Math 4520 **Exam 1 Review Sheet** Conley

Here are summaries of the text sections to be covered:

1.2. Be able to perform arithmetic in $\mathbb{C}$: addition, subtraction, multiplication, division. Know $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$. Know conjugation, $\overline{z}$, and its properties, *e.g.,* $\overline{zw} = \overline{z}\,\overline{w}$.

1.3. Know $|z|$, the triangle inequality, and the vector pictures of addition, subtraction, $\overline{z}$, and $|z|$.

1.4. Know the polar form $z = r \operatorname{cis} \theta$ of z, *i.e.,* how to compute the polar form from the rectangular form and vice versa. Know arg and Arg. Know the geometry of multiplication, and its variations for division and taking exponents: multiplication in $\mathbb{C}$ multiplies lengths and adds angles.

1.5. Be able to compute z^n and z^{-n} for large n, using the polar form. Be able to compute all n n^{th} roots of any complex number.

1.6. Parametrized curves, $D_\epsilon(z)$, $\overline{D}_\epsilon(z)$, $D^*_\epsilon(z)$, $C_\epsilon(z)$, open and closed sets, connected and bounded sets, regions, and domains.

2.1. Functions from $\mathbb{C}$ to $\mathbb{C}$, and their real and imaginary parts $f = u + iv$, linear maps.

2.2. The behaviour of the functions z^n and $z^{1/n}$, especially z^2 and $z^{1/2}$. Know the images of annular wedges under all of these functions. Know the images and preimages of horizontal and vertical lines (and the regions to their right and left) under z^2 and $z^{1/2}$.

2.3. Limits and continuous functions. Be able to compute $\lim_{z \to z_0} f(z)$ for problems like those on the homework. Know the properties of limits and continuous functions stated in the theorems in the section.

2.4. Branches (skip Riemann surfaces).

2.5. The effect of $h(z) = z^{-1}$ on lines and circles and their interiors and exteriors.

3.1. Differentiation: the definition of $f'(z)$. Know that all the calculus rules still hold. Know the definitions of differentiable, analytic, and entire, and L'Hôpital's rule.

3.2. The Cauchy-Riemann equations: $f = u + iv$ is differentiable at z if and only if $u_x = v_y$ and $u_y = -v_x$ at z. Know that if f is analytic on a domain and either $|f|$ is constant or $f' \equiv 0$, then f is constant. Skip Theorem 3.5.

3.3. Harmonic functions, harmonic conjugates, and their relation to analytic functions. Be able to find harmonic conjugates.

Here are some practice problems, in roughly the same format as the exam. On the exam, you should show all work, and when asked to draw a picture or a graph, label its intercepts and corners. You will be allowed to do the problems in any order, so when you get the exam, look through it, decide which ones look the easiest, and start with those!

Problem 1. Let $z = -8(1 + i\sqrt{3})$.

(a) Find the polar form of z.

(b) Compute z^{-20} in both polar and rectangular form. (Leave powers of 2 in your answer.)

(c) Find the polar and rectangular forms of all fourth roots of z.

Problem 2. Find linear functions on $\mathbb{C}$ with the following properties:

(a) $L_1(z)$ is twice as far from 0 as z, and has the same argument.

(b) $L_2(z)$ is z rotated $3\pi/4$ clockwise around 0.

(c) $L_3(z)$ is z moved a distance $\sqrt{2}$ "northeast".

(d) $L(z)$ magnifies $\mathbb{C}$ by 2 (fixing 0), then rotates it $3\pi/4$ clockwise, then moves it $\sqrt{2}$ northeast.

Problem 3. Let $p(x+iy) = x^2 - y^2 + iy$.

(a) Where is p differentiable?

(b) Where is p analytic?

(c) Find $p'(z_0)$ at all points z_0 where p is differentiable.

Problem 4. (a) Which function is harmonic?

$$u_1(x,y) = y^3 - 3x^2y + x - y, \quad u_2(x,y) = y^3 - 3xy^2 + x^2 - y.$$

(b) Call the harmonic one u, and find v such that $u + iv$ is analytic.

Problem 5. Let $W = \{r \operatorname{cis} \theta : 0 < r < 2,\ -\frac{\pi}{2} < \theta < \frac{\pi}{3}\}$.

(a) Draw W and $-W$ (specify which is which).

(b) Let $s(z) = z^2$. On separate plots, draw $s(W)$ and $s(-W)$.

(c) Let $r(z)$ be the prinicipal root $z^{1/2}$. On separate plots, draw $r(W)$ and $r(-W)$.

Problem 6. Let $s(z)$ and $r(z)$ be as in Problem 5. Let $H = \{z : \operatorname{Re}(z) > 2\}$.

(a) Draw $s(H)$ and give its equation.

(b) Draw $r(H)$ and give its equation.

Problem 7. Let $R(z) = 1/z$. Let W and H be as in Problems 5 and 6.

(a) Draw $R(W)$.

(b) Draw $R(H)$.

(c) Draw $R\big(D_1(2)\big)$.

Extra Credit.

(a) Draw $r(-H)$.

(b) Draw $R\big(H \cap D_1(2)\big)$.

(c) Draw $s\big(D_{\sqrt{2}}(2)\big)$. Hint: first draw $D_{\sqrt{2}}(2)$ together with $y = \pm x$.

(d) Draw $(L_1)^3$ and find its angle of self-intersection.