Math 4500 **Exam 1 Review Sheet** Conley, Spring 2002

The exam is on Tuesday, Feb. 19. Below are summaries of the text sections to be covered, with some practice problems. I suggest that you work backwards, first doing the problems from the sections covered last, as the test will cover those sections more thoroughly than the earlier ones. We will spend all of Thursday reviewing, so try the practice problems before then and ask questions. Also go over your old problem sets and ask questions.

Announcement. In order to give you time to look at this sheet by Thursday, I am changing the due date of Problem Set 4 to the day of the exam.

1. Sets: unions, intersections, products. Statements: contrapositives, converses, negations. Practice: 2bfn, 5c, 7E, 10d.

2. Functions: surjectivity, injectivity, composition, inverses. The notion of images and inverse images is very important. Practice: 2g, 4e, 5c.

3. Equivalence and order relations. Know what the LUB property is! Practice: 5a, 6, first question in 15b.

4. Properties of the complete ordered field $\mathbb{R}$ and its positive integers $\mathbb{Z}_+$. Practice: 6 (try to word your induction argument as cleanly as you can).

5. Finite and countable Cartesian products. Practice: 2a, 4e.

6. Finite sets. This is a good warm-up for Section 7. Practice: 4, 6a.

7. This is an important section. You should be able to prove that sets are countable without "getting your hands dirty" with actual bijections, using Results 7.1, 7.4, and 7.6. Theorems 7.7 and 7.8 give important examples of uncountable sets. Practice: 5bcdh.

9. The axiom of choice. Know Theorem 9.1, its proof, and the statement of Lemma 9.2 (you can skip its proof and the actual axiom). Practice: 2ac

10. Well-ordered sets: the well-ordering theorem. Be able to do problems like proving that the sets in Problem 10.3 are well-ordered. Practice: Show $\mathbb{Z}_+ \times \mathbb{Z}_+$ is well-ordered in the dictionary order, and then do 4. Spend some time on 4, it is tricky. Hint: suppose that $B \subset A$ has no least element. Choose $b_{-1} \in B$. Then choose $b_{-2} \in B$ less than b_{-1} (and say why this is possible). Proceed by induction. As always, try to word your proof cleanly.

12-13. The definitions of topologies and bases. The first three lemmas of Section 13 are very useful. Know what is meant by "at least as fine as" and "strictly finer." Practice: 2, 4a.

14-16. The order, product, and subspace topologies. You should know their definitions and a little about the relations between them. Practice: 2, 5a, 10. Like Problem 10.4, Problem 16.10 is worth some time. Hint: by considering open sets containing $(\frac{1}{2}, 1)$ in each topology, I think you can show that the first two are incomparable and the third is strictly finer than both of them. The following "warm-up" problem may help: compare the dictionary order topology, subspace topology (coming from $\mathbb{R} \times \mathbb{R}$ in the standard product topology), and discrete topology on $\mathbb{Z} \times \mathbb{Z}$.

Math 4500 **Exam 2 Review Sheet** Conley, Spring 2002

The exam is on Tuesday, April 2. This Thursday will be a review day. Below are summaries of the text sections to be covered. For review problems you should go over Problem Sets 5-7 and redo as many as you can, in addition to doing Problem Set 8. I suggest the following from Sets 5-7: Section 17: 9, 11, 12, 16; Section 18: 7a, 8, 10; Section 20: 2, 3a.

17 Closed sets and limit points. You should know all of this section: the definition and properties of closed sets, the properties of the closure $\overline{A}$ and the limit point set A', convergent sequences, and T_1 and T_2 (Hausdorff) spaces.

18 Continuous functions. Know all of this section: the various definitions of C^0 (continuous) functions, rules for constructing them, C^0 functions into products, and homeomorphisms.

20 The metric topology. Know the definitions of metric spaces and their topologies, and the standard examples of metrics on $\mathbb{R}$ and $\mathbb{R}^2$. You may skip Lemma 20.2 and everything after Theorem 20.3.

21 More on the metric topology. This section relates the metric topology to other topological properties we have studied. You should know all of it up through Theorem 21.5, particularly Theorem 21.1 on ϵ-δ continuity and Lemma 21.2 and Theorem 21.3 on convergent sequences.

Math 4500 **Final Exam Review Sheet** Conley, Spring 2002

The final exam is on Tuesday, May 7, from 10:30 to 12:30. You may come by my office with questions at any time before then. Below are summaries of the text sections to be covered. Those sections not listed will not be covered explicitly, all though some of them are prerequisites for later sections. Practice problems drawn from old problem sets are given. Problem Set 12 will do as practice for the last three sections; you may turn it in when you take the final if you wish.

Note: Some of the exam problems will ask you to prove some of the easier results from the book (in which case you may quote any other book result). The results below are all at about the right level, so they would be good to review. There will be some problems dealing with the specific spaces $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}_Z$, and $\mathbb{R}_\ell$.

Results to review: 13.1, 15.1, 16.2, 17.2, 17.6, 17.10, 18.3, 20.1, 21.2, 23.3, 24.3 for $\mathbb{R}$, 26.2, 26.5, 26.8, 27.3 for $\mathbb{R}$, 28.1.

12-13. The definitions of topologies and bases. The first three lemmas of Section 13 are very useful. Know what is meant by "at least as fine as" and "strictly finer."

Practice: 8.

15-16. Know the product and subspace topologies and the relations between them. The only order topology on the exam will be the usual one on $\mathbb{R}$.

Practice: 3, 4, 6.

17 Closed sets and limit points. You should know all of this section except Theorem 17.11: the definition and properties of closed sets, the properties of the closure $\overline{A}$ and the limit point set A', convergent sequences, and T_1 and T_2 (Hausdorff) spaces.

Practice: 2, 3, 6b, 13, 16 for $\mathbb{R}$, $\mathbb{R}_Z$, and $\mathbb{R}_\ell$.

18 Continuous functions. Know all of this section: the various definitions of C^0 (continuous) functions, rules for constructing them, C^0 functions into products, and homeomorphisms.

Practice: 4, 8, 10.

20 The metric topology. Know the definitions of metric spaces and their topologies, and the standard examples of metrics on $\mathbb{R}$ and $\mathbb{R}^2$. You may skip Lemma 20.2 and everything after Theorem 20.3.

Practice: 1a.

21 More on the metric topology. This section relates the metric topology to other topological properties we have studied. You should know all of it up through Theorem 21.5, particularly Theorem 21.1 on ϵ-δ continuity and Lemma 21.2 and Theorem 21.3 on convergent sequences.

Practice: 8, 10.

23 Connected spaces. Know all of this section except Theorem 23.4. It defines connectedness, derives some basic properties of the definition, proves that continuous functions map connected sets to connected sets, and proves that products of connected sets are connected.

Practice: 3.

24 Connected subspaces of $\mathbb{R}$. You should know all of this section, except that you should consider only $\mathbb{R}$ rather than arbitrary ordered spaces. The main results are the fact that intervals in $\mathbb{R}$ are connected and the intermediate value theorem (IVT).

Practice: 2.

26 Compact spaces. You should know all of this up through Lemma 26.8. It begins with the definition of compactness and relations between compact sets and closed sets. Then it proves that continuous functions map compact sets to compact sets and products of compact sets are compact. Among Results 26.6-8, the most important one for you to know how to prove is 26.8, the tube lemma.

27 Compact subspaces of $\mathbb{R}$. You should know all of this up through Theorem 27.6, except that as in Section 24 you should consider only $\mathbb{R}$ rather than arbitrary ordered spaces. It proves that subsets of $\mathbb{R}^n$ are compact if and only if they are closed and bounded, continuous functions mapping compact spaces into $\mathbb{R}$ assume their extreme values, open covers of compact metric spaces have Lebesgue numbers, and continuous functions from compact metric spaces to arbitrary metric spaces are uniformly continuous.

28 Limit point compactness. You should know the definition of limit point and sequential compactness, the fact that compactness implies limit point compactness, and the fact that in a metric space, limit point compactness implies sequential compactness.

Math 4500 **Exam 1** Conley, Feb. 19, 2002

There are 11 problems, 5 from each chapter and one (difficult) extra one. Do any 7 of the first 10 problems, 3 from one chapter and 4 from the other. Each of your best 7 problems will count for 15 points. Any work on additional problems will count as extra credit. Notes, books, and calculators are not allowed. Throughout the test,

$$I \text{ denotes the closed unit interval } [0,1] \subset \mathbb{R}.$$

Important: *All answers must be proven,* but you may quote any results from the book in your proofs unless the problem says otherwise. This test is very long, even with only 7 of the 10 problems to do! Do not lose your morale - if you do 4 problems completely correctly you have done well. The easier ones in each section come first, so begin by looking over the whole test and choosing the problems you want to try.

Chapter II: Problems 1-5.

Problem 1. Which of the following is a basis for a topology on $\mathbb{R}$?

(a) $\{[a, a+1) : a \in \mathbb{R}\}$, (b) $\{[n, n+1) : n \in \mathbb{Z}\}$,

(c) $\{[n, n+1] : n \in \mathbb{Z}\}$, (d) $\{(n, n+1) : n \in \mathbb{Z}\}$.

Problem 2. Recall that $\mathbb{R}_Z$ is $\mathbb{R}$ with the topology consisting of $\emptyset$ and all sets A such that $\mathbb{R} - A$ is finite. Prove that the collection of sets $\{\mathbb{R} - \{a\} : a \in \mathbb{R}\}$ is a ~~basis~~ subbasis of $\mathbb{R}_Z$.

Problem 3. Let $a \in \mathbb{R}$ be arbitrary. Which of the following sets is open in $\mathbb{R}_\ell$? (Recall that intervals of the form $[a, b)$ are a basis of $\mathbb{R}_\ell$.)

(a) $[a, \infty)$, (b) $(-\infty, a]$, (c) (a, ∞), (d) $(-\infty, a)$.

Problem 4. Recall that the collection of all open rectangles $(a, b) \times (c, d)$ in $\mathbb{R}^2$ is a basis of the standard product topology on $\mathbb{R}^2$. For any point $x = (x_1, x_2) \in \mathbb{R}^2$ and any $r > 0$ in $\mathbb{R}$, let $B_r(x)$ be the open disc around x of radius r (the points y such that $|y - x| < r$). Prove that the collection of all open balls $B_r(x)$ is also a basis of the product topology on $\mathbb{R}^2$ (you need not include much algebraic detail).

Problem 5. Consider $(\mathbb{R} \times \mathbb{R})_<$, the dictionary order topology on $\mathbb{R}^2$. The subset $\mathbb{R} \times I$ of $\mathbb{R}^2$ has both the subspace topology $(\mathbb{R} \times I)_{\text{sub}}$ and the order topology $(\mathbb{R} \times I)_<$. Compare them. Is one strictly finer than the other?

(OVER)

Chapter I: Problems 6-10.

Problem 6. Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^2$.

(a) Find the preimage $f^{-1}(I)$ of I.

(b) Find the equivalence classes of the equivalence relation "$\sim$" on $\mathbb{R}$ defined by $x \sim y$ if $f(x) = f(y)$.

Problem 7. Consider the following ordered sets (the second two in the dictionary order). Which have the LUB property?

$$\text{(a)}\ \ (0,1), \qquad \text{(b)}\ \ (0,1)^2, \qquad \text{(c)}\ \ I^2.$$

Problem 8. Consider $\mathbb{Z}_+ \times \mathbb{Z}_+$ in the dictionary order.

(a) Is $\mathbb{Z}_+ \times \mathbb{Z}_+$ well-ordered?

(b) Do $\mathbb{Z}_+$ and $\mathbb{Z}_+ \times \mathbb{Z}_+$ have the same order type?

Problem 9. You may assume that $\mathbb{Z}^n$ is countable. Let $\mathbb{Z}_0^\omega$ be the subset of $\mathbb{Z}^\omega$ of sequences ending in an infinite string of zeroes, and let $\mathbb{Z}_c^\omega$ be the subset of sequences ending in an infinite constant string.

(a) Find a surjection from $\cup_{n=1}^{\infty} \mathbb{Z}^n$ to $\mathbb{Z}_0^\omega$. Then use it to prove that $\mathbb{Z}_0^\omega$ is countable.

(b) Prove that $\mathbb{Z}_c^\omega$ is countable.

(c) Prove directly that $\mathbb{Z}^\omega$ is uncountable, without using any theorems from the book.

Problem 10. Let A be a well-ordered set.

(a) Prove that all elements of A have an immediate successor.

(b) Prove that if all elements of A have an immediate predecessor, then A has the order type of $\mathbb{Z}_+$. Hint: define an injection from $\mathbb{Z}_+$ to A, and prove by contradiction that it is a surjection.

(c) Prove that the minimal uncountable well-ordered set S_Ω has an element without an immediate predecessor. You may use part (b) for this even if you could not do it.

Extra Problem.

(a) Repeat Problem 5 for $I \times \mathbb{R}$. You may not quote Theorem 16.4.

(b) Compare the topologies $\mathbb{R}_\ell \times \mathbb{R}$ and $(\mathbb{R} \times \mathbb{R})_<$.

(c) Prove that S_Ω contains uncountably many elements with no immediate predecessor.

Math 4500 **Exam 2** Conley, April 2, 2002

There are 10 regular problems and 3 extra credit problems. Do any 2 from Section 17, any 3 from Section 18, and any 2 from Sections 20-21. Additional work will count as extra credit. Notes, books, and calculators are not allowed. All answers must be proven. In problems which ask you to prove results from the book, you must rewrite the proof.

SECTION 17: Closed sets. Do any 2.

Problem 1. Let X be a topological space containing an open set U and a closed set A. Prove that $A - U$ is closed.

Problem 2. Let $S = \{\frac{1}{2}, -\frac{1}{2}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{4}, -\frac{3}{4}, \ldots\}$. Find the closures of S in the following spaces:

(a) $\mathbb{R}$, (b) $\mathbb{R}_\ell$, (c) $\mathbb{R}_Z$.

Problem 3.

(a) Give the definition of T_2 (Hausdorff) spaces.

(b) Let X be a T_2 space. Prove that the diagonal $\Delta = \{(x, x) : x \in X\}$ is closed in X^2.

SECTION 18: C^0 (continuous) functions. Do any 3.

Problem 4. Let $f : \mathbb{R}_\ell \to \mathbb{R}$ be a C^0 function. Given $x \in \mathbb{R}_\ell$ and $\epsilon > 0$, prove that there exists $\delta > 0$ such that $f([x, x + \delta)) \subset (f(x) - \epsilon, f(x) + \epsilon)$.

Problem 5. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be C^0 functions.

(a) Prove that $A = \{x \in \mathbb{R} : f(x) \geq g(x)\}$ is closed. Hint: consider the function $f - g$.

(b) Prove that the function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \max\{f(x), g(x)\}$ is C^0.

Problem 6. Prove that a function $f : \mathbb{R}_Z \to \mathbb{R}_Z$ is C^0 if and only if $f^{-1}(\{x\})$ is finite for all $x \in \mathbb{R}_Z$.

Problem 7. Prove that a function $f : \mathbb{R}_Z \to \mathbb{R}$ is C^0 if and only if it is constant.

(OVER)

SECTIONS 20-21: Metric spaces. Do any 2.

Problem 8. Define $d : \mathbb{R}^2 \to \mathbb{R}$ to be $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$.

(a) Prove that d is a metric, and draw $B_d(0,1)$.

(b) Prove that the metric topology on $\mathbb{R}^2$ induced by d is the same as the product topology.

Problem 9. Let X be a metric space.

(a) Prove that X is T_2.

(b) Let A be a subset of X and suppose $x \in \overline{A}$. Prove that there exists a sequence $x_1, x_2, \ldots$ in A converging to x. Be sure to prove that your sequence converges to x.

Problem 10. Let X, d and Y, e be metric spaces. Prove that $f : X \to Y$ is continuous if and only if given $x \in X$ and $\epsilon > 0$, there exists $\delta > 0$ such that $f(B_d(x,\delta)) \subset B_e(f(x),\epsilon)$.

EXTRA CREDIT PROBLEMS.

Problem 11. Describe the closed sets in $\mathbb{R}^2_Z$. What is the closure of the diagonal Δ?

Problem 12. Let $f : \mathbb{R} \to \mathbb{R}$ be a bijective order-preserving function. Prove that it is a homeomorphism.

Problem 13. Define a function $\chi : \mathbb{Q} \to \mathbb{Q}$ as follows: $\chi(0) = 0$, and whenever m and n are odd, $\chi(2^k \frac{m}{n}) = 2^{-k}$.

(a) Prove that $d(p,q) = \chi(p-q)$ is a metric on $\mathbb{Q}$. Hint: For the triangle inequality, first show that it is enough to check that $\chi(a+b) \leq \chi(a) + \chi(b)$.

(b) Prove the "ultra-triangle inequality:" $\chi(a+b) \leq \max\{\chi(a), \chi(b)\}$.

(c) Describe the balls $B_d(0, 2^{-k})$ for $k \in \mathbb{Z}_+$.

HOMEWORK 9: for future reference, here is Homework 9. It is due Thursday, April 11.

Section 21: 6, 8 for $X = Y = \mathbb{R}$. *Section 23:* 2, 4 for $\mathbb{R}_Z$, 5, 7, 9, 12.

There are 10 regular problems and 3 extra credit problems. The regular problems are split into 5 groups of two. You should do a total of 7. Do at least 1 from each group and at least 3 from the last two groups, which are on connectedness and compactness. Additional work will count as extra credit. Notes, books, and calculators are not allowed. All answers must be proven. In problems which ask you to prove results from the book, you must rewrite the proof.

SECTIONS 12-16: Topology: Definitions and Examples

Problem 1.

(a) Give the definition of a basis $\mathcal{B}$ of a topology $\mathcal{T}$ on a set X.

(b) Let $\mathcal{B}$ be a basis of a topology $\mathcal{T}$ on X. Prove that a subset U of X is open in $\mathcal{T}$ if and only if it is a union of elements of $\mathcal{B}$.

Problem 2. Let X and Y be topological spaces.

(a) Given a subset S of X, define the subspace topology on S.

(b) Define the product topology on $X \times Y$.

(c) Let $[0,\infty)_{\mathrm{sub}}$ be $[0,\infty)$ in the subspace topology induced by $\mathbb{R}$, and let $([0,\infty)^2)_{\mathrm{sub}}$ be $[0,\infty)^2$ in the subspace topology induced by $\mathbb{R}^2$. Prove that the product topology $([0,\infty)_{\mathrm{sub}})^2$ and the subspace topology $([0,\infty)^2)_{\mathrm{sub}}$ both have basis

$$\{(a,b)\times(c,d) : 0 < a < b, 0 < c < d\} \cup \{[0,b)\times(c,d) : 0 < b, 0 < c < d\} \cup \{(a,b)\times[0,d) : 0 < a < b, 0 < d\}.$$

SECTIONS 17-18: Closed Sets, Limit Points, and Continuous Functions

Problem 3. Let Y be a subset of a topological space X. Prove that a subset B of Y is closed in the subspace topology on Y if and only if it is the intersection of Y with a closed subset C of X.

Problem 4. Let f and g be continuous functions from $\mathbb{R}^2$ to $\mathbb{R}$. Assume that $f = g$ on the unit circle $C = \{(x,y) : x^2 + y^2 = 1\}$. Define $h : \mathbb{R}^2 \to \mathbb{R}$ by $h(x) = f(x)$ for $x^2 + y^2 \leq 1$ and $h(x) = g(x)$ for $x^2 + y^2 > 1$. Prove that h is continuous.

SECTIONS 20-21: Metric Topologies

Problem 5. Let X be a space containing a subset A.

(a) Give the definition of convergent sequences in X.

(b) Suppose that x_n is a sequence in A converging to a point $\overline{x}$ in X. Prove that $\overline{x}$ is in $\overline{A}$.

(c) Assume that X is metric and $\overline{x}$ is in $\overline{A}$. Prove that A contains a sequence converging to $\overline{x}$.

Problem 6. Define a function $d : \mathbb{R}^2 \to \mathbb{R}$ as follows:

$$d(x, y) = y - x \text{ if } x \leq y \leq x + 1 \text{ and } d(x, y) = 1 \text{ otherwise.}$$

(a) Prove that the set $B_d(x, \epsilon) = \{y \in \mathbb{R} : d(x, y) < \epsilon\}$ is $[x, x + \epsilon)$ is $0 < \epsilon < 1$ and $\mathbb{R}$ otherwise.

(b) Prove that d satisfies the triangle inequality.

(c) Why doesn't this prove that $\mathbb{R}_\ell$ is a metric space?

SECTIONS 23-24: Connectedness

Problem 7. Suppose that A and B are connected subsets of a space X with non-empty intersection. Prove that $A \cup B$ is connected.

Problem 8. Prove that the image of a continuous function $f : \mathbb{R} - \{0\} \to \mathbb{R}_\ell$ is a set of either 1 or 2 elements. Hint: write $\mathbb{R} - \{0\}$ as a union of connected subsets.

SECTIONS 26-28: Compactness

Problem 9. Let $f : X \to Y$ be continuous, X compact. Prove that $f(X)$ is a compact subspace of Y.

Problem 10.

(a) Prove that $[0, 1]$ is not compact in $\mathbb{R}_\ell$.

(b) Prove that a set A in $\mathbb{R}$ is compact if and only if it is finite.

EXTRA CREDIT PROBLEMS.

Problem 11. What can you say about continuous functions from $\mathbb{R}_Z - \{0\}$ to $\mathbb{R}_\ell$?

Problem 12. Let B be a closed bounded subset of R^2. What can you say about the image of a continuous function from B to $\mathbb{R}_\ell$?

Problem 13. Which of the following sets in $\mathbb{R}^2_\ell$ is open? Which is closed? All of them are open unit discs with part of the boundary included.