathcal{S}_n A^*$, but rather are maps from E_{n-1} to E_n and $\mathcal{S}_{n-1} A^*$ to $\mathcal{S}_n A^*$, respectively. First, a direct argument like the one used to prove that $z(N_k) \subset N_k$ shows that $z(E_{n-1}) \subset \mathcal{S}_{n-1} A^* \otimes E_{n-1}$, and so both sides map E_{n-1} to $\mathcal{S}_n A^* \otimes E_n$. Let us write ρ for both of the projections $E_n^* \rightarrow E_{n-1}^*$ and $\mathcal{S}_n A \rightarrow \mathcal{S}_{n-1} A$. Then we find that $(z \circ \sigma_X)^T : \mathcal{S}_n A \otimes E_n^* \rightarrow E_{n-1}^*$ acts on $\Omega \otimes \epsilon$ to give $\sigma_X^T \circ \Sigma^T(\Omega)\epsilon$, while $z^T \circ (\phi \otimes \tilde{\sigma})_X^T = z^T \circ (\phi_X^T \circ \rho + \rho \circ \sigma_X^T)$ acts on it to give $\Sigma^T(\phi_X^T \Omega) \circ \rho \epsilon + \Sigma^T(\rho \Omega) \circ \sigma_X^T \epsilon$. Thus we need to check that $\Sigma(\Omega) \circ \sigma_X$ and $\Sigma(\phi_X^T \Omega) + \sigma_X \circ \Sigma(\rho \Omega)$ are equal as maps from E_{n-1} to E_n . Here $\phi_X = \ell_X + m_{Xp_0}$ gives $\phi_X^T \Omega = -\rho \ell_X^* \Omega + \partial_{Xp_0} \Omega$, and note that since $\Sigma(A^n)E_{n-1} = 0$, Ω and $\rho \Omega$ have the same action on E_{n-1} under Σ . Hence we come down to proving that $[\sigma_X, \Sigma_\Omega] = \Sigma(-\ell_X^* \Omega + \partial_{Xp_0} \Omega)$. Since both $\text{ad}(\sigma_X) \circ \Sigma$ and $\Sigma \circ (\partial_{Xp_0} - \ell_X^*)$ are Σ -derivations from $\mathcal{S}_n A$ to $\text{End}(E_n)$, it is enough to check the equality for $\Omega = \alpha \in \mathfrak{a}$, where it is easy. \square

The following variation on Theorem 8.1 can be more practical than the theorem for computations. For example, if $\mathcal{U} = \mathcal{V}^\pi$ then $\sigma = \mathcal{F}\mathcal{U} = \phi \otimes \pi$, and so Theorem 8.1 exhibits \mathcal{U} as a subquotient of $\mathcal{V}^{\tilde{\phi} \otimes \pi}$, where $\tilde{\phi}$ is the representation of H from Proposition 7.4. Here \mathcal{U} is actually equal to \mathcal{V}^π , a much simpler representation than $\mathcal{V}^{\tilde{\phi} \otimes \pi}$, and it would be nice to have an improvement of Theorem 8.1 which would sense this and exhibit \mathcal{U} as a subquotient of \mathcal{V}^π . Corollary 8.3 does this, as the restriction of $\phi \otimes \pi$ to a representation of S^∞ on $\mathcal{S}_n A^* \otimes V$ has $\pi|_{S^\infty}$ as a quotient. To see this, divide by the S^∞ -subrepresentation $\langle A^* \rangle \otimes V$, and use Proposition 7.4 to check that the quotient is $\pi|_{S^\infty}$ and that it has the property required by Corollary 8.3.

Corollary 8.3 *Given \mathcal{U} in $\text{Ext}_C^{\mathcal{C}} \mathcal{O}$, let $\sigma = \mathcal{F}\mathcal{U}$. Suppose that the restriction of σ to a representation $\tilde{\sigma}$ of S^∞ on $E_n \supset E_{n-1}$ has a quotient S^∞ -representation $\bar{\sigma}$ on a flag $X_n \supset X_{n-1}$, such that the subspace X_n^* of E_n^* generates E_n^* under the action of the representation $\sigma^*|_A$. If there is a representation π of H on a space V such that $\bar{\sigma}$ is a subquotient of $\pi|_{S^\infty}$, then \mathcal{U} is a subquotient of \mathcal{V}^π .*

Proof. We must show that σ is a \mathcal{C} -subquotient of $\phi \otimes \pi$. Since $\bar{\sigma}$ is a S^∞ -subquotient of π , $\phi \otimes \bar{\sigma}$ is a \mathcal{C} -subquotient of $\phi \otimes \pi$ by Lemma 4.3, and

so we need only show that σ is a \mathcal{C} -subquotient of $\phi \otimes \bar{\sigma}$. Let $\rho : E_n \rightarrow X_n$ be the quotient map intertwining $\bar{\sigma} = \sigma|_{S^\infty}$ with $\bar{\sigma}$, let $z : E_n \rightarrow \mathcal{S}_n A^* \otimes E_n$ be the \mathcal{C} -map from the proof of Theorem 8.1, and let $\bar{z} : E_n \rightarrow \mathcal{S}_n A^* \otimes X_n$ be $(\mathbf{1} \otimes \rho) \circ z$. By Lemma 4.3, $\mathbf{1} \otimes \rho$ is a \mathcal{C} -map from $\phi \otimes \bar{\sigma}$ to $\phi \otimes \bar{\sigma}$, and so \bar{z} is a \mathcal{C} -map from σ to $\phi \otimes \bar{\sigma}$. Now $\bar{z}^T : \mathcal{S}_n A \otimes X_n^* \rightarrow E_n^*$ is just z^T , and so $\bar{z}^T(\mathcal{S}_n A \otimes X_n^*) = \Sigma^T(\mathcal{S}_n A)X_n^*$. The assumption that $\sigma^*(\mathfrak{U}(\mathfrak{a}))X_n^* = E_n^*$ therefore gives that \bar{z}^T is surjective, and so \bar{z} is injective and σ is actually a \mathcal{C} -subobject of $\phi \otimes \bar{\sigma}$. \square

9 An Example

We will now apply our main theorem to the example of Rideau's mentioned in the introduction. Let $A = \mathbb{R}^{1,3}$, and let $H = \mathrm{SL}_2\mathbb{C}$ act on A via the double cover of the identity component of $\mathrm{SO}(1, 3)$ by H . Then G is the Poincare group, and we take \mathcal{O} to be the forward light cone. Let F_0 be the trivial bundle $\mathcal{O} \times \mathbb{C}$, let F_1 be the complexified tangent bundle $T_c\mathcal{O}$, and let \mathcal{U}^i be the canonical representation of G in $\mathcal{D}F_i$ for $i = 0, 1$. Let σ_i be the little representation of S in $F_i^*(p_0)$, so that \mathcal{U}^i is induced from the representation $\sigma^* \otimes e^{ip_0}$ of S' . In [9], Rideau studied indecomposable representations \mathcal{U} of G acting in $\mathcal{D}F_0 \oplus \mathcal{D}F_1$ that leave $\mathcal{D}F_1$ invariant and define \mathcal{U}^0 and \mathcal{U}^1 in the subquotients. He found that there is a bijection from the set of equivalence classes of these representations to the projective complex plane \mathbb{P}^1 , which is described in the notation of this paper on page 663 of [4]. There we classified the objects σ (actually, $\eta(\sigma)$) of \mathcal{C} such that $\sigma = \mathcal{F}\mathcal{U}$ for some such \mathcal{U} , indexing them by elements (p, x) of \mathbb{C}^2 . Write $\sigma_{p,x}$ for the object of \mathcal{C} with index (p, x) , and let $\mathcal{U}_{p,x}$ be a representation in $\mathrm{Ext}_G^{\mathcal{C}}\mathcal{O}$ with $\mathcal{F}\mathcal{U}_{p,x} = \sigma_{p,x}$. We found that $\mathcal{U}_{p,x} \cong \mathcal{U}_{p',x'}$ if and only if $[p, x] = [p', x']$ in \mathbb{P}^1 or both indices are $(0, 0)$, and that $\mathcal{U}_{p,x}$ decomposes if and only if $(p, x) = (0, 0)$. Furthermore, we saw that $\mathcal{U}_{0,x}$ is induced and is isomorphic to the natural representation on $\mathcal{D}\mathcal{O} \otimes A$ if $x \neq 0$, and that $\mathcal{U}_{p,x}$ is not induced if $p \neq 0$. Here we will prove that for all (p, x) , $\mathcal{U}_{p,x}$ has the type of geometric realization sought after in this paper. This result was first obtained by Cassinelli, Truini, and Varadarajan [2].

Proposition 9.1 *For all (p, x) in \mathbb{C}^2 , $\mathcal{U}_{p,x}$ is isomorphic to a subquotient of $\mathcal{V}^{(1)} \otimes (\pi \oplus \mathbf{1})$, where $\mathbf{1}$ is the trivial representation of H on \mathbb{C} and π is the representation of H on A .*

Proof. We will use Corollary 8.3, and for brevity we will use the notation from Section 6 of [4]. Fix (p, x) , and write σ for $\sigma_{p,x}$ and η for $\eta(\sigma_{p,x})$. Since $E_{1,1}$ is linearly isomorphic to $M_{1,1}$ and σ acts in $E_{1,1}$ (see Section 2 of this paper), we may transport σ to $M_{1,1}$. Here $M_{1,1} = L_0 \oplus (L_0 \otimes \mathfrak{t}) \oplus L_1$, where L_0 has basis f_0^0 , L_1 has basis f_1^{-2}, f_1^0, f_1^2 , and \mathfrak{t} has basis M_+, M_-, M_3 .

In order to describe the S'^∞ -action of σ , we must give the actions of \mathfrak{s}' and \mathfrak{t} . The space $E_{0,0} = M_{0,0}$ is just L_0 , and it is only here that $\sigma|_{\mathfrak{t}}$ is defined. A

review of the definition of the S'^∞ -structure of objects of \mathcal{C} shows that $M \in \mathfrak{r}$ acts by right multiplication: $\sigma_M f_0^0 = -f_0^0 \otimes M$. The \mathfrak{s}' -action follows from the formulae for the action of η given in [4]; we will simply state the relevant facts. The flag $L_0 \otimes \mathfrak{r} \supset 0$ is an S^∞ -subrepresentation of the representation $\sigma|_{S^\infty}$ on $M_{1,1} \supset L_0$ which has the quotient property. Keep in mind that when we speak of S'^∞ , S' is being viewed as a subgroup of G , while when we speak of S^∞ , S is being viewed as subgroup of H . Writing $X_1 \supset X_0$ for the quotient flag, we find that the quotient S^∞ -representation is isomorphic to one of two representations of S^∞ , depending on whether x is 0 or not. In both cases, \mathfrak{h} acts by 0 on X_0 . If $x \neq 0$, the S -action on X_1 is isomorphic to π , and if $x = 0$ it is isomorphic to $\sigma_0 \oplus \sigma_1$. In either case, it follows from Lemma 1.2 that $X_1 \subset X_0$ is isomorphic to a subquotient of the restriction of $\pi \oplus \mathbf{1}$ to a representation of S^∞ . We leave the reader to use the formulae for $\eta|_{\mathfrak{a}}$ from [4] to check that X_1^* generates E_1^* under the action of A , and then apply Corollary 8.3 to complete the proof. \square

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