Complex Analysis Fact Sheet

(§6.3) ① If C a closed curve, f C^ω on and in C, then $\oint_C f\,dz = 0$.

② If C, C' curves from a to b, f C^ω on and in $C - C'$, then $\int_C f\,dz = \int_{C'} f\,dz$.

③ If C simple closed positive, f C^ω on and in C except at $z_1, \cdots, z_r$ in C,

then $$\oint_C f\,dz = \sum_{k=1}^{r} \oint_{C_\varepsilon(z_k)} f\,dz.$$

(§6.4) ④ If f is C^ω on a simply connected domain D (D has no holes) and F is any antiderivative of f that is C^ω on D ($F' = f$), then

$$\int_C f\,dz = F(b) - F(a)$$ for <u>all</u> curves C from a to b in D.

(§6.5) ⑤ If C is a simple closed positive curve around α and f is C^ω on and in C, then

$$\oint_C \frac{f(z)\,dz}{z-\alpha} = 2\pi i\, f(\alpha), \qquad \oint_C \frac{f(z)\,dz}{(z-\alpha)^{n+1}} = 2\pi i\, \frac{f^{(n)}(\alpha)}{n!}.$$

(§6.6) ⑥ If f C^ω on and in $C_R(\alpha)$, then $f(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha + Re^{i\theta})\,d\theta$.

⑦ If f is C^ω on a domain D, then $|f|$ has no interior maxima in D.

⑧ If f C^ω on and in $C_R(\alpha)$, then $|f^{(n)}(\alpha)| \le \frac{n!}{R^n} \max_{C_R(\alpha)} |f|$.

⑨ Bounded entire $\Rightarrow$ constant.

(§7.2) ⑩ f C^ω on $D_\varepsilon(\alpha) \Rightarrow f = \sum_0^\infty c_k (z-\alpha)^k$, $c_k = \frac{f^{(k)}(\alpha)}{k!}$.

ROC is R = Distance from α to nearest genuine singularity.
Formula valid on $D_R(\alpha)$.

(§7.3) ⑪ f C^ω on $\{z : r < |z-\alpha| < R\} \Rightarrow f = \sum_{-\infty}^{\infty} c_k (z-\alpha)^k$ there.

(§7.4) ⑫ Suppose f C^ω on $D^*_\varepsilon(\alpha)$, $f = \sum_{-\infty}^{\infty} c_k (z-\alpha)^k$ there.

- If $c_k = 0$ for $k < n$, $c_n \ne 0$, f has an order n zero at α.
  (An order $-n$ zero is an ord n pole.)

- If $c_k \neq 0$ for ∞ly many negative k, f has an essential singularity.

- If f, g have ord n and m zeroes at α, then fg has an ord $n+m$ zero and $1/f$ has an order $-n$ zero.

- If f has an ord n zero, then $f = (z-\alpha)^n g$, $g(\alpha) \neq 0$, g C^ω on $D_\varepsilon(\alpha)$

(13) Suppose f C^ω at α, $f(\alpha) = 0$. Then the zero at α is of order k, where k is the least pos integer such that $f^{(k)}(\alpha) \neq 0$.

(§8.1) (14) If f C^ω on $\bar{D}^*_\varepsilon(\alpha)$, $f = \sum_{-\infty}^{\infty} c_k (z-\alpha)^k$, then $\operatorname{Res}_\alpha f = c_{-1}$. In this case

$$\oint_{C_\varepsilon(\alpha)} f\,dz = 2\pi i \operatorname{Res}_\alpha f.$$

If f is C^ω on C and in C except at $\alpha_1, \dots, \alpha_r$, then $\oint_C f\,dz = 2\pi i \sum_1^r \operatorname{Res}_{\alpha_k} f$. ($C$ simple pos.)

(15) If f has an order k pole at α, $\operatorname{Res}_\alpha f = \dfrac{1}{(k-1)!} \dfrac{d^k}{dz^k}\Big|_\alpha \left((z-\alpha)^k f\right)$.

(§8.2) (16)
$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta = \oint_{C_1(0)} F\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz}.$$

(§8.3-4) (17)
$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \oint_{-R \frown R} f(z)\,dz$$
if f C^ω and $\left|\frac{f}{z}\right| \to 0$ as $\underbrace{|z| \to \infty}_{\text{Upper half plane}}$.

Series $\dfrac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$, $e^z = 1 + z + \dfrac{z^2}{2!} + \dfrac{z^3}{3!}$, $\sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - \dfrac{z^7}{7!} + \cdots$,

$\cos z = 1 - \dfrac{z^2}{2} + \dfrac{z^4}{4!} - \cdots$