Math 4520 **The Complex Numbers** Conley

The set of complex numbers $\mathbf{C}$ is defined to be $\{a+ib : a, b \in \mathbf{R}\}$. Here $\mathbf{R}$ denotes the real numbers and the symbol "$\in$" means "in," so a and b are any real numbers. As usual, $i = \sqrt{-1}$. You should think of the complex number $a+ib$ as the point in the plane with coordinates (a, b). This allows us to draw pictures of complex numbers in the "complex plane."

Definitions. Let $z = a + ib$ be in $\mathbf{C}$.

1. The *real part* of z is $\mathrm{Re}(z) = a$, and its *imaginary part* is $\mathrm{Im}(z) = b$. If $b = 0$ we say that z is real; note that then it lies on the horizontal axis in the complex plane. If $a = 0$ we say that z is purely imaginary; here it lies on the vertical axis.
2. The *conjugate* of z is $\overline{z} = a - ib$.
3. The *magnitude* of z is $|z| = \sqrt{a^2 + b^2}$. I will often use the words "length" or "absolute value" in place of "magnitude." Note that $|z|$ is the distance from the point (a, b) to the origin $(0, 0)$.
4. We will write $\angle(z)$ for the angle from the horizontal axis to the point (a, b), measured counter clockwise, *in radians*. This is not standard notation, but it will be convenient for us. Note that $\tan(\angle z) = b/a$.

Facts. Let $z = a + ib$ and $w = x + iy$ be in $\mathbf{C}$.

1. Complex numbers can be multiplied using the "foil" process and the fact that $i^2 = -1$:

$$zw = (a + ib)(x + iy) = (ax - by) + i(ay + bx).$$

   You should check that $|zw| = |z||w|$.

2. Here are some very important properties of conjugation which you should check:
   a) $\overline{\overline{z}} = z$, i.e., conjugating twice gives you z back.
   b) $\overline{zw} = \overline{z} \cdot \overline{w}$.
   c) $z\overline{z} = |z|^2$, i.e., $z\overline{z} = a^2 + b^2$.
   d) $\overline{z^n} = \overline{z}^n$ for any $n \in \mathbf{Z}$. Here $\mathbf{Z}$ denotes the set of integers $\{\ldots, -1, 0, 1, 2, \ldots\}$. As usual, z^{-n} means $1/z^n$; see item 3 for a review of complex division.
   e) $\mathrm{Re}(z) = (z + \overline{z})/2$ and $\mathrm{Im}(z) = (z - \overline{z})/2i$.

3. The complex fraction z/w may be rewritten as $z\overline{w}/w\overline{w}$. This is useful because the new denominator $w\overline{w}$ is the real number $|w|^2 = x^2 + y^2$. For example, $1/z$ may be written as $\overline{z}/|z|^2$.

4. Here is a geometric way to think of complex multiplication. We will see in Problem Set 1 that if z is the complex number of length r and angle θ, and w is the complex number of length s and angle ϕ, then zw is the complex number of length rs and angle $\theta + \phi$.

5. Here are some extremely important facts about the complex exponential function. Remember that if x is any real number, $e^x = \sum_{n=0}^{\infty} x^n/n!$, i.e., $e^x = 1 + x + x^2/2 + x^3/3! + \cdots$. This series converges for any x, because $n!$ grows so rapidly (for a proof, apply the ratio test). We define the *complex exponential* e^z using the same series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots.$$

   Now it is not so important that you be comfortable with this power series, but you will need to know the following properties of e^z. When convenient, we write $\exp(z)$ instead of e^z.

a) If $\theta \in \mathbf{R}$, $e^{i\theta} = \cos\theta + i\sin\theta$. This is the famous formula of DeMoivre.

b) $e^{z+w} = e^z e^w$, $e^{-z} = 1/e^z$, and $(e^z)^w = e^{zw}$, just as for ordinary exponential functions.

c) $re^{i\theta}$ is the complex number whose length is r and whose angle is θ. Any complex number can be written in this form; here r and θ are called its polar coordinates.

d) If $z = a + ib$, then $|e^z| = e^a$, $\angle(e^z) = b$, and $\overline{e^z} = e^{\overline{z}}$.

e) $\overline{e^{i\theta}} = e^{-i\theta} = \cos\theta - i\sin\theta$, so we get

$$\cos\theta = \mathrm{Re}(e^{i\theta}) = (e^{i\theta} + e^{-i\theta})/2, \qquad \sin\theta = \mathrm{Im}(e^{i\theta}) = (e^{i\theta} - e^{-i\theta})/2i.$$

6. Every complex number has exactly n distinct n^{th} roots. The n^{th} roots of 1 are the powers $1, \delta_n, \delta_n^2, \delta_n^3, \ldots, \delta_n^{n-1}$, where $\delta_n = \exp(2\pi i/n)$. The n^{th} roots of $re^{i\theta}$ are $\delta_n^k r^{1/n}\exp(i\theta/n)$, for $k = 0, \ldots, n-1$.

7. Sequences and series of complex numbers behave a lot like sequences and series of real numbers.

a) If $z_1, z_2, z_3, \ldots$ is a *sequence,* we say that it *converges* to the complex number z if $\lim_{n\to\infty} |z_n - z| = 0$. Sequences which do not converge to anything are said to *diverge.*

b) If $\sum_{n=0}^{\infty} z_n$ is a *series,* i.e., an infinite sum, we say that it converges to the complex number s if the sequence of its *partial sums* $s_n = \sum_{k=0}^{n} z_k$ converges to s. If the partial sums s_n do not converge, we say that the series diverges.

c) The *ratio test* sometimes tells you if a series converges or diverges. It works like this: if you have a series $\sum_{n=0}^{\infty} z_n$, let $\rho_n = |z_n|/|z_{n-1}|$, and let $\rho = \lim_{n\to\infty} \rho_n$ if the limit exists. Then the series converges if $\rho < 1$ and it diverges if $\rho > 1$. However, the test gives no information if $\rho = 1$ or ρ does not exist.

d) One type of series that comes up often is the "p-series" $\sum_{n=1}^{\infty} 1/n^p$, where p is a positive integer. Here the ratio test gives no information because $\rho = 1$, but the integral test from Calculus tells us that the series diverges for $p = 1$ and converges for $p = 2, 3, \ldots$.

8. Throughout this course we will be studying *complex-valued* functions $f : \mathbf{R} \to \mathbf{C}$, mapping the real numbers to the complex numbers. Since complex numbers can be graphed as points in the complex plane, these functions can be graphed parametrically. For example you should check that the graph of $f(t) = e^{it}$ is the unit circle.

Such functions can also be differentiated and integrated according to the usual rules. For example, if $\lambda \in \mathbf{R}$ then

$$\frac{d}{dt}e^{i\lambda t} = i\lambda e^{i\lambda t}, \text{ and } \qquad \int_0^x e^{i\lambda t}dt = \left.\frac{e^{i\lambda t}}{i\lambda}\right|_0^x = \frac{e^{i\lambda x} - 1}{i\lambda}.$$